Samadi et al. *Journal of Inequalities and Applications* (2022) 2022:123
https://doi.org/10.1186/s13660-022-02861-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Existence analysis on a coupled multiorder system of FBVPs involving integro-differential conditions

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**Abstract**

In this research study, we investigate the existence and uniqueness of solutions for a coupled multiorder system of fractional differential equations involving coupled integro-differential boundary conditions in the Riemann–Liouville setting. The presented results are obtained via classical Banach principle along with Leray–Schauder and Krasnosel'skiĭ's fixed-point theorems. Examples are included to support the effectiveness of the obtained results.

**MSC:** 26A33; 34A08; 34A60; 34B15

**Keywords:** Boundary value problems; Riemann–Liouville fractional derivative; Caputo fractional derivative; Systems of fractional differential equation; Fixed-point theory

## 1 Introduction and preliminaries

Fractional calculus is a powerful tool to express real-world problems than the integer-order differentiation. Consequently, fractional calculus has been used in several areas such as mathematics, physics, control systems, and other sciences; see, for example, [1–10] and related references therein. Indeed, fractional operators associated with fractional differential equations can describe some phenomena of nature more accurately than those of integer order. Accordingly, many researchers have studied FDEs with boundary/initial conditions, and the most important subject in the study of this field is showing the existence and uniqueness of a solution; see [3, 4, 6, 11–19]. Furthermore, an important class of applied analysis is presented by boundary value problems of FDEs. Many authors have considered the FDEs by using the Caputo or Riemann–Liouville derivative. Indeed, some new models have been developed by engineers and scientists that involve FDEs with Caputo or Riemann–Liouville derivatives. Besides, fractional differential equations involving a coupled system have nonlocal nature and applications in many real-world processes and synchronization of chaotic systems [20–27]. Hence many authors have studied coupled systems of fractional differential equations, and in this regard, we can find a lot of monographs [23–25, 27].

Niyom et al. [28] considered the following Riemann–Liouville fractional differential equation:

$$\begin{cases} \lambda D^{\overline{k}}(\check{p}(b)) + (1-\lambda)D^{\theta}(\check{p}(b)) = \widehat{\Upsilon}(b,\check{p}(b)), & b\in[0,T], \overline{k}\in(1,2], \\ \check{p}(0)=0, \quad \mu_1 D^{\gamma_1}\check{p}(T) + (1-\mu_1)D^{\gamma_2}\check{p}(T) = \delta_1, \end{cases} \tag{1.1}$$

and obtained some related results on the existence of a solution. Ntouyas et al. [29] changed this problem to the following form with integral conditions:

$$\begin{cases} \lambda D^{\overline{k}}(\check{p}(b)) + (1-\lambda)D^{\theta}(\check{p}(b)) = \widehat{\Upsilon}(b,\check{p}(b)), & b\in[0,T], \overline{k}\in(1,2], \\ \check{p}(0)=0, \quad \mu_2 I^{q_1}\check{p}(T) + (1-\mu_2)I^{q_2}\check{p}(T) = \delta_2. \end{cases} \tag{1.2}$$

Under similar assumptions, they proved some results in this regard. Recently, Chikh et al. [30], by mixing the above ideas and using standard fixed-point methods have studied the following multiorder Riemann–Liouville fractional differential equations in the context of the boundary conditions in the form of the linear combinations of unknown function:

$$\begin{cases} \lambda D^{\overline{k}}(\check{p}(b)) + (1-\lambda)D^{\theta}(\check{p}(b)) = \widehat{\Upsilon}(b,\check{p}(b)), & b\in[0,T], \overline{k}\in[2,3), \\ \check{p}(0)=0, \quad \mu_1 D^{\gamma_1}\check{p}(T) + (1-\mu_1)D^{\gamma_2}\check{p}(T) = \delta_1, \\ \mu_2 I^{q_1}\check{p}(T) + (1-\mu_2)I^{q_2}\check{p}(T) = \delta_1. \end{cases} \tag{1.3}$$

Inspired by the published papers on problems (1.1)–(1.3), in this direction, we investigate the existence and uniqueness of solutions for the multiorder Riemann–Liouville fractional BVPs (RL-FBVP) as the coupled fractional systems of the form

$$\begin{cases} \lambda_1 D^{\overline{k}_1}(\check{p}(b)) + (1-\lambda_1)D^{\theta_1}(\check{p}(b)) = f(b,\check{p}(b),\check{q}(b)), & b\in[0,1], \\ \lambda_2 D^{\overline{k}_2}(\check{q}(b)) + (1-\lambda_2)D^{\theta_2}(\check{q}(b)) = g(b,\check{\check{p}}(b),\check{q}(b)), & b\in[0,1], \\ \check{p}(0)=0, \quad \check{q}(0)=0, \quad \check{q}(1)=\lambda D^{\gamma_1}\check{p}(\eta), \quad \check{p}(1)=\gamma I^{q_1}\check{q}(\xi), \end{cases} \tag{1.4}$$

where $D^{\theta}$ ($\theta\in\{\overline{k}_1,\overline{k}_2,\theta_1,\theta_2,\gamma_1\}$) and $I^{q_1}$ are the Riemann–Liouville derivative and integral, respectively, $1<\overline{k}_1,\overline{k}_2\leq 2$, $1<\theta_1,\theta_2<\min\{\overline{k}_1,\overline{k}_2\}$, $0<\lambda_1,\lambda_2\leq 1$, $f,g\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$, $\eta,\xi\in(0,1)$, $\lambda,\gamma\in\mathbb{R}$, $q_1\in\mathbb{R}_+$, and $0\leq\gamma_1<\min\{\overline{k}_1-\theta_1,\overline{k}_2-\theta_2\}$.

The structure of this paper is as follows. First, we recall some main concepts in Sect. 2. The classical Banach, Leray–Schauder, and Krasnosel'skii fixed point theorems are applied to obtain the results on the existence and uniqueness in Sect. 3. Besides, to support the obtained results, we include some examples.

For a function $\overline{g}:(0,\infty)\longrightarrow\mathbb{R}$, the Riemann–Liouville fractional integral of order $\overline{\eta}$ has been defined by $I^{\overline{\eta}}\overline{g}(b)=\int_0^b\frac{(b-s)^{\overline{\eta}-1}}{\Gamma(\overline{\eta})}\overline{g}(s)\,ds$, where the right-hand side is defined pointwise on $(0,\infty)$ [2]. Besides, for a function $g$, the Riemann–Liouville fractional derivative of order $\beta$ is defined as $D^{\beta}g(t)=\frac{1}{\Gamma(n-\beta)}(\frac{d}{dt})^n\int_0^t\frac{g(s)}{(t-s)^{\beta-n+1}}\,ds$ with $n=[\beta]+1$ [2].

The following lemma is the main tool to present the main result in this paper.

**Lemma 1.1** ([2]) *Let $\check{p}\in C_{\mathbb{R}}(0,1)$. Then, for $\rho>0$, we have*

$$I^{\rho}D^{\rho}\check{p}(b)=\check{p}(b)+C_1 b^{\rho-1}+C_2 b^{\rho-2}+\cdots+c_n b^{\rho-n},$$

*where $n-1<\rho<n$, and $C_1,C_2,\ldots,C_n$ are some real constants.*

The following three theorems due to Banach, Leray and Schauder, and Krasnosel'skii play the main role in our arguments. $X$ is a Banach space in both theorems.

