Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Characterizing small spheres in a unit sphere by Fischer–Marsden equation

Nasser Bin Turki[1], Sharief Deshmukh[1] and Gabriel-Eduard Vîlcu[2,3,4*]

*Correspondence:
gvilcu@upg-ploiesti.ro
[2]Department of Mathematics and Informatics, Faculty of Applied Sciences, Politehnica of Bucharest, Splaiul Independenţei 313, 060042 Bucharest, Romania
[3]"Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie Nr. 13, 050711 Bucharest, Romania
Full list of author information is available at the end of the article

Abstract

We use a nontrivial concircular vector field **u** on the unit sphere $\mathbf{S}^{n+1}$ in studying geometry of its hypersurfaces. An orientable hypersurface M of the unit sphere $\mathbf{S}^{n+1}$ naturally inherits a vector field **v** and a smooth function ρ. We use the condition that the vector field **v** is an eigenvector of the de-Rham Laplace operator together with an inequality satisfied by the integral of the Ricci curvature in the direction of the vector field **v** to find a characterization of small spheres in the unit sphere $\mathbf{S}^{n+1}$. We also use the condition that the function ρ is a nontrivial solution of the Fischer–Marsden equation together with an inequality satisfied by the integral of the Ricci curvature in the direction of the vector field **v** to find another characterization of small spheres in the unit sphere $\mathbf{S}^{n+1}$.

MSC: 53C20; 53C99; 58J99

Keywords: Sphere; de-Rham Laplace operator; Fischer–Marsden differential equation; Small sphere

1 Introduction

The study of the geometry of hypersurfaces in a sphere is a captivating subject in differential geometry that has been investigated by many researchers (see, e.g., [4, 7, 8, 11, 12, 20–23, 26, 31, 32, 35]), one of the most interesting problems in this field, still unsolved, being the famous *Chern Conjecture for isoparametric hypersurfaces* (see [39, Problem 105] and also the remarkable review paper [28]). We would like to emphasize that several notable results have been established in this field over time. For instance, Okumura [24] provided a criterion for a hypersurface of constant mean curvature in an odd-dimensional sphere to be totally umbilical. Later, do Carmo and Warner [13], as well as Wang and Xia [34], investigated the rigidity of hypersurfaces in spheres, while Chen characterized minimal hypersurfaces in the same ambient space [6]. Some global pinching results concerning minimal hypersurfaces in spheres were obtained by Shen [30]. Other interesting pinching theorems were derived in [1, 18, 36–38]. Recent results on the geometry of hypersurfaces in spheres were obtained in [2, 3, 27, 29, 40].

One of the interesting but challenging problems in submanifold geometry is characterizing small spheres (non-totally geodesic totally umbilical spheres) in a unit sphere $\mathbf{S}^{n+1}$ (see [19]). On a Riemannian manifold (M, g), the Ricci operator T is defined using Ricci tensor S, namely $S(X,Y) = g(TX,Y)$, $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. Similarly, the rough Laplace operator on the Riemannian manifold (M,g), $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$\Delta X = \sum_{i=1}^{m} (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X), \quad X \in \mathfrak{X}(M),$$

where ∇ is the Riemannian connection and $\{e_1, \dots, e_m\}$ is a local orthonormal frame on M, $m = \dim M$. The rough Laplace operator is used in finding characterizations of spheres as well as of Euclidean spaces (cf. [15, 17]). Recall that the de-Rham Laplace operator $\square : \mathfrak{X}(M) \to \mathfrak{X}(M)$ on a Riemannian manifold (M,g) is defined by (cf. [14], p.83)

$$\square = T + \Delta \tag{1}$$

and is used to characterize a Killing vector field on a compact Riemannian manifold. It is known that if ξ is a Killing vector field on a Riemannian manifold (M,g) or soliton vector field of a Ricci soliton (M,g,ξ,λ), then $\square\xi = 0$ (cf. [10]). Also, Fischer and Marsden considered in [16] the following differential equation on a Riemannian manifold (M,g):

$$(\Delta f)g + fS = Hess(f), \tag{2}$$

where $Hess(f)$ is the Hessian of a smooth function f and Δ is the Laplace operator acting on smooth functions of M. They conjectured that if a compact Riemannian manifold admits a nontrivial solution of the differential equation (2), then it must be an Einstein manifold. Recent investigations on manifolds satisfying the Fischer–Marsden equation were done in [5, 9, 25, 33].

