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Adapting step size algorithms for solving split equilibrium problems with applications to signal recovery

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Abstract

Recent developments in split equilibrium problems (SEPs) have found practical applications in convex optimization problems, information theory, and signal processing. In this paper, we present three novel algorithms with no prior knowledge of the operator norm of a bounded linear operator to approximated solutions for SEPs. Strong convergence results are well presented under appropriate conditions. In addition, we illustrate our main results by providing various numerical examples. Their computational performances are compared with those previously studied in the literature, and the results are presented by showing numerical implementation of the sparse sensor signal recovery problem.

Keywords: Fixed point problem; Self-adaptive step size; Equilibrium problem; Signal recovery

1 Introduction

Throughout this work, let two nonempty closed convex subsets C and D be in two real Hilbert spaces H_1 and H_2 , respectively. This paper focuses on the split equilibrium problems (SEPs) introduced by He [1], which are used to solve a point $w^* \in C$ that

$$f(w^*, p) \geq 0, \quad \forall p \in C, \quad (1)$$

and a point $v^* = Aw^* \in D$ solves

$$g(v^*, q) \geq 0, \quad \forall q \in D, \quad (2)$$

where f and g are nonlinear bi-functions on $C \times C$ and $D \times D$, respectively, and an operator $A : H_1 \rightarrow H_2$ is bounded linear with its adjoint operator A^* . The equilibrium problem plays a vital role in various branches of science, optimization, and economics (see [2–6] for details). We provide a set of all solutions for the split equilibrium problems (SEPs) (1)–(2)

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as follows:

$$\Omega_{\text{SEPs}} = \{u \in \text{EP}(f) : Au \in \text{EP}(g)\}, \quad (3)$$

where $\text{EP}(f)$ and $\text{EP}(g)$ denote the solution sets of classical equilibrium problems (1) and (2), respectively.

Over a decade, many methods have been constructed and developed to find solutions for SEPs (1)–(2). The development of algorithms to solve the SEPs is stated as follows. Kazmi and Rizvi [7], in 2013, devised an iterative technique to examine a solution for the SEPs. Let a real number L_A be the spectral radius of A^*A , P_C be an orthogonal projection onto C , and a map $E : C \rightarrow H_1$ be α -inverse strongly monotone. Their algorithm was defined as follows:

$$\begin{cases} w_1 \in H_1, & \forall n \geq 1, \\ v_n = T_{r_n}^f(I + \gamma A^*(T_{r_n}^g - I)A)w_n, \\ u_n = P_C(I - \lambda_n E)v_n, \\ w_{n+1} = \mu_n u + \rho_n w_n + \sigma_n W u_n, \end{cases} \quad (4)$$

where $\gamma \in (0, \frac{1}{L_A})$, $r_n > 0$, $\lambda_n \in (2, 2\alpha)$, and real sequences $\{\mu_n\}$, $\{\rho_n\}$, and $\{\sigma_n\}$ are in $(0, 1)$. Under suitable conditions of all parameters, the iterative method (4) provided the strong convergence for the SEPs.

In 2016, Suantai [8] presented an iterative algorithm for solving the SEPs as follows:

$$\begin{cases} x_1 \in C, & \forall n \geq 1, \\ u_n = T_{r_n}^f(I + \gamma A^*(T_{r_n}^g - I)A)x_n, \\ x_{n+1} \in \mu_n x_n + (1 - \mu_n)W u_n, \end{cases} \quad (5)$$

where W is a $\frac{1}{2}$ -nonspreading multivalued mapping, $\gamma \in (0, \frac{1}{L_A})$, $\{r_n\} \subseteq (0, \infty)$, and $\{\mu_n\} \subseteq (0, 1)$. Suantai also provided a weak convergence result.

By using Suantai's ideas, Onjai-uea and Phuengrattana [9] constructed and developed an algorithm to solve the SEPs in 2017 as follows:

$$\begin{cases} x_1 \in C, & \forall n \geq 1, \\ u_n = T_{r_n}^f(I + \gamma A^*(T_{r_n}^g - I)A)x_n, \\ y_n \in \alpha_n x_n + (1 - \alpha_n)W u_n, \\ x_{n+1} \in \mu_n x_n + (1 - \mu_n)W y_n, \end{cases} \quad (6)$$

where W is a multivalued λ -hybrid mapping, $\gamma \in (0, \frac{1}{L_A})$, $\{r_n\} \subseteq (0, \infty)$, and $\{\alpha_n\}, \{\mu_n\} \subseteq (0, 1)$. Under some appropriate conditions, Onjai-uea and Phuengrattana presented that algorithm (6) weakly converges to a solution for the SEPs. There are many methods to solve the SEPs that are not mentioned above. The reader can refer to [1, 10–12] for details.

Note that the iterative methods (4), (5), and (6) are usually used with the stepsize γ that depends on the operator norm $\|A\|$. Hence this paper is focused on overcoming this difficulty by constructing algorithms with no prior knowledge of operator norm $\|A\|$. For

more details about these ideas, the reader is directed to [13, 14]. For any $r_n > 0$, we define

$$\psi(x) := \frac{1}{2} \|(I - T_{r_n}^f)x\|^2; \quad (7)$$

$$\omega(x) := \frac{1}{2} \|(I - T_{r_n}^g)Ax\|^2. \quad (8)$$

Consider in case that f and g are indicator functions for the closed convex subsets C and Q of Hilbert spaces, then $T_{r_n}^f = P_C$ and $T_{r_n}^g = P_Q$. Recall that the squared distance functions $\|x - P_C x\|^2$ and $\|x - P_Q x\|^2$ are differentiable, we have the gradients $\nabla \psi$ and $\nabla \omega$ of operators ω and ψ , respectively, as follows:

$$\nabla \psi(x) := (I - T_{r_n}^f)x; \quad (9)$$

$$\nabla \omega(x) := A^*(I - T_{r_n}^g)Ax. \quad (10)$$

However, gradients (9) and (10) are not true in general.

By using operators (8)–(9), we construct a new step size by using self-adaptive method to avoid computing the operator norm $\|A\|$ as follows:

$$\tau_n := \begin{cases} \frac{\rho_n \omega(u_n)}{\|\nabla \omega(u_n)\|^2 + \|\nabla \psi(u_n)\|^2} & \|\nabla \omega(u_n)\|^2 + \|\nabla \psi(u_n)\|^2 \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

where $0 < \rho_n < 4$.

Inspired by the above works, the aims of this papers are as follows: (1) To construct and propose three modified step size algorithms by using new self-adaptive techniques; (2) to obtain theoretical results of strong convergence for the SEPs based on the proposed algorithms; (3) to show the performance of our proposed algorithms by comparing them to the previously developed algorithms by Kazmi and Rizvi [7], Suantai [8], Onjai-uea and Phuengrattana [9]; and (4) to demonstrate the presented results by showing numerical implementation of the sparse sensor signal recovery problem.

The remainder of this paper is organized as follows. Section 2 establishes preliminaries used in this work, and related definitions and lemma are presented. Section 3 introduces the proposed step size algorithms for solving the SEPs with three modifications, and the main theorems are proved. Section 4 then demonstrates the numerical examples of the proposed algorithms and shows theoretical applications in split feasibility problem and optimization problem. Section 5 then gives conclusions of this work.

2 Preliminaries

In this section, we give a preliminary result that is necessary for our consequent analysis. In the weak topology, the set of all limit points is given by $\omega_w(u_n)$ for a sequence $\{u_n\}$ in a real Hilbert space. Let $f : C \times C \rightarrow \mathbb{R}$ and $g : D \times D \rightarrow \mathbb{R}$ be nonlinear bi-functions. Suppose the following assumptions for a bi-function f :

- (A1) $f(s, s) \geq 0$;
- (A2) $f(s, t) + f(t, s) \leq 0$ (f is monotone);
- (A3) $\limsup_{\zeta \rightarrow 0} f(\zeta s + (1 - \zeta)t, w) \leq f(t, w)$;
- (A4) $s \mapsto f(t, s)$ is a lower semicontinuous and convex function;

for all $s, t, w \in C$. Let $T_r^f : H_1 \rightarrow C$ be a mapping given by

$$T_r^f(w) = \left\{ s \in C : f(s, t) + \frac{1}{r} \langle t - s, s - w \rangle \geq 0, \forall t \in C \right\} \quad (12)$$

for $r > 0$ and for all $w \in H_1$.

Definition 2.1 For any $v, w \in C$, the mapping $S : C \rightarrow H_1$ is said to be

1. *monotone* if $\langle v - w, Sv - Sw \rangle \geq 0$;
2. *ϑ -inverse strongly monotone* (for short ϑ -ism) if $\langle w - v, Sw - Sv \rangle \geq \vartheta \|Sw - Sv\|^2$ for some $\vartheta > 0$;
3. *L -Lipschitz continuous* if $\|Sw - Sv\| \leq L\|w - v\|$ for some $L \in [0, 1]$;
4. *nonexpansive* if $\|Sw - Sv\| \leq \|w - v\|$;
5. *firmly nonexpansive* if $\|Sw - Sv\|^2 \leq \langle Sw - Sv, w - v \rangle$.

Recall that every nonexpansive mapping S satisfies the following inequalities:

1. $\langle (v - Sv) - (w - Sw), Sw - Sv \rangle \leq \frac{1}{2} \|(Sv - v) - (Sw - w)\|^2$;
2. $\langle v - Sv, w - Sv \rangle \leq \frac{1}{2} \|Sv - v\|^2$.

Lemma 2.2 For any $v, w \in H_1$ and $\theta \in [0, 1]$, the following relationships hold:

1. $\|w + u\|^2 \leq \|w\|^2 + 2\langle w, u \rangle$;
2. $\|w - u\|^2 = \|w\|^2 + \|u\|^2 - 2\langle w, u \rangle$;
3. $\|\theta w + (1 - \theta)u\|^2 = \theta\|w\|^2 + (1 - \theta)\|u\|^2 - \theta(1 - \theta)\|w - u\|^2$.

Lemma 2.3 For the metric projection P_C , the following relationships hold:

1. P_C is a nonexpansive mapping;
2. $\|P_C v - P_C w\|^2 \leq \langle v - w, P_C v - P_C w \rangle$;
3. $\langle v - P_C v, w - P_C w \rangle \leq 0$;
4. $\|v - P_C v\|^2 + \|w - P_C v\|^2 \leq \|v - w\|^2$;
5. $\|v - w\|^2 - \|P_C v - P_C w\|^2 \leq \|(v - w) - (P_C v - P_C w)\|^2$;

for any $v, w \in H_1$.

Lemma 2.4 ([3]) Assume that a bi-function f satisfies assumptions (A1)–(A4) and the mapping T_r^f is defined by equation (12). Then:

1. T_r^f is nonempty, single-valued, and firmly nonexpansive;
2. $\text{EP}(f) = \text{Fix}(T_r^f)$;
3. $\text{EP}(f)$ is a closed convex set.

Moreover, see in [15], the mapping T_r^f satisfies the following inequality:

$$\|T_s^f v - T_r^f w\| \leq \|v - w\| + \left| \frac{s - r}{s} \right| \|T_s^f w - w\| \quad (13)$$

for any $v, w \in H_1$ and $r, s > 0$.

Lemma 2.5 ([16]) Let $\{w_n\}$ and $\{v_n\}$ be two bounded sequences and a real sequence $\{\zeta_n\} \subseteq [0, 1]$ such that

$$w_{n+1} = (1 - \zeta_n)v_n + \zeta_n w_n, \quad \forall n \geq 1.$$

If the following statements hold:

1. $\limsup_{n \rightarrow \infty} (\|v_{n+1} - v_n\| - \|w_{n+1} - w_n\|) \leq 0$;
2. $0 < \liminf_{n \rightarrow \infty} \zeta_n \leq \limsup_{n \rightarrow \infty} \zeta_n < 1$.