**Theorem 1.1** ([31]) *Assume that $\overline{D}$ is a closed set in $\overline{X}$ and $\hat{T}:\overline{D}\longrightarrow\overline{D}$ satisfies*

$$|\hat{T}\overline{x}-\hat{T}\overline{y}|\leq\lambda|\overline{x}-\overline{y}|\quad\forall\overline{x},\overline{y}\in\overline{D}$$

*for some $\lambda\in(0,1)$. Then $\overline{T}$ admits a unique fixed point in $\overline{D}$.*

**Theorem 1.2** ([32]) *Suppose that $\overline{\Omega}$ is a closed bounded convex in $\overline{X}$ and $O$ is an open set contained in $\overline{\Omega}$ with $0\in O$. Let $\overline{T}:\overline{U}\to\overline{\Omega}$ be continuous and compact. Then*

($\mathfrak{a}$) *$T$ has a fixed point in $\overline{U}$, or*

($\mathfrak{aa}$) *$\exists\overline{u}\in\overline{U}$ and $\hat{\mu}\in(0,1)$ such that $\overline{u}=\hat{\mu}\overline{T}(\overline{u})$.*

**Lemma 1.2** (Krasnosel'skii fixed point theorem [33]) *Assume that $\overline{M}$ is a closed, bounded, convex, and nonempty subset of a Banach space $\overline{X}$. Besides, assume that $\overline{A}$, $\overline{B}$ are two operators such that* (i) *$\overline{A}x+\overline{B}y\in\overline{M}$ for $x,y\in\overline{M}$,* (ii) *$\overline{A}$ is compact and continuous, and* (iii) *$\overline{B}$ is a contraction mapping. Then $\overline{A}+\overline{B}$ has a fixed point in $\overline{M}$.*

## 2 An auxiliary result

The following lemma is a key result, which will be applied to verify the fundamental results.

**Lemma 2.1** *Let $h_1,h_2\in C([0,1],\mathbb{R})$. Then the pair $(\check{p},\check{q})$ is a solution to the linear multi-order RL-FBVPs*

$$\begin{cases}\lambda_1D^{k_1}(\check{p}(b))+(1-\lambda_1)D^{\theta_1}(\check{p}(b))=h_1(b), & b\in[0,1],\\ \lambda_2D^{k_2}(\check{q}(b))+(1-\lambda_2)D^{\theta_2}(\check{q}(b))=h_2(b), & b\in[0,1],\\ \check{p}(0)=0,\qquad \check{q}(0)=0,\qquad \check{q}(1)=\lambda D^{\gamma_1}\check{p}(\eta),\qquad \check{p}(1)=\gamma I^{q_1}q(\xi),\end{cases}\tag{2.1}$$

*if and only if*

$$\begin{aligned}\check{p}(b)={}&\frac{\lambda_1-1}{\lambda_1}\times\frac{1}{\Gamma(k_1-\theta_1)}\int_0^b(b-s)^{k_1-\theta_1-1}\check{p}(s)\,ds\\&+\frac{1}{\lambda_1}\times\frac{1}{\Gamma(k_1)}\int_0^b(b-s)^{k_1-1}h_1(s)\,ds\\&+\frac{b^{k_1-1}}{\Theta}\biggl[\frac{\gamma(\lambda_2-1)}{\lambda_2}I^{k_2-\theta_2+q_1}\check{q}(\xi)+\frac{\gamma}{\lambda_2}I^{k_2+q_1}h_2(\xi)\\&+\frac{\Lambda_1\lambda(\lambda_1-1)}{\lambda_1}I^{k_1-\theta_1-\gamma_1}\check{p}(\eta)+\frac{\Lambda_1\lambda}{\lambda_1}I^{k_1-\gamma_1}h_1(\eta)\\&-\frac{(\lambda_2-1)\Lambda_1}{\lambda_2}I^{k_2-\theta_2}\check{q}(1)-\frac{\Lambda_1}{\lambda_2}I^{k_2}h_2(1)\\&-\frac{\lambda_1-1}{\lambda_1}I^{k_1-\theta_1}\check{p}(1)-\frac{1}{\lambda_1}I^{k_1}h_1(1)\biggr]\end{aligned}$$

*and*

$$\begin{aligned}
\check{q}(b) &= \frac{\lambda_2 - 1}{\lambda_2} \times \frac{1}{\Gamma(k_2 - \theta_2)} \int_0^b (b-s)^{k_2-\theta_2-1} \check{q}(s)\, ds \\
&\quad + \frac{1}{\lambda_2} \times \frac{1}{\Gamma(k_2)} \int_0^t (b-s)^{k_2-1} h_2(s)\, ds + \frac{b^{k_2-1}}{|\Theta|} \biggl[ \frac{\lambda(\lambda_1 - 1)}{\lambda_1} I^{k_1-\theta_1-\gamma_1} \check{p}(\eta) \\
&\quad + \frac{\lambda}{\lambda_1} I^{k_1-\gamma_1} h_1(\eta) - \frac{(\lambda_2 - 1)}{\lambda_2} I^{k_2-\theta_2} \check{q}(1) - \frac{1}{\lambda_2} I^{k_2} h_2(1) \\
&\quad + \Lambda_2 ( \frac{\gamma(\lambda_2 - 1)}{\lambda_2} I^{k_2-\theta_2+q_1} \check{q}(\xi) + \frac{\gamma}{\lambda_2} I^{k_2+q_1} h_2(\xi) \\
&\quad - \frac{(\lambda_1 - 1)}{\lambda_1} I^{k_1-\theta_1} \check{p}(1) - \frac{1}{\lambda_1} h_1(1) \biggr],
\end{aligned}$$

*where*

$$\begin{aligned}
\Lambda_1 &= \gamma \xi^{k_2-1+q_1} \frac{\Gamma(k_2)}{\Gamma(k_2 + q_1)}, \\
\Lambda_2 &= \frac{\Gamma(k_1)}{\Gamma(k_1 - \gamma_1)} \eta^{k_1-\gamma_1-1}, \qquad \Theta = 1 - \Lambda_1 \Lambda_2 \neq 0.
\end{aligned}$$

*Proof* Taking the operator $I^{k_1}$ on both sides of the first equation in (2.1), by Lemma 1.1 we deduce that

$$\begin{aligned}
\check{p}(b) &= \frac{\lambda_1 - 1}{\lambda_1} \frac{1}{\Gamma(k_1 - \theta_1)} \int_0^b (b-s)^{k_1-\theta_1-1} \check{p}(s)\, ds \\
&\quad + \frac{1}{\lambda_1} \frac{1}{\Gamma(k_1)} \int_0^b (b-s)^{k_1-1} h_1(s)\, ds + b^{k_1-1} C_1 + b^{k_1-2} C_2.
\end{aligned} \tag{2.2}$$

Similarly, taking the operator $I^{k_2}$ on both sides of the second equation in (2.1), we get

$$\begin{aligned}
\check{q}(b) &= \frac{\lambda_2 - 1}{\lambda_2} \frac{1}{\Gamma(k_2 - \theta_2)} \int_0^b (b-s)^{k_2-\theta_2-1} \check{q}(s)\, ds \\
&\quad + \frac{1}{\lambda_2} \frac{1}{\Gamma(k_2)} \int_0^b (b-s)^{k_2-1} h_2(s)\, ds + C_3 b^{k_2-1} + C_4 b^{k_2-2},
\end{aligned} \tag{2.3}$$

where $C_1, C_2, C_3 \in \mathbb{R}$. From $\check{p}(0) = 0$ and $\check{q}(0) = 0$ we obtain $C_2 = C_4 = 0$. Consequently,

$$\begin{aligned}
\check{p}(b) &= \frac{\lambda_1 - 1}{\lambda_1} \frac{1}{\Gamma(k_1 - \theta_1)} \int_0^b (b-s)^{k_1-\theta_1-1} \check{p}(s)\, ds \\
&\quad + \frac{1}{\lambda_1} \frac{1}{\Gamma(k_1)} \int_0^b (b-s)^{k_1-1} h_1(s)\, ds + b^{k_1-1} C_1,
\end{aligned}$$

and

$$\begin{aligned}
\check{q}(b) &= \frac{\lambda_2 - 1}{\lambda_2} \frac{1}{\Gamma(k_2 - \theta_2)} \int_0^b (b-s)^{k_2-\theta_2-1} \check{q}(s)\, ds \\
&\quad + \frac{1}{\lambda_2} \frac{1}{\Gamma(k_2)} \int_0^b (b-s)^{k_2-1} h_2(s)\, ds + C_3 b^{k_2-1}.
\end{aligned}$$

