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Factorization of Hilbert operators

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Dedicated to Prof. Maryam Mirzakhani who, in spite of a short lifetime, left a long standing impact on mathematics.

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Abstract

In this research, we introduce some factorization for Hilbert operators of order n based on two important classes of Hausdorff operators, Cesàro and gamma. These factorizations lead us to some new inequalities, one is a generalized version of Hilbert's inequality. Moreover, as an application of our factorization, we compute the norm of Hilbert operators on some matrix domains.

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1 Introduction

Let ω denote the set of all real-valued sequences. Any linear subspace of ω is called a sequence space. The Banach space ℓ_p is the set of all real sequences $x = (x_k)_{k=0}^{\infty} \in \omega$ such that

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} < \infty \quad (1 \le p < \infty).$$

We consider infinite matrices $\mathfrak{M} = (m_{j,k})$, where all the indices j and k are nonnegative. The matrix domain associated with $\mathfrak{M}$ is defined as

$$\mathfrak{M}_S = \{\mathbf{x} \in \omega : \mathfrak{M}\mathbf{x} \in S\}. \tag{1.1}$$

In the special case $S = \ell_p$, we use the notation $\mathfrak{M}_p$ instead of $\mathfrak{M}_{\ell_p}$. It is rather trivial that $I_p = \ell_p$, where I is the infinite identity matrix. This concept has inspired many researchers to seek and define new Banach spaces as the domain of an infinite matrix. See [18, 20, 22] and the textbook [1].

Let Σ be a matrix with nonnegative entries, which maps ℓ_p into itself and satisfies the inequality

$$\|\Sigma x\|_{\ell_p} \le \rho \|x\|_{\ell_p},$$

for the constant ρ not depending on x and for every $x \in \ell_p$. The norm of Σ is the smallest possible value of ρ. The problem of finding the norm and the lower bound of operators on the matrix domains has been investigated in some of the references [7, 8, 10–12, 19, 21].

Hilbert matrix For a nonnegative integer n, the Hilbert matrix of order n, $\mathcal{H}_n$, is defined by

$$[\mathcal{H}_n]_{j,k} = \frac{1}{j+k+n+1}.$$

Evidently, for $n = 0$, $\mathcal{H}_0 = \mathcal{H}$ is the well-known Hilbert matrix

$$\mathcal{H} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which was introduced by David Hilbert in 1894. More examples:

$$\mathcal{H}_1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \mathcal{H}_2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Hilbert matrix is a bounded operator on ℓ_p ([6], Theorem 323) and

$$\|\mathcal{H}\|_{\ell_p \to \ell_p} = \Gamma(1/p)\Gamma\bigl(1/p^*\bigr) = \pi \csc(\pi/p),$$

where p^* is the conjugate of p i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$.

Hausdorff matrices The Hausdorff matrix H_μ is one of the most important examples of summability matrices defined by

$$\bigl[H_\mu\bigr]_{j,k} = \begin{cases} \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k}\, d\mu(\theta) & 0 \le k \le j, \\ 0 & \text{otherwise}, \end{cases}$$

where μ is a probability measure on $[0,1]$. While obtaining the ℓ_p-norm of operators is a hard endeavor, for Hausdorff matrices, we luckily have Hardy's formula [5, Theorem 216] which states that this matrix is a bounded operator on ℓ_p if and only if

$$\int_0^1 \theta^{\frac{-1}{p}}\, d\mu(\theta) < \infty, \quad 1 \le p < \infty.$$

In fact,

$$\bigl\|H_\mu\bigr\|_{\ell_p \to \ell_p} = \int_0^1 \theta^{\frac{-1}{p}}\, d\mu(\theta). \tag{1.2}$$

Hausdorff operators have the interesting norm separating property.