Consider the sphere $\mathbf{S}^{n+1}$ as hypersurface of the Euclidean space $\mathbf{R}^{n+2}$ with unit normal ξ and shape operator $B = -\sqrt{c}I$, where I denotes the identity operator. For the constant vector field $\overrightarrow{a} = \frac{\partial}{\partial x^1}$ on the Euclidean space $\mathbf{R}^{n+2}$, where $x^1, \dots, x^{n+2}$ are Euclidean coordinates on $\mathbf{R}^{n+2}$, we denote by $\mathbf{u}$ the tangential projection of $\overrightarrow{a}$ on the unit sphere $\mathbf{S}^{n+1}$. Then we have

$$\overrightarrow{a} = \mathbf{u} + \overline{f}\xi,$$

where $\overline{f} = \langle \overrightarrow{a}, \xi \rangle$, $\langle , \rangle$ is the Euclidean metric on $\mathbf{R}^{n+2}$. Taking covariant derivative in the above equation with respect to a vector field X on the unit sphere $\mathbf{S}^{n+1}$ and using Gauss–Weingarten formulae for hypersurface, we conclude

$$\overline{\nabla}_X \mathbf{u} = -\overline{f}X, \qquad \operatorname{grad}\overline{f} = \mathbf{u}, \tag{3}$$

where $\overline{\nabla}$ is the Riemannian connection on the unit sphere $\mathbf{S}^{n+1}$ with respect to the canonical metric g and $\operatorname{grad}\overline{f}$ is the gradient of the smooth function $\overline{f}$ on $\mathbf{S}^{n+1}$. Thus, $\mathbf{u}$ is a concircular vector field on the unit sphere $\mathbf{S}^{n+1}$. Now consider the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ defined by

$$\mathbf{S}^n\left(\frac{1}{c^2}\right) = \left\{ (x^1, \dots x^{n+2}) : \sum_{i=1}^{n+1} (x^i)^2 = c^2, x^{n+2} = \sqrt{1-c^2}, 0 < c < 1 \right\}.$$

Then it follows that $\mathbf{S}^n(\frac{1}{c^2})$ is a hypersurface of the unit sphere $\mathbf{S}^{n+1}$ with unit normal vector field N given by

$$N = \left(-\frac{\sqrt{1-c^2}}{c}x^1,\ldots,-\frac{\sqrt{1-c^2}}{c}x^{n+1},c\right).$$

We denote by the same letter g the induced metric on the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ and denote by ∇ the Riemannian connection with respect to the induced metric g. Then, by a straightforward computation, we find that

$$\overline{\nabla}_X N = -\frac{\sqrt{1-c^2}}{c}X, \quad X \in \mathfrak{X}\left(\mathbf{S}^n\left(\frac{1}{c^2}\right)\right). \tag{4}$$

Thus, the shape operator A of the hypersurface $\mathbf{S}^n(\frac{1}{c^2})$ is given by

$$A = \frac{\sqrt{1-c^2}}{c}I = \alpha I, \tag{5}$$

where α is the mean curvature of the hypersurface $\mathbf{S}^n(\frac{1}{c^2})$. It is clear that α is a nonzero constant as $0 < c < 1$. Now, denote by $\mathbf{v}$ the tangential projection of the vector field $\mathbf{u}$ to the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ and define $\rho = g(\mathbf{u},N)$. Then we have

$$\mathbf{u} = \mathbf{v} + \rho N. \tag{6}$$

However, we can easily see using the definitions of $\mathbf{u}$ and N that

$$g(\mathbf{u},N) = -\frac{\sqrt{1-c^2}}{c}f,$$

where f is the restriction of $\overline{f}$ to $\mathbf{S}^n(\frac{1}{c^2})$. Thus, $\rho = -\alpha f$. Taking covariant derivative in equation (6) and using Gauss–Weingarten formulae for hypersurface, we conclude on using equations (3) and (5) by equating tangential components that

$$\nabla_X \mathbf{v} = -(1+\alpha^2)fX, \qquad \operatorname{grad}\rho = -\alpha\mathbf{v}, \tag{7}$$

for $X \in \mathfrak{X}(\mathbf{S}^n(\frac{1}{c^2}))$. Also, we have $\operatorname{grad} f = \mathbf{v}$. Thus, the rough Laplace operator Δ acting on $\mathbf{v}$ and the Laplace operator acting on the smooth function ρ are respectively given by

$$\Delta\mathbf{v} = -(1+\alpha^2)\mathbf{v}, \qquad \Delta\rho = -n(1+\alpha^2)\rho. \tag{8}$$