Now, applying the operators $I^{q_1}$ and $D^{\gamma_1}$, we get

$$\begin{aligned}
I^{q_1}\breve{q}(b) &= \frac{\lambda_2-1}{\Gamma(k_2-\theta_2+q_1)\lambda_2}\int_0^b (b-s)^{k_2-\theta_2+q_1-1}\breve{q}(s)\,ds \\
&\quad + \frac{1}{\lambda_2\Gamma(k_2+q_1)}\int_0^b (b-s)^{k_2+q_1-1}h_2(s)\,ds + C_3 b^{k_2-1+q_1}\frac{\Gamma(k_2)}{\Gamma(k_2+q_1)}, \\
D^{\gamma_1}\breve{p}(b) &= \frac{\lambda_1-1}{\lambda_1\Gamma(k_1-\theta_1-\gamma_1)}\int_0^b (b-s)^{k_1-\theta_1-\gamma_1-1}\breve{p}(s)\,ds \\
&\quad + \frac{1}{\lambda_1\Gamma(k_1-\gamma_1)}\int_0^b (b-s)^{k_1-\gamma_1-1}h_1(s)\,ds + b^{k_1-\gamma_1-1}\frac{C_1\Gamma(k_1)}{\Gamma(k_1-\gamma_1)}.
\end{aligned}$$

Hence, in view of $\breve{q}(1) = \lambda D^{\gamma_1}\breve{p}(\eta)$ and $\breve{p}(1) = \gamma I^{q_1}\breve{q}(\xi)$, we have

$$\begin{aligned}
&C_1 - C_3\Lambda_1 = d_1, \\
&C_3 - C_1\Lambda_2 = d_2,
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\gamma(\lambda_2-1)}{\lambda_2}I^{k_2-\theta_2+q_1}\breve{q}(\xi) + \frac{\gamma}{\lambda_2}I^{k_2+q_1}h_2(\xi) - \frac{(\lambda_1-1)}{\lambda_1}I^{k_1-\theta_1}\breve{p}(1) - \frac{1}{\lambda_1}h_1(1), \\
d_2 &== \frac{\lambda(\lambda_1-1)}{\lambda_1}I^{k_1-\theta_1-\gamma_1}\breve{p}(\eta) + \frac{\lambda}{\lambda_1}I^{k_1-\gamma_1}h_1(\eta) - \frac{(\lambda_2-1)}{\lambda_2}I^{k_2-\theta_2}\breve{q}(1) - \frac{1}{\lambda_2}I^{k_2}h_2(1).
\end{aligned}$$

Accordingly, $C_1 = \frac{d_1+d_2\Lambda_1}{\Theta}$ and $C_3 = \frac{d_2+\Lambda_2 d_1}{\Theta}$. By inserting the values of $C_1$ and $C_3$ into (2.2) and (2.3) the conclusion follows. The converse is obtained by direct computation. □

## 3 Results concerning the existence and uniqueness

In this subsection, by applying some classical fixed point theorems, we present some results. First, we apply the Banach fixed point theorem to obtain our first existence result.

Let $\overline{\mathfrak{X}} = C([0,1],\mathbb{R})$ be the Banach space of all continuous mappings on $[0,1]$ with norm

$$\|\breve{x}\| = \sup\bigl\{\bigl|\breve{x}(b)\bigr|; b\in[0,1]\bigr\}.$$

It is clear that the space $\overline{\mathfrak{X}}\times\overline{\mathfrak{X}}$ with norm $\|(\breve{x},\breve{y})\| = \|\breve{x}\| + \|\breve{y}\|$ is a Banach space. In the following theorem, we apply the classical Banach fixed point theorem to obtain the uniqueness result for the RLFBVP (1.4). For convenience, we set

$$\begin{aligned}
A_1 &= \frac{1}{\lambda_1\Gamma(k_1+1)} + \frac{\Lambda_1\lambda}{\Theta\lambda_1\Gamma(k_1-\gamma_1+1)} + \frac{1}{\Theta\lambda_1\Gamma(k_1+1)}, \\
A_2 &= \frac{\gamma}{\Theta\lambda_2\Gamma(k_2+q_1+1)} + \frac{\Lambda_1}{\Theta\lambda_2\Gamma(k_2+1)}, \\
A_3 &= \frac{|\lambda_1-1|}{\lambda_1\Gamma(k_1-\theta_1+1)} + \frac{|\lambda_1-1|}{\Theta\lambda_1\Gamma(k_1-\theta_1+1)} + \frac{\Lambda_1\lambda|\lambda_1-1|}{\Theta\lambda_1\Gamma(k_1-\theta_1-\gamma_1+1)}, \\
A_4 &= \frac{\gamma|\lambda_2-1|}{\Theta\lambda_2\Gamma(k_2-\theta_2+q_1+1)} + \frac{\Lambda_1|\lambda_2-1|}{\lambda_2\Theta\Gamma(k_2-\theta_2+1)}, \\
B_1 &= \frac{\lambda}{\lambda_1\Gamma(k_1-\gamma_1+1)} + \frac{\Lambda_2}{\Theta\lambda_1\Gamma(k_1+1)},
\end{aligned}$$

$$
\begin{aligned}
B_2 &= \frac{1}{\Theta\lambda_2\Gamma(k_2+1)} + \frac{\Lambda_2\gamma}{\Theta\lambda_2\Gamma(k_2+q_1+1)} + \frac{1}{\lambda_2\Gamma(k_2+1)}, \\
B_3 &= \frac{\lambda|\lambda_1-1|}{\Theta\lambda_1\Gamma(k_1-\theta_1-\gamma_1+1)} + \frac{\Lambda_2|\lambda_1-1|}{\Theta\lambda_1\Gamma(k_1-\theta_1+1)} + \frac{|\lambda_2-1|}{\lambda_2\Gamma(k_2-\theta_2+1)}, \\
B_4 &= \frac{|\lambda_2-1|}{\Theta\lambda_2\Gamma(k_2-\theta_2+1)} + \frac{\Lambda_2\gamma|\lambda_2-1|}{\Theta\lambda_2\Gamma(k_2-\theta_2+q_1+1)}.
\end{aligned}
\tag{3.1}
$$

**Theorem 3.1** *Let $f,g:[0,1]\times\mathbb{R}^2\longrightarrow R$ be two continuous functions such that for all $(\breve{u}_1,\breve{v}_1),(\breve{u}_2,\breve{v}_2)\in\mathbb{R}^2$,*

$$
\begin{aligned}
&\bigl|f(b,\breve{u}_1,\breve{v}_1)-f(b,\breve{u}_2,\breve{v}_2)\bigr| \le \ell_1\bigl(|\breve{u}_1-\breve{u}_2|+|\breve{v}_1-\breve{v}_2|\bigr), \\
&\bigl|g(b,\breve{u}_1,\breve{v}_1)-g(b,\breve{u}_2,\breve{v}_2)\bigr| \le \ell_2\bigl(|\breve{u}_1-\breve{u}_2|+|\breve{v}_1-\breve{v}_2|\bigr),
\end{aligned}
\tag{3.2}
$$

*where $\ell_1,\ell_2\in\mathbb{R}$. Besides, assume that $\ell_1(A_1+B_1)+\ell_2(A_2+B_2)+(A_3+A_4+B_3+B_4)<1$ where $A_i$, $B_i$ are given in (3.1). Then the RL-FBVP (1.4) admits a unique solution on* $[0,1]$.