Then $\lim_{n \rightarrow \infty} \|v_n - w_n\| = 0$.

Lemma 2.6 ([17]) *The Hilbert space H satisfies the Opial condition, if $w_n \rightharpoonup w$ for any sequence $\{w_n\} \in H$ and element $v \in H$ with $v \neq w$, then*

$$\liminf_{n \rightarrow \infty} \|w_n - w\| < \liminf_{n \rightarrow \infty} \|w_n - v\|.$$

Lemma 2.7 ([18]) *Let $\{w_n\} \subseteq (0, \infty)$ be a sequence of real numbers with*

$$w_{n+1} \leq (1 - \mu_n)w_n + \mu_n\delta_n + \xi_n$$

for all $n \geq 1$, where

1. $\{\mu_n\} \subset [0, 1]$ and $\sum \mu_n = \infty$,
2. $\limsup \delta_n \leq 0$,
3. $\xi_n \geq 0$ and $\sum \xi_n < \infty$.

Then $\lim_{n \rightarrow \infty} w_n = 0$.

Lemma 2.8 ([19]) *Suppose that a real number sequence $\{\Gamma_m\}$ does not decrease at infinity. This means that there is $\{\Gamma_{m_i}\} \subseteq \{\Gamma_m\}$ with $\{\Gamma_{m_i}\} < \{\Gamma_{m_i+1}\} \forall i \geq 0$. Let a sequence of integers $\{\sigma(m)\}_{m \geq m_0}$ be given by*

$$\sigma(m) = \max\{l \leq m : \Gamma_l \leq \Gamma_{l+1}\}.$$

Thus $\{\sigma(m)\}_{m \geq m_0}$ is a nondecreasing sequence that verifies $\lim_{m \rightarrow \infty} \sigma(m) = \infty$,

$$\max\{\Gamma_{\sigma(m)}, \Gamma_m\} \leq \Gamma_{\sigma(m)+1}$$

for all $m \geq m_0$.

Lemma 2.9 *Assume that $\psi : H_1 \rightarrow \mathbb{R}$ and $\omega : H_2 \rightarrow \mathbb{R}$ are two functions given by equations (7) and (8), respectively. Then the gradients $\nabla\omega$ and $\nabla\psi$ of the functions ω and ψ are Lipschitz continuous.*

Proof Since $\nabla\omega(p) = A^*(I - T_{r_n}^g)Ap$, we have

$$\begin{aligned} & \|\nabla\omega(p) - \nabla\omega(q)\|^2 \\ &= \langle A^*((I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq), A^*((I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq) \rangle \\ &= \langle (I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq, AA^*((I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq) \rangle \\ &\leq L_A \|(I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq\|^2, \end{aligned} \quad (14)$$

where $L_A = \|A\|^2$. On the other hand,

$$\langle \nabla\omega(p) - \nabla\omega(q), p - q \rangle = \langle A^*((I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq), p - q \rangle$$

$$\begin{aligned}
&= \langle (I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq, A(p - q) \rangle \\
&\geq \| (I - T_{r_n}^g)Ap - (I - T_{r_n}^g)Aq \|^2.
\end{aligned} \tag{15}$$

By combining (14) with (15), we get that

$$\langle \nabla \omega(p) - \nabla \omega(q), p - q \rangle \geq \frac{1}{L_A} \|\nabla \omega(p) - \nabla \omega(q)\|^2.$$

Thus $\nabla \omega$ is $\frac{1}{L_A}$ -inverse strongly monotone. Moreover,

$$\langle \nabla \omega(p) - \nabla \omega(q), p - q \rangle \leq \|p - q\| \|\nabla \omega(p) - \nabla \omega(q)\|.$$

Hence $\|\nabla \omega(p) - \nabla \omega(q)\| \leq L_A \|p - q\|$. Similarly, $\nabla \psi$ is Lipschitz continuous. \square

3 Results

This section presents and analyzes our three iterative algorithms generated sequences that strongly converge to a solution of SEPs (1)–(2) and the fixed point problems of a nonexpansive mapping. In the sequel, the solution set is given by

$$\Psi = \Omega_{\text{SEPs}} \cap \text{Fix}(S). \tag{16}$$

Before starting the results, assume that two bi-functions f and g satisfy assumptions (A1)–(A4) and g is upper semicontinuous. Let a mapping S be nonexpansive on C and $\Psi \neq \emptyset$. Now, we constructed and then analyzed three algorithms as follows.

Condition 3.1 Suppose that $\{r_n\} \subseteq (0, \infty)$, a is a constant, $\{\rho_n\}$ is any positive sequence, and real parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\kappa_n\}$ satisfy $\alpha_n, \beta_n, \gamma_n, \kappa_n \in (0, 1)$, and the following conditions hold:

- (C1) $\alpha_n + \beta_n + \gamma_n = 1$;
- (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C3) $\lim_{n \rightarrow \infty} \kappa_n = 0$ and $\lim_{n \rightarrow \infty} \frac{\kappa_n}{\alpha_n} = 0$;
- (C4) $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$ and $0 < a \leq r_n$;
- (C5) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C6) $\lim_{n \rightarrow \infty} (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$;
- (C7) $\inf \rho_n(4 - \rho_n) > 0$ and $0 < \rho_n < 4$.

Theorem 3.1 The sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to a solution $s \in \Psi$, where $s = P_{\Psi}u$.

Proof Since Ψ is a nonempty set, take each $c \in \Psi$, this implies $c = T_{r_n}^f(c)$ and $Ac = T_{r_n}^g(Ac)$. Thus $(I - T_{r_n}^g)Ac = Ac - Ac = 0$. Since $\nabla \omega(x_n) = A^*(T_{r_n}^g - I)Ax_n$ and $T_{r_n}^g - I$ is a firmly nonexpansive mapping, we find that

$$\begin{aligned}
\langle \nabla \omega(x_n), x_n - c \rangle &= \langle A^*(T_{r_n}^g - I)Ax_n, x_n - c \rangle \\
&= \langle (T_{r_n}^g - I)Ax_n - (T_{r_n}^g - I)As, Ax_n - Ac \rangle
\end{aligned}$$

Algorithm 1 The first algorithm to solve split equilibrium problems

- **Initialization:** Let $x_1, u \in H_1$ be arbitrarily chosen. Choose some positive sequences $\{\rho_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\kappa_n\}$ satisfying Condition 3.1. Put $n = 1$.
- **Iterative step:** Given the current iterate x_n , for $r_n > 0$, compute

$$u_n = T_{r_n}^f(x_n + \tau_n A^*(T_{r_n}^g - I)Ax_n),$$

where

$$\tau_n = \begin{cases} \frac{\rho_n \omega(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} & \|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2 \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S[\kappa_n v + (1 - \kappa_n)u_n].$$

- **Stopping criterion:** If $x_{n+1} = x_n = u_n$, stop. If not, set $n = n + 1$ and go to **Iterative step**.
-

$$\begin{aligned} &\geq \|(T_{r_n}^g - I)Ax_n\|^2 \\ &= 2\omega(x_n). \end{aligned} \tag{17}$$

By using the fact that $T_{r_n}^f(I + \tau_n A^*(T_{r_n}^g - I)A)$ is a nonexpansive mapping, we can easily verify that

$$\begin{aligned} \|u_n - c\|^2 &= \|T_{r_n}^f(I + \tau_n A^*(T_{r_n}^g - I)A)x_n - T_{r_n}^f c\|^2 \\ &\leq \|(I + \tau_n A^*(T_{r_n}^g - I)A)x_n - c\|^2 \\ &= \|x_n - \tau_n \nabla \omega(x_n) - c\|^2 \\ &\leq \|x_n - c\|^2 + \tau_n^2 \|\nabla \omega(x_n)\|^2 - 2\tau_n \langle x_n - c, \nabla \omega(x_n) \rangle \\ &\leq \|x_n - c\|^2 + \tau_n^2 \|\nabla \omega(x_n)\|^2 - 4\tau_n \omega(x_n) \\ &= \|x_n - c\|^2 + \frac{\rho_n^2 \omega^2(x_n)}{(\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2)^2} \|\nabla \omega(x_n)\|^2 \\ &\quad - 4 \frac{\rho_n \omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \\ &\leq \|x_n - c\|^2 + \frac{\rho_n^2 \omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \\ &\quad - 4 \frac{\rho_n \omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \\ &= \|x_n - c\|^2 - \rho_n(4 - \rho_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2}. \end{aligned} \tag{18}$$

This implies that

$$\|u_n - c\| \leq \|x_n - c\|. \tag{19}$$

Now, let $v \in C$ be fixed and $y_n = \kappa_n v + (1 - \kappa_n)u_n$. We find that

$$\begin{aligned}\|y_n - c\| &= \|\kappa_n v + (1 - \kappa_n)u_n - c\| \\ &\leq (1 - \kappa_n)\|u_n - c\| + \kappa_n\|v - c\| \\ &\leq (1 - \kappa_n)\|x_n - c\| + \kappa_n\|v - c\|.\end{aligned}\quad (20)$$

Also, we find that

$$\begin{aligned}\|y_n - c\|^2 &= \|(1 - \kappa_n)(u_n - c) + \kappa_n(v - c)\|^2 \\ &\leq (1 - \kappa_n)\|u_n - c\|^2 + 2\kappa_n\langle v - c, y_n - c \rangle.\end{aligned}\quad (21)$$

By setting $\varrho_n = (\alpha_n + \kappa_n\gamma_n)$ and using (18) and (20), we obtain

$$\begin{aligned}\|x_{n+1} - c\| &= \|\alpha_n u + \beta_n x_n + \gamma_n S y_n - c\| \\ &= \beta_n\|x_n - c\| + \|\alpha_n(u - c) + \gamma_n(S y_n - c)\| \\ &\leq \beta_n\|x_n - c\| + \alpha_n\|u - c\| + \gamma_n\|y_n - c\| \\ &\leq \alpha_n\|u - c\| + \beta_n\|x_n - c\| + \gamma_n\{\kappa_n\|v - c\| + (1 - \kappa_n)\|x_n - c\|\} \\ &= (\beta_n + \gamma_n - \kappa_n\gamma_n)\|x_n - c\| + \alpha_n\|u - c\| + \kappa_n\gamma_n\|v - c\| \\ &\leq (1 - \varrho_n)\|x_n - c\| + \varrho_n \max\{\|u - c\|, \|v - c\|\} \\ &\leq \max\{\|x_n - c\|, \|u - c\|, \|v - c\|\}.\end{aligned}\quad (22)$$

It concludes that $\{x_n\}$ is bounded, and two sequences $\{u_n\}$, $\{y_n\}$ are also bounded. Because $c = T_{r_n}^f c$ and $T_{r_n}^f$ is a firmly nonexpansive mapping, thus

$$\begin{aligned}\|u_n - c\|^2 &= \|T_{r_n}^f(x_n + \tau_n A^*(T_{r_n}^g - I)Ax_n) - c\|^2 \\ &\leq \langle x_n + \tau_n A^*(T_{r_n}^g - I)Ax_n - c, u_n - c \rangle \\ &= \frac{1}{2}\{\|x_n - s\|^2 - \|x_n - u_n\|^2 - \tau_n^2\|\nabla\omega(x_n)\|^2 + \|u_n - c\|^2 \\ &\quad + 2\tau_n\langle \nabla\omega(x_n), x_n - u_n \rangle\} \\ &\leq \|x_n - c\|^2 - \|x_n - u_n\|^2 + 2\tau_n\|x_n - u_n\|\|\nabla\omega(x_n)\|.\end{aligned}\quad (23)$$