The Ricci operator T of the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ is given by

$$TX = (n-1)(1+\alpha^2)X.$$

Thus, we observe that the vector field $\mathbf{v}$ on the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ satisfies

$$\Box\mathbf{v} = (n-2)(1+\alpha^2)\mathbf{v}. \tag{9}$$

Also, using equation (8), we see that the Hessian of ρ is given by

$$\begin{aligned} Hess(\rho)(X,Y) &= g(\nabla_X \operatorname{grad}\rho, Y) \\ &= \alpha\left(1+\alpha^2\right)fg(X,Y) \\ &= -\left(1+\alpha^2\right)\rho g(X,Y) \end{aligned}$$

for $X, Y \in \mathfrak{X}(\mathbf{S}^n(\frac{1}{c^2}))$, and using the above equation with expression for Ricci tensor and equation (8), we see that the function ρ on the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ satisfies the Fischer–Marsden equation

$$(\Delta\rho)g + \rho S = Hess(\rho). \tag{10}$$

Thus, in view of equations (9) and (10), the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ admits a vector field $\mathbf{v}$ that is an eigenvector of the de-Rham Laplace operator with eigenvalue $(n-2)(1+\alpha^2)$, and it admits a smooth function ρ that is a solution of the Fischer–Marsden differential equation. These raise two questions: (i) Given a compact hypersurface M of the unit sphere $\mathbf{S}^{n+1}$ that admits a vector field $\mathbf{v}$, which is the eigenvector of de-Rham Laplace operator $\square$ corresponding to positive eigenvalue, is this hypersurface necessarily isometric to a small sphere? (ii) Given a compact hypersurface M admitting a vector field $\mathbf{v}$ and a smooth function ρ with gradient $\operatorname{grad}\rho = -A\mathbf{v}$ a nontrivial solution of the Fischer–Marsden differential equation, is this hypersurface necessarily isometric to a small sphere? In this paper, we answer these questions (cf. Theorem 3.1 and Theorem 3.2).

2 Preliminaries

Let M be an orientable hypersurface of the unit sphere $\mathbf{S}^{n+1}$ with unit normal vector field N and shape operator A. We denote the canonical metric on $\mathbf{S}^{n+1}$ by g and denote by the same letter g the induced metric on the hypersurface M. Let $\overline{\nabla}$ and ∇ be the Riemannian connections on the unit sphere $\mathbf{S}^{n+1}$ and on the hypersurface M, respectively. Then we have the following fundamental equations of the hypersurface:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX,Y)N, \qquad \overline{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M). \tag{11}$$

The curvature tensor field R, the Ricci tensor S, and the scalar curvature τ of the hypersurface M are given by

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY, \tag{12}$$

$$S(X,Y) = (n-1)g(X,Y) + n\alpha g(AX,Y) - g(AX,AY), \tag{13}$$

and

$$\tau = n(n-1) + n^2\alpha^2 - \|A\|^2, \tag{14}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $\alpha = \frac{1}{n}\operatorname{Tr}A$ is the mean curvature of the hypersurface M and $\|A\|^2 = \operatorname{Tr}A^2$. The Codazzi equation of hypersurface gives

$$(\nabla A)(X,Y) = (\nabla A)(Y,X), \quad X, Y \in \mathfrak{X}(M), \tag{15}$$

where

$$(\nabla A)(X,Y) = \nabla_X AY - A(\nabla_X Y).$$

Taking a local orthonormal frame $\{e_1,\dots,e_n\}$ on the hypersurface M, equation (15) yields

$$n\,\mathrm{grad}\,\alpha = \sum_{i=1}^{n}(\nabla A)(e_i,e_i). \tag{16}$$

Let $\mathbf{u}$ be the concircular vector field on the unit sphere $\mathbf{S}^{n+1}$ considered in the previous section, which satisfies equation (3), where $\overline{f}$ is the function defined on $\mathbf{S}^{n+1}$ by $\overline{f} = \langle \overrightarrow{a}, \xi\rangle$. We denote the restriction of $\overline{f}$ to the hypersurface M by f and the tangential projection of the vector field $\mathbf{u}$ on M by $\mathbf{v}$. Then we have