*Proof* Consider $\sup_{t\in[0,1]}|g(t,0,0)|=N$ and $\sup_{t\in[0,1]}|f(t,0,0)|=M$ coupled with

$$
\mathbb{B}_r=\bigl\{(\breve{u},\breve{v})\in\overline{\mathfrak{X}}\times\overline{\mathfrak{X}};\bigl\|(\breve{u},\breve{v})\bigr\|\le r\bigr\},
$$

where

$$
r\ge\frac{M(A_1+B_1)+N(A_2+B_2)}{1-[\ell_1(A_1+B_1)+\ell_2(A_2+B_2)+A_3+A_4+B_3+B_4]}.
$$

Now by Lemma 2.1 we define the operator $F:\overline{\mathfrak{X}}\times\overline{\mathfrak{X}}\to\overline{\mathfrak{X}}\times\overline{\mathfrak{X}}$ by

$$
F(\breve{u},\breve{v})(b)=\bigl(F_1(\breve{u},\breve{v})(b),F_2(\breve{u},\breve{v})(b)\bigr),
$$

where

$$
\begin{aligned}
F_1(\breve{u},\breve{v})(b) &= \frac{\lambda_1-1}{\lambda_1}\times\frac{1}{\Gamma(k_1-\theta_1)}\int_0^b(b-s)^{k_1-\theta_1-1}\breve{u}(s)\,ds \\
&\quad+\frac{1}{\lambda_1}\times\frac{1}{\Gamma(k_1)}\int_0^b(b-s)^{k_1-1}f\bigl(s,\breve{u}(s),\breve{v}(s)\bigr)\,ds \\
&\quad+\frac{b^{k_1-1}}{\Theta}\biggl[\frac{\gamma(\lambda_2-1)}{\lambda_2}I^{k_2-\theta_2+q_1}\breve{v}(\xi)+\frac{\gamma}{\lambda_2}I^{k_2+q_1}g\bigl(\xi,\breve{u}(\xi),\breve{v}(\xi)\bigr) \\
&\quad+\frac{\Lambda_1\lambda(\lambda_1-1)}{\lambda_1}I^{k_1-\theta_1-\gamma_1}\breve{u}(\eta)+\frac{\Lambda_1\lambda}{\lambda_1}I^{k_1-\gamma_1}f\bigl(\eta,\breve{u}(\eta),\breve{v}(\eta)\bigr) \\
&\quad-\frac{(\lambda_2-1)\Lambda_1}{\lambda_2}I^{k_2-\theta_2}\breve{v}(1)-\frac{\Lambda_1}{\lambda_2}I^{k_2}g\bigl(1,\breve{u}(1),\overline{v}(1)\bigr) \\
&\quad-\frac{\lambda_1-1}{\lambda_1}I^{k_1-\theta_1}\breve{u}(1)-\frac{1}{\lambda_1}I^{k_1}f\bigl(1,\breve{u}(1),\breve{v}(1)\bigr)\biggr]
\end{aligned}
$$

and

$$
F_2(\breve{u},\breve{v})(b)=\frac{\lambda_2-1}{\lambda_2}\times\frac{1}{\Gamma(k_2-\theta_2)}\int_0^b(b-s)^{k_2-\theta_2-1}\breve{v}(s)\,ds
$$

$$
\begin{aligned}
&+ \frac{1}{\lambda_2} \times \frac{1}{\Gamma(k_2)} \int_0^b (b-s)^{k_2-1} g\big(\big(s, \check{u}(s), \check{v}(s)\big)\, ds \\
&+ \frac{b^{k_2-1}}{|\Theta|}\bigg[\frac{\lambda(\lambda_1-1)}{\lambda_1} I^{k_1-\theta_1-\gamma_1}\check{u}(\eta) + \frac{\lambda}{\lambda_1} I^{k_1-\gamma_1} f\big(\eta, \check{u}(\eta), \check{v}(\eta)\big) \\
&- \frac{(\lambda_2-1)}{\lambda_2} I^{k_2-\theta_2}\check{v}(1) - \frac{1}{\lambda_2} I^{k_2} g\big(1, \check{\check{u}}(1), \check{v}(1)\big) \\
&+ \Lambda_2\bigg(\frac{\gamma(\lambda_2-1)}{\lambda_2} I^{k_2-\theta_2+q_1}\check{v}(\xi) + \frac{\gamma}{\lambda_2} I^{k_2+q_1} g\big(\xi, \check{u}(\xi), \check{v}(\xi)\big) \\
&- \frac{(\lambda_1-1)}{\lambda_1} I^{k_1-\theta_1}\check{u}(1) - \frac{1}{\lambda_1} f\big(1, \check{u}(1), \check{v}(1)\big)\bigg)\bigg].
\end{aligned}
$$

First, we prove that $F(\mathbb{B}_r) \subseteq \mathbb{B}_r$. For all $(\check{u}, \check{v}) \in \mathbb{B}_r$ and $b \in [0,1]$, we have

$$
\begin{aligned}
\big|f\big(b, \check{u}(b), \check{v}(b)\big)\big| &\leq \big|f\big(b, \check{u}(b), \check{v}(b)\big) - f(b,0,0)\big| + \big|f(b,0,0)\big| \\
&\leq \ell_1\big(\big|\overline{u}(t)\big| + \big|\overline{v}(t)\big|\big) + M \\
&\leq \ell_1 r + M.
\end{aligned}
$$

Similarly, we have

$$
\big|g\big(b, \check{u}(b), \check{v}(b)\big)\big| \leq \ell_2 r + N.
$$

Consequently,

$$
\begin{aligned}
&\big|F_1(\check{u}, \check{v})(b)\big| \\
&\quad\leq (\ell_1 r + M)\bigg[\frac{1}{\lambda_1\Gamma(k_1+1)} + \frac{\Lambda_1\lambda}{\Theta\lambda_1\Gamma(k_1-\gamma_1+1)} + \frac{1}{\Theta\lambda_1\Gamma(k_1+1)}\bigg] \\
&\qquad+ (\ell_2 r + N)\bigg[\frac{\gamma}{\Theta\lambda_2\Gamma(k_2+q_1+1)} + \frac{\Lambda_1}{\Theta\lambda_2\Gamma(k_2+1)}\bigg] \\
&\qquad+ r\bigg[\frac{|\lambda_1-1|}{\lambda_1\Gamma(k_1-\theta_1+1)} + \frac{\gamma|\lambda_2-1|}{\Theta\lambda_2\Gamma(k_2-\theta_2+q_1+1)} + \frac{\Lambda_1\lambda|\lambda_1-1|}{\Theta\lambda_1\Gamma(k_1-\theta_1-\gamma_1+1)} \\
&\qquad+ \frac{|\lambda_2-1|\Lambda_1}{\Theta\lambda_2\Gamma(k_2-\theta_2+1)} + \frac{|\lambda_1-1|}{\lambda_1\Theta\Gamma(k_1-\theta_1+1)}\bigg].
\end{aligned}
$$

Thus

$$
\big\|F_1(\check{u}, \check{v})\big\| \leq (\ell_1 r + M)A_1 + (\ell_2 r + N)A_2 + r(A_3 + A_4). \tag{3.3}
$$

Similarly, we have

$$
\begin{aligned}
&\big\|F_2(\check{u}, \check{v})\big\| \\
&\quad\leq (\ell_1 r + M)\bigg[\frac{\lambda}{\Theta\lambda_1\Gamma(k_1-\gamma_1+1)} + \frac{\Lambda_2}{\Theta\lambda_1\Gamma(k_1+1)}\bigg] \\
&\qquad+ (\ell_2 r + N)\bigg[\frac{1}{\Theta\lambda_2\Gamma(k_2+1)} + \frac{\Lambda_2\gamma}{\Theta\lambda_2\Gamma(k_2+q_1+1)} + \frac{1}{\lambda_2\Gamma(k_2+1)}\bigg]
\end{aligned}
$$

$$+ r\left[\frac{|\lambda_2 - 1|}{\lambda_2\Gamma(k_2 - \theta_2 + 1)} + \frac{\lambda|\lambda_1 - 1|}{\Theta\lambda_1\Gamma(k_1 - \theta_1 - \gamma_1 + 1)} + \frac{|\lambda_2 - 1|}{\Theta\lambda_2\Gamma(k_2 - \theta_2 + 1)}\right.$$
$$\left. + \frac{\Lambda_2\gamma|\lambda_2 - 1|}{\Theta\lambda_2\Gamma(k_2 - \theta_2 + q_1 + 1)} + \frac{\Lambda_2|\lambda_1 - 1|}{\Theta\lambda_1\Gamma(k_1 - \theta_1 + 1)}\right],$$

which leads to

$$\left\|F_2(\breve{u}, \breve{v})\right\| \le (\ell_1 r + M)B_1 + (\ell_2 r + N)B_2 + r(B_3 + B_4). \tag{3.4}$$

Using (3.3) and (3.4), we infer that

$$\begin{aligned}\left\|F(\breve{u}, \breve{v})\right\| &\le \left\|F_1(\breve{u}, \breve{v})\right\| + \left\|F_2(\breve{u}, \breve{v})\right\| \\ &\le (\ell_1 r + M)(A_1 + B_1) + (\ell_2 r + N)(A_2 + B_2) \\ &\quad + r(A_3 + A_4 + B_3 + B_4) \\ &\le r.\end{aligned}$$