Thus, by using inequalities (21) and (23), we get that

$$\begin{aligned}\|x_{n+1} - c\|^2 &\leq \beta_n\|x_n - c\|^2 + \alpha_n\|u - c\|^2 + \gamma_n\|S y_n - c\|^2 \\ &\leq \beta_n\|x_n - c\|^2 + \alpha_n\|u - c\|^2 + \kappa_n\gamma_n\|v - c\|^2 + \gamma_n(1 - \kappa_n)\|u_n - c\|^2 \\ &\leq \alpha_n\|u - c\|^2 + \beta_n\|x_n - c\|^2 + \kappa_n\gamma_n\|v - c\|^2 \\ &\quad + (\gamma_n - \kappa_n\gamma_n)\{\|x_n - c\|^2 - \|x_n - u_n\|^2 + 2\tau_n\|x_n - u_n\|\|\nabla\omega(x_n)\|\} \\ &\leq \alpha_n\|u_n - c\|^2 + (1 - \alpha_n - \kappa_n\gamma_n)\|x_n - c\|^2 + \kappa_n\gamma_n\|v - c\|^2 \\ &\quad + (\kappa_n\gamma_n - \gamma_n)\|x_n - u_n\|^2 + 2\rho_n\gamma_n(1 - \kappa_n)\omega(x_n)\|x_n - u_n\|\end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \|u - c\|^2 + \|x_n - c\|^2 + \kappa_n \gamma_n \|v - c\|^2 - (\gamma_n + \kappa_n \gamma_n) \|x_n - u_n\|^2 \\ &\quad + 2\rho_n (\gamma_n - \kappa_n \gamma_n) \omega(x_n) \|x_n - u_n\|. \end{aligned} \quad (24)$$

This implies that

$$\begin{aligned} (\gamma_n - \kappa_n \gamma_n) \|x_n - u_n\|^2 &\leq \|x_n - c\|^2 + \alpha_n \|u - c\|^2 - \|x_{n+1} - c\|^2 \\ &\quad + 2\rho_n (\gamma_n - \kappa_n \gamma_n) \omega(x_n) \|u_n - x_n\|. \end{aligned} \quad (25)$$

By setting $s = P_\Psi u$, we next prove that $\|x_{n+1} - x_n\| \rightarrow 0$ and $x_n \rightarrow s$. By using inequality (18), we note that

$$\begin{aligned} \|x_{n+1} - s\|^2 &= \|\alpha_n u + \beta_n x_n + \gamma_n S y_n - s\|^2 \\ &\leq \alpha_n \|u - s\|^2 + \beta_n \|x_n - s\|^2 + \kappa_n \gamma_n \|v - s\|^2 + \gamma_n (1 - \kappa_n) \|u_n - s\|^2 \\ &\leq \alpha_n \|u - s\|^2 + \beta_n \|x_n - s\|^2 + \kappa_n \gamma_n \|v - s\|^2 + (\gamma_n - \kappa_n \gamma_n) \left[\|x_n - s\|^2 \right. \\ &\quad \left. - (4\rho_n - \rho_n^2) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \right] \\ &\leq \alpha_n \|u - s\|^2 + \|x_n - s\|^2 + \kappa_n \gamma_n \|v - s\|^2 \\ &\quad - \gamma_n (1 - \kappa_n) (4\rho_n - \rho_n^2) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2}. \end{aligned} \quad (26)$$

This implies that

$$\begin{aligned} &\gamma_n (1 - \kappa_n) \rho_n (4 - \rho_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \\ &\leq \|x_n - s\|^2 + \alpha_n \|u - s\|^2 + \kappa_n \gamma_n \|v - s\|^2 - \|x_{n+1} - s\|^2. \end{aligned} \quad (27)$$

Next, we consider two possible cases to show $\|x_n - s\| \rightarrow 0$ as $n \rightarrow \infty$.

Case 1. Assume that, for existing $n_0 \in \mathbb{N}$, a sequence $\{\|x_n - s\|^2\}$ is nonincreasing. Then the limit of a sequence $\lim_{n \rightarrow \infty} \|x_n - s\|$ exists and

$$\lim_{n \rightarrow \infty} (\|x_{n+1} - s\| - \|x_n - s\|) = 0.$$

Since the sequences α_n and κ_n tend to 0, then by using inequality (27), we get that

$$\rho_n (4 - \rho_n) \frac{\gamma_n \omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \rightarrow 0.$$

Since $\nabla \omega$ and $\nabla \psi$ are Lipschitz continuous, and $\liminf \gamma_n > 0$ and $\inf \rho_n (4 - \rho_n) > 0$. Thus we obtain $\lim_{n \rightarrow \infty} \omega^2(x_n) = 0$, and also $\omega(x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Next we prove $\|x_n - s\| \rightarrow 0$ as $n \rightarrow \infty$ by dividing the proof into three steps as follows.

Step 1. First, we must show that $\lim_{n \rightarrow \infty} \|S y_n - y_n\| = 0$. By using the fact that $T_{r_{n+1}}^f$ and $T_{r_{n+1}}^g$ are firmly nonexpansive mappings, and the mapping $T_{r_{n+1}}^f (I + \tau_n A^* (T_{r_{n+1}}^g - I) A)$ is a nonexpansive mapping, we find that

$$\|u_{n+1} - u_n\|$$

$$\begin{aligned}
&\leq \|T_{r_{n+1}}^f(x_{n+1} + \tau_{n+1}A^*(T_{r_{n+1}}^gAx_{n+1} - Ax_{n+1})) - T_{r_{n+1}}^f(x_n + \tau_nA^*(T_{r_{n+1}}^gAx_n - Ax_n))\| \\
&\quad + \|T_{r_{n+1}}^f(x_n + \tau_nA^*(T_{r_{n+1}}^gAx_n - Ax_n)) - T_{r_n}^f(x_n + \tau_nA^*(T_{r_n}^gAx_n - Ax_n))\| \\
&\leq \|x_{n+1} - x_n\| + \|(x_n + \tau_nA^*(T_{r_{n+1}}^gAx_n - Ax_n)) - (x_n + \tau_nA^*(T_{r_n}^gAx_n - Ax_n))\| \\
&\quad + \left|1 - \frac{r_n}{r_{n+1}}\right| \|T_{r_{n+1}}^f(x_n + \tau_nA^*(T_{r_{n+1}}^gAx_n - Ax_n)) - (x_n + \tau_nA^*(T_{r_{n+1}}^gAx_n - Ax_n))\| \\
&\leq \|x_{n+1} - x_n\| + |\tau_n|\|A\| \|(T_{r_{n+1}}^gAx_n - Ax_n) - (T_{r_n}^gAx_n - Ax_n)\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \zeta_n \\
&\leq \|x_{n+1} - x_n\| + \left|\frac{r_{n+1} - r_n}{r_{n+1}}\right| (|\tau_n|\|A\|\sigma_n + \zeta_n) \\
&= \|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|B,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\sigma_n &:= \|T_{r_{n+1}}^gAx_n - Ax_n\|, \\
\zeta_n &:= \|T_{r_{n+1}}^f(x_n + \tau_nA^*(T_{r_{n+1}}^g - I)Ax_n) - (x_n + \tau_nA^*(T_{r_{n+1}}^g - I)Ax_n)\|,
\end{aligned}$$

and $B = \sup_{n \geq 1} (|\tau_n|\|A\|\sigma_n + \zeta_n)$. Next, by using inequality (28), we get that

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \|\kappa_{n+1}v + (1 - \kappa_{n+1})u_{n+1} - \kappa_nv - (1 - \kappa_n)u_n\| \\
&\leq |\kappa_{n+1} - \kappa_n|\|v - u_n\| + (1 - \kappa_{n+1})\|u_{n+1} - u_n\| \\
&\leq (1 - \kappa_{n+1})\|x_{n+1} - x_n\| + |\kappa_{n+1} - \kappa_n|K + \frac{1}{a}|r_{n+1} - r_n|M,
\end{aligned} \tag{29}$$

where $K = \sup_{n \geq 1} \|v - u_n\|$. If we put $x_{n+1} = \beta_nx_n + (1 - \beta_n)d_n$, then by using Algorithm 1, we get that $d_n = (\alpha_nu + \gamma_nSy_n)/(1 - \beta_n)$. Further, we have that

$$\begin{aligned}
d_{n+1} - d_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}Sy_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_nu + \gamma_nSy_n}{1 - \beta_n} \\
&= \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right)u + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right)Sy_n \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(Sy_{n+1} - Sy_n).
\end{aligned} \tag{30}$$

Thus, by combining the above inequality with (29), we find that

$$\begin{aligned}
&\|d_{n+1} - d_n\| \\
&\leq \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right| \|u\| + \left|\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right| \|Sy_n\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|y_{n+1} - y_n\| \\
&\leq \|x_{n+1} - x_n\| + |\kappa_{n+1} - \kappa_n|K + \frac{1}{a}|r_{n+1} - r_n|M
\end{aligned}$$

$$+ \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \|u\| + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| \|Sy_n\|. \quad (31)$$

This implies that

$$\begin{aligned} \|d_{n+1} - d_n\| - \|x_{n+1} - x_n\| &\leq |\kappa_{n+1} - \kappa_n|K + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| \|Sy_n\| \\ &\quad + \frac{1}{a} |r_{n+1} - r_n|M + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \|u\|. \end{aligned} \quad (32)$$

Thus, by using conditions (C2)–(C6), we have

$$\limsup_{n \rightarrow \infty} (\|d_{n+1} - d_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (33)$$

Using Lemma 2.5, and from inequality (33), we find that $\lim_{n \rightarrow \infty} \|d_n - x_n\| = 0$. Moreover, we can conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (34)$$

Now, by combining the above information with inequality (25), we have

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (35)$$

Since $\|y_n - u_n\| = \kappa_n \|u_n - v\|$ and $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$, we find that

$$\lim_{n \rightarrow \infty} \|y_n - u_n\| = 0. \quad (36)$$

We next consider

$$\begin{aligned} \|x_n - Sy_n\| &\leq \|x_{n+1} - Sy_n\| + \|x_n - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n \|Sy_n - u\| + \beta_n \|Sy_n - x_n\|. \end{aligned} \quad (37)$$

Then it is easy to see that

$$\|x_n - Sy_n\| \leq \frac{1}{1-\beta_n} \|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n} \|Sy_n - u\|. \quad (38)$$

Since $\|x_{n+1} - x_n\| \rightarrow 0$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. By using condition (C5), we get that

$$\lim_{n \rightarrow \infty} \|Sy_n - x_n\| = 0. \quad (39)$$

Note that

$$\|y_n - Sy_n\| \leq \|y_n - u_n\| + \|u_n - x_n\| + \|x_n - Sy_n\|. \quad (40)$$

Thus, by using inequalities (35), (36), and (39), we obtain

$$\lim_{n \rightarrow \infty} \|y_n - Sy_n\| = 0. \quad (41)$$

Step 2. We will prove that $\limsup_{n \rightarrow \infty} \langle u - s, s - Sy_n \rangle \geq 0$ with $s = P_\Psi u$. Then we choose $\{y_{n_l}\} \subseteq \{y_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - s, Sy_n - s \rangle = \lim_{l \rightarrow \infty} \langle u - s, Sy_{n_l} - s \rangle.$$

By using the fact that $\{y_{n_l}\}$ is a bounded sequence, we can find the $\{y_{n_l}\} \subseteq \{y_{n_l}\}$ that weakly converges to y . Thus we can assume that $y_{n_l} \rightharpoonup y$ without losing a generality. So, we have that $Sy_{n_l} \rightharpoonup y$ since $\|Sy_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Next, claim that $y \in \text{Fix}(S)$, suppose by contradiction that $y \notin \text{Fix}(S)$. By using Lemma 2.6, we find that

$$\begin{aligned} \liminf_{l \rightarrow \infty} \|y_{n_l} - y\| &< \liminf_{l \rightarrow \infty} \|y_{n_l} - Sy\| \\ &\leq \liminf_{l \rightarrow \infty} (\|y_{n_l} - Sy_{n_l}\| + \|Sy_{n_l} - Sy\|) \\ &\leq \liminf_{l \rightarrow \infty} \|y_{n_l} - y\|, \end{aligned} \quad (42)$$

it implies a contradiction. Hence, $y \in \text{Fix}(S)$.

