Kim *Journal of Inequalities and Applications* (2022) 2022:119
https://doi.org/10.1186/s13660-022-02856-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Mixture and interpolation of the parameterized ordered means

Sejong Kim[1*]

*Correspondence:
skim@chungbuk.ac.kr
[1]Department of Mathematics, Chungbuk National University, Cheongju 361-763, Korea

Abstract

Loewner partial order plays a very important role in metric topology and operator inequality on the open convex cone of positive invertible operators. In this paper, we consider a family $G = \{G_n\}_{n\in\mathbb{N}}$ of the ordered means for positive invertible operators equipped with homogeneity and properties related to the Loewner partial order such as the monotonicity, joint concavity, and arithmetic-G-harmonic weighted mean inequalities. Similar to the resolvent average, we construct a parameterized ordered mean and compare two types of mixtures of parameterized ordered means in terms of the Loewner order. We also show a relation between two families of parameterized ordered means associated with the power mean monotonic interpolating given two parameterized ordered means.

MSC: Primary 47B65; secondary 15B48

Keywords: Ordered mean; Parameterized ordered mean; Mixture; Interpolation; Power mean; Kantorovich constant

1 Introduction

As the proximal average of proper convex lower semicontinuous functions in the context of convex analysis and optimization, the weighted resolvent mean, which is a parameterized harmonic mean. has been introduced in [3] and extended to monotone operators in [2]:

$$\mathcal{R}^{\mu}(\omega;\mathbf{A}) := \left[\sum_{i=1}^{n} w_i(A_i+\mu I)^{-1}\right]^{-1} - \mu I, \quad \mu \geq 0,$$

where $\omega = (w_1,\dots,w_n)$ is a positive probability vector in $\mathbb{R}^n$, and $\mathbf{A} = (A_1,\dots,A_n)$ is an n-tuple of positive definite Hermitian matrices. As a symmetrized version of the weighted resolvent mean and a unique minimizer of the weighted sum of Kullback–Leibler divergence, a parameterized weighted arithmetic-geometric-harmonic mean (simply the weighted $\mathcal{A}\#\mathcal{H}$ mean) has been introduced in [9]:

$$\mathcal{L}^{\mu}(\omega;\mathbf{A}) := \left[\sum_{i=1}^{n} w_i(A_i+\mu I)\right]\#\left[\sum_{i=1}^{n} w_i(A_i+\mu I)^{-1}\right]^{-1} - \mu I, \quad \mu \geq 0,$$

© The Author(s) 2022. **Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

where $A\#B$ is the midpoint of the Riemannian geodesic $A\#_pB = A^{1/2}(A^{-1/2}BA^{-1/2})^pA^{1/2}$, $p \in [0,1]$, of positive definite Hermitian matrices A and B for the Riemannian trace metric $\delta(A,B) = \|\log A^{-1/2}BA^{-1/2}\|_2$. Interesting results of these means are that they interpolate the weighted harmonic mean and arithmetic mean, the weighted $\mathcal{A}\#\mathcal{H}$ mean is the limit of the mean iteration of the two-variable arithmetic mean and resolvent mean, and they satisfy the monotonicity for parameter μ and the nonexpansiveness for the Thompson part metric $d_T(A,B) = \|\log A^{-1/2}BA^{-1/2}\|$, where $\|\cdot\|$ denotes the operator norm. Recently, a generalization of the parameterized version of weighted means including the Cartan mean, which is the unique minimizer of the weighted sum of Riemannian trace distances to given variables, to contractive barycentric maps of probability measures has been developed in [14].

On the open convex cone of positive invertible (positive definite) bounded linear operators as the infinite-dimensional setting, we consider a family $G = \{G_n\}$ of the n-variable weighted means equipped with homogeneity and properties related to the Loewner partial order for each $n \in \mathbb{N}$: the monotonicity, joint concavity, and arithmetic-G-harmonic mean inequalities. It includes many multivariate means such as the resolvent mean, power mean, Karcher mean [12], and we call it the *ordered mean*. Similar to the weighted resolvent mean and the weighted $\mathcal{A}\#\mathcal{H}$ mean, we construct the parameterized ordered mean from given ordered mean G:

$$G^{\mu}(\omega;\mathbf{A}) := G^{\mu}(\omega;A_1+\mu I,\dots,A_n+\mu I)-\mu I, \quad \mu \geq 0,$$

and $G^{\mu}(\omega;\mathbf{A}) := G^{-\mu}(\omega;\mathbf{A}^{-1})^{-1}$ for $\mu < 0$, where $\mathbf{A}^{-1} := (A_1^{-1},\dots,A_n^{-1})$. In Sect. 3, we first investigate properties of the parameterized ordered mean additionally to those in [14] and then compare two mixed means of parameterized ordered means: for the n-by-k block matrix $\mathbb{A} = [A_{ij}]$ whose block entries are positive definite operators and for a positive probability vector $\lambda \in \mathbb{R}^k$,

$$G_n^{\nu}\big(\omega; G_k^{\mu_1}\big(\lambda;\mathbb{A}^1\big),\dots,G_k^{\mu_n}\big(\lambda;\mathbb{A}^n\big)\big) \quad \text{and} \quad G_k^{\sum\omega_i\mu_i}\big(\lambda; G_n^{\nu}(\omega;\mathbb{A}_1),\dots,G_n^{\nu}(\omega;\mathbb{A}_k)\big),$$

where $\mathbb{A}^i$ and $\mathbb{A}_j$ denote the tuples of the ith row and jth column of $\mathbb{A}$. They coincide when the variables A_{ij} commute, but this does not hold in general. We obtain interesting inequalities associated with the Kantorovich constant.

Furthermore, in Sect. 4, we consider two families of parameterized ordered means

$$\big\{G^{P_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A})\big\}_{t\in[0,1]} \quad \text{and} \quad \big\{P_p\big(1-t,t;G^{\mu}(\omega;\mathbf{A}),G^{\nu}(\omega;\mathbf{A})\big)\big\}_{t\in[0,1]}$$

for given parameters $\mu,\nu > 0$ and any $p \in [0,1]$. Note that $P_p(\omega;\mathbf{A})$ is the weighted power mean of positive definite operators, which is the unique positive definite solution X of the nonlinear equation $X = \sum_{i=1}^n w_iX\#_pA_i$. The interesting fact of these families is that they monotonically interpolate two parameterized ordered means $G^{\mu}(\omega;\mathbf{A})$ and $G^{\nu}(\omega;\mathbf{A})$ due to the monotonicities of parameterized ordered means on parameters and power means on variables. We show their relation with respect to the Loewner order and provide a generalization to the multivariate power means, so that we obtain an interesting chain of inequalities for positive parameters. Finally, in Sect. 5, we give two open problems about the interpolation of parameterized ordered means for the generalized means (Hölder means) instead of the power means and the contractive barycentric maps of probability measures with compact support.

2 Ordered means

Let $B(\mathcal{H})$ be the Banach space of all bounded linear operators on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$, and let $S(\mathcal{H}) \subset B(\mathcal{H})$ be the real vector space of all self-adjoint operators. We call $A \in S(\mathcal{H})$ positive semidefinite (positive definite) if $\langle x, Ax\rangle \geq (>)0$ for all (nonzero, respectively) vectors $x \in \mathcal{H}$. We denote by $\mathbb{P} \subset S(\mathcal{H})$ the open convex cone of all positive definite operators. For self-adjoint operators A, B, we write $A \leq (<)B$ if $B - A$ is positive semidefinite (positive definite, respectively). This is known as the Loewner partial order.

Since Kubo and Ando [10] established two-variable means of positive definite matrices and operators, many different kinds of construction schemes of n-variable means have been developed. Especially, Ando, Li, and Mathias [1] suggested ten desired properties for extended geometric means. We consider a family of the weighted means of positive definite operators with homogeneity and properties only related to the Loewner order and call it the ordered mean. In the following, Δ_n is the simplex of positive probability vectors in $\mathbb{R}^n$ convexly spanned by the unit coordinate vectors.