$$\mathbf{u} = \mathbf{v} + \rho N, \qquad \rho = g(\mathbf{u}, N). \tag{17}$$

We call the vector field $\mathbf{v}$ the induced vector field on the hypersurface M. We also call the functions ρ and f the support function and the associated function, respectively, of the hypersurface M. Note that $\mathrm{grad} f$ is the tangential component of $\mathrm{grad}\overline{f}$, i.e.,

$$\mathrm{grad} f = [\mathrm{grad}\overline{f}]^T,$$

while the normal component of $\mathrm{grad} f$ is

$$\begin{aligned}[\mathrm{grad}\overline{f}]^{\perp} &= g(\mathrm{grad}\overline{f}, N)N \\ &= g(\mathbf{u}, N)N \\ &= \rho N,\end{aligned}$$

that is, on using equations (3) and (17), we have

$$\mathrm{grad} f = \mathbf{v}. \tag{18}$$

Taking covariant derivative in equation (17) and using equations (3) and (11), we get on equating tangential and normal components

$$\nabla_X \mathbf{v} = -fX + \rho AX, \qquad \mathrm{grad}\,\rho = -A\mathbf{v}, \quad X \in \mathfrak{X}(M). \tag{19}$$

Lemma 2.1 *Let M be a compact hypersurface of the unit sphere $\mathbf{S}^{n+1}$ with induced vector field $\mathbf{v}$, support function ρ, and associated function f. Then*

$$\int_M \|\mathbf{v}\|^2 = n\int_M \left(f^2 - f\rho\alpha\right).$$

Proof Using equation (19), we have

$$\mathrm{div}\,\mathbf{v} = n(-f + \rho\alpha),$$

and using equation (18), we get

$$\operatorname{div}(f\mathbf{v}) = \|\mathbf{v}\|^2 + nf(-f + \rho\alpha).$$

Integrating the above equation, we get the result. □

Lemma 2.2 *Let M be a compact hypersurface of the unit sphere $\mathbf{S}^{n+1}$ with induced vector field $\mathbf{v}$, support function ρ, and associated function f. Then*

$$\int_M \rho\mathbf{v}(\alpha) = \int_M \big[\alpha g(A\mathbf{v},\mathbf{v}) + nf\rho\alpha - n\rho^2\alpha^2\big].$$

Proof Note that we have

$$\begin{aligned}\operatorname{div}\big(\alpha(\rho\mathbf{v})\big) &= \rho\mathbf{v}(\alpha) + \alpha\operatorname{div}(\rho\mathbf{v})\\ &= \rho\mathbf{v}(\alpha) + \alpha\big[\mathbf{v}(\rho) + n\rho(-f + \rho\alpha\big].\end{aligned}$$

Integrating this equation and using the second equation in (19), we get the result. □

3 Characterizations of small spheres

Let $\mathbf{u}$ be the concircular vector field on the unit sphere $\mathbf{S}^{n+1}$ and M be its orientable non-totally geodesic hypersurface with mean curvature α and induced vector field $\mathbf{v}$, potential function ρ, and associated function f. In this section we find different characterizations of the small spheres in $\mathbf{S}^{n+1}$.

Theorem 3.1 *Let M be an orientable non-totally geodesic compact and connected hypersurface of the unit sphere $\mathbf{S}^{n+1}$, $n \geq 2$, with induced vector field $\mathbf{v}$, nonzero potential function ρ, and associated function f. Then $\square\mathbf{v} = \lambda\mathbf{v}$ for a constant λ, and the inequality*

$$\int_M S(\mathbf{v},\mathbf{v}) \leq n\int_M (f - \rho\alpha)\big[(\lambda + 1)f - \rho\alpha\big]$$

holds if and only if α is a constant and M is isometric to the small sphere $\mathbf{S}^m(1+\alpha^2)$.

Proof Suppose that $\mathbf{v}$ satisfies

$$\square\mathbf{v} = \lambda\mathbf{v}, \tag{20}$$

where λ is a constant. Using equation (13), we have

$$T(\mathbf{v}) = (n-1)\mathbf{v} + n\alpha A\mathbf{v} - A^2\mathbf{v}. \tag{21}$$