Next, we will prove that $y \in \Omega_{\text{SEP}_S}$. Because y is a weak limit point of the sequence $\{y_n\}$, by using inequalities (39) and (41), there is a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ that $x_{n_k} \rightharpoonup y$. By using the fact that ω is lower semicontinuous, we find that

$$0 \leq \omega(y) \leq \lim_{k \rightarrow \infty} \omega(x_{n_k}) = \lim_{n \rightarrow \infty} \omega(x_n) = 0.$$

This implies that $\omega(y) = \frac{1}{2} \|(I - T_{r_n}^g)Ay\|^2 = 0$. Therefore $Ay \in \text{EP}(f)$. We then consider

$$\|u_n - T_{r_n}^f x_n\| \leq |\tau_n| \|A^*(T_{r_n}^g - I)Ax_n\| \quad (43)$$

and

$$\|x_n - T_{r_n}^f x_n\| \leq \|x_n - u_n\| + \|u_n - T_{r_n}^f x_n\|. \quad (44)$$

Since we have $\omega(x_n) \rightarrow 0$ as $n \rightarrow \infty$, we get by using inequality (43) that $\|u_n - T_{r_n}^f x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Moreover, we obtain by applying (35) and $\|u_n - T_{r_n}^f x_n\| \rightarrow 0$ into (43) that $\|x_n - T_{r_n}^f x_n\| \rightarrow 0$. Since ψ is lower semicontinuous, we find that

$$0 \leq \psi(y) \leq \liminf_{k \rightarrow \infty} \psi(x_{n_k}) = \liminf_{n \rightarrow \infty} \psi(x_n) = 0.$$

This implies that $\frac{1}{2} \|(I - T_{r_n}^f)y\|^2 = 0$. Therefore $y \in \text{EP}(g)$. Thus $y \in \Omega_{\text{SEP}_S}$. We now conclude that $y \in \Psi$.

Next, by using the fact that $y \in \Psi$, $s = P_\Psi u$, $\langle u - P_\Psi u, y - P_\Psi u \rangle \leq 0$, and inequality (39). Thus

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - s, x_{n+1} - s \rangle &= \limsup_{n \rightarrow \infty} \langle u - s, Sy_n - s \rangle \\ &= \lim_{l \rightarrow \infty} \langle u - s, Sy_{n_l} - s \rangle \end{aligned}$$

$$\begin{aligned}
 &= \langle u - s, y - s \rangle \\
 &\leq 0.
 \end{aligned} \tag{45}$$

Step 3. We show that $x_n \rightarrow s = P_\Psi u$. We observe that

$$\begin{aligned}
 &\|x_{n+1} - s\|^2 \\
 &= \langle x_{n+1} - s, \alpha_n u + \beta_n x_n + \gamma_n S y_n - s \rangle \\
 &\leq \frac{\beta_n}{2} [\|x_{n+1} - s\|^2 + \|x_n - s\|^2] + \frac{\gamma_n}{2} [\|x_{n+1} - s\|^2 + \|S y_n - s\|^2] \\
 &\quad + \alpha_n \langle x_{n+1} - s, u - s \rangle \\
 &\leq \frac{\beta_n}{2} [\|x_{n+1} - s\|^2 + \|x_n - s\|^2] + \frac{\gamma_n}{2} [\|x_{n+1} - s\|^2 + 2\kappa_n \|y_n - s\| \|v - s\| \\
 &\quad + \|x_n - s\|^2] + \alpha_n \langle x_{n+1} - s, u - s \rangle.
 \end{aligned} \tag{46}$$

And we observe that

$$\|x_{n+1} - s\|^2 \leq (1 - \alpha_n) \|x_n - s\|^2 + 2\alpha_n \langle x_{n+1} - s, u - s \rangle + \kappa_n \|v - s\| \|y_n - s\|. \tag{47}$$

Thus, by using inequality (45), conditions (C2)–(C3), the boundedness of $\{y_n\}$, and Lemma 2.7, we find that $x_n \rightarrow s = P_\Psi u$.

Case 2. Assume that $\{\|x_n - s\|^2\}$ is an increasing sequence. From inequality (25), $\kappa_n \rightarrow 0$, and $\alpha_n \rightarrow 0$, it is easy to see that $\omega(x_n) \rightarrow 0$. By using a similar way as Case 1, we get $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Next, let a mapping σ be defined on integers $n > n_0$ (where n_0 is large enough) by

$$\sigma(n) = \max \{ m \leq n : \|x_m - s\| \leq \|x_{m+1} - s\| \}. \tag{48}$$

Then we see that $\sigma(n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\|x_{\sigma(n)} - s\| \leq \|x_{\sigma(n)+1} - s\|. \tag{49}$$

Thus $\{\|x_{\sigma(n)} - s\|\}$ is a nondecreasing sequence. By virtue of Case 1, we find that

$$\lim_{n \rightarrow \infty} \frac{\omega^2(x_{\sigma(n)})}{\|\nabla \omega(x_{\sigma(n)})\|^2 + \|\nabla \psi(x_{\sigma(n)})\|^2} = 0. \tag{50}$$

Consequently, we have $\lim_{n \rightarrow \infty} \omega(x_{\sigma(n)}) = 0$. Moreover, we find that

$$\lim_{n \rightarrow \infty} \|S y_{\sigma(n)} - x_{\sigma(n)}\| = 0$$

and

$$\limsup_{n \rightarrow \infty} \langle u - s, x_{\sigma(n)+1} - s \rangle \leq 0.$$

From (47) we get that

$$\|x_{\sigma(n)} - s\|^2 \leq 2\langle u - s, x_{\sigma(n)+1} - s \rangle + \frac{\kappa_{\sigma(n)}}{\alpha_{\sigma(n)}} \|v - s\| \|y_n - s\|. \quad (51)$$

This inequality yields

$$\lim_{n \rightarrow \infty} \|x_{\sigma(n)} - s\| = 0.$$

By using inequality (47), we have

$$\limsup_{n \rightarrow \infty} \|x_{\sigma(n)+1} - s\|^2 = \limsup_{n \rightarrow \infty} \|x_{\sigma(n)} - s\|^2 = 0.$$

Therefore $\lim_{n \rightarrow \infty} \|x_{\sigma(n)+1} - s\| = 0$. Next, by using Lemma 2.8, we have

$$0 \leq \|x_n - s\| \leq \max\{\|x_{\sigma(n)} - s\|, \|x_n - s\|\} \leq \|x_{\sigma(n)+1} - s\| \rightarrow 0, \quad (52)$$

which implies that the sequence $x_n \rightarrow s = P_{\Psi}u$. \square

Now, let us set the following conditions for some parameters used in Algorithm 2 and Algorithm 3.

Condition 3.2 Suppose that $\{r_n\} \subseteq (0, \infty)$, a is a constant, $\{\rho_n\}$ is any positive sequence, and $\{\alpha_n\}$ and $\{\kappa_n\}$ are real sequences that $0 < \alpha_n, \kappa_n < 1$ and satisfy:

- (C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\lim_{n \rightarrow \infty} \kappa_n = 0$ and $\lim_{n \rightarrow \infty} \frac{\kappa_n}{\alpha_n} = 0$;
- (C3) $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$ and $0 < a \leq r_n$;
- (C4) $\inf \rho_n(4 - \rho_n) > 0$ and $0 < \rho_n < 4$.

Algorithm 2 The second algorithm to solve split equilibrium problems

- **Initialization:** Let $x_1, u \in H_1$ be arbitrarily chosen. Choose some positive sequences $\{\rho_n\}, \{\alpha_n\}$, and $\{\kappa_n\}$ satisfying **Condition 3.2**. Put $n = 1$.
- **Iterative step:** Given the current iterate x_n , for $r_n > 0$, compute

$$u_n = T_{r_n}^f(x_n + \tau_n A^*(T_{r_n}^g - I)Ax_n),$$

where

$$\tau_n = \begin{cases} \frac{\rho_n \omega(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} & \|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2 \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S[\kappa_n v + (1 - \kappa_n) u_n].$$

- **Stopping criterion:** If $x_{n+1} = x_n = u_n$, stop. If not, set $n = n + 1$ and go to **Iterative step**.
-

Theorem 3.2 *The sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to a solution $s \in \Psi$, where $s = P_\Psi u$.*

Proof Since Ψ is a nonempty set, take each $c \in \Psi$. By using inequalities (20), (19) and proving similar to inequality (22), we obtain that three sequences $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ are bounded. Next, we note that

$$\begin{aligned}\|x_{n+1} - Sy_n\| &= \|\kappa_n v + (1 - \kappa_n)Sy_n - Sy_n\| \\ &= \kappa_n \|v - Sy_n\|.\end{aligned}$$

Since $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} \|x_{n+1} - Sy_n\| = 0. \quad (53)$$

By putting $s = P_\Psi u$ and using inequality (21), we find that

$$\begin{aligned}\|x_{n+1} - s\|^2 &= \|\alpha_n u + (1 - \alpha_n)Sy_n - s\|^2 \\ &\leq 2\alpha_n \langle x_{n+1} - s, u - s \rangle + (1 - \alpha_n) \|y_n - s\|^2 \\ &\leq 2\alpha_n \langle x_{n+1} - s, u - s \rangle + (1 - \alpha_n) \left[(1 - \kappa_n) \|u_n - s\|^2 + 2\kappa_n \langle v - s, y_n - s \rangle \right] \\ &\leq 2\alpha_n \langle x_{n+1} - s, u - s \rangle + (1 - \alpha_n) \|x_n - s\|^2 + 2\kappa_n \langle v - s, y_n - s \rangle \\ &\quad - \rho_n (4 - \rho_n) (1 - \alpha_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2}.\end{aligned} \quad (54)$$

This implies that

$$\begin{aligned}\rho_n (4 - \rho_n) (1 - \alpha_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \\ \leq 2\alpha_n \langle u - s, x_{n+1} - s \rangle + 2\kappa_n \langle v - s, y_n - s \rangle + \|x_n - s\|^2 - \|x_{n+1} - s\|^2.\end{aligned} \quad (55)$$

Next, to show that $\lim_{n \rightarrow \infty} \|x_n - s\| = 0$, we divide it into two possible cases.