Theorem 1.1 ([3], Theorem 9) *Let $p \geq 1$ and H_μ, H_φ, and H_ν be Hausdorff matrices such that $H_\mu = H_\varphi H_\nu$. Then H_μ is bounded on ℓ_p if and only if both H_φ and H_ν are bounded on ℓ_p. Moreover, we have*

$$\|H_\mu\|_{\ell_p \to \ell_p} = \|H_\varphi\|_{\ell_p \to \ell_p} \|H_\nu\|_{\ell_p \to \ell_p}.$$

The Hausdorff matrix contains several famous classes of matrices. For positive integer n, two of these classes are as follows.

Cesàro matrix The measure $d\mu(\theta) = n(1-\theta)^{n-1}\, d\theta$ gives the Cesàro matrix of order n, C_n, which is defined by

$$[C_n]_{j,k} = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, according to (1.2), C_n has the ℓ_p-norm

$$\|C_n\|_{\ell_p \to \ell_p} = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}.$$

Note that $C_0 = I$, where I is the identity matrix, and

$$C_1 = C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the classical Cesàro matrix, which has ℓ_p-norm $\|C\|_{\ell_p \to \ell_p} = \frac{p}{p-1}$. The matrix domain associated with C_n is defined by

$$C_n(p) = \left\{ \mathbf{x} \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p < \infty \right\},$$

having the norm

$$\|\mathbf{x}\|_{C_n(p)} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p \right)^{1/p}$$

is a Banach space. The author investigated the Cesàro matrix domain in [15, 17].

Gamma matrix The measure $d\mu(\theta) = n\theta^{n-1}\, d\theta$ gives the gamma matrix of order n, $\mathcal{G}_n$, for which

$$[\mathcal{G}_n]_{j,k} = \begin{cases} \frac{\binom{n+k-1}{k}}{\binom{n+j}{j}} & 0 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Hardy's formula, $\mathcal{G}_n$ has the ℓ_p-norm

$$\|\mathcal{G}_n\|_{\ell_p\to\ell_p} = \frac{np}{np-1}.$$

The sequence space associated with $\mathcal{G}_n$ is the set

$$\mathcal{G}_n(p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p < \infty \right\},$$

which is called the gamma space of order n. The space $\mathcal{G}_n(p)$ is a Banach space with the norm

$$\|x\|_{\mathcal{G}_n(p)} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p \right)^{\frac{1}{p}}.$$

Note that $\mathcal{G}_1$ is the classical Cesàro matrix C, and we show the gamma sequence space $\mathcal{G}_1(p)$ by the notation $C(p)$. For more information about the gamma matrix domain, the eager readers can refer to [9, 16].

For finding the norm of a transpose of an operator, we use a helpful theorem also known as Hellinger–Toeplitz theorem.

Theorem 1.2 ([4], Proposition 7.2) *Suppose that $1 < p, q < \infty$. A matrix Σ with nonnegative entries maps ℓ_p into ℓ_q if and only if the transposed matrix Σ^t maps ℓ_{q^*} into ℓ_{p^*}. Then we have*

$$\|\Sigma\|_{\ell_p\to\ell_q} = \left\|\Sigma^t\right\|_{\ell_{q^*}\to\ell_{p^*}}.$$

Motivation The infinite Hilbert operator is one of the most complicated operators which is used in cryptography because of its complexity. Recently the author [13, 14] has introduced some factorizations for the infinite Hilbert operator based on Cesàro and gamma operators. Through this study, the author not only has generalized its previous results to the Hilbert operators of any order, but has introduced some factorizations that result in several new inequalities.

2 Factorization of the Hilbert operator

Bennett in [2] introduced a factorization of the form $\mathcal{H} = UC$, where C is the Cesàro operator and the matrix U is defined by

$$[U]_{j,k} = \frac{k+1}{(j+k+1)(j+k+2)}. \tag{2.1}$$

The matrix U is a bounded operator on ℓ_p and $\|U\|_{\ell_p\to\ell_p} = \Gamma(1/p)\Gamma(1+1/p^*)$, ([2], Proposition 2). In the sequel, we generalize this result for all Hilbert operators, but first we need the following lemma also known as Schur's lemma.