On the other hand, for all $(\breve{u}_1, \breve{v}_1), (\breve{u}_2, \breve{v}_2) \in \mathbb{B}_r$ and $b \in [0, 1]$, we get

$$\begin{aligned}&\left|F_1(\breve{u}_1, \breve{v}_1)(b) - F_1(\breve{u}_2, \breve{v}_2)(b)\right| \\ &\quad\le \frac{|\lambda_1 - 1|}{\lambda_1\Gamma(k_1 - \theta_1 + 1)}\left(\|\breve{u}_1 - \breve{u}_2\|\right) \\ &\qquad + \frac{\ell_1}{\lambda_1\Gamma(k_1 + 1)}\left(\|\breve{u}_1 - \breve{u}_2\| + \|\breve{v}_1 - \breve{v}_2\|\right) \\ &\qquad + \frac{\gamma|\lambda_2 - 1|}{\Theta\lambda_2\Gamma(k_2 - \theta_2 + q_1 + 1)}\left(\|\breve{v}_1 - \breve{v}_2\|\right) \\ &\qquad + \frac{\gamma}{\Theta\lambda_2\Gamma(k_2 + q_1 + 1)}\ell_2\left(\|\breve{u}_1 - \breve{u}_2\| + \|\breve{v}_1 - \breve{v}_2\|\right) \\ &\qquad + \frac{\Lambda_1\lambda(|\lambda_1 - 1)|}{\Theta\lambda_1\Gamma(k_1 - \theta_1 - \gamma_1 + 1)}\|\breve{u}_1 - \breve{u}_2\| \\ &\qquad + \frac{\Lambda_1\lambda}{\Theta\lambda_1\Gamma(k_1 - \gamma_1 + 1)}\ell_1\left(\|\breve{u}_1 - \breve{u}_2\| + \|\breve{v}_1 - \breve{v}_2\|\right) \\ &\qquad + \frac{\Lambda_1|\lambda_2 - 1|}{\lambda_2\Theta\Gamma(k_2 - \theta_2 + 1)}\|\breve{v}_1 - \breve{v}_2\| + \frac{\Lambda_1}{\Theta\lambda_2\Gamma(k_2 + 1)}\ell_2\left(\|\breve{u}_1 - \breve{u}_2\| + \|\breve{v}_1 - \breve{v}_2\|\right) \\ &\qquad + \frac{|\lambda_1 - 1|}{\Theta\lambda_1\Gamma(k_1 - \theta_1 + 1)}\left(\|\breve{u}_1 - \breve{u}_2\|\right) + \frac{1}{\Theta\lambda_1\Gamma(k_1 + 1)}\ell_1\left(\|\overline{u}_1 - \overline{u}_2\| + \|\overline{v}_1 - \overline{v}_2\|\right) \\ &\quad= (\ell_1 A_1 + \ell_2 A_2)\left(|\breve{u}_1 - \breve{u}_2\| + |\breve{v}_1 - \breve{v}_2\|\right) + A_3\|\breve{u}_1 - \breve{u}_2\| + A_4\|\breve{v}_1 - \breve{v}_2\| \\ &\quad\le \left((\ell_1 A_1 + \ell_2 A_2) + A_3 + A_4\right)\left(|\breve{u}_1 - \breve{u}_2\| + |\breve{v}_1 - \breve{v}_2\|\right).\end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\left\|F_2(\breve{u}_1, \breve{v}_1)(b) - F_2(\breve{u}_2, \breve{v}_2)(b)\right\| \le \left((\ell_1 B_1 + \ell_2 B_2) + B_3 + B_4\right)\left(\|\breve{u}_1 - \breve{u}_2\| + \|\breve{v}_1 - \check{\breve{v}}_2\|\right). \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$\begin{aligned}&\left\|F(\check{u}_1,\check{v}_1)-F(\check{u}_2,\check{u}_2)\right\|\\&\quad\le\bigl(\ell_1(A_1+B_1)+\ell_2(A_2+B_2)+(A_3+A_4+B_3+B_4)\bigr)\bigl(|\check{u}_1-\check{u}_2\|+|\check{v}_1-\check{v}_2\|\bigr).\end{aligned}$$

As $\ell_1(A_1+B_1)+\ell_2(A_2+B_2)+(A_3+A_4+B_3+B_4)<1$, $F$ is a contraction, and Theorem 1.1 yields the desired result on the existence of a unique fixed-point for $F$ and, accordingly, of a unique solution to the multiorder RL-FBVP (1.4). □

In the following theorem, we apply Theorem 1.2 to prove the second result.

**Theorem 3.2** *Let $f,g:[0,1]\times\mathbb{R}^2\longrightarrow R$ be continuous functions such that for all $(\check{u}_1,\check{v}_1)\in\mathbb{R}^2$,*

$$\begin{aligned}&\left|f(b,\check{u}_1,\check{v}_1)\right|\le\alpha_0+\alpha_1|\check{u}_1|+\alpha_2|\check{v}_1|,\\&\left|g(b,\check{u}_1,\check{v}_1)\right|\le\beta_0+\beta_1|\check{u}_1|+\beta_2|\check{v}_1|\end{aligned}\tag{3.7}$$

*for some $\alpha_i,\beta_i\ge 0$ $(i=1,2)$ and $\alpha_0,\beta_0>0$. Moreover, assume that $(A_1+B_1)\alpha_1+(A_2+B_2)\beta_1+A_3+B_3<1$ and $(A_1+B_1)\alpha_2+(A_2+B_2)\beta_2+A_4+B_4<1$. Then the multiorder RL-FBVP system (1.4) admits a solution.*

*Proof* First, we claim that $F:\overline{\mathfrak{X}}\times\overline{\mathfrak{X}}\to\overline{\mathfrak{X}}\times\overline{\mathfrak{X}}$ is completely continuous. Indeed, by the continuity of $f$ and $g$ we infer the continuity of $F_1$ and $F_2$. Further, we show that $F$ is uniformly bounded. For a bounded subset $\mathbb{B}_r=\{(\check{x},\check{y})\in\overline{\mathfrak{X}}\times\overline{\mathfrak{X}};\|(\check{x},\check{y})\|\le r\}\subseteq\overline{\mathfrak{X}}\times\overline{\mathfrak{X}}$, there exist positive constants $M_1,M_2$ such that $|f(b,\check{u}(t),\check{v}(b))|\le M_1$ and $|g(b,\check{u}(b),\check{v}(b))|\le M_2$. Then for all $(\check{u},\check{v})\in\mathbb{B}_r$, we write

$$\begin{aligned}&\left|F_1(\check{u},\check{\bar{v}})(b)\right|\\&\quad\le\frac{|\lambda_1-1|}{\lambda_1\Gamma(k_1-\theta_1+1)}r+\frac{M_1}{\lambda_1\Gamma(k_1+1)}+\frac{\gamma|\lambda_2-1|}{\Theta\lambda_2\Gamma(k_2-\theta_2+q_1+1)}r\\&\qquad+\frac{\gamma M_2}{\Theta\lambda_2\Gamma(k_2+q_1+1)}+\frac{\Lambda_1\lambda|\lambda_1-1|}{\Theta\lambda_1\Gamma(k_1-\theta_1-\gamma_1+1)}r+\frac{\Lambda_1\lambda}{\Theta\lambda_1\Gamma(k_1-\gamma_1+1)}M_1\\&\qquad+\frac{|\lambda_2-1|\Lambda_1}{\Theta\lambda_2\Gamma(k_2-\theta_2+1)}r+\frac{\Lambda_1}{\Theta\lambda_2\Gamma(k_2+1)}M_2+\frac{|\lambda_1-1|}{\Theta\lambda_1\Gamma(k_1-\theta_1+1)}r\\&\qquad+\frac{M_1}{\Theta\lambda_1\Gamma(k_1+1)}.\end{aligned}$$

Hence

$$\left\|F_1(\check{u},\check{v})\right\|\le A_1M_1+A_2M_2+(A_3+A_4)r.\tag{3.8}$$

Similarly, we obtain

$$\left\|F_2(\check{u},\check{v})\right\|\le B_1M_1+B_2M_2+(B_3+B_4)r.\tag{3.9}$$

Due to the last two inequalities, we get

$$\|F(\breve{u},\breve{v})\| \le (A_1+B_1)M_1+(A_2+B_2)M_2+r(A_3+A_4+B_3+B_4),$$

so that $F$ is uniformly bounded.