Case 1. Let a sequence $\{\|x_n - s\|^2\}$ be nonincreasing. By virtue of Theorem 3.1 (Case 1), we obtain

$$\lim_{n \rightarrow \infty} (\|x_{n+1} - s\| - \|x_n - s\|) = 0.$$

By applying $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \kappa_n = 0$ into inequality (55), we have

$$\lim_{n \rightarrow \infty} \rho_n (4 - \rho_n) (1 - \alpha_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} = 0.$$

Since $\inf \rho_n (4 - \rho_n) > 0$, and $\nabla \omega$ and $\nabla \psi$ are Lipschitz continuous, then $\omega(x_n) \rightarrow 0$ as $n \rightarrow \infty$. Observe that

$$\|x_{n+1} - x_n\| \leq |\alpha_n - \alpha_{n-1}| \|u\| + |\alpha_n - \alpha_{n-1}| \|Sy_{n-1}\| + (1 - \alpha_n) \|Sy_n - Sy_{n-1}\|$$

$$\leq |\alpha_n - \alpha_{n-1}|M + (1 - \alpha_n)\|y_n - y_{n-1}\|, \quad (56)$$

where $M := \sup_{n \geq 1} \|u - Sy_{n-1}\|$. We observe that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\kappa_n v + (1 - \kappa_n)u_n - \kappa_{n-1}v - (1 - \kappa_{n-1})u_{n-1}\| \\ &\leq (1 - \kappa_n)\|u_n - u_{n-1}\| + |\kappa_n - \kappa_{n-1}|K, \end{aligned} \quad (57)$$

where $K := \sup_{n \geq 1} \sup \|v - u_{n-1}\|$. By virtue of inequality (28), we find that

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{a}|r_{n+1} - r_n|B, \quad (58)$$

where

$$\begin{aligned} B &= \sup_{n \geq 1} (|\tau_n| \|A\| \sigma_n + \zeta_n), \\ \sigma_n &:= \|T_{r_{n+1}}^g Ax_n - Ax_n\| \quad \text{and} \\ \zeta_n &:= \|T_{r_{n+1}}^f (x_n + \tau_n A^* (T_{r_{n+1}}^g - I) Ax_n) - (x_n + \tau_n A^* (T_{r_{n+1}}^g - I) Ax_n)\|. \end{aligned}$$

By combining inequality (56) with inequalities (57) and (58), we find that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \alpha_n)\|x_n - x_{n-1}\| + \frac{1}{a}|r_{n+1} - r_n|M + |\kappa_n - \kappa_{n-1}|M + |\alpha_n - \alpha_{n-1}|K. \end{aligned} \quad (59)$$

Thus, by using Lemma 2.7, we get that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (60)$$

Moreover, by virtue of formula (39), we conclude that

$$\|Sy_n - x_n\| \leq \|Sy_n - x_{n+1}\| + \|x_{n+1} - x_n\| \rightarrow 0 \quad (61)$$

as $n \rightarrow \infty$. Also,

$$\|Sy_n - y_n\| = \|Sy_n - x_n\| + \|x_n - x_{n+1}\| \rightarrow 0 \quad (62)$$

as $n \rightarrow \infty$. By following the proof in Theorem 3.1 (Case 1), we have $w_w(x_n) \subset \Psi$. Moreover, by the property of metric projection P_Ψ ,

$$\limsup_{n \rightarrow \infty} \langle u - s, x_{n+1} - x \rangle = \max_{w \in w_w(x_n)} \langle u - P_\Psi u, w - P_\Psi u \rangle \leq 0. \quad (63)$$

Next, by following inequality (54), we find that

$$\begin{aligned} \|x_{n+1} - s\|^2 &\leq (1 - \alpha_n)\|x_n - s\|^2 + 2\alpha_n \left[\langle u - s, x_{n+1} - s \rangle + \frac{\kappa_n}{\alpha_n} \|v - s\| \|y_n - s\| \right]. \end{aligned} \quad (64)$$

Algorithm 3 The third algorithm to solve split equilibrium problems

- **Initialization:** Let $x_1 \in H_1$ be arbitrarily chosen. Choose some positive sequences $\{\rho_n\}$, $\{\alpha_n\}$, and $\{\kappa_n\}$ satisfying **Condition 3.2**. Put $n = 1$.
- **Iterative step:** Given the current iterate x_n , for $r_n > 0$, compute

$$u_n = T_{r_n}^f(x_n + \tau_n A^*(T_{r_n}^g - I)Ax_n),$$

where

$$\tau_n = \begin{cases} \frac{\rho_n \omega(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} & \|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2 \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S[\kappa_n v + (1 - \kappa_n) u_n].$$

- **Stopping criterion:** If $x_{n+1} = x_n = u_n$, stop. If not, set $n = n + 1$ and go to **Iterative step**.

By applying Lemma 2.7, inequality (63), and condition (C2) into inequality (64), we find that $x_n \rightarrow s = P_\Psi u$.

Case 2. Assume that $\{\|x_n - s\|^2\}$ is an increasing sequence. From inequality (55), $\alpha_n \rightarrow 0$ and $\kappa_n \rightarrow 0$, it is easy to see that $\omega(x_n) \rightarrow 0$ as $n \rightarrow \infty$. By following the proof in Case 1, we find that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Suppose that the map σ is given by Theorem 3.1. By using formula (64), we complete the proof for this case in the same way as in the proof of Case 2 in Theorem 3.1. \square

Theorem 3.3 *The sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a solution $s \in \Psi$, where $s = P_\Psi v$.*

Proof By using the fact that Ψ is a nonempty set, we then take each $c \in \Psi$. Moreover, by using inequality (20), we find that

$$\begin{aligned} \|x_{n+1} - c\| &\leq \alpha_n \|x_n - c\| + (1 - \alpha_n) \|Sy_n - c\| \\ &\leq \alpha_n \|x_n - c\| + (1 - \alpha_n) [\kappa_n \|v - c\| + (1 - \kappa_n) \|x_n - c\|] \\ &\leq \alpha_n \|x_n - c\| + (1 - \alpha_n) \max\{\|v - c\|, \|x_n - c\|\} \\ &\leq \max\{\|x_n - c\|, \|v - c\|\}. \end{aligned} \quad (65)$$

This inequality implies $\{x_n\}$ is bounded. Thus, from inequalities (19) and (20), we then find that $\{u_n\}$ and $\{y_n\}$ are also bounded.

Setting $s = P_\Psi v$, we now show $\|x_{n+1} - x_n\| \rightarrow 0$ and $x_n \rightarrow s$ as $n \rightarrow \infty$. By inequality (21), we obtain

$$\|x_{n+1} - s\|^2 = \|\alpha_n x_n + (1 - \alpha_n) Sy_n - s\|^2$$

$$\begin{aligned}
&\leq \alpha_n \|x_n - s\|^2 + (1 - \alpha_n) [\kappa_n \|v - s\|^2 + \|u_n - s\|^2] \\
&\leq \alpha_n \|x_n - s\|^2 + \kappa_n \|v - s\|^2 + (1 - \kappa_n) \left[\|x_n - s\|^2 \right. \\
&\quad \left. - \rho_n (4 - \rho_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \right].
\end{aligned} \tag{66}$$

This inequality implies that

$$\begin{aligned}
&(1 - \kappa_n) \rho_n (4 - \rho_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} \\
&\leq \|x_n - s\|^2 - \|x_{n+1} - s\|^2 + \kappa_n \|v - s\|^2.
\end{aligned} \tag{67}$$

Next, to prove that $\lim_{n \rightarrow \infty} \|x_n - s\| = 0$, we divide it into two possible cases.

Case 1. Assume that a sequence $\{\|x_n - s\|^2\}$ is nonincreasing. By virtue of Theorem 3.1 (Case 1), we have

$$\lim_{n \rightarrow \infty} (\|x_{n+1} - s\| - \|x_n - s\|) = 0.$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \kappa_n = 0$, by following inequality (55), we find that

$$\lim_{n \rightarrow \infty} \rho_n (4 - \rho_n) \frac{\omega^2(x_n)}{\|\nabla \omega(x_n)\|^2 + \|\nabla \psi(x_n)\|^2} = 0.$$

Since $\nabla \omega$ and $\nabla \psi$ are Lipschitz continuous, and $\inf \rho_n (4 - \rho_n) > 0$, we obtain that $\omega(x_n) \rightarrow 0$ as $n \rightarrow \infty$. We then observe

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 - \alpha_n) \|Sy_n - Sy_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_n\| + |\alpha_n - \alpha_{n-1}| \|Sy_{n-1}\| \\
&\leq (1 - \alpha_n) \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| K,
\end{aligned} \tag{68}$$

where $K = \sup_{n \geq 1} \{\|x_n\| + \|Sy_{n-1}\|\}$. By a similar argument to the proof of inequality (59), we combine the above inequality with inequalities (57) and (58). Then

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Next, consider

$$\begin{aligned}
&\|x_{n+1} - s\|^2 \\
&= \alpha_n \langle x_{n+1} - s, x_n - s \rangle + (1 - \alpha_n) \langle x_{n+1} - s, Sy_n - s \rangle \\
&= \frac{\alpha_n}{2} [\|x_{n+1} - s\|^2 + \|x_n - s\|^2] + \frac{(1 - \alpha_n)}{2} [\|x_{n+1} - s\|^2 + \|y_n - s\|^2] \\
&\leq \frac{1}{2} [\alpha_n \|x_n - s\|^2 + \alpha_n \|x_{n+1} - s\|^2 + (1 - \alpha_n) \|x_{n+1} - s\|^2] \\
&\quad + \frac{(1 - \alpha_n)(1 - \kappa_n)}{2} \|x_n - s\|^2 + \kappa_n (1 - \alpha_n) \langle v - s, y_n - s \rangle.
\end{aligned} \tag{69}$$

Consequently,

$$\|x_{n+1} - s\|^2 \leq (1 - (1 - \alpha_n)\kappa_n)\|x_n - s\|^2 + 2\kappa_n(1 - \alpha_n)\langle v - s, y_n - s \rangle. \quad (70)$$

By a similar argument to the proof of Theorem 3.1, we have $y \in \Psi$ and

$$\limsup_{n \rightarrow \infty} \langle v - s, y_n - s \rangle \leq 0. \quad (71)$$

By applying the above inequality, Lemma 2.7, conditions (C1)–(C2) into inequality (70), we find that $x_n \rightarrow s = P_\Psi v$.

Case 2. Assume that $\{\|x_n - z\|^2\}$ is an increasing sequence. From inequality (67), $\alpha_n \rightarrow 0$, and $\kappa_n \rightarrow 0$, it is easy to see that $\omega(x_n) \rightarrow 0$ as $n \rightarrow \infty$. By following the proof in Case 1, we find that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Suppose that the map σ is given by Theorem 3.1. By using formula (70), we complete the proof for this case in the same way as in the proof of Case 2 in Theorem 3.1. \square

Remark 3.3 Because of the conditions of all parameters, our three proposed theorems are entirely different.

4 Numerical examples and applications

In this section, we implement our proposed algorithms to show their performance of computation in three different numerical examples. The proposed Algorithms 1, 2, and 3 are coded with MATLAB (R2021a) programming language. CPU time and numbers of iteration are considered the computational performance of the iteration algorithms.

Example 4.1 In this example, we illustrate our main results. We consider the numerical behavior of the proposed algorithms by comparing them with different ρ_n .

Let $H_1 = H_2 = \mathbb{R}^N$ and $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be defined by $Ax := \frac{1}{2}(x_1, x_2, \dots, x_N)$ with its adjoint operator $A^*x := \frac{1}{2}(x_1, x_2, \dots, x_N)$ for all $x = \{x_i\}_{i=1}^N \in \mathbb{R}^N$. We define the sets $C = D = \mathbb{R}^N$. Suppose that $f(p, q) = p^2 - pq$ for all $p, q \in C$. Then we derive the resolvent function $T_{r_n}^f$ as follows: find $s \in C$ such that $f(s, q) + \frac{1}{r_n}\langle q - s, s - x \rangle \geq 0$ for all $q \in C$ and $x \in H_1$. We observe that

$$\begin{aligned} f(s, q) + \frac{1}{r_n}\langle q - s, s - x \rangle \geq 0 &\Leftrightarrow s(s - q) + \frac{1}{r_n}\langle q - s, s - x \rangle \geq 0 \\ &\Leftrightarrow r_n s(s - q) + (q - s)(s - x) \geq 0 \\ &\Leftrightarrow (q - s)[(1 - r_n)s - x] \geq 0. \end{aligned}$$

It follows from Lemma 2.4 that $T_{r_n}^f$ is single-valued; therefore we get that $s = \frac{x}{1 - r_n}$. This implies that

$$T_{r_n}^f(x) = \frac{x}{1 - r_n}.$$

Let $g(v, w) = v^2 + vw - 2w^2$ for all $v, w \in D$. Then we derive the resolvent function $T_{r_n}^g$ as follows: find $u \in C$ such that $g(u, w) + \frac{1}{r_n} \langle w - u, u - y \rangle \geq 0$ for all $w \in C$ and $y \in H_2$. We observe that

$$\begin{aligned} g(u, w) + \frac{1}{r_n} \langle w - u, u - y \rangle &\geq 0 \\ \Leftrightarrow r_n u^2 + r_n u w - 2r_n w^2 + u w - w y - u^2 + u y &\geq 0 \\ \Leftrightarrow -2r_n w^2 + r_n u w + u w - w y + r_n u^2 - u^2 + u y &\geq 0 \\ \Leftrightarrow -2r_n w^2 + (r_n u + u - y) w + r_n u^2 - u^2 + u y &\geq 0. \end{aligned}$$