Schur Lemma ([6], Theorem 275) *Let $p > 1$ and Σ be a matrix with nonnegative entries. Suppose that $\mathcal{S}$, $\mathcal{R}$ are two positive numbers such that*

$$\sum_{j=0}^{\infty}[\Sigma]_{j,k} \leq \mathcal{S} \quad \forall k, \qquad \sum_{k=0}^{\infty}[\Sigma]_{j,k} \leq \mathcal{R} \quad \forall j.$$

Then

$$\|\Sigma\|_{\ell_p\to\ell_p} \leq \mathcal{R}^{1/p^*}\mathcal{S}^{1/p}.$$

Theorem 2.1 *Let n and m be two nonnegative integers. The Hilbert matrix of order n has a factorization of the form $\mathcal{H}_n = \mathcal{R}_{n,m}C_m$, where the matrix $\mathcal{R}_{n,m}$ has the entries*

$$\begin{aligned}[\mathcal{R}_{n,m}]_{j,k} &= \frac{(k+1)\cdots(k+m)}{(j+k+n+1)\cdots(j+k+n+m+1)} \\ &= \binom{m+k}{k}\beta(j+k+n+1, m+1)\end{aligned}$$

and is a bounded operator on ℓ_p with

$$\|\mathcal{R}_{n,m}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}.$$

In particular,

(a) *the Hilbert matrix of order n has a factorization of the form $\mathcal{H}_n = \mathcal{R}_nC_n$, where the matrix $\mathcal{R}_n$ has the entries*

$$\begin{aligned}[\mathcal{R}_n]_{j,k} &= \frac{(k+1)\cdots(k+n)}{(j+k+n+1)\cdots(j+k+2n+1)} \\ &= \binom{n+k}{k}\beta(j+k+n+1, n+1)\end{aligned}$$

and is a bounded operator on ℓ_p with $\|\mathcal{R}_n\|_{\ell_p\to\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}$.

(b) *the Hilbert matrix has a factorization of the form $\mathcal{H} = B_nC_n$, where the matrix U_n has the entries*

$$[B_n]_{j,k} = \frac{(k+1)\cdots(k+n)}{(j+k+1)\cdots(j+k+n+1)} = \binom{n+k}{k}\beta(j+k+1, n+1)$$

with ℓ_p-norm $\|B_n\|_{\ell_p\to\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}$.

Proof Let Δ_n be the backward difference matrix of order n. That is a lower triangle matrix with the entries

$$\left[\Delta_n\right]_{j,k} = (-1)^{(j-k)}\binom{n}{j-k} \quad k \leq j \leq n+k,$$

which is invertible. We use the notation Δ_n^{-1} as its inverse which is defined by

$$\left[\Delta_n^{-1}\right]_{j,k} = \binom{n+j-k-1}{j-k} \quad j \geq k.$$

Note that for $n = 0$, the backward difference matrix $\Delta_0 = I$, where I is the identity matrix. It can be easily seen that the Cesàro matrix of order n can be represented by the backward difference operator of the form

$$[C_n]_{j,k} = \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} = \frac{[\Delta_n^{-1}]_{j,k}}{\binom{n+j}{j}}.$$

On the other hand,

$$\mathcal{H}_n\Delta_0 = \mathcal{H}_n = \frac{1}{j+k+n+1} = \beta(j+k+n+1,1)$$

and

$$\mathcal{H}_n\Delta_1 = \frac{1}{(j+k+n+1)(j+k+n+2)} = \beta(j+k+n+1,2).$$

By induction, we can prove that

$$\mathcal{H}_n\Delta_m = \beta(j+k+n+1,m+1).$$

Now, by a simple calculation, we deduce that

$$\begin{aligned}\mathcal{R}_{n,m}C_m &= \sum_{i=k}^{\infty}\binom{m+i}{i}\beta(j+i+n+1,m+1)\frac{\binom{m+i-k-1}{i-k}}{\binom{m+i}{i}}\\ &= \big(\mathcal{H}_n\Delta_m\big)\Delta_m^{-1} = \mathcal{H}_n.\end{aligned}$$