Definition 2.1 The *ordered mean* is a family $G = \{G_n\}_{n\in\mathbb{N}}$ such that for each n, a map $G_n : \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ satisfies the following properties: for $\mathbf{A} = (A_1, \dots, A_n)$, $\mathbf{B} = (B_1, \dots, B_n) \in \mathbb{P}^n$, $\omega = (w_1, \dots, w_n) \in \Delta_n$, and a positive real number a,

(P1) (homogeneity) $G_n(\omega; a\mathbf{A}) = aG_n(\omega; \mathbf{A})$;

(P2) (monotonicity) $G_n(\omega; \mathbf{B}) \leq G_n(\omega; \mathbf{A})$ whenever $B_i \leq A_i$ for all $1 \leq i \leq n$;

(P3) (joint concavity) $G_n(\omega; (1-s)\mathbf{A} + s\mathbf{B}) \geq (1-s)G_n(\omega; \mathbf{A}) + sG_n(\omega; \mathbf{B})$ for $0 \leq s \leq 1$;

(P4) (arithmetic-G-harmonic weighted mean inequalities)

$$\mathcal{H}(\omega; \mathbf{A}) := \left[\sum_{i=1}^{n} w_i A_i^{-1}\right]^{-1} \leq G_n(\omega; \mathbf{A}) \leq \sum_{i=1}^{n} w_i A_i =: \mathcal{A}(\omega; \mathbf{A}).$$

By the arithmetic-G-harmonic weighted mean inequalities (P4) we can see that the ordered mean G is idempotent, that is, $G_n(\omega; A, \dots, A) = A$ for all $A \in \mathbb{P}$ and $n \in \mathbb{N}$.

Remark 2.2 Many multivariate means of positive definite matrices and operators, including the Ando–Li–Mathias mean [1], Bini-Meini–Poloni mean [5], resolvent average [3], arithmetic-geometric-harmonic mean [9], power mean [15], and Karcher mean [12], fulfill the definition of ordered means. Moreover, every ordered mean $G = \{G_n\}_{n\in\mathbb{N}}$ is the multivariate Lie–Trotter mean since it satisfies the arithmetic-G-harmonic weighted mean inequalities, that is, for each n

$$\lim_{s\to 0} G_n\big(\omega; \gamma_1(s), \dots, \gamma_n(s)\big)^{1/s} = \exp\left[\sum_{i=1}^{n} w_j \gamma_i'(0)\right],$$

where $\omega \in \Delta_n$, and $\gamma_1, \dots, \gamma_n$ are differentiable curves on $\mathbb{P}$ with $\gamma_i(0) = I$ for all i. See [6] for more detail.

Remark 2.3 By [11, Proposition 2.3] the ordered mean G is nonexpansive for the Thompson metric d_T. In other words, let $\mathbf{A} = (A_1, \dots, A_n)$, $\mathbf{B} = (B_1, \dots, B_n) \in \mathbb{P}^n$, and $\omega =$

$(w_1,\dots,w_n)\in\Delta_n$. Then for each $n\in\mathbb{N}$,

$$d_T\big(G_n(\omega;\mathbf{A}),G_n(\omega;\mathbf{B})\big)\le\max_{1\le i\le n}d_T(A_i,B_i),$$

where $d_T(A,B)=\|\log A^{-1/2}BA^{-1/2}\|$ for $A,B\in\mathbb{P}$ and the operator norm $\|\cdot\|$. This provides a generalization to the contractive barycentric map of probability measures [13, 14] and the continuity of the ordered mean G_n.

Let

$$\mathbb{A}=[A_{ij}]=\begin{pmatrix}A_{11}&A_{12}&\cdots&A_{1k}\\A_{21}&A_{22}&\cdots&A_{2k}\\\vdots&\vdots&\ddots&\vdots\\A_{n1}&A_{n2}&\cdots&A_{nk}\end{pmatrix}$$

be an n-by-k block matrix with block entries $A_{ij}\in\mathbb{P}$. We denote by $\mathbb{A}^i:=(A_{i1},A_{i2},\dots,A_{ik})\in\mathbb{P}^k$ and $\mathbb{A}_j:=(A_{1j},A_{2j},\dots,A_{nj})\in\mathbb{P}^n$, respectively, the tuples of the ith row and jth column of $\mathbb{A}$. Also, we denote by $A_1\oplus A_2\oplus\cdots\oplus A_n$ the n-by-n block diagonal matrix with block entries $A_i\in\mathbb{P}$.

Given $\omega=(w_1,\dots,w_n)\in\Delta_n$, let

$$\Phi(\mathbb{A})=\sum_{i=1}^n w_iA_{ii}$$

for an n-by-n block matrix $\mathbb{A}=[A_{ij}]$. Then it is strictly positive and unital linear map. Assume that $0<mI\le A_i\le MI$ for all $i=1,\dots,n$ and some constants $M,m>0$. Applying [4, Proposition 2.7.8] to Φ with $A_1\oplus A_2\oplus\cdots\oplus A_n$, we obtain the reverse inequality of arithmetic-harmonic weighted mean inequality:

$$\sum_{i=1}^n w_iA_i\le\frac{(M+m)^2}{4Mm}\left[\sum_{i=1}^n w_iA_i^{-1}\right]^{-1}.$$

Here the value $K=\frac{(M+m)^2}{4Mm}$ is known as the Kantorovich constant. By the G-harmonic weighted mean inequality in (P4) we have

$$\sum_{i=1}^n w_iA_i\le\frac{(M+m)^2}{4Mm}G_n(\omega;A_1,\dots,A_n).\tag{2.1}$$

For each n, consider a multivariate geometric mean G_n satisfying the consistency with scalars, that is,

$$G_n(\omega;\mathbf{A})=\prod_{i=1}^n A_i^{w_i}$$

when the A_i commute, where $\mathbf{A}=(A_1,\dots,A_n)\in\mathbb{P}^n$. Then the following holds:

$$G_n\big(\omega;G_k\big(\lambda;\mathbb{A}^1\big),\dots,G_k\big(\lambda;\mathbb{A}^n\big)\big)=\prod_{i,j}A_{ij}^{w_i\lambda_j}=G_k\big(\lambda;G_n(\omega;\mathbb{A}_1),\dots,G_n(\omega;\mathbb{A}_k)\big)$$

when the A_{ij} commute, where $\lambda = (\lambda_1, \dots, \lambda_k) \in \Delta_k$, and $\mathbb{A} = [A_{ij}]$ is the n-by-k block matrix with $A_{ij} \in \mathbb{P}$ for all i, j. Although it does not hold in general, we have the following inequality.

Theorem 2.4 *Let $\mathbb{A} = [A_{ij}]$ be an n-by-k block matrix with $A_{ij} \in \mathbb{P}$ for all i, j. Assume that $0 < mI \le A_{ij} \le MI$ for all i, j, where $M, m > 0$ are some constants. Then the ordered mean $G = \{G_n\}_{n\in\mathbb{N}}$ satisfies that for all $\lambda \in \Delta_k$ and $\omega \in \Delta_n$,*

$$G_n\bigl(\omega; G_k\bigl(\lambda; \mathbb{A}^1\bigr), \dots, G_k\bigl(\lambda; \mathbb{A}^n\bigr)\bigr) \le KG_k\bigl(\lambda; G_n(\omega; \mathbb{A}_1), \dots, G_n(\omega; \mathbb{A}_k)\bigr),$$

where $K = \frac{(M+m)^2}{4Mm}$.