Let $M(w) = -2r_n w^2 + (r_n u + u - y) w + r_n u^2 - u^2 + u y$. Then M is a quadratic function of w with coefficient $a = -2r_n$, $b = r_n u + u - y$, $c = r_n u^2 - u^2 + u y$. We observe that the discriminant of $M(w)$ can be computed as follows:

$$\begin{aligned} b^2 - 4ac &= (r_n u + u - y)(r_n u + u - y) - 4(-2r_n)(r_n u^2 - u^2 + u y) \\ &= r_n^2 u^2 + r u^2 - r_n u y + r_n u^2 + u^2 - u y - r_n u y - u y + y^2 \\ &\quad + 8r_n^2 u^2 - 8r_n u^2 + 8r_n u y \\ &= 9r_n^2 u^2 - 6r_n u^2 + 6r_n u y - 2u y + u^2 + y^2 \\ &= y^2 + 6r_n u y - 2u y + 9r_n^2 u^2 - 6r_n u^2 + u^2 \\ &= y^2 - 2((1 - 3r_n)u)y + ((1 - 3r_n)u)^2 \\ &= (y - (1 - 3r_n)u)^2. \end{aligned} \tag{72}$$

Thus $b^2 - 4ac \geq 0$ for all $y \in H_2$. It follows from Lemma 2.4 that $T_{r_n}^g$ is single-valued; therefore we get that

$$T_{r_n}^g(x) = \frac{x}{1 - 3r_n}.$$

It is easy to check that assumptions (A1)–(A4) are satisfied. We take $S : C \rightarrow C$ by $\frac{1}{2}(x_1, x_2, \dots, x_N)$ for all $x = \{x_i\}_{i=1}^N \in \mathbb{R}^N$ and choose $\alpha_n = \frac{1}{2n+2}$, $\kappa_n = \frac{1}{(2n+2)^2}$, $\beta_n = \frac{n}{2n+2}$, $\gamma_n = \frac{1}{2}$, $r_n = \frac{1}{2^n}$, $u = 0.1$, $v = 0.5$.

We test our proposed algorithms for different values of the parameter ρ_n by using $E(n) = \frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-5}$, $N = 3$, and the initial value $x_1 \in ([-5, 5], 1, N)$ (randomly generated vector in \mathbb{R}^N) as follows:

Case I: $\rho_n = 0.01$;

Case II: $\rho_n = 2.00$;

Case III: $\rho_n = 3.99$.

In addition, Fig. 1–Fig. 3 show the error $E(n)$ and behavior $\|x_{n+1} - x_n\|$ and Table 1 presents the CPU times and the number of iterations in three cases for testing suitable values of the parameter ρ_n .

Remark 4.2 By the results of Example 4.1, we observed that:

- the third algorithm was more efficient than other proposed algorithms regarding CPU time;

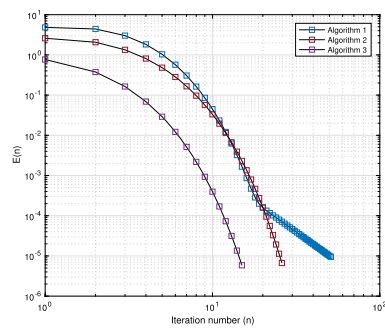
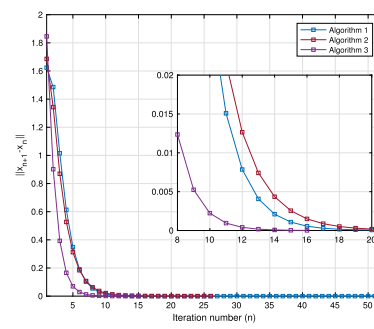

(a) Convergence behavior of error $E(n)$.

(b) Convergence behavior of $\|x_{n+1} - x_n\|$.

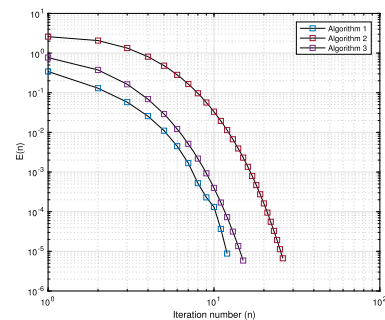
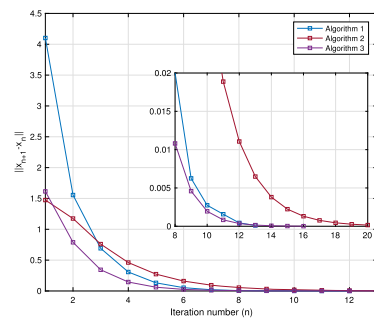
Figure 1 Computational results for Example 4.1 Case I

(a) Convergence behavior of error $E(n)$.

(b) Convergence behavior of $\|x_{n+1} - x_n\|$.

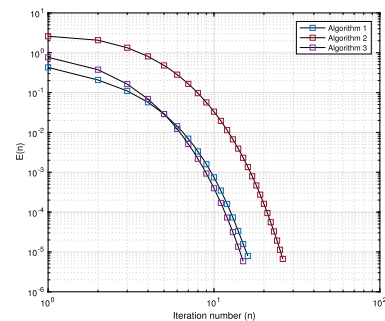
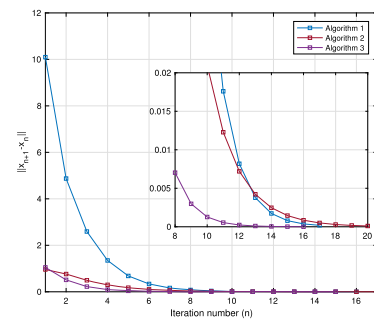
Figure 2 Computational results for Example 4.1 Case II

(a) Convergence behavior of error $E(n)$.

(b) Convergence behavior of $\|x_{n+1} - x_n\|$.

Figure 3 Computational results for Example 4.1 Case III

(b) in case II ($\rho_n = 2.00$), the first algorithm was more efficient than other proposed algorithms regarding the number of iterations.

Example 4.3 In this example, we show our performance of the proposed algorithms by comparing them with algorithms (4)–(6) in the literature.

Table 1 Computational results for Example 4.1

	Case I		Case II		Case III	
	CPU time (S)	No. of iter.	CPU time (S)	No. of iter.	CPU time (S)	No. of iter.
Algorithm 1	0.0299	52	0.00162	13	0.00212	17
Algorithm 2	0.0185	27	0.0192	27	0.0213	27
Algorithm 3	0.00192	16	0.00183	16	0.00195	16

Let $H_1 = H_2 = l_2(\mathbb{R})$ and $A : l_2 \rightarrow l_2$ be defined by $Ax := \frac{1}{4}(x_1, x_2, \dots, x_i, \dots)$ with its adjoint operator $A^*x := \frac{1}{4}(x_1, x_2, \dots, x_i, \dots)$ for all $x = \{x_i\}_{i=1}^\infty \in l_2$. We define the sets $C = D := \{z \in l_2 : \|z\| \leq 1\}$. Suppose that $f(p, q) = -11p^2 + pq + 10q^2$ for all $p, q \in C$. Then we derive the resolvent function $T_{r_n}^f$ as follows: find $s \in C$ such that $f(s, q) + \frac{1}{r_n}\langle q - s, s - x \rangle \geq 0$ for all $q \in C$ and $x \in H_1$. We observe that

$$\begin{aligned}
 f(s, q) + \frac{1}{r_n}\langle q - s, s - x \rangle &\geq 0 \\
 \Leftrightarrow -11r_ns^2 + r_nsq + 10r_nq^2 + sq - qx - s^2 + sx &\geq 0 \\
 \Leftrightarrow 10r_nq^2 + r_nsq + sq - qx - 11r_ns^2 - s^2 + sx &\geq 0 \\
 \Leftrightarrow 10r_nq^2 + (r_ns + s - x)q + (-11r_ns^2 - s^2 + sx) &\geq 0.
 \end{aligned}$$

Let $P(q) = 10r_nq^2 + (r_ns + s - x)q + (-11r_ns^2 - s^2 + sx)$. Then P is a quadratic function of q with coefficient $a = 10r_n$, $b = r_ns + s - x$, $c = -11r_ns^2 - s^2 + sx$. We observe that the discriminant of $P(q)$ can be computed as follows:

$$\begin{aligned}
 b^2 - 4ac &= (r_ns + s - x)(r_ns + s - x) - 4(10r_n)(-11r_ns^2 - s^2 + sx) \\
 &= r_n^2s^2 + rs^2 - r_nsx + r_ns^2 + s^2 - sx - r_nsx - sx + x^2 \\
 &\quad + 440r_n^2s^2 + 40r_ns^2 - 40r_nsx \\
 &= 441r_n^2s^2 + 42r_ns^2 - 42r_nsx - 2sx + s^2 + x^2 \\
 &= x^2 - 42r_nsx - 2sx + 441r_n^2s^2 + 42r_ns^2 + s^2 \\
 &= x^2 - 2((1 + 21r_n)s)y + ((1 + 21r_n)s)^2 \\
 &= (x - (1 + 21r_n)s)^2.
 \end{aligned} \tag{73}$$

Thus $b^2 - 4ac \geq 0$ for all $x \in H_1$. It follows from Lemma 2.4 that $T_{r_n}^f$ is single-valued; therefore we get that

$$T_{r_n}^f(x) = \frac{x}{1 + 21r_n}.$$

Let $g(v, w) = -15v^2 + vw + 14w^2$ for all $v, w \in D$. Then we derive the resolvent function $T_{r_n}^g$ as follows: find $u \in C$ such that $g(u, w) + \frac{1}{r_n}\langle w - u, u - y \rangle \geq 0$ for all $w \in C$ and $y \in H_2$. We observe that

$$\begin{aligned}
 g(u, w) + \frac{1}{r_n}\langle w - u, u - y \rangle &\geq 0 \\
 \Leftrightarrow -15r_nu^2 + r_nuw + 14r_nw^2 + uw - wy - u^2 + uy &\geq 0
 \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow 14r_n w^2 + r_n u w + u w - w y - 15r_n u^2 - u^2 + u y \geq 0 \\ &\Leftrightarrow 14r_n w^2 + (r_n u + u - y)w + (-15r_n u^2 - u^2 + u y) \geq 0. \end{aligned}$$

Let $M(w) = 14r_n w^2 + (r_n u + u - y)w + (-15r_n u^2 - u^2 + u y)$. Then M is a quadratic function of w with coefficient $a = 14r_n$, $b = r_n u + u - y$, $c = -15r_n u^2 - u^2 + u y$. We observe that the discriminant of $M(w)$ can be computed as follows:

$$\begin{aligned} b^2 - 4ac &= (r_n u + u - y)(r_n u + u - y) - 4(14r_n)(-15r_n u^2 - u^2 + u y) \\ &= r_n^2 u^2 + r u^2 - r_n u y + r_n u^2 + u^2 - u y - r_n u y - u y + y^2 \\ &\quad + 840r_n^2 u^2 + 58r_n u^2 - 58r_n u y \\ &= 841r_n^2 u^2 - 58r_n u^2 + 58r_n u y - 2u y + u^2 + y^2 \\ &= y^2 - 58r_n u y - 2u y + 841r_n^2 u^2 + 58r_n u^2 + u^2 \\ &= y^2 - 2((1 + 29r_n)u)y + ((1 + 29r_n)u)^2 \\ &= (y - (1 + 29r_n)u)^2. \end{aligned} \quad (74)$$

Thus $b^2 - 4ac \geq 0$ for all $y \in H_2$. It follows from Lemma 2.4 that $T_{r_n}^g$ is single-valued; therefore we get that

$$T_{r_n}^g(x) = \frac{x}{1 + 29r_n}.$$

It is easy to check that assumptions (A1)–(A4) are satisfied. We take $S : C \rightarrow C$ by $\frac{1}{2}(x_1, x_2, \dots, x_i, \dots)$ for all $x = \{x_i\}_{i=1}^\infty \in l_2$ and choose $\alpha_n = \frac{1}{2n+2}$, $\kappa_n = \frac{1}{(2n+2)^2}$, $\beta_n = \frac{n}{2n+2}$, $\gamma_n = \frac{1}{2}$, $r_n = \frac{1}{2n}$, $u = 0.1$, $v = 0.5$, $\rho_n = 2.00$. For algorithms (4)–(6), we take $D \equiv I$, $Sx = \{\frac{1}{2}(x_1, x_2, \dots, x_i, \dots)\}$ for all $x = \{x_i\}_{i=1}^\infty \in l_2$, $\gamma = 0.01$, $\alpha_n = \frac{1}{2n+2}$, and $\beta_n = \frac{n}{2n+2}$.