For computing the ℓ_p-norm of $\mathcal{R}_{n,m}$, we show first

$$\|\mathcal{R}_{0,m}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}. \tag{2.2}$$

For convenience, let $\mathcal{R}_{0,m} = \mathcal{R}_m$. We introduce a family of matrices, $\mathcal{R}_m(s)$, $0 < s \leq 1$, given by

$$\big[\mathcal{R}_m(s)\big]_{j,k} = \binom{j+k}{k}s^j(1-s)^{m+k}.$$

Since

$$\begin{aligned}\sum_{k=0}^{\infty}\big[\mathcal{R}_m(s)\big]_{j,k} &= s^j(1-s)^m\sum_{k=0}^{\infty}\binom{j+1+k-1}{k}(1-s)^k\\ &= s^j(1-s)^m\big[1-(1-s)\big]^{-(j+1)}\\ &= \frac{(1-s)^m}{s}\end{aligned}$$

and

$$\sum_{j=0}^{\infty}\big[\mathcal{R}_m(s)\big]_{j,k} = (1-s)^{m+k}\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}s^j = (1-s)^{m-1},$$

the row sums and the column sums are $\frac{(1-s)^m}{s}$ and $(1-s)^{m-1}$, respectively. Thus Schur's lemma results in

$$\left\|\mathcal{R}_m(s)\right\|_{\ell_p\to\ell_p} \leq (1-s)^{m-1/p}s^{-1/p^*}.$$

On the other hand,

$$\begin{aligned}\int_0^1 \left[\mathcal{R}_m(s)\right]_{j,k} ds &= \binom{j+k}{k}\int_0^1 s^j(1-s)^{m+k}\,ds \\ &= \binom{j+k}{k}\beta(j+1,m+k+1) \\ &= \binom{m+k}{k}\beta(j+k+1,m+1) = \mathcal{R}_m.\end{aligned}$$

Now,

$$\begin{aligned}\|\mathcal{R}_m\|_{\ell_p\to\ell_p} &= \left\|\int_0^1 \mathcal{R}_m(s)\,ds\right\|_{\ell_p\to\ell_p} \leq \int_0^1 \left\|\mathcal{R}_m(s)\right\|_{\ell_p\to\ell_p} ds \\ &\leq \int_0^1 (1-s)^{m-1/p}s^{-1/p^*}\,ds = \beta\left(m+1/p^*,1/p\right) \\ &= \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}.\end{aligned}$$

The other side of the above inequality will result from the factorization $\mathcal{H} = \mathcal{R}_{0,m}C_m = \mathcal{R}_mC_m$. Now, suppose that

$$\|\mathcal{R}_{n,m}\|_{\ell_p\to\ell_p} = f(n,m),$$

where f is a nonnegative function. As a result of equality (2.2), we conclude that

$$\|\mathcal{R}_{n,m}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}g(n),$$

where $g(n)$ is a function of n. Now, let $m = 0$. Since $\mathcal{R}_{n,0} = \mathcal{H}_n$ is the Hilbert operator of order n, hence $g(n) = 1$ and

$$\|\mathcal{R}_{n,m}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}.$$

So we have the desired result. □

Corollary 2.2 *The Hilbert operator of order n admits a factorization of the form $\mathcal{H}_n = U_nC$, where C is the classical Cesàro matrix and U_n is defined by*

$$[U_n]_{j,k} = \frac{k+1}{(j+k+n+1)(j+k+n+2)},$$

and has the ℓ_p-norm

$$\|U_n\|_{\ell_p\to\ell_p} = \Gamma(1/p)\Gamma\left(1+1/p^*\right) = \pi/p^*\csc(\pi/p).$$

In particular, Hilbert operator has the factorization $\mathcal{H} = UC$, where $\|U\|_{\ell_p\to\ell_p} = \Gamma(1/p)\times\Gamma(1+1/p^*)$.