Now, using equation (18), we get

$$\nabla_X\nabla_X\mathbf{v} - \nabla_{\nabla_X X}\mathbf{v} = -X(f)X + X(\rho)AX + \rho(\nabla A)(X,Z),$$

which gives the rough Laplace operator acting on the vector field $\mathbf{v}$ as

$$\Delta\mathbf{v} = -\operatorname{grad} f + A(\operatorname{grad}\rho) + n\rho\operatorname{grad}\alpha,$$

where we have used equation (16). The above equation in view of equations (18) and (19) becomes

$$\Delta \mathbf{v} = -\mathbf{v} - A^2\mathbf{v} + n\rho \operatorname{grad} \alpha. \tag{22}$$

Thus, equations (20), (21), and (22) imply

$$(n-2-\lambda)\mathbf{v} - 2A^2\mathbf{v} + n\alpha A\mathbf{v} + n\rho \operatorname{grad} \alpha = 0.$$

Taking the inner product in the above equation with $\mathbf{v}$, we get

$$(n-2-\lambda)\|\mathbf{v}\|^2 - 2\|A\mathbf{v}\|^2 + n\alpha g(A\mathbf{v},\mathbf{v}) + n\rho \mathbf{v}(\alpha) = 0.$$

By integrating the above equation and using Lemma 2.2, we conclude

$$\int_M \big[(n-2-\lambda)\|\mathbf{v}\|^2 - 2\|A\mathbf{v}\|^2 + 2n\alpha g(A\mathbf{v},\mathbf{v}) + n^2 f\rho\alpha - n^2\rho^2\alpha^2\big] = 0.$$

Now, using equation (13) in the above equation, we arrive at

$$\int_M \big[-(n+\lambda)\|\mathbf{v}\|^2 + 2S(\mathbf{v},\mathbf{v}) + n^2 f\rho\alpha - n^2\rho^2\alpha^2\big] = 0,$$

which in view of Lemma 2.1 gives

$$\int_M S(\mathbf{v},\mathbf{v}) = \int_M \big[n^2(-f+\rho\alpha)^2 - nf^2 - \rho^2\|A\|^2 + 2nf\rho\alpha\big].$$

Therefore, we derive

$$\int_M \big[-n(n+\lambda)f^2 + n(2n+\lambda)f\rho\alpha - n^2\rho^2\alpha^2 + 2S(\mathbf{v},\mathbf{v})\big] = 0. \tag{23}$$

Note that equation (18) implies

$$S(\mathbf{v},\mathbf{v}) = S(\operatorname{grad} f, \operatorname{grad} f)$$

and Bochner's formula gives

$$\int_M S(\mathbf{v},\mathbf{v}) = \int_M \big[(\Delta f)^2 - Hess(f)^2\big]. \tag{24}$$

Using equation (18), we have

$$\Delta f = n(-f+\rho\alpha)$$

and

$$Hess(f)(X,Y) = g(\nabla_X \operatorname{grad} f, Y)$$

$$= -fg(X,Y) + \rho g(AX,Y).$$

Hence we derive

$$Hess(f)^2 = nf^2 + \rho^2\|A\|^2 - 2nf\rho\alpha.$$

Thus, from equation (24), we have

$$\int_M S(\mathbf{v},\mathbf{v}) = \int_M \left[n^2(-f+\rho\alpha)^2 - nf^2 - \rho^2\|A\|^2 + 2nf\rho\alpha\right],$$

that is,

$$\int_M S(\mathbf{v},\mathbf{v}) = \int_M \left[n(n-1)f^2 + n^2\rho^2\alpha^2 - \rho^2\|A\|^2 - 2n(n-1)f\rho\alpha\right]. \tag{25}$$

Combining equations (23) and (25) (retaining out of $2S(\mathbf{v},\mathbf{v})$ one term in (24)), we get

$$\int_M \rho^2\left(\|A\|^2 - n\alpha^2\right) = \int_M \left[-n\left[(\lambda+1)f^2 - (\lambda+2)f\rho\alpha + \rho^2\alpha^2\right] + S(\mathbf{v},\mathbf{v})\right].$$

The above equation gives immediately

$$\int_M \rho^2\left(\|A\|^2 - n\alpha^2\right) = \int_M \left[S(\mathbf{v},\mathbf{v}) - n(f-\rho\alpha)\left((\lambda+1)f - \rho\alpha\right)\right].$$

Using the condition in the statement in the above equation, we get

$$\rho^2\left(\|A\|^2 - n\alpha^2\right) = 0.$$

However, as the support function $\rho \neq 0$, we get $\|A\|^2 = n\alpha^2$, and this equality in view of Schwartz's inequality holds if and only if

$$A = \alpha I. \tag{26}$$

Using a local orthonormal frame $\{e_1,\dots,e_n\}$ in the above equation, we get

$$\sum_{i=1}^n (\nabla A)(e_i,e_i) = \operatorname{grad}\alpha,$$

and combining the above equation with equation (16), we get

$$(n-1)\operatorname{grad}\alpha = 0.$$

As $n \geq 2$, we conclude that the mean curvature α is a constant, and by equation (26) we see that M is totally umbilical hypersurface. Hence, by equation (12), we see that M is isometric to the small sphere $\mathbf{S}^n(1+\alpha^2)$.