We test all of the algorithms for different values of initial values x_1 by using $E(n) = \frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-5}$ as follows:

Case I: $x_1 = (1, 1, 1, \dots, 1, 0_{200}, \dots)$;

Case II: $x_1 = (1.8, -2.56, 0.6, 0.6, \dots, 0.6_{100}, 0, \dots)$;

Case III: $x_1 = (5.3, 1, 1.02, 0, 0, \dots)$;

Case IV: $x_1 = (4, -2, 6, -1, 1, 1, \dots, 1_{100}, 0, \dots)$.

In addition, Fig. 4–Fig. 7 show the error $E(n)$ and behavior $\|x_{n+1} - x_n\|$ and Table 2 presents the CPU times and the number of iterations in four cases for comparing three proposed algorithms with algorithms (4)–(6).

Remark 4.4 By examining the outcomes of Example 4.3, we found that our proposed algorithms were more efficient than other algorithms in the literature.

4.1 Application to the optimization problem

Let $P : C \rightarrow \mathbb{R}$ and $Q : D \rightarrow \mathbb{R}$ be two functions. Given $f(v, w) = P(w) - P(v)$ for all $v, w \in C$, and $g(y, z) = Q(z) - Q(y)$ for all $y, z \in D$. The optimization problem is to find a point $s \in C$ such that

$$F(s) \leq F(\bar{s}), \quad \forall \bar{x} \in C, \quad (75)$$

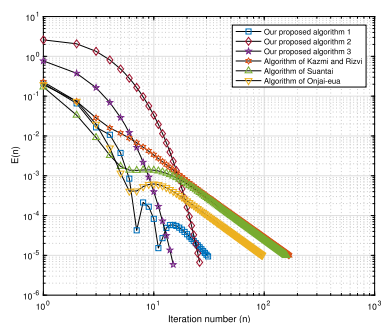
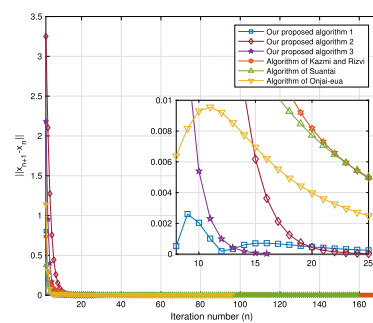

(c) Convergence behavior of error $E(n)$.

(d) Convergence behavior of $\|x_{n+1} - x_n\|$.

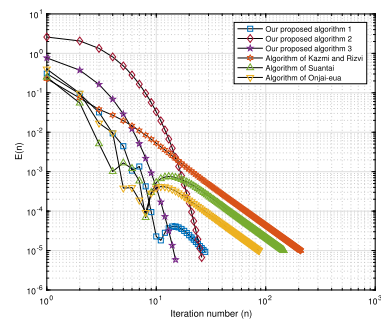
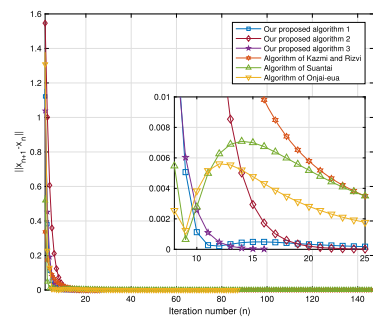
Figure 4 Computational results for Example 4.3 Case I

(a) Convergence behavior of error $E(n)$.

(b) Convergence behavior of $\|x_{n+1} - x_n\|$.

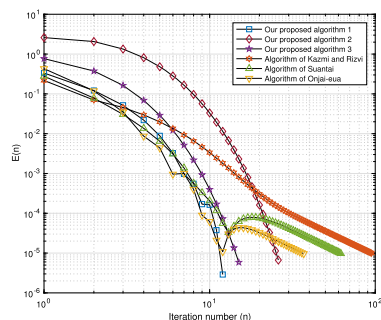
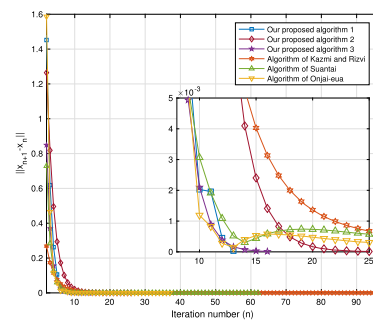
Figure 5 Computational results for Example 4.3 Case II

(a) Convergence behavior of error $E(n)$.

(b) Convergence behavior of $\|x_{n+1} - x_n\|$.

Figure 6 Computational results for Example 4.3 Case III

and a point $t = As \in D$ solves

$$G(t) \leq G(\bar{t}), \quad \forall \bar{t} \in D. \quad (76)$$

A solution set of the optimization problem (75)–(76) is defined by Φ , and assume that $\Phi \neq \emptyset$. It is easy to check that assumptions (A1)–(A4) are satisfied. Clearly, $\Phi = \Psi$. Theo-

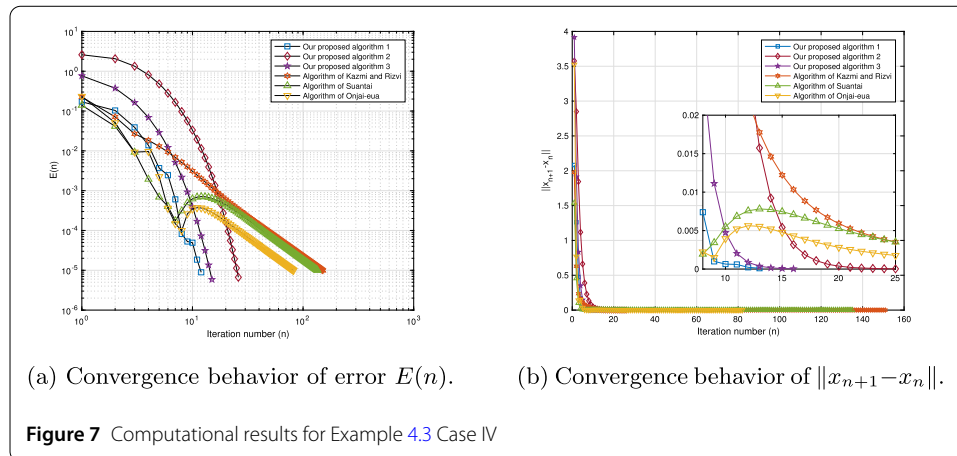


Table 2 Computational results for Example 4.3

	Case I		Case II		Case III		Case IV	
	CPU time (S)	No. of iter.	CPU time (S)	No. of iter.	CPU time (S)	No. of iter.	CPU time (S)	No. of iter.
Algorithm 1	0.0269	32	0.0224	29	0.00165	13	0.00167	13
Algorithm 2	0.0215	27	0.0211	27	0.0219	27	0.0221	27
Algorithm 3	0.00184	16	0.00191	16	0.00188	16	0.00190	16
Algorithm of Kazmi and Rizvi	0.0678	169	0.0916	210	0.0560	96	0.0613	152
Algorithm of Suantai	0.0614	160	0.0568	147	0.0416	63	0.0592	136
Algorithm of Onjai-eua	0.0437	97	0.0419	88	0.0311	38	0.0513	83

lems 3.1–3.3 can be reduced to the strong convergence theorems for approximating the common solution of split minimization problems and fixed point problems of a nonexpansive mapping.

4.2 Application to the split feasibility problem

The well-known split feasibility problem (SFP) was introduced in 1994 by Censor and Elfving [20]. This problem was defined as follows: find a point

$$v \in C \quad \text{such that } Av \in D. \quad (77)$$

The SFP has attracted many researchers due to its applications in a large variety of problems such as in image reconstruction, signal processing, intensity-modulation radiation therapy treatment planning (IMRT). The reader can refer to [21–24] for details.

Denote a solution set of (77) by Φ and assume that $\Phi \neq \emptyset$. Let $f(p, q) = i_C(q) - i_C(p)$ for all $p, q \in C$, and $g(v, w) = i_D(w) - i_D(v)$ for all $v, w \in D$, where i_C and i_D are indicator functions of the subsets C and D , respectively. It is easy to see that all assumptions are satisfied. Clearly, $\Phi = \Psi$. By putting $T_{r_n}^f = P_C$ and $T_{r_n}^g = P_D$ into Theorems 3.1–3.3, Theorems 3.1–3.3 can be reduced to the strong convergence theorems for approximating the common solution of split feasibility problems and fixed point problems of a nonexpansive mapping. Moreover, by these settings, we present the following numerical implementation to solve the LASSO problem.

Example 4.5 In this example, we show the performance of our algorithms for the sparse signal recovery problem in compressed sensing. This problem can be considered as the following linear inverse problem:

$$b = Ax + \epsilon, \quad (78)$$

where $A : \mathbb{R}^M \rightarrow \mathbb{R}^N$, ϵ is the observed data with noisy in \mathbb{R}^M , x is the signal to recover in \mathbb{R}^N . The linear inverse problem (78) can be solved by transforming it into the following LASSO problem:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2} \|b - Ax\|^2 \quad \text{subject to } \|x\|_1 \leq q, \quad (79)$$

where $q > 0$ is a constant. LASSO problem (79) can be considered as the split feasibility problem (SFP) (77) if the sets $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq q\}$ and $D = \{b\}$. Thus we can apply our algorithms to solve (79).

In all algorithms, we take $S \equiv I$ for all $x = \{x_i\}_{i=1}^N \in \mathbb{R}^N$ and choose $\alpha_n = \frac{1}{2n+2}$, $\kappa_n = \frac{1}{(2n+2)^2}$, $\beta_n = \frac{n}{2n+2}$, $\gamma_n = \frac{1}{2}$, $r_n = \frac{1}{2n}$, $u = 0.1$, $v = 0.5$, and $\rho_n = 2$.

In this example, the matrix A is generated by the normal distribution with $\mu = 0$ and $\sigma = 1$, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-1, 1]$ with k ($0 < k \ll N$), and the observation y is generated by the white Gaussian noise with SNR = 40.

We test all of the algorithms for different dimension N and different sparsity k with $q = k$, the initial point $x_1 = \text{zeros}(N, 1)$, and the restoration accuracy as the mean squared error $\text{MSE} = \frac{1}{N} \|x^* - x_n\|^2 < 10^{-4}$, where x^* is the original signal, as follows:

Case I: $N = 400$, $k = 10$;

Case II: $N = 400$, $k = 30$;

Case III: $N = 1000$, $k = 50$;

Case IV: $N = 1000$, $k = 100$.