Now we prove the equicontinuity of $F$. Let $\breve{t}_1,\breve{t}_2\in[0,1]$ with $\breve{t}_1<\breve{t}_2$. For all $(\breve{u},\breve{v})\in\mathbb{B}_r$, we have

$$\begin{aligned}
&\left|F_1(\breve{u},\breve{v})(\breve{t}_2)-F_1(\breve{u},\breve{v})(\breve{t}_1)\right| \\
&\quad\le \frac{r|\lambda_1-1|}{\Gamma(k_1-\theta_1+1)\lambda_1}\left[2(\breve{t}_2-\breve{t}_1)^{k_1-\theta_1}+\left|\breve{t}_2^{k_1-\theta_1}-\breve{t}_1^{k_1-\theta_1}\right|\right] \\
&\qquad+\frac{M_1}{\lambda_1\Gamma(k_1+1)}\left[2(\breve{t}_2-\breve{t}_1)^{k_1-\theta_1}+\left|\breve{t}_2^{k_1-\theta_1}-\breve{t}_1^{k_1-\theta_1}\right|\right] \\
&\qquad+\left(\breve{t}_2^{k_1-1}-\breve{t}_1^{k_1-1}\right)\left[\frac{\gamma|\lambda_2-1|}{\Theta\lambda_2\Gamma(k_2-\theta_2+q_1+1)}r+\frac{\gamma}{\Theta\lambda_2\Gamma(k_2+q_1+1)}M_2\right. \\
&\qquad+\frac{\Lambda_1\lambda|\lambda_1-1|}{\lambda_1\Theta\Gamma(k_1-\gamma_1-\theta_1+1)}r+\frac{\Lambda_1\lambda}{\Theta\lambda_1\Gamma(k_1-\gamma_1+1)}M_1 \\
&\qquad+\frac{|\lambda_2-1|\Lambda_1}{\lambda_2\Theta\Gamma(k_2-\theta_2+1)}r+\frac{\Lambda_1}{\Theta\lambda_2\Gamma(k_2+1)}M_1 \\
&\qquad\left.+\frac{|\lambda_1-|}{\Theta\lambda_1\Gamma(k_1-\theta_1+1)}r+\frac{1}{\Theta\lambda_1\Gamma(k_1+1)}M_1\right].
\end{aligned}$$

Accordingly, $|F_1(\breve{u},\breve{v})(\breve{t}_2)-F_1(\breve{u},\breve{v})(\breve{t}_2)|\longrightarrow 0$ as $\breve{t}_2\longrightarrow\breve{t}_1$ Analogously, we conclude that $|F_2(\breve{u},\breve{v})(\breve{t}_2)-F_2(\breve{u},\breve{v})(\breve{t}_1)|\longrightarrow 0$ as $\breve{t}_2\longrightarrow\breve{t}_1$. Thus $F$ is equicontinuous. Finally, we show the boundedness of the set

$$D=\left\{(\breve{u},\breve{v})\in\overline{\mathfrak{X}}\times\overline{\mathfrak{X}};(\breve{u},\breve{v})=\eta F(\breve{u},\breve{v}),0<\eta<1\right\}.$$

Suppose that $(\breve{u},\breve{v})\in D$. For $b\in[0,1]$, we have $\breve{u}(b)=\eta F_1(\breve{u},\breve{v})(b)$, $\breve{v}(b)=\eta F_2(\breve{u},\breve{v})(b)$. Therefore, using condition (3.7), we get

$$\begin{aligned}
\|\breve{u}\| &\le \left(\alpha_0+\alpha_1\|\breve{u}\|+\alpha_2\|\breve{v}\|\right)A_1+\left(\beta_0+\beta_1\|\breve{u}\|+\beta_2\|\breve{v}\|\right)A_2+\|\overline{u}\|A_3+\|\overline{v}\|A_4, \\
\|\breve{v}\| &\le \left(\alpha_0+\alpha_1\|\breve{u}\|+\alpha_2\|\breve{v}\|\right)B_1+\left(\beta_0+\beta_1\|\breve{u}\|+\beta_2\|\breve{v}\|\right)B_2+\|\breve{u}\|B_3+\|\breve{v}\|B_4.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|\breve{u}\|+\|\breve{v}\| \\
&\quad\le (A_1+B_1)\alpha_0+(A_2+B_2)\beta_0 \\
&\qquad+\left[(A_1+B_1)\alpha_1+(A_2+B_2)\beta_1+A_3+B_3\right]\|\breve{u}\| \\
&\qquad+\left[(A_1+B_1)\alpha_2+(A_2+B_2)\beta_2+(A_4+B_4)\right]\|\breve{v}\|.
\end{aligned}$$

Hence

$$\|(\breve{u},\breve{v})\| \le \frac{(A_1+B_1)\alpha_0+(A_2+B_2)\beta_0}{D^*},$$

where

$$D^* = \min\bigl\{1 - (A_1 + B_1)\alpha_1 - (A_2 + B_2)\beta_1 - (A_3 + B_3),$$
$$1 - (A_1 + B_1)\alpha_2 - (A_2 + B_2)\beta_2 - (A_4 + B_4)\bigr\}.$$

Therefore $D$ is bounded, and by Lemma 1.2 the operator $F$ has a solution that is a solution of system (1.4). □

Now we apply Krasnosel'skii's fixed point theorem (Lemma 1.2) to present another existence result.

**Theorem 3.3** *Let that $f, g : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous functions satisfying condition (3.2) in Theorem 3.1. Besides, let $K_1$, $K_2$ be two positive constants such that for all $t \in [0,1]$ and $\breve{u}_i, \breve{v}_i \in \mathbb{R}$, $i = 1, 2$, we have*

$$\bigl|f(b, \breve{u}_1, \breve{v}_1)\bigr| \le K_1,$$
$$\bigl|g(b, \breve{u}_1, \breve{v}_1)\bigr| \le K_2. \tag{3.10}$$

*Moreover, assume that $A_3 + A_4 < 1$, $B_3 + B_4 < 1$, and $[\frac{1}{\lambda_1\Gamma(k_1+1)}\ell_1 + \frac{1}{\lambda_2\Gamma(k_2+1)}\ell_2] < 1$. Then problem (1.4) admits a solution on $[0,1]$.*