Conversely, if (M,g) is isometric to the sphere $\mathbf{S}^m(1+\alpha^2)$, then choosing positive constant c such that

$$c^2 = \frac{1}{1+\alpha^2},$$

it is clear that $0 < c < 1$. We know by equation (9) that potential function ρ on the small sphere $\mathbf{S}^n(\frac{1}{c^2})$ satisfies

$$\Box \mathbf{v} = \lambda \mathbf{v}, \qquad \lambda = (n-2)\bigl(1+\alpha^2\bigr), \tag{27}$$

where λ is obviously a constant. Also, we have the Ricci curvature

$$S(\mathbf{v},\mathbf{v}) = (n-1)\bigl(1+\alpha^2\bigr)\|\mathbf{v}\|^2,$$

and, in view of Lemma 2.1 and $\rho = -\alpha f$ for the small sphere, we deduce

$$\int_M S(\mathbf{v},\mathbf{v}) = n(n-1)\bigl(1+\alpha^2\bigr)\int_M f^2. \tag{28}$$

Also, on using

$$\rho = -\alpha f, \qquad \lambda = (n-2)\bigl(1+\alpha^2\bigr),$$

we have

$$n\int_M (f-\rho\alpha)\bigl[(\lambda+1)f-\rho\alpha\bigr] = n(n-1)\bigl(1+\alpha^2\bigr)\int_M f^2. \tag{29}$$

Thus, equations (27), (28), and (29) imply that the conditions in the statement of Theorem hold. Finally, observe that if $\rho = 0$ on the small sphere $\mathbf{S}^m(1+\alpha^2)$ with constant $\alpha \neq 0$, we get $f = 0$, and consequently $\mathbf{v} = 0$. Then, by equation (6), we get $\mathbf{u} = 0$, and equation (3) implies $\overline{f} = 0$. Thus, with assumption $\rho = 0$, we reach $\overrightarrow{a} = 0$, hence a contradiction to the fact that $\overrightarrow{a}$ is a constant unit vector field on the Euclidean space $\mathbf{R}^{n+2}$. Hence all the requirements in the statement are met. □

Recall that if an n-dimensional Riemannian manifold (M,g) admits a nontrivial solution of the Fischer–Marsden differential equation (2), $n > 2$, then the scalar curvature τ is a constant (cf. [16]) and the nontrivial solution f satisfies

$$\Delta f = -\frac{\tau}{n-1} f. \tag{30}$$

Theorem 3.2 *Let M be an orientable non-totally geodesic compact and connected hypersurface of the unit sphere $\mathbf{S}^{n+1}$, $n > 2$, with induced vector field $\mathbf{v}$, nonzero potential function ρ, and associated function f. Then the potential function ρ is a nontrivial solution of the Fischer–Marsden equation (2) and the inequality*

$$\int_M S(\mathbf{v},\mathbf{v}) \geq \frac{n-1}{n}\int_M (\operatorname{div}\mathbf{v})^2$$

holds if and only if α is a constant and M is isometric to the small sphere $\mathbf{S}^m(1+\alpha^2)$.

Proof Let M be an orientable non-totally geodesic compact and connected hypersurface of the unit sphere $\mathbf{S}^{n+1}$, $n > 2$, with induced vector field $\mathbf{v}$, nonzero potential function ρ, and associated function f. Suppose that ρ is the nontrivial solution of the Fischer–Marsden equation (2). Then, by equation (30), we have

$$\Delta\rho = -\frac{\tau}{n-1}\rho. \tag{31}$$

Using equations (16) and (19), we find

$$\operatorname{div} A\mathbf{v} = -nf\alpha + \rho\|A\|^2 + n\mathbf{v}(\alpha),$$

and consequently, equation (19) implies

$$\Delta\rho = nf\alpha - \rho\|A\|^2 - n\mathbf{v}(\alpha). \tag{32}$$

Using equation (31) with the above equation, we get

$$\rho^2\left(\|A\|^2 - n\alpha^2\right) = nf\rho\alpha + \frac{\tau}{n-1}\rho^2 - n\rho\mathbf{v}(\alpha) - n\rho^2\alpha^2.$$