Proof Let $\omega = (w_1, \dots, w_n) \in \Delta_n$. Then

$$\begin{aligned} G_n\bigl(\omega; G_k\bigl(\lambda; \mathbb{A}^1\bigr), \dots, G_k\bigl(\lambda; \mathbb{A}^n\bigr)\bigr) &\le \sum_{i=1}^n w_i G_k\bigl(\lambda; \mathbb{A}^i\bigr) \\ &\le G_k\left(\lambda; \sum_{i=1}^n w_i \mathbb{A}^i\right) = G_k\left(\lambda; \sum_{i=1}^n w_i A_{i1}, \dots, \sum_{i=1}^n w_i A_{ik}\right) \\ &\le G_k\bigl(\lambda; KG_n(\omega; \mathbb{A}_1), \dots, KG_n(\omega; \mathbb{A}_k)\bigr) \\ &= KG_k\bigl(\lambda; G_n(\omega; \mathbb{A}_1), \dots, G_n(\omega; \mathbb{A}_k)\bigr). \end{aligned}$$

The first inequality follows from the arithmetic-G weighted mean inequality in (P4), the second from the joint concavity (P3), the third from (2.1) together with the monotonicity (P2), and the last equality from the homogeneity (P1). □

Theorem 2.5 *Let $\mathbb{A} = [A_{ij}]$ be the n-by-k block matrix with $A_{ij} \in \mathbb{P}$ for all i, j. Assume that $0 < mI \le A_{ij} \le MI$ for all i, j, where $M, m > 0$ are some constants. Then*

$$G_n\bigl(\omega; G_k\bigl(\lambda; \mathbb{A}^1\bigr), \dots, G_k\bigl(\lambda; \mathbb{A}^n\bigr)\bigr) \le tG_k\bigl(\lambda; G_n(\omega; \mathbb{A}_1), \dots, G_n(\omega; \mathbb{A}_k)\bigr) + \rho_{M,m}(t)I,$$

where

$$\rho_{M,m}(t) = \begin{cases} (1-t)m, & t \ge M/m, \\ M + m - 2\sqrt{tMm}, & m/M \le t \le M/m, \\ (1-t)M, & t \le m/M. \end{cases}$$

Proof It has been shown in [7, Theorem 2.2] that

$$\Psi(A) - t\Psi\bigl(A^{-1}\bigr)^{-1} \le \Psi(A) - t\left[-\frac{1}{Mm}\Psi(A) + \frac{M+m}{Mm}I\right]^{-1}$$

for any positive unital linear map Ψ and any $t > 0$, where $A \in \mathbb{P}$ with $0 < mI \le A \le MI$. We can easily see that the function $f(x) = x - \frac{tMm}{M+m-x}$ has only one critical point $x_0 = M + m - \sqrt{tMm}$ and $f''(x) < 0$ in the closed interval $[m, M]$. Thus by fundamental calculation we obtain, as above,

$$\rho_{M,m}(t) = \max_{x\in[m,M]} f(x)$$

and

$$\Psi(A) \leq t\Psi\left(A^{-1}\right)^{-1} + \rho_{M,m}(t)I$$

for all $t > 0$ and any $A \in \mathbb{P}$ such that $0 < mI \leq A \leq MI$. Then by the arithmetic-G-harmonic weighted mean inequalities in (P4) we have

$$\begin{aligned} &G_n\left(\omega; G_k\left(\lambda; \mathbb{A}^1\right), \dots, G_k\left(\lambda; \mathbb{A}^n\right)\right) - tG_k\left(\lambda; G_n(\omega; \mathbb{A}_1), \dots, G_n(\omega; \mathbb{A}_k)\right) \\ &\quad \leq \sum_{i,j} w_i \lambda_j A_{ij} - t\left[\sum_{i,j} w_i \lambda_j A_{ij}^{-1}\right]^{-1} = \Phi(\widehat{\mathbb{A}}) - t\Phi\left(\widehat{\mathbb{A}}^{-1}\right)^{-1} \leq \rho_{M,m}(t)I, \end{aligned}$$

where $\Phi(\widehat{\mathbb{A}}) = \sum_{i,j} w_i \lambda_j A_{ij}$ for

$$\widehat{\mathbb{A}} := A_{11} \oplus \cdots \oplus A_{1k} \oplus A_{21} \oplus \cdots \oplus A_{2k} \oplus \cdots \oplus A_{n1} \oplus \cdots \oplus A_{nk}$$

is the positive unital linear map for given probability vectors $\omega = (w_1, \dots, w_n)$ and $\lambda = (\lambda_1, \dots, \lambda_k)$. □

Remark 2.6 For $t = 1$ in Theorem 2.5,

$$G_n\left(\omega; G_k\left(\lambda; \mathbb{A}^1\right), \dots, G_k\left(\lambda; \mathbb{A}^n\right)\right) \leq G_k\left(\lambda; G_n(\omega; \mathbb{A}_1), \dots, G_n(\omega; \mathbb{A}_k)\right) + (\sqrt{M} - \sqrt{m})^2 I.$$

3 Parameterized ordered means

For given ordered mean $G = \{G_n\}$, we define the parameterized ordered means $G^\mu : \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ as

$$G^\mu(\omega; \mathbf{A}) := G(\omega; \mathbf{A} + \mu\mathbf{I}) - \mu I, \quad \mu \geq 0, \tag{3.1}$$

where $\mathbf{I} = (I, \dots, I) \in \mathbb{P}^n$ with identity operator I, and

$$G^\mu(\omega; \mathbf{A}) := G^{-\mu}\left(\omega; \mathbf{A}^{-1}\right)^{-1}, \quad \mu < 0. \tag{3.2}$$

We also denote

$$G^\infty(\omega; \mathbf{A}) = \lim_{\mu \to \infty} G^\mu(\omega; \mathbf{A}) \ \text{ and } \ G^{-\infty}(\omega; \mathbf{A}) = \lim_{\mu \to -\infty} G^\mu(\omega; \mathbf{A}).$$

Remark 3.1 We recall the strong (operator) topology on the Banach space $B(\mathcal{H})$ of bounded linear operators as the topology of pointwise convergence. If a net of positive semidefinite operators A_α converges strongly to A, then the nonnegative values $\langle x, A_\alpha x\rangle$ converge to a nonnegative value $\langle x, Ax\rangle$. So the cone $\{A : A \geq 0\}$ is strongly closed, and hence the partial order $\{(A,B) \in S(\mathcal{H}) \times S(\mathcal{H}) : A \leq B\}$ is also strongly closed. Note from [18, Theorem 4.28(b)] that any decreasing (increasing) bounded below (above) net of self-adjoint operators possesses an infimum (supremum, respectively) to which it strongly converges.

Using the arithmetic-G-harmonic weighted mean inequalities in (P4) we have

$$G^\infty(\omega; \mathbf{A}) = \mathcal{A}(\omega; \mathbf{A})$$

under the strong topology. By (3.2) we also have

$$G^{-\infty}(\omega;\mathbf{A}) = \mathcal{H}(\omega;\mathbf{A}).$$

Lim [14] has established many remarkable properties of parameterized ordered means including a stochastic approximation and L^1 ergodic theorem for the parameterized Cartan (Karcher) mean. From [14, Proposition 5.3] we have the following properties of the parameterized ordered means G^μ induced from those of ordered means G.