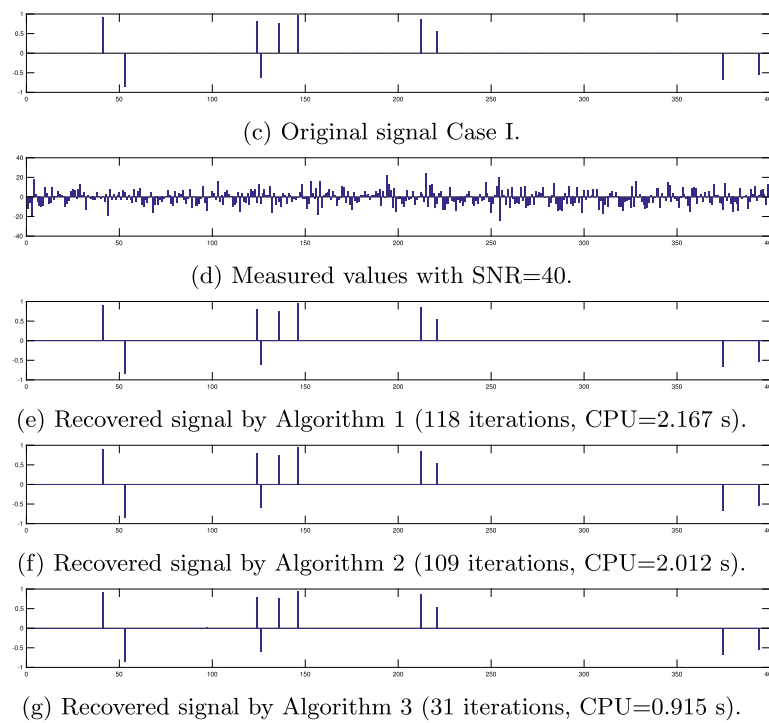
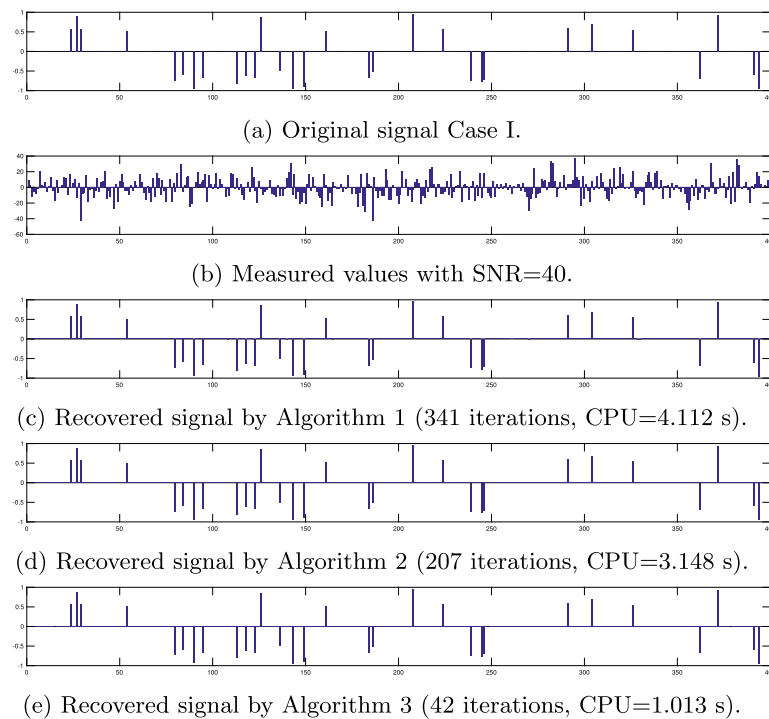
The MSE values against the number of iterations are presented in Fig. 8–Fig. 12. From the numerical results in Fig. 8–Fig. 12, we can summarize that the proposed algorithms outperform the other previous algorithms in [7–9] with greater efficiency and faster computations. Considering the proposed algorithms, the implemented results show that Algorithm 3 solves the sparse sensor signal recovery problem with the fastest computation time, as shown in Fig. 12.

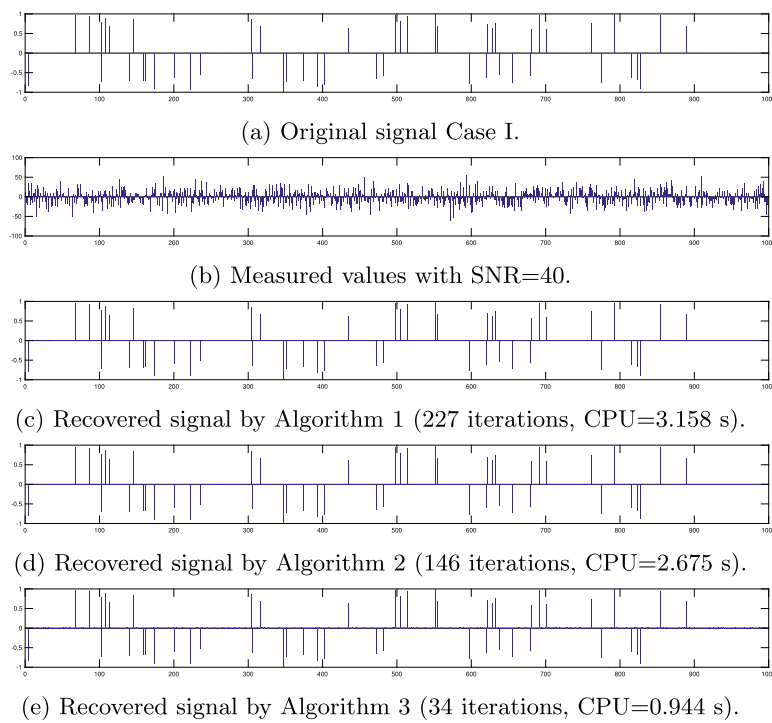
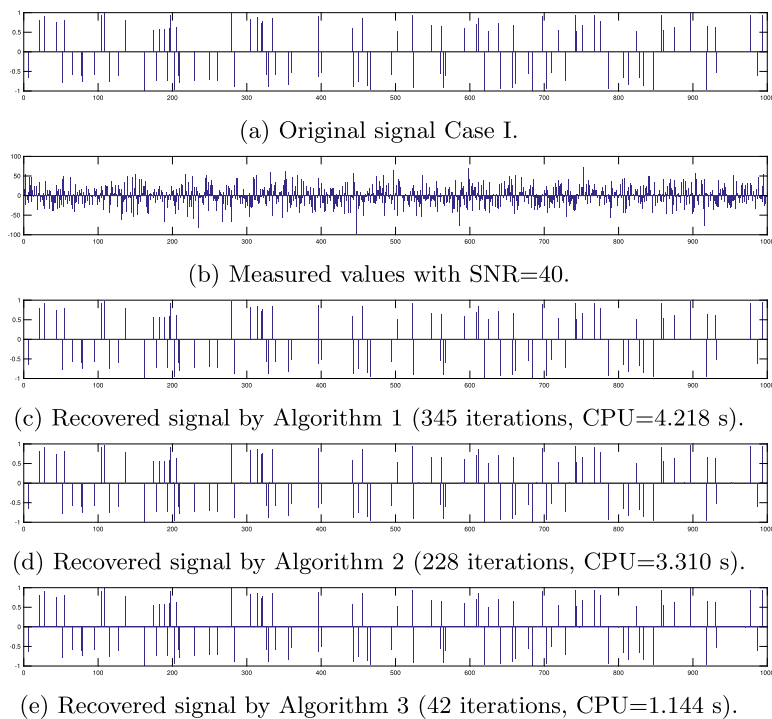
Remark 4.6 By observing the outcomes of Example 4.5, we obtain that:

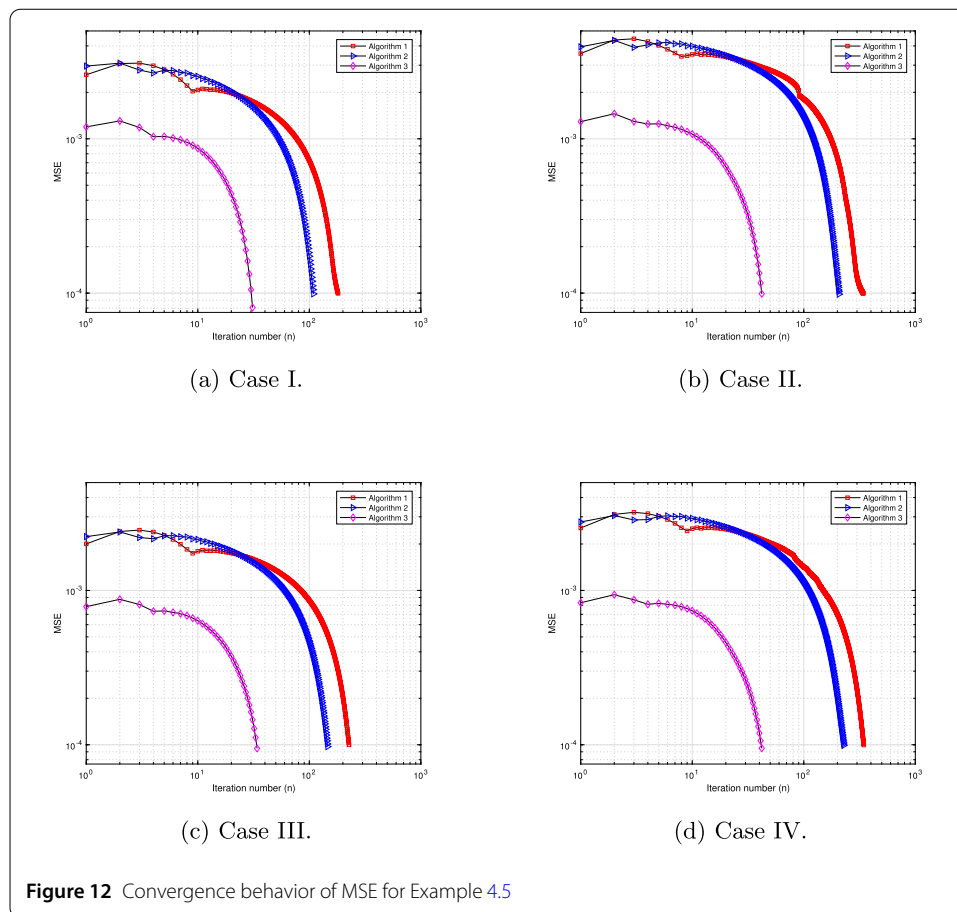
- As shown in Fig. 8–Fig. 11, the signal x can be recovered by our proposed algorithms. However, it is revealed that among these methods, Algorithm 3 has the smallest number of iterations and also the shortest CPU time for all cases;
- In Fig. 12, we plotted the mean squared error (MSE) value per iteration. It is evident that the errors obtained by Algorithm 3 decrease faster than for the other proposed algorithms.

5 Conclusions

This work proposed three modified step size algorithms to solve the SEPs. These iterative algorithms were developed to approximate solutions for the SEPs with no prior knowl-

**Figure 8** Computational results for Example 4.5 Case I**Figure 9** Computational results for Example 4.5 Case II

**Figure 10** Computational results for Example 4.5 Case III**Figure 11** Computational results for Example 4.5 Case IV



edge. Strong convergence theorems are well established and proved under appropriate conditions. We gave some of the applications to solve the split feasibility problem and optimization problem. Numerical examples were presented to show the usefulness of our main results. Moreover, we showed the computational performance of the proposed algorithms by comparing them with the existing algorithms. The compared results show that the proposed algorithms outperform the state-of-the-art algorithms.

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Declarations

Competing interests

The authors declare that they have no competing interests.

Author contribution

Conceptualization, KS, NK; Formal analysis, KS, NK; Investigation, KS, AJ; Software, SM, NK; Validation, SM, KS; Writing the original draft, KS, NK, AJ. All authors read and approved the final manuscript.

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