Corollary 2.3 *The following inequalities hold*:

$$\|\mathcal{H}_n x\|_{\ell_p} \le \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}\|C_m x\|_{\ell_p}.$$

In particular,

$$\|\mathcal{H}x\|_{\ell_p} \le \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}\|C_n x\|_{\ell_p}$$

and

$$\|\mathcal{H}_n x\|_{\ell_p} \le \Gamma(1/p)\Gamma\bigl(1+1/p^*\bigr)\|Cx\|_{\ell_p}.$$

More explicitly,

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p \le \left[\frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}\right]^p\sum_{j=0}^{\infty}\left|\sum_{k=0}^{j}\frac{\binom{m+j-k-1}{j-k}x_k}{\binom{m+j}{j}}\right|^p,$$

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p \le \left[\frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}\right]^p\sum_{j=0}^{\infty}\left|\sum_{k=0}^{j}\frac{\binom{n+j-k-1}{j-k}x_k}{\binom{n+j}{j}}\right|^p,$$

and

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p \le \left[\frac{\pi}{p^*}\csc(\pi/p)\right]^p\sum_{j=0}^{\infty}\left|\sum_{k=0}^{n}\frac{x_k}{j+1}\right|^p.$$

Corollary 2.4 *The Hilbert operator of order n, $\mathcal{H}_n$, is a bounded operator from $C_m(p)$ into ℓ_p and*

$$\|\mathcal{H}_n\|_{C_m(p)\to\ell_p} = \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}.$$

In particular, the Hilbert operator $\mathcal{H}$ is a bounded operator from $C(p)$ into ℓ_p and $\|\mathcal{H}\|_{C(p)\to\ell_p} = \frac{\pi}{p^*}\csc(\pi/p)$.

Proof Since the map $C_m(p)\to\ell_p$, $x\to C_m x$ is an isomorphism between these two spaces, according to Theorem 2.1, we have

$$\begin{aligned}\|\mathcal{H}_n\|_{C_m(p)\to\ell_p} &= \sup_{x\in C_m(p)}\frac{\|\mathcal{H}_n x\|_{\ell_p}}{\|x\|_{C_m(p)}} = \sup_{C_m x\in\ell_p}\frac{\|\mathcal{H}_n x\|_{\ell_p}}{\|C_m x\|_{\ell_p}}\\ &= \sup_{C_m x\in\ell_p}\frac{\|\mathcal{R}_{n,m}C_m x\|_{\ell_p}}{\|C_m x\|_{\ell_p}} = \sup_{y\in\ell_p}\frac{\|\mathcal{R}_{n,m}y\|_{\ell_p}}{\|y\|_{\ell_p}}\\ &= \|\mathcal{R}_{n,m}\|_{\ell_p\to\ell_p} = \frac{\Gamma(m+1/p^*)\Gamma(1/p)}{\Gamma(m+1)}.\end{aligned}$$

□

Theorem 2.5 *The Hilbert operator has a factorization of the form $\mathcal{H} = U'C^t$, where U' is a bounded operator that has the ℓ_p-norm*

$$\|U'\|_{\ell_p\to\ell_p} = \Gamma(1+1/p)\Gamma\big(1/p^*\big).$$

Proof By a simple calculation, U' is an operator with the matrix representation

$$U' = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/6 & 1/12 & \cdots \\ 1/3 & 1/6 & 1/10 & \cdots \\ 1/4 & 3/20 & 1/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

or $U' = \binom{e_1}{U^t}$, where $e_1 = (1,0,0,\dots)$ and U^t is the transpose of the matrix U defined in relation (2.1). Obviously, U' has the ℓ_p-norm same as U^t. Hence

$$\|U'\|_{\ell_p\to\ell_p} = \|U^t\|_{\ell_p\to\ell_p} = \|U\|_{\ell_{p^*}\to\ell_{p^*}} = \Gamma(1+1/p)\Gamma\big(1/p^*\big). \qquad \square$$

As a result of the above theorem, we have the following inequality.