Proposition 3.2 *Let* $\mathbf{A} = (A_1,\dots,A_n)$, $\mathbf{B} = (B_1,\dots,B_n) \in \mathbb{P}^n$ *and* $\omega = (w_1,\dots,w_n) \in \Delta_n$. *The parameterized ordered mean* G^μ *for* $\mu \in [-\infty,\infty]$ *has the following properties*:

(1) (*homogeneity*) *For a positive real number* a,

$$\begin{cases} G^\mu(\omega;a\mathbf{A}) = aG^{\frac{\mu}{a}}(\omega;\mathbf{A}), & \mu \in [0,\infty]; \\ G^\mu(\omega;a\mathbf{A}) = aG^{a\mu}(\omega;\mathbf{A}), & \mu \in [-\infty,0); \end{cases}$$

(2) (*monotonicity on variables*) *If* $B_i \le A_i$ *for all* $1 \le i \le n$, *then*

$$G^\mu(\omega;\mathbf{B}) \le G^\mu(\omega;\mathbf{A});$$

(3) (*joint concavity*) *For* $\mu \in [0,\infty]$ *and* $0 \le s \le 1$,

$$G^\mu\big(\omega;(1-s)\mathbf{A} + s\mathbf{B}\big) \ge (1-s)G^\mu(\omega;\mathbf{A}) + sG^\mu(\omega;\mathbf{B});$$

(4) (*arithmetic-*G^μ*-harmonic weighted mean inequalities*)

$$\left[\sum_{i=1}^n w_i A_i^{-1}\right]^{-1} \le G^\mu(\omega;\mathbf{A}) \le \sum_{i=1}^n w_i A_i;$$

(5) (*monotonicity on parameters*) *For* $0 \le \nu \le \mu \le \infty$,

$$\mathcal{H} = G^{-\infty} \le \cdots \le G^{-\mu} \le G^{-\nu} \le \cdots \le G^0 = G \le \cdots \le G^\nu \le G^\mu \le \cdots \le G^\infty = \mathcal{A};$$

(6) (*nonexpansiveness*) G^μ *is nonexpansive for the Thompson metric*, *that is*,

$$d_T\big(G^\mu(\omega;\mathbf{A}), G^\mu(\omega;\mathbf{B})\big) \le \max_{1\le i\le n} d_T(A_i,B_i).$$

Proof Most of properties have been proved in [14, Proposition 5.3]. Especially, the arithmetic-G^μ-harmonic weighted mean inequalities (4) is derived from

$$\mathcal{H}(\omega;\mathbf{A}) \le \mathcal{R}^\mu(\omega;\mathbf{A}) \le G^\mu(\omega;\mathbf{A}) \le \mathcal{A}(\omega;\mathbf{A}),$$

where $\mathcal{R}^\mu(\omega;\mathbf{A}) := [\sum_{i=1}^n w_i(A_i+\mu I)^{-1}]^{-1} - \mu I$ is the resolvent mean [9].

We show the homogeneity (1).

(1) Let $a > 0$. For $\mu \geq 0$, by the homogeneity of ordered means (P1) we have

$$\begin{aligned} G^{\mu}(\omega; a\mathbf{A}) &= G(\omega; aA_1 + \mu I, \ldots, aA_n + \mu I) - \mu I \\ &= aG\bigl(\omega; A_1 + (\mu/a)I, \ldots, A_n + (\mu/a)I\bigr) - \mu I = aG^{\frac{\mu}{a}}(\omega; \mathbf{A}). \end{aligned}$$

For $\mu < 0$, similarly, by using the above result together with (3.2) we have

$$G^{\mu}(\omega; a\mathbf{A}) = G^{-\mu}\bigl(\omega; a^{-1}\mathbf{A}^{-1}\bigr)^{-1} = \bigl[a^{-1}G^{-a\mu}\bigl(\omega; \mathbf{A}^{-1}\bigr)\bigr]^{-1} = aG^{a\mu}(\omega; \mathbf{A}). \qquad \square$$

Proposition 3.3 *Let $G = \{G_n\}$ be the ordered mean such that for all n and $\omega = (w_1, \ldots, w_n) \in \Delta_n$,*

(i) *G_n is invariant under permutation, that is, for any permutation σ on n letters,*

$$G_n(\omega_\sigma; \mathbf{A}_\sigma) = G_n(\omega; \mathbf{A}),$$

where $\omega_\sigma = (w_{\sigma(1)}, \ldots, w_{\sigma(n)})$ and $\mathbf{A}_\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(n)})$,

(ii) *G_n is invariant under repetition, that is, for all $k \in \mathbb{N}$,*

$$G_{nk}\bigl(\omega^{(k)}; \underbrace{A_1, \ldots, A_n}, \ldots, \underbrace{A_1, \ldots, A_n}\bigr) = G_n(\omega; A_1, \ldots, A_n),$$

where $\omega^{(k)} = \frac{1}{k}(\underbrace{w_1, \ldots, w_n}, \ldots, \underbrace{w_1, \ldots, w_n}) \in \Delta_{nk}$,

(iii) *G_n is invariant under congruence transformation, that is, for any invertible operator $S \in B(\mathcal{H})$,*

$$G_n\bigl(\omega; S^*\mathbf{A}S\bigr) = S^*G_n(\omega; \mathbf{A})S,$$

where $S^*\mathbf{A}S = (S^*A_1S, \ldots, S^*A_nS)$,

(iv) *$G_n(\omega; A_1, \ldots, A_{n-1}, X) = X$ if and only if $X = G_{n-1}(\hat{\omega}; A_1, \ldots, A_{n-1})$, where $\hat{\omega} = \frac{1}{1-w_n}(w_1, \ldots, w_{n-1}) \in \Delta_{n-1}$,*

(v) *$\Phi(G_n(\omega; A_1, \ldots, A_n)) \leq G_n(\omega; \Phi(A_1), \ldots, \Phi(A_n))$ for any positive unital linear map Φ.*

Then the corresponding parameterized ordered mean G^μ for $\mu \in [-\infty, \infty]$ has the same properties (i), (ii), (iii) *for any unitary operator S and* (iv). *Moreover,*

$$\begin{cases} \Phi(G_n^{\mu}(\omega; A_1, \ldots, A_n)) \leq G_n^{\mu}(\omega; \Phi(A_1), \ldots, \Phi(A_n)), & \mu \in [0, \infty], \\ \Phi(G_n^{\mu}(\omega; A_1, \ldots, A_n)) \geq G_n^{\mu}(\omega; \Phi(A_1^{-1})^{-1}, \ldots, \Phi(A_n^{-1})^{-1}), & \mu \in [-\infty, 0). \end{cases}$$

Proof It is obvious from properties (i)–(iv) of the ordered means G that the corresponding parameterized ordered mean G^μ for $\mu \in [-\infty, \infty]$ has the same properties.

For $\mu \in [0, \infty]$, applying (v) with the positive unital linear map Φ, we have

$$\begin{aligned} \Phi\bigl(G_n^{\mu}(\omega; A_1, \ldots, A_n)\bigr) &\leq G_n\bigl(\omega; \Phi(A_1 + \mu I), \ldots, \Phi(A_n + \mu I)\bigr) - \mu I \\ &= G_n\bigl(\omega; \Phi(A_1) + \mu I, \ldots, \Phi(A_n) + \mu I\bigr) - \mu I \\ &= G_n^{\mu}\bigl(\omega; \Phi(A_1), \ldots, \Phi(A_n)\bigr). \end{aligned}$$

Similarly, for $\mu \in [-\infty, 0)$,

$$\begin{aligned}\Phi\big(G_n^{\mu}(\omega; A_1, \dots, A_n)\big) &= \Phi\big(G_n^{-\mu}\big(\omega; A_1^{-1}, \dots, A_n^{-1}\big)^{-1}\big) \\ &\geq \Phi\big(G_n^{-\mu}\big(\omega; A_1^{-1}, \dots, A_n^{-1}\big)\big)^{-1} \\ &\geq G_n^{-\mu}\big(\omega; \Phi\big(A_1^{-1}\big), \dots, \Phi\big(A_n^{-1}\big)\big)^{-1} \\ &= G_n^{\mu}\big(\omega; \Phi\big(A_1^{-1}\big)^{-1}, \dots, \Phi\big(A_n^{-1}\big)^{-1}\big).\end{aligned}$$

The first equality follows from definition (3.2), the second inequality from Choi's inequality in [4, Theorem 2.3.6], the third inequality from (v) and the order reversing of inversion, and the last equality again follows from definition (3.2). □

Theorem 3.4 *Let $\mathbb{A} = [A_{ij}]$ be the n-by-k block matrix with $A_{ij} \in \mathbb{P}$ for all i, j. Let $\omega = (w_1, \dots, w_n) \in \Delta_n$ and $\lambda = (\lambda_1, \dots, \lambda_k) \in \Delta_k$. Then for any $\mu_1, \dots, \mu_n \geq 0$,*

$$\sum_{i=1}^{n} w_i G_k^{\mu_i}\big(\lambda; \mathbb{A}^i\big) \leq G_k^{\omega \bullet \mu}\left(\lambda; \sum_{i=1}^{n} w_i \mathbb{A}^i\right),$$

where $\omega \bullet \mu = \sum_{i=1}^{n} w_i \mu_i$ for $\mu := (\mu_1, \dots, \mu_n)$.