Integrating the above equation and using Lemma 2.2, we get

$$\int_M \rho^2\left(\|A\|^2 - n\alpha^2\right) = \int_M \left[-n(n-1)f\rho\alpha + n(n-1)\rho^2\alpha^2 + \frac{\tau}{n-1}\rho^2 - n\alpha g(A\mathbf{v},\mathbf{v})\right]. \tag{33}$$

Note that τ is a constant and equations (19) and (31) imply

$$\int_M \|Av\|^2 = \int_M \|\operatorname{grad}\rho\|^2 = \frac{\tau}{n-1}\int_M \rho^2. \tag{34}$$

Also, equation (13) gives

$$\int_M \left[\|Av\|^2 - n\alpha g(A\mathbf{v},\mathbf{v})\right] = \int_M \left[(n-1)\|\mathbf{v}\|^2 - S(\mathbf{v},\mathbf{v})\right],$$

which in view of equation (34) and Lemma 2.1 implies

$$\int_M \left[\frac{\tau}{n-1}\rho^2 - n\alpha g(A\mathbf{v},\mathbf{v})\right] = \int_M \left[n(n-1)\left(f^2 - f\rho\alpha\right) - S(\mathbf{v},\mathbf{v})\right].$$

Combining the above equation with equation (33), we arrive at

$$\int_M \rho^2\left(\|A\|^2 - n\alpha^2\right) = \int_M \left[n(n-1)(-f + \rho\alpha)^2 - S(\mathbf{v},\mathbf{v})\right].$$

Now, using

$$\operatorname{div}\mathbf{v} = n(-f + \rho\alpha)$$

in the above equation, we get

$$\int_M \rho^2\left(\|A\|^2 - n\alpha^2\right) = \int_M \left[\frac{(n-1)}{n}(\operatorname{div}\mathbf{v})^2 - S(\mathbf{v},\mathbf{v})\right]. \tag{35}$$

Using now the hypothesis

$$\int_M S(\mathbf{v},\mathbf{v}) \geq \frac{n-1}{n}\int_M (\operatorname{div}\mathbf{v})^2$$

in equation (35), we conclude

$$\rho^2\left(\|A\|^2 - n\alpha^2\right) = 0.$$

However, as the function $\rho \neq 0$ on connected M, we have $\|A\|^2 = n\alpha^2$. But, in view of Schwartz's inequality, this equality holds if and only if $A = \alpha I$. Hence, M being non-totally geodesic hypersurface and $n > 2$, M is isometric to the small sphere $\mathbf{S}^n(1+\alpha^2)$.

Conversely, as we have seen in the introduction, on the small sphere $\mathbf{S}^n(1+\alpha^2)$, the function ρ is a solution of Fischer–Marsden equation (cf. equation (10)). Now, the Ricci curvature

$$S(\mathbf{v},\mathbf{v}) = (n-1)\left(1+\alpha^2\right)\|\mathbf{v}\|^2$$

together with Lemma 2.1 and $\rho = -f\alpha$ implies

$$\int_M S(\mathbf{v},\mathbf{v}) = n(n-1)\left(1+\alpha^2\right)\int_M f^2. \tag{36}$$

Also, we have

$$\begin{aligned}\operatorname{div}\mathbf{v} &= n(-f+\rho\alpha)\\ &= n\left(1+\alpha^2\right)(-f),\end{aligned}$$

and we derive

$$\frac{n-1}{n}\int_M (\operatorname{div}\mathbf{v})^2 = n(n-1)\left(1+\alpha^2\right)\int_M f^2. \tag{37}$$

As seen in the proof of Theorem 3.1, we have that the function $\rho \neq 0$. Thus, by equations (36) and (37), we can see immediately that all the requirements are met in the statement for the small sphere $\mathbf{S}^n(1+\alpha^2)$. □

Acknowledgements
Not applicable.

Funding
This work is supported by the Researchers Supporting Project number (RSP2022R413), King Saud University, Riyadh, Saudi Arabia.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Science, King Saud University, P.O. Box-2455, 11451, Riyadh, Saudi Arabia. [2]Department of Mathematics and Informatics, Faculty of Applied Sciences, Politehnica of