Corollary 2.6 *The following statement holds*:

$$\|\mathcal{H}x\|_{\ell_p} \le \frac{\pi}{p}\csc(\pi/p)\|C^t x\|_{\ell_p}.$$

More explicitly,

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p \le \left[\frac{\pi}{p}\csc(\pi/p)\right]^p \sum_{j=0}^{\infty}\left|\sum_{k=j}^{\infty}\frac{x_k}{k+1}\right|^p.$$

Theorem 2.7 *For $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, a Hilbert operator of order n, $\mathcal{H}_n$, has a factorization of the form $\mathcal{H}_n = \mathcal{S}_{n,m}\mathcal{G}_m$, where $\mathcal{S}_{n,m}$ is defined by*

$$[\mathcal{S}_{n,m}]_{j,k} = \frac{(1-1/m)(j+n+1)+(k+1)}{(j+k+n+1)(j+k+n+2)}$$

and is a bounded operator on ℓ_p with

$$\|\mathcal{S}_{n,m}\|_{\ell_p\to\ell_p} = (1-1/mp)\Gamma(1/p)\Gamma\big(1/p^*\big).$$

Proof At first we prove the factorization. Let $\alpha = (k+1)(k+2)\dots(k+m-1)$, we have

$$\begin{aligned} \mathcal{S}_{n,m}\mathcal{G}_m &= \sum_{i=k}^{\infty}\frac{[(1-1/m)(j+n+1)+(i+1)]}{(j+i+n+1)(j+i+n+2)}\frac{\binom{m+k-1}{k}}{\binom{m+i}{i}} \\ &= \binom{m+k-1}{k}\sum_{i=k}^{\infty}\frac{[(m-1)(j+n+1)+m(i+1)]}{(j+i+n+1)(j+i+n+2)}\frac{i!(m-1)!}{(m+i)!} \end{aligned}$$

$$
\begin{aligned}
&= \alpha \sum_{i=k}^{\infty} \frac{(m-1)(j+n+1)+m(i+1)}{(i+1)\cdots(i+m)(j+i+n+1)(j+i+n+2)} \\
&= \alpha \sum_{i=k}^{\infty} \frac{(i+m)(j+i+n+2)-(i+1)(j+i+n+1)}{(i+1)\cdots(i+m)(j+i+n+1)(j+i+n+2)} \\
&= \alpha \sum_{i=k}^{\infty} \left\{ \frac{1}{(i+1)\cdots(i+m-1)(j+i+n+1)} - \frac{1}{(i+2)\cdots(i+m)(j+i+n+2)} \right\} \\
&= \frac{1}{j+k+n+1} = \mathcal{H}_n.
\end{aligned}
$$

For obtaining the norm of $\mathcal{S}_{n,m}$, consider that $\mathcal{S}_{n,m} = (1-1/m)\mathcal{U}_n^t + \mathcal{U}_n$, where the matrix $\mathcal{U}_n$ defined in the Corollary 2.2. Hence by applying the Hellinger–Toeplitz theorem

$$
\begin{aligned}
\|\mathcal{S}_{n,m}\|_{\ell_p\to\ell_p} &\le (1-1/m)\|\mathcal{U}_n^t\|_{\ell_p\to\ell_p} + \|\mathcal{U}_n\|_{\ell_p\to\ell_p} \\
&= (1-1/m)\Gamma\big(1/p^*\big)\Gamma(1+1/p) + \Gamma(1/p)\Gamma\big(1+1/p^*\big) \\
&= (1-1/mp)\Gamma(1/p)\Gamma\big(1/p^*\big),
\end{aligned}
$$

which completes the proof. □

Corollary 2.8 *The Hilbert matrix has a factorization of the form* $\mathcal{H} = \mathcal{S}_n\mathcal{G}_n$, *where the matrix* $\mathcal{S}_n$ *has the entries*

$$
[\mathcal{S}_n]_{j,k} = \frac{(1-1/n)(j+1)+(k+1)}{(j+k+1)(j+k+2)}
$$

and

$$
\|\mathcal{S}_n\|_{\ell_p\to\ell_p} = (1-1/np)\Gamma(1/p)\Gamma\big(1/p^*\big).
$$

In particular, the Hilbert matrix has Bennett's factorization $\mathcal{H} = UC$, *where the matrix* U *is a bounded operator and* $\|U\|_{\ell_p\to\ell_p} = \Gamma(1/p)\Gamma(1+1/p^*)$.