Proof By the joint concavity of ordered means (P3) we have

$$\begin{aligned}\sum_{i=1}^{n} w_i G_k^{\mu_i}\big(\lambda; \mathbb{A}^i\big) &= \sum_{i=1}^{n} w_i G_k\big(\lambda; \mathbb{A}^i + \mu_i I\big) - \sum_{i=1}^{n} w_i \mu_i I \\ &\leq G_k\left(\lambda; \sum_{i=1}^{n} w_i\big(\mathbb{A}^i + \mu_i I\big)\right) - \sum_{i=1}^{n} w_i \mu_i I = G_k^{\omega \bullet \mu}\left(\lambda; \sum_{i=1}^{n} w_i \mathbb{A}^i\right).\end{aligned}$$ □

Remark 3.5 For $n = 2$, taking $\mu_1 = \mu_2 = \nu (\geq 0)$ and $\omega = (1-t, t)$ for $t \in [0, 1]$ in Theorem 3.4 yields the joint concavity in Proposition 3.2(3):

$$(1-t)G^{\nu}(\lambda; \mathbf{A}) + tG^{\nu}(\lambda; \mathbf{B}) \leq G^{\nu}\big(\lambda; (1-t)\mathbf{A} + t\mathbf{B}\big).$$

So Theorem 3.4 is a multivariate extension of the joint concavity.

Theorem 3.6 *Let $\mathbb{A} = [A_{ij}]$ be the n-by-k block matrix such that $0 < mI \leq A_{ij} \leq MI$ for some constants $M, m > 0$. Let $\omega = (w_1, \dots, w_n) \in \Delta_n$ and $\lambda = (\lambda_1, \dots, \lambda_k) \in \Delta_k$. Then*

(i) *for any $\mu_1, \dots, \mu_n, \nu \geq 0$,*

$$G_n^{\nu}\big(\omega; G_k^{\mu_1}\big(\lambda; \mathbb{A}^1\big), \dots, G_k^{\mu_n}\big(\lambda; \mathbb{A}^n\big)\big) \leq K G_k^{\omega \bullet \mu}\big(\lambda; G_n^{\nu}(\omega; \mathbb{A}_1), \dots, G_n^{\nu}(\omega; \mathbb{A}_k)\big);$$

(ii) *for any $\mu_1, \dots, \mu_n, \nu < 0$,*

$$G_n^{\nu}\big(\omega; G_k^{\mu_1}\big(\lambda; \mathbb{A}^1\big), \dots, G_k^{\mu_n}\big(\lambda; \mathbb{A}^n\big)\big) \leq K^{-1} G_k^{\omega \bullet \mu}\big(\lambda; G_n^{\nu}(\omega; \mathbb{A}_1), \dots, G_n^{\nu}(\omega; \mathbb{A}_k)\big),$$

where $K = \frac{(M+m)^2}{4Mm}$.

Proof Assume that $0 < mI \leq A_{ij} \leq MI$ for some constants $M, m > 0$, where $\mathbb{A} = [A_{ij}]$ is an n-by-k block matrix.

(i) Since the parameterized ordered mean G^{ν} satisfies the arithmetic-G^{ν}-harmonic weighted mean inequalities in Proposition 3.2 (4), from (2.1) we have

$$\sum_{i=1}^{n} w_i A_i \leq K G_n^{\nu}(\omega; A_1, \ldots, A_n). \tag{3.3}$$

Then for $\mu_1, \ldots, \mu_n, \nu \geq 0$,

$$\begin{aligned} G_n^{\nu}\big(\omega; G_k^{\mu_1}(\lambda; \mathbb{A}^1), \ldots, G_k^{\mu_n}(\lambda; \mathbb{A}^n)\big) &\leq \sum_{i=1}^{n} w_i G_k^{\mu_i}(\lambda; \mathbb{A}^i) \\ &\leq G_k^{\omega \bullet \mu}\left(\lambda; \sum_{i=1}^{n} w_i \mathbb{A}^i\right) \\ &= G_k^{\omega \bullet \mu}\left(\lambda; \sum_{i=1}^{n} w_i A_{i1}, \ldots, \sum_{i=1}^{n} w_i A_{ik}\right) \\ &\leq G_k^{\omega \bullet \mu}\big(\lambda; K G_n^{\nu}(\omega; \mathbb{A}_1), \ldots, K G_n^{\nu}(\omega; \mathbb{A}_k)\big) \\ &= K G_k^{\frac{\omega \bullet \mu}{K}}\big(\lambda; G_n^{\nu}(\omega; \mathbb{A}_1), \ldots, G_n^{\nu}(\omega; \mathbb{A}_k)\big) \\ &\leq K G_k^{\omega \bullet \mu}\big(\lambda; G_n^{\nu}(\omega; \mathbb{A}_1), \ldots, G_n^{\nu}(\omega; \mathbb{A}_k)\big). \end{aligned}$$

The first inequality follows from the arithmetic-G^{ν} weighted mean inequality in Proposition 3.2(4), the second inequality from Theorem 3.4, the third inequality from (3.3), the second equality from the homogeneity in Proposition 3.2(1), and the last inequality from the monotonicity of parameterized ordered means for parameters in Proposition 3.2(5) since $K \geq 1$.

(ii) For $\mu_1, \ldots, \mu_n, \nu < 0$, we have

$$\begin{aligned} &G_n^{\nu}\big(\omega; G_k^{\mu_1}(\lambda; \mathbb{A}^1), \ldots, G_k^{\mu_n}(\lambda; \mathbb{A}^n)\big) \\ &\quad = G_n^{-\nu}\big(\omega; G_k^{-\mu_1}\big(\lambda; (\mathbb{A}^1)^{-1}\big), \ldots, G_k^{-\mu_n}\big(\lambda; (\mathbb{A}^n)^{-1}\big)\big)^{-1} \\ &\quad \geq K^{-1} G_k^{\omega \bullet (-\mu)}\big(\lambda; G_n^{-\nu}\big(\omega; (\mathbb{A}_1)^{-1}\big), G_n^{-\nu}\big(\omega; (\mathbb{A}_k)^{-1}\big)\big)^{-1} \\ &\quad = K^{-1} G_k^{-\omega \bullet \mu}\big(\lambda; G_n^{\nu}(\omega; \mathbb{A}_1)^{-1}, G_n^{\nu}(\omega; \mathbb{A}_k)^{-1}\big)^{-1} \\ &\quad = K^{-1} G_k^{\omega \bullet \mu}\big(\lambda; G_n^{\nu}(\omega; \mathbb{A}_1), \ldots, G_n^{\nu}(\omega; \mathbb{A}_k)\big). \end{aligned}$$

The first equality follows from (3.2), the inequality from (i), and the order reversing of inversion, and the second and last equalities again follow from (3.2). □

4 Interpolation with power means

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an n-tuple of given positive real numbers, and let $\omega = (w_1, \ldots, w_n)$ be a positive probability vector. The generalized means, also called the Hölder mean, with exponent $p \in \mathbb{R}$ are a family of functions

$$\mathfrak{M}_p(\omega; \mathbf{a}) := \left(\sum_{i=1}^{n} w_i a_i^p\right)^{\frac{1}{p}}, \quad p \neq 0,$$

and

$$\mathfrak{M}_0(\omega;\mathbf{a}) := \lim_{p\to 0}\mathfrak{M}_p(\omega;\mathbf{a}) = \prod_{i=1}^n a_i^{w_i}.$$