*Proof* First, we decompose the operator $F$ in Theorem 3.1, into four operators

$$\begin{aligned}
H_1(\breve{u},\breve{v})(b) &= \frac{\lambda_1 - 1}{\lambda_1} \times \frac{1}{\Gamma(k_1 - \theta_1)} \int_0^b (b-s)^{k_1-\theta_1-1}\breve{u}(s)\,ds \\
&\quad + \frac{b^{k_1-1}}{\Theta}\biggl[\frac{\gamma(\lambda_2-1)}{\lambda_2} I^{k_2-\theta_2+q_1}\breve{v}(\xi) + \frac{\gamma}{\lambda_2} I^{k_2+q_1} g\bigl(\xi, \breve{u}(\xi), \breve{v}(\xi)\bigr) \\
&\quad + \frac{\Lambda_1\lambda(\lambda_1-1)}{\lambda_1} I^{k_1-\theta_1-\gamma_1}\breve{u}(\eta) + \frac{\Lambda_1\lambda}{\lambda_1} I^{k_1-\gamma_1} f\bigl(\eta, \breve{u}(\eta), \breve{v}(\eta)\bigr) \\
&\quad - \frac{(\lambda_2-1)\Lambda_1}{\lambda_2} I^{k_2-\theta_2}\breve{v}(1) - \frac{\Lambda_1}{\lambda_2} I^{k_2} g\bigl(1, \breve{u}(1), \breve{v}(1)\bigr) \\
&\quad - \frac{\lambda_1-1}{\lambda_1} I^{k_1-\theta_1}\breve{u}(1) - \frac{1}{\lambda_1} I^{k_1} f\bigl(1, \breve{u}(1), \breve{v}(1)\bigr)\biggr], \\
H_2(\breve{u},\breve{v})(b) &= \frac{1}{\lambda_1} \times \frac{1}{\Gamma(k_1)} \int_0^b (b-s)^{k_1-1} f\bigl(s, \breve{u}(s), \breve{v}(s)\bigr)\,ds = \frac{1}{\lambda_1} I^{k_1} f_{\breve{u},\breve{v}}(b), \\
H_3(\breve{u},\breve{v})(b) &= \frac{\lambda_2 - 1}{\lambda_2} \times \frac{1}{\Gamma(k_2 - \theta_2)} \int_0^b (b-s)^{k_2-\theta_2-1}\breve{v}(s)\,ds \\
&\quad + \frac{b^{k_2-1}}{|\Theta|}\biggl[\frac{\lambda(\lambda_1-1)}{\lambda_1} I^{k_1-\theta_1-\gamma_1}\breve{u}(\eta) + \frac{\lambda}{\lambda_1} I^{k_1-\gamma_1} f\bigl(\eta, \breve{u}(\eta), \breve{v}(\eta)\bigr) \\
&\quad - \frac{(\lambda_2-1)}{\lambda_2} I^{k_2-\theta_2}\breve{v}(1) - \frac{1}{\lambda_2} I^{k_2} g\bigl(1, \breve{u}(1), \breve{\bar{v}}(1)\bigr) \\
&\quad + \Lambda_2\biggl(\frac{\gamma(\lambda_2-1)}{\lambda_2} I^{k_2-\theta_2+q_1}\breve{v}(\xi) + \frac{\gamma}{\lambda_2} I^{k_2+q_1} g\bigl(\xi, \breve{u}(\xi), \breve{v}(\xi)\bigr) \\
&\quad - \frac{(\lambda_1-1)}{\lambda_1} I^{k_1-\theta_1}\breve{u}(1) - \frac{1}{\lambda_1} f\bigl(1, \breve{u}(1), \breve{v}(1)\bigr)\biggr)\biggr],
\end{aligned}$$

$$H_4(\check{u},\check{v})(t) = \frac{1}{\lambda_2} \times \frac{1}{\Gamma(k_2)} \int_0^b (b-s)^{k_2-1} g\big(s,\check{u}(s),\check{v}(s)\big) = \frac{1}{\lambda_2} I^{k_2} g_{\check{u},\check{v}}(b). \tag{3.11}$$

Accordingly, $F_1(\check{u},\check{v})(b) = H_1(\check{u},\check{v})(b) + H_2(\check{u},\check{v})(b)$ and $F_2(\check{u},\check{v})(b) = H_3(\check{u},\check{v})(b) + H_4(\check{u},\check{v})(b)$. Define $D_\varepsilon = \{(\check{u},\check{v}) \in \overline{\mathfrak{X}} \times \overline{\mathfrak{X}}; \|(\check{u},\check{v})\| \le \varepsilon\}$ with

$$\varepsilon \ge \max\left\{\frac{A_1K_1 + A_2K_2}{1-(A_3+A_4)}, \frac{B_1K_1 + B_2K_2}{1-(B_3+B_4)}\right\}.$$

We claim that $F_1(\check{u},\check{v}) + F_2(\check{u},\check{v}) \in D_\varepsilon$. For all $(\check{u},\check{v}) \in D_\varepsilon$, we have

$$\begin{aligned} &\big|H_1(\check{u},\check{v})(b) + H_2(\check{u},\check{v})(b)\big| \le A_1K_1 + A_2K_2 + (A_3+A_4)\varepsilon \le \varepsilon, \\ &\big|H_3(\check{x},\check{y})(b) + H_4(\check{x},\check{y})(b)\big| \le B_1K_1 + B_2K_2 + (B_3+B_4)\varepsilon \le \varepsilon. \end{aligned} \tag{3.12}$$

Consequently, $F_1(\check{u},\check{v}) + F_2(\check{u},\check{v}) \in D_\varepsilon$, and we obtain condition (i) of Lemma 1.2. Now we show that the operator $(H_2,H_4)$ is a contraction. For $(\check{x}_1,\check{y}_1),(\check{u}_1,\check{v}_1) \in D_\varepsilon$, we have

$$\begin{aligned} &\big|H_2(\check{x}_1,\check{y}_1)(b) - H_2(\check{u}_1,\check{v}_1)(b)\big| \\ &\quad\le \frac{1}{\lambda_1} I^{k_1} |f_{\check{x}_1,\check{y}_1} - f_{\check{u}_1,\check{v}_1}|(b) \\ &\quad\le \frac{\ell_1}{\lambda_1}\big(\|\check{x}_1-\check{u}_1\| + \|\check{y}_1-\check{v}_1\|\big) I^{k_1}(1)(1) \\ &\quad\le \frac{\ell_1}{\lambda_1\Gamma(k_1+1)}\big(\|\check{x}_1-\check{u}_1\| + \|\check{y}_1-\check{v}_1\|\big) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\big|H_4(\check{x}_1,\check{y}_1)(b) - H_4(\check{u}_1,\check{v}_1)(b)\big| \\ &\quad\le \frac{1}{\lambda_2} I^{k_2} |g_{\check{x}_1,\check{y}_1} - g_{\check{u}_1,\check{v}_1}|(b) \\ &\quad\le \frac{\ell_2}{\lambda_2\Gamma(k_2)}\big(\|\check{x}_1-\check{u}_1\| + \|\check{y}_1-\check{v}_1\|\big) I^{k_2}(1)(1) \\ &\quad\le \frac{\ell_2}{\lambda_2\Gamma(k_2+1)}\big(\|\check{x}_1-\check{u}_1\| + \|\check{y}_1-\check{v}_1\|\big). \end{aligned} \tag{3.14}$$

Due to (3.13) and (3.14), we get

$$\begin{aligned} &\big\|(H_2,H_4)(\check{x}_1,\check{y}_1) - (H_2,H_4)(\check{u}_1,\check{v}_1)\big\| \\ &\quad\le \left[\frac{\ell_1}{\lambda_1\Gamma(k_1+1)} + \frac{\ell_2}{\lambda_2\Gamma(k_2+1)}\right]\big(\|\check{x}_1-\check{u}_1\| + \|\check{y}_1-\check{v}_1\|\big). \end{aligned} \tag{3.15}$$

Since $\frac{\ell_1}{\lambda_1\Gamma(k_1+1)} + \frac{\ell_2}{\lambda_2\Gamma(k_2+1)} < 1$, the operator $(H_2,H_4)$ is a contraction, and condition (iii) of Lemma 1.2 is satisfied. Now we verify condition (ii) of Lemma 1.2 for the operator $(H_1,H_3)$. Since $f, g$ are continuous, the operator $(H_1,H_3)$ is also continuous. Moreover, for any $(\check{u},\check{v}) \in D_\varepsilon$, similarly to the proof of Theorem 3.1, we have

$$\big|H_1(\check{u},\check{v})(b)\big| \le \left(A_1 - \frac{1}{\lambda_1\Gamma(k_1+1)}\right)K_1 + A_2K_2 + (A_3+A_4)\varepsilon = G^*,$$

$$\left|H_3(\breve{u},\breve{v})(b)\right| \le B_1K_1 + \left(B_2 - \frac{1}{\lambda_2\Gamma(k_2+1)}\right)K_2 + (B_3+B_4)\varepsilon = L^*. \tag{3.16}$$

Accordingly, we have $\|(H_1,H_3)(\breve{u},\breve{v})\| \le G^* + L^*$, which implies that $(H_1,H_3)D_\varepsilon$ is uniformly bounded. Finally, we show that the set $(H_1,H_3)D_\varepsilon$ is equicontinuous. Let $\tau_1,\tau_2 \in [0,1]$ be such that $\tau_1 < \tau_2$. For any $(\breve{u},\breve{v}) \in D_\varepsilon$, as in the proof of the existence in Theorem 1.2, we can show that $|H_1(\breve{u},\breve{v})(\tau_2) - H_1(\breve{u},\breve{v})(\tau_1)|$, $|H_3(\breve{u},\breve{v})(\tau_2) - H_3(\breve{u},\breve{v})(\tau_1)| \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$. Consequently, the set $(H_1,H_3)D_\varepsilon$ is equicontinuous, and by the Arzelà–Ascoli theorem $(H_1,H_3)$ is compact on $D_\varepsilon$. Therefore by Lemma 1.2 problem (1.4) has a solution on $[0,1]$. This completes the proof. □