Corollary 2.9 *The following inequalities hold*:

$$
\|\mathcal{H}_n x\|_{\ell_p} \le (1-1/mp)\Gamma(1/p)\Gamma\big(1/p^*\big)\|\mathcal{G}_m x\|_{\ell_p}.
$$

In particular,

$$
\|\mathcal{H} x\|_{\ell_p} \le (1-1/np)\Gamma(1/p)\Gamma\big(1/p^*\big)\|\mathcal{G}_n x\|_{\ell_p}
$$

and

$$
\|\mathcal{H}_n x\|_{\ell_p} \le \Gamma(1/p)\Gamma\big(1+1/p^*\big)\|Cx\|_{\ell_p}.
$$

More explicitly,

$$
\sum_{j=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{x_k}{j+k+n+1} \right|^p \le \big[(1-1/mp)\Gamma(1/p)\Gamma\big(1/p^*\big)\big]^p \sum_{j=0}^{\infty} \left| \sum_{k=0}^{j} \frac{\binom{m+k-1}{k} x_k}{\binom{m+j}{j}} \right|^p,
$$

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right|^p \le \left[(1-1/np)\Gamma(1/p)\Gamma\left(1/p^*\right)\right]^p \sum_{j=0}^{\infty}\left|\sum_{k=0}^{j}\frac{\binom{n+k-1}{k}x_k}{\binom{n+j}{j}}\right|^p,$$

and

$$\sum_{j=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{x_k}{j+k+n+1}\right|^p \le \left[\Gamma(1/p)\Gamma\left(1+1/p^*\right)\right]^p \sum_{j=0}^{\infty}\left|\sum_{k=0}^{n}\frac{x_k}{j+1}\right|^p.$$

Corollary 2.10 *The Hilbert operator of order n, $\mathcal{H}_n$, is a bounded operator from $\mathcal{G}_m(p)$ into ℓ_p and*

$$\|\mathcal{H}_n\|_{\mathcal{G}_m(p)\to\ell_p} = (1-1/mp)\Gamma(1/p)\Gamma\left(1/p^*\right).$$

In particular, the Hilbert operator $\mathcal{H}$ is a bounded operator from $C(p)$ into ℓ_p and $\|\mathcal{H}\|_{C(p)\to\ell_p} = \Gamma(1/p)\Gamma(1+1/p^*)$.

Proof The map $\mathcal{G}_m(p) \to \ell_p, x \to \mathcal{G}_m x$ is an isomorphism between these two spaces, hence according to Theorem 2.7 we have

$$\begin{aligned}
\|\mathcal{H}_n\|_{\mathcal{G}_m(p)\to\ell_p} &= \sup_{x\in\mathcal{G}_m(p)}\frac{\|\mathcal{H}_n x\|_{\ell_p}}{\|x\|_{\mathcal{G}_m(p)}} = \sup_{\mathcal{G}_m x\in\ell_p}\frac{\|\mathcal{H}_n x\|_{\ell_p}}{\|\mathcal{G}_m x\|_{\ell_p}} \\
&= \sup_{\mathcal{G}_m x\in\ell_p}\frac{\|\mathcal{S}_{n,m}\mathcal{G}_m x\|_{\ell_p}}{\|\mathcal{G}_m x\|_{\ell_p}} = \sup_{y\in\ell_p}\frac{\|\mathcal{S}_{n,m} y\|_{\ell_p}}{\|y\|_{\ell_p}} \\
&= \|\mathcal{S}_{n,m}\|_{\ell_p\to\ell_p} = (1-1/mp)\Gamma(1/p)\Gamma\left(1/p^*\right). \qquad \square
\end{aligned}$$

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Declarations

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