Note that

$$\mathfrak{M}_\infty(\omega;\mathbf{a}) := \lim_{p\to \infty}\mathfrak{M}_p(\omega;\mathbf{a}) = \max\{a_1,\dots,a_n\},$$
$$\mathfrak{M}_{-\infty}(\omega;\mathbf{a}) := \lim_{p\to -\infty}\mathfrak{M}_p(\omega;\mathbf{a}) = \min\{a_1,\dots,a_n\}.$$

One of the interesting properties for the generalized means is the monotonicity for exponents, that is,

$$\mathfrak{M}_p(\omega;\mathbf{a}) \le \mathfrak{M}_q(\omega;\mathbf{a}) \quad \text{if } p \le q. \tag{4.1}$$

Via the theory of power means of positive definite Hermitian matrices in [15], the power means of positive invertible operators have been successfully defined and developed in [12]. The power mean $P_p(\omega;\mathbf{A})$ of $\mathbf{A} = (A_1,\dots,A_n) \in \mathbb{P}^n$ for $p \in (0,1]$ is the unique solution $X \in \mathbb{P}$ of the nonlinear equation

$$X = \sum_{i=1}^n w_i X \#_p A_i,$$

and $P_p(\omega;\mathbf{A}) = P_{-p}(\omega;\mathbf{A}^{-1})^{-1}$ for $p \in [-1,0)$. Here $X\#_p A_i = X^{1/2}(X^{-1/2}A_iX^{-1/2})^pX^{1/2}$ is known as the p-weighted geometric mean of X and A_i. It is the operator version of generalized mean $\mathfrak{M}_p$ of positive scalars; in other words, $P_p(\omega;\mathbf{A}) = \mathfrak{M}_p(\omega;\mathbf{A})$ if the A_i commute. The most interesting results shown in [12] are that the power means converge to the Karcher mean under the strong operator topology, i.e.,

$$\lim_{p\to 0} P_p(\omega;\mathbf{A}) = \Lambda(\omega;\mathbf{A}),$$

where the Karcher mean $\Lambda(\omega;\mathbf{A})$ is the unique solution $X \in \mathbb{P}$ of the Karcher equation $\sum_{i=1}^n w_i \log X^{1/2}A_i^{-1}X^{1/2} = 0$, and for $0 < p \le q \le 1$,

$$\mathcal{H} = P_{-1} \le \cdots \le P_{-q} \le P_{-p} \le \cdots \le P_0 = \Lambda \le \cdots \le P_p \le P_q \le \cdots \le P_1 = \mathcal{A}. \tag{4.2}$$

For two parameters $\mu,\nu > 0$ and $t \in [0,1]$, the generalized means $\mathfrak{M}_p := \mathfrak{M}_p(1-t,t;\mu,\nu)$ with exponent $p \in \mathbb{R}$ have the following chain by (4.1):

$$\mathfrak{M}_{-\infty} \le \mathfrak{M}_{-q} \le \mathfrak{M}_{-p} \le \mathfrak{M}_p \le \mathfrak{M}_q \le \mathfrak{M}_\infty$$

for $0 \le p \le q$. Thus Proposition 3.2(5) provides the following chain of parameterized ordered means $G^{\mathfrak{M}_p} := G^{\mathfrak{M}_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A})$ for $p \in \mathbb{R}$:

$$G^{\mathfrak{M}_{-\infty}} \le G^{\mathfrak{M}_{-q}} \le G^{\mathfrak{M}_{-p}} \le G^{\mathfrak{M}_p} \le G^{\mathfrak{M}_q} \le G^{\mathfrak{M}_\infty}. \tag{4.3}$$

For fixed $p \in [-1,1]$ and $\mu,\nu > 0$, we consider the following two families of parameterized ordered means:

$$\left\{G^{P_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A})\right\}_{t\in[0,1]}, \quad \left\{P_p\left(1-t,t;G^{\mu}(\omega;\mathbf{A}),G^{\nu}(\omega;\mathbf{A})\right)\right\}_{t\in[0,1]},$$

which are continuous curves in $\mathbb{P}$ connecting two parameterized ordered means $G^{\mu}(\omega;\mathbf{A})$ at $t=0$ and $G^{\nu}(\omega;\mathbf{A})$ at $t=1$.

Remark 4.1 Two families $\{G^{P_p(1-t,t;\mu,\nu)}\}_{t\in[0,1]}$ and $\{P_p(1-t,t;G^{\mu},G^{\nu}\}_{t\in[0,1]}$ are interpolating monotonically two parameterized ordered means $G^{\mu}(\omega;\mathbf{A})$ and $G^{\nu}(\omega;\mathbf{A})$ depending on the size of μ and ν. Indeed, without loss of generality, assume that $0<\mu\le\nu$. Then the generalized mean with fixed exponent $p\in[-1,1]$

$$P_p(1-t,t;\mu,\nu) = \begin{cases} \mu[1-t+tx^p]^{\frac{1}{p}}, & p\neq 0, \\ \mu x^t, & p=0, \end{cases}$$

for $x=\frac{\nu}{\mu}\ge 1$ is an increasing function on $t\in[0,1]$. So the family $\{G^{P_p(1-t,t;\mu,\nu)}\}_{t\in[0,1]}$ is increasing on t by the monotonicity of parameterized ordered means on parameters in Proposition 3.2(5). Moreover, $G^{\mu}\le G^{\nu}$ again by Proposition 3.2(5), and

$$P_p\left(1-t,t;G^{\mu},G^{\nu}\right) = \begin{cases} G^{\mu}\#_{\frac{1}{p}}[(1-t)G^{\mu}+tG^{\mu}\#_p G^{\nu}], & p\in(0,1], \\ G^{\mu}\#_t G^{\nu}, & p=0, \\ G^{\mu}\#_{-\frac{1}{p}}[(1-t)(G^{\mu})^{-1}+t(G^{\mu})^{-1}\#_p(G^{\nu})^{-1}]^{-1}, & p\in[-1,0), \end{cases}$$

by [15, Proposition 3.8]. So the family $\{P_p(1-t,t;G^{\mu},G^{\nu}\}_{t\in[0,1]}$ is also increasing on t.

The following shows the relation between the above families of parameterized ordered means for $p=1$.

Theorem 4.2 *Let* $\mathbf{A}=(A_1,\dots,A_n)\in\mathbb{P}^n$ *and* $\omega=(w_1,\dots,w_n)\in\Delta_n$. *Then for all* $t\in[0,1]$,
(i) $(1-t)G^{\mu}(\omega;\mathbf{A})+tG^{\nu}(\omega;\mathbf{A})\le G^{(1-t)\mu+t\nu}(\omega;\mathbf{A})$ *for all* $\mu,\nu\ge 0$, *and*
(ii) $(1-t)G^{\mu}(\omega;\mathbf{A})+tG^{\nu}(\omega;\mathbf{A})\ge G^{(1-t)\mu+t\nu}(\omega;\mathbf{A})$ *for all* $\mu,\nu<0$.

Proof Taking the n-by-n block matrix

$$\mathbb{A} = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ A_1 & A_2 & \cdots & A_n \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_n \end{pmatrix}$$

in Theorem 3.4 and using the arithmetic-G weighted mean inequality, we obtain (i).
For any $\mu,\nu<0$,

$$\begin{aligned}(1-t)G^{\mu}(\omega;\mathbf{A})+tG^{\nu}(\omega;\mathbf{A}) &= (1-t)G^{-\mu}\left(\omega;\mathbf{A}^{-1}\right)^{-1}+tG^{-\nu}\left(\omega;\mathbf{A}^{-1}\right)^{-1} \\ &\ge \left[(1-t)G^{-\mu}\left(\omega;\mathbf{A}^{-1}\right)+tG^{-\nu}\left(\omega;\mathbf{A}^{-1}\right)\right]^{-1} \\ &\ge G^{-(1-t)\mu-t\nu}\left(\omega;\mathbf{A}^{-1}\right)^{-1} = G^{(1-t)\mu+t\nu}(\omega;\mathbf{A}).\end{aligned}$$

The first equality follows from (3.2), the first inequality from the convexity of inversion, and the second inequality from (i) and the order reversing of inversion. □

Theorem 4.3 *Let* $\mathbf{A} \in \mathbb{P}^n$, $\omega \in \Delta_n$, *and* $\mu, \nu > 0$. *Then for all* $t \in [0,1]$ *and* $p \in [-1,1)$,

$$G^{P_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A}) \geq P_p\big(1-t,t;G^{\frac{\mu}{K}}(\omega;\mathbf{A}),G^{\frac{\nu}{K}}(\omega;\mathbf{A})\big),$$

where $K = \frac{(\mu+\nu)^2}{4\mu\nu}$.