## 4 Example

*Example* 4.1 Define the system of BVPs with integro-differential conditions

$$\begin{cases} (\frac{1}{2}D^2\breve{u}(\breve{z}) + \frac{1}{2}D^{\frac{3}{2}}\breve{u}(\breve{z})) = f(\breve{z},\breve{u}(\breve{z}),\breve{v}(\breve{z})), \\ z \quad (\frac{1}{3}D^{\frac{1}{2}}\breve{v}(\breve{z}) + \frac{2}{3}D^{\frac{1}{4}}\breve{v}(\breve{z})) = g(\breve{z},\breve{u}(\breve{z}),\breve{v}(\breve{z})), \\ \breve{\breve{u}}(0) = 0, \qquad \breve{v}(0) = 0, \qquad \breve{v}(1) = 2D^{\frac{1}{4}}\breve{u}(\frac{1}{4}), \qquad \breve{\breve{u}}(1) = -10I^1\breve{v}(\frac{2}{5}). \end{cases} \tag{4.1}$$

Here $k_1 = 2, \theta_1 = 1.3, k_2 = \frac{3}{2}, \theta_2 = 1.4, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda = 2, \gamma_1 = \frac{1}{4}, \eta = \frac{1}{4}, \gamma = -10, \xi = \frac{2}{5}, q_1 = 1$, $\Lambda_1 \approx -1.686738$, $\Lambda_2 \approx 0.034043$, $\Theta \approx 1.040554$, $A_1 \approx -1.533673$, $A_2 \approx -12.333418$, $A_3 \approx -1.46116$, $A_4 \approx -7.597032$, $B_1 \approx 2.413185$, $B_2 \approx 4.217126$, $B_3 \approx 2.182248$, $B_4 \approx 3.245973$.

*(i)* If the functions $f, g$ are taken as

$$f(\breve{z},\breve{u},\breve{v}) = \frac{e^{-\breve{z}}}{10^3}\sin\bigl(\left|\breve{u}(\breve{z}) - \breve{v}(\breve{z})\right|\bigr), \qquad g(\breve{z},\breve{u},\breve{v}) = \frac{e^{-\breve{z}}}{10^2}\cos\bigl(\left|\breve{u}(\breve{z}) - \breve{v}(\breve{z})\right|\bigr),$$

then we can verify that

$$\begin{aligned} \left|f(\breve{z},\breve{u}_1,\breve{v}_1) - f(\breve{z},\breve{u}_2,\breve{v}_2)\right| &\le \frac{1}{10^3}\bigl(|\breve{u}_1 - \breve{\breve{u}}_2| + |\breve{v}_1 - \breve{v}_2|\bigr), \\ \left|f(\breve{z},\breve{u}_1,\breve{v}_1) - f(\breve{z},\breve{u}_2,\breve{v}_2)\right| &\le \frac{1}{10^2}\bigl(|\breve{u}_1 - \breve{u}_2| + |\breve{v}_1\breve{v}_2|\bigr). \end{aligned}$$

Hence $f, g$ satisfy the Lipschitz conditions in Theorem 3.1 with $\ell_1 = \frac{1}{10^3}$ and $\ell_2 = \frac{1}{10^2}$. Moreover,

$$\ell_1(A_1+B_1) + \ell_2(A_2+B_2) + (A_3+A_4+B_3+B_4) \approx -3.710254 < 1.$$

Hence all conditions of Theorem 3.1 are satisfied, and as a conclusion, we can infer that the coupled system (4.1) has a unique solution on $[0,1]$.

*(ii)* Now define the functions $f, g$ as

$$\begin{aligned} f(\breve{z},\breve{u},\breve{v}) &= \frac{1}{\breve{z}+1} + \frac{1}{3}\left(\frac{|\breve{u}|}{8+|\breve{u}|}\right) + \frac{1}{5}e^{-\breve{v}}|\breve{v}|, \\ g(\breve{z},\breve{u},\breve{v}) &= \frac{|\cos\breve{u}|+1}{4} + \frac{1}{10}\bigl(1+|\sin\breve{v}|\bigr)|\breve{u}| + \frac{|\breve{v}|}{2(2+|\breve{v}|)}. \end{aligned}$$

Accordingly,

$$\bigl|f(\check{z},\check{u},\check{v})\bigr| \le 1 + \frac{1}{3}|\check{u}| + \frac{1}{5}|\check{v}|,$$
$$\bigl|g(\check{z},\check{u},\check{v})\bigr| \le \frac{1}{2} + \frac{1}{5}|\check{u}| + \frac{1}{2}|\check{v}|.$$

Thus in view of Theorem 3.2, the constants $\check{u}_i$, $\check{v}_i$ are obtained by $\alpha_0 = 1$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{5}$, $\beta_0 = \frac{1}{2}$, $\beta_1 = \frac{1}{5}$, $\beta_2 = \frac{1}{2}$. On the other hand, we have

$$(A_1 + B_1)\alpha_1 + (A_2 + B_2)\beta_1 + A_3 + B_3 \approx -0.609 < 1,$$
$$(A_1 + B_1)\alpha_2 + (A_2 + B_2)\beta_2 + A_4 + B_4 \approx -8.219972 < 1.$$

Consequently, by Theorem 3.2 we conclude that the coupled system (4.1) has a solution on $[0,1]$.

*(iii)* Now we take the nonlinear bounded functions $f$, $g$ defined as

$$f(\check{z},\check{u},\check{v}) = \frac{1}{3} + \frac{1}{4}\check{z} + \frac{1}{12}\left(\frac{|\check{u}|}{1+|\check{u}|}\right) + \frac{1}{2}\cos|\check{v}|,$$
$$g(\check{z},\check{u},\check{v}) = 1 + \frac{4}{44}\tan^{-1}|\check{u}| + \frac{4}{84}\left(\frac{|\check{v}|}{1+|\check{v}|}\right).$$

We easily see that these functions are bounded since

$$\bigl|f(\check{z},\check{u},\check{v})\bigr| \le \frac{9}{6}, \qquad \bigl|g(\check{z},\check{u},\check{v})\bigr| \le \frac{96}{84}.$$

Besides, we conclude condition $(H_1)$ of Theorem 3.1 with $\ell_1 = \frac{1}{2}$ and $\ell_2 = \frac{4}{44}$. Consequently, $[\frac{1}{\lambda_1\Gamma(k_1+1)}\ell_1 + \frac{1}{\lambda_2\Gamma(k_2+1)}\ell_2] = \frac{1}{2} + \frac{12}{44\Gamma(\frac{5}{2})} < 1$. On the other hand, we have $A_3 + A_4 \approx -9.058863 < 1$ and $B_3 + B_4 \approx 5.428221 > 1$. Therefore we cannot apply Theorem 3.3 to obtain a solution of problem (4.1) on $[0,1]$.

## 5 Conclusions

In this paper, we investigated a coupled system of fractional differential equations involving integro-differential conditions. Firstly, we presented a lemma that was the basic tool in proving the main results. After that, by applying Banach's fixed point theorem we investigated the existence of a unique solution for system (1.4). We applied the Leray–Schauder alternative and Krasnosel'skii's fixed point theorems to obtain the existence of solutions for system (1.4). Moreover, our main results were supported by presented examples. The standard methods have been used to prove the main results, but new formation is constructed on problem (1.4).

**Acknowledgements**
The second and third authors were supported by Azarbaijan Shahid Madani University.

**Funding**
Not applicable.

**Availability of data and materials**
Data sharing not applicable to this paper no datasets were generated or analyzed during the current study.

## Declarations

### Competing interests
The authors declare no competing interests.

### Author contributions
AS and JM prepared the original draft and MM checked, edited and prepared the final draft of the manuscript. All authors read and approved the final manuscript.

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## Publisher's Note


Received: 29 May 2022 Accepted: 12 September 2022 Published online: 20 September 2022

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