Proof For two parameters $\mu, \nu > 0$,

$$\begin{aligned} G^{P_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A}) &\geq G^{\frac{(1-t)\mu+t\nu}{K}}(\omega;\mathbf{A}) = \frac{1}{K}G^{(1-t)\mu+t\nu}(\omega;K\mathbf{A}) \\ &\geq \frac{1}{K}\big[(1-t)G^{\mu}(\omega;K\mathbf{A}) + tG^{\nu}(\omega;K\mathbf{A})\big] \\ &\geq \frac{1}{K}P_p\big(1-t,t;G^{\mu}(\omega;K\mathbf{A}),G^{\nu}(\omega;K\mathbf{A})\big) \\ &= P_p\big(1-t,t;G^{\frac{\mu}{K}}(\omega;\mathbf{A}),G^{\frac{\nu}{K}}(\omega;\mathbf{A})\big). \end{aligned}$$

The first inequality follows from Proposition 3.2(5) with

$$(1-t)\mu + t\nu \leq K\big[(1-t)\mu^{-1} + t\nu^{-1}\big]^{-1} \leq KP_p(1-t,t;\mu,\nu)$$

for $t \in [0,1]$, the first equality from the homogeneity in Proposition 3.2(1), the second inequality from Theorem 4.2(i), the third inequality from the arithmetic-power mean inequality in (4.2), and the last equality from the homogeneities of power mean and parameterized ordered mean, respectively, in [12, Proposition 3.6] and Proposition 3.2(1). □

Remark 4.4 Theorem 4.3 shows the order relation between two families

$$\big\{G^{P_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A})\big\}_{t\in[0,1]} \quad \text{and} \quad \big\{P_p\big(1-t,t;G^{\frac{\mu}{K}}(\omega;\mathbf{A}),G^{\frac{\nu}{K}}(\omega;\mathbf{A})\big)\big\}_{t\in[0,1]}$$

for $p \in [-1,1)$. Note that

$$P_p\big(1-t,t;G^{\mu}(\omega;\mathbf{A}),G^{\nu}(\omega;\mathbf{A})\big) \geq P_p\big(1-t,t;G^{\frac{\mu}{K}}(\omega;\mathbf{A}),G^{\frac{\nu}{K}}(\omega;\mathbf{A})\big),$$

since $G^{\mu} \geq G^{\frac{\mu}{K}}$ and $G^{\nu} \geq G^{\frac{\nu}{K}}$ by the monotonicity of parameterized ordered means for parameters in Proposition 3.2(5) and the power mean P_p is monotonic on variables. So it is open problem to compare $\{G^{P_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A})\}_{t\in[0,1]}$ and $\{P_p(1-t,t;G^{\mu}(\omega;\mathbf{A}),G^{\nu}(\omega;\mathbf{A}))\}_{t\in[0,1]}$.

Theorem 4.3 can be extended to the multivariable power means, and its proof follows similarly to that of Theorem 4.3 for the multivariable power means.

Proposition 4.5 *Let* $\mathbf{A} \in \mathbb{P}^n$, $\omega \in \Delta_n$, *and* $\lambda \in \Delta_k$. *Then for all* $\mu_1,\ldots,\mu_k > 0$ *and* $p \in [-1,1)$,

$$G^{P_p(\lambda;\mu_1,\ldots,\mu_k)}(\omega;\mathbf{A}) \geq P_p\big(\lambda;G^{\frac{\mu_1}{K}}(\omega;\mathbf{A}),\ldots,G^{\frac{\mu_k}{K}}(\omega;\mathbf{A})\big),$$

where $K = \frac{(\mu_{\max}+\mu_{\min})^2}{4\mu_{\max}\mu_{\min}}$ *for* $\mu_{\max} = \max\{\mu_1,\ldots,\mu_k\}$ *and* $\mu_{\min} = \min\{\mu_1,\ldots,\mu_k\}$.

The following theorem shows a relation between two families of parameterized ordered means associated with power means for parameters p and q with $-1 \leq p \leq 1 \leq q$.

Theorem 4.6 *Let $\mathbf{A} \in \mathbb{P}^n$, $\omega \in \Delta_n$, and $\lambda = (\lambda_1, \dots, \lambda_k) \in \Delta_k$. Then for all $\mu_1, \dots, \mu_k > 0$ and $-1 \leq p \leq 1 \leq q$,*

$$G^{P_q(\lambda;\mu_1,\dots,\mu_k)}(\omega; \mathbf{A}) \geq P_p\big(\lambda; G^{\mu_1}(\omega; \mathbf{A}), \dots, G^{\mu_k}(\omega; \mathbf{A})\big).$$

Proof Let $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{R}^k$ with positive components. Note that $P_q(\lambda; \mu_1, \dots, \mu_k) = \mathfrak{M}_q(\lambda; \mu_1, \dots, \mu_k)$ for any $q \geq 1$. Then

$$G^{P_q(\lambda;\mu)}(\omega; \mathbf{A}) \geq G^{\lambda\bullet\mu}(\omega; \mathbf{A}) \geq \sum_{i=1}^{k} \lambda_i G^{\mu_i}(\omega; \mathbf{A}) \geq P_p\big(\lambda; G^{\mu_1}(\omega; \mathbf{A}), \dots, G^{\mu_k}(\omega; \mathbf{A})\big),$$

where $\lambda \bullet \mu$ denotes the Euclidean inner product of λ and μ or, alternatively, the weighted arithmetic mean of μ with probability vector λ. The first inequality follows from the monotonicity on parameters in Proposition 3.2(5) with (4.1), the second from Theorem 3.4, and the third from the monotonicity of power means in (4.2). □

Remark 4.7 By (4.2), Theorem 4.6, and (4.3) we obtain a new chain of inequalities

$$\Lambda\big(\lambda; G^{\mu_1}, \dots, G^{\mu_k}\big) \leq P_{\frac{1}{q}}\big(\lambda; G^{\mu_1}, \dots, G^{\mu_k}\big) \leq P_{\frac{1}{p}}\big(\lambda; G^{\mu_1}, \dots, G^{\mu_k}\big) \leq G^{P_p(\lambda;\mu)} \leq G^{P_q(\lambda;\mu)}$$

for $1 \leq p \leq q$, where $G^\nu := G^\nu(\omega; \mathbf{A})$ for $\nu \geq 0$. For negative parameters, the reverse inequalities in the above chain hold, but it remains to show that

$$P_{-\frac{1}{p}}\big(\lambda; G^{\mu_1}, \dots, G^{\mu_k}\big) \geq G^{P_{-p}(\lambda;\mu)}$$

for $p \geq 1$.

5 Final remarks and open problems

The generalized mean $\mathfrak{M}_p$ of positive real numbers can be naturally defined for positive definite operators $A_1, \dots, A_n$ as

$$\mathfrak{M}_p(\omega; \mathbf{A}) := \left(\sum_{i=1}^{n} w_i A_i^p\right)^{\frac{1}{p}}, \quad p \neq 0.$$

There are many properties analogous to those for positive real numbers, but there are some different ones. For instance, from [8] we have

$$\mathfrak{M}_0(\omega; \mathbf{A}) := \lim_{p\to 0} \mathfrak{M}_p(\omega; \mathbf{A}) = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right),$$

where the right-hand side is known as the log-Euclidean mean, and

$$\mathfrak{M}_{-q} \leq \mathfrak{M}_{-p} \leq \mathfrak{M}_{-1} = \mathcal{H} \leq \mathfrak{M}_1 = \mathcal{A} \leq \mathfrak{M}_p \leq \mathfrak{M}_q$$

for $1 \leq p \leq q$. We can find more information from [8] by taking the finitely supported measure $\sum_{i=1}^{n} w_i \delta_{A_i}$, where δ_A is the point measure at $A \in \mathbb{P}$.

Similarly to Sect. 4, we can consider the following two families of parameterized ordered means:

$$\left\{G^{\mathfrak{M}_p(1-t,t;\mu,\nu)}(\omega;\mathbf{A})\right\},\ \left\{\mathfrak{M}_p\big(1-t,t;G^{\mu}(\omega;\mathbf{A}),G^{\nu}(\omega;\mathbf{A})\big)\right\}$$

for fixed $p \in \mathbb{R}$ and $\mu, \nu > 0$. By Theorem 4.2(i) and Theorem 4.3 with $p = -1$ we obtain

$$\begin{aligned}
&G^{\mathcal{A}(1-t,t;\mu,\nu)}(\omega;\mathbf{A}) \geq \mathcal{A}\big(1-t,t;G^{\mu}(\omega;\mathbf{A}),G^{\nu}(\omega;\mathbf{A})\big),\\
&G^{\mathcal{H}(1-t,t;\mu,\nu)}(\omega;\mathbf{A}) \geq \mathcal{H}\big(1-t,t;G^{\frac{\mu}{K}}(\omega;\mathbf{A}),G^{\frac{\nu}{K}}(\omega;\mathbf{A})\big).
\end{aligned}$$

Note that this may not hold for $p \in (-1,1)$, since the monotonicity of generalized means including the log-Euclidean mean does not hold. So it is an interesting question to compare two families for general $p \in \mathbb{R}$.

At last, we explain the background to extending a multivariate mean to a barycenter of probability measures and give some open problems from the results in this paper. Let $B(X)$ be the algebra of Borel sets on a metric space (X,d). Let $\mathcal{P}(X)$ be the set of all probability measures on $(X,B(X))$ with separable support, and let $\mathcal{P}^p(X) \subset \mathcal{P}(X)$ for $p \geq 1$ be the set of all probability measures with finite p-moment: for some $y \in X$,

$$\int_X d^p(x,y)\,d\mu(x) < \infty.$$

We denote by $\mathcal{P}^{\infty}(X)$ the set of probability measures on $(X,B(X))$ with compact support. For $p \geq 1$, the p-Wasserstein distance on $\mathcal{P}^p(X)$ is defined by

$$d_p^W(\rho,\sigma) := \left[\inf_{\pi\in\Pi(\rho,\sigma)} \int_{X\times X} d^p(x,y)\,d\pi(x,y)\right]^{1/p},$$

where $\Pi(\rho,\sigma)$ denotes the set of all couplings for $\rho,\sigma \in \mathcal{P}^p(X)$. Moreover, the ∞-Wasserstein distance on $\mathcal{P}^{\infty}(X)$ is given by

$$d_{\infty}^W(\rho,\sigma) = \lim_{p\to\infty} d_p^W(\rho,\sigma) = \inf_{\pi\in\Pi(\rho,\sigma)} \sup\big\{d(x,y) : (x,y) \in \operatorname{supp}(\pi)\big\}.$$

For more details and information, see [16, 17].

For each natural number n, in general, a mean G_n on a set X is a map $G_n : X^n \to X$ satisfying the idempotency. An intrinsic mean G_n is the mean with invariance under permutation and repetition. By [13, Proposition 2.7] a nonexpansive intrinsic mean $G = \{G_n\}_{n\in\mathbb{N}}$ on a complete metric space X uniquely extends to a d_{∞}^W-contractive barycentric map $\beta_G : \mathcal{P}^{\infty}(X) \to X$, where β_G is d_{∞}^W-contractive if and only if

$$d\big(\beta_G(\rho),\beta_G(\sigma)\big) \leq d_{\infty}^W(\rho,\sigma)$$

for all $\rho,\sigma \in \mathcal{P}^{\infty}(X)$. Thus the parameterized ordered mean $G^{\mu} = \{G_n^{\mu}\}$ with invariance under permutation and repetition can be extended to a d_{∞}^W-contractive barycentric map $\beta_{G^{\mu}}$ by Proposition 3.2 (6). It is also an interesting problem to generalize results in Sect. 3 and Sect. 4 to the d_{∞}^W-contractive barycentric map $\beta_{G^{\mu}}$.

Acknowledgements
I deeply thank the anonymous reviewer for improving this paper.

Funding
This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (No. 2018R1C1B6001394, 2022R1A2C4001306).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that they have no competing interests.

Author contribution
The author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 7 January 2021 Accepted: 2 September 2022 Published online: 09 September 2022

References

1. Ando, T., Li, C.-K., Mathias, R.: Geometric means. Linear Algebra Appl. **385**, 305–334 (2004)
2. Bartz, S., Bauschke, H.H., Moffat, S.M., Wang, X.: The resolvent average of monotone operators: dominant and recessive properties. SIAM J. Optim. **26**(1), 602–634 (2016)
3. Bauschke, H.H., Moffat, S.M., Wang, X.: The resolvent average for positive semidefinite matrices. Linear Algebra Appl. **432**, 1757–1771 (2010)
4. Bhatia, R.: Positive Definite Matrices. Princeton Series in Applied Mathematics. Princeton University Press, Princeton (2007)
5. Bini, D., Meini, B., Poloni, F.: An effective matrix geometric mean satisfying the Ando–Li–Mathias properties. Math. Comput. **79**, 437–452 (2010)
6. Hwang, J., Kim, S.: Lie–Trotter means of positive definite operators. Linear Algebra Appl. **531**, 268–280 (2017)
7. Kian, M., Moslehian, M.S., Seo, Y.: Variants of Ando–Hiai type inequalities for deformed means and applications. Glasg. Math. J. (2020). https://doi.org/10.1017/S0017089520000403
8. Kim, S.: The quasi-arithmetic means and Cartan barycenters of compactly supported measures. Forum Math. **30**(3), 753–765 (2018)
9. Kim, S., Lawson, J., Lim, Y.: The matrix geometric mean of parameterized, weighted arithmetic and harmonic means. Linear Algebra Appl. **435**, 2114–2131 (2011)
10. Kubo, F., Ando, T.: Means of positive linear operators. Math. Ann. **246**(3), 205–224 (1979/80)
11. Kum, S., Lim, Y.: Nonexpansiveness of the resolvent average. J. Math. Anal. Appl. **432**, 918–927 (2015)
12. Lawson, J., Lim, Y.: Karcher means and Karcher equations of positive definite operators. Trans. Am. Math. Soc. Ser. B **1**, 1–22 (2014)
13. Lawson, J., Lim, Y.: Contractive barycentric maps. J. Oper. Theory **77**, 87–107 (2017)
14. Lim, Y.: Contractive barycentric maps and L^1 ergodic theorems on the cone of positive definite matrices. J. Math. Anal. Appl. **459**, 291–306 (2018)
15. Lim, Y., Palfia, M.: The matrix power means and the Karcher mean. J. Funct. Anal. **262**(4), 1498–1514 (2012)
16. Villani, C.: Topics in Optimal Transportation. Graduate Studies in Mathematics, vol. 58. Springer, Berlin (2003)
17. Villani, C.: Optimal Transport: Old and New. Lecture Notes, École d'Été de Probabilités de Saint-Flour. Springer, New York (2008)
18. Weidmann, J.: Linear Operators in Hilbert Spaces. Graduate Texts in Mathematics, vol. 68. Springer, New York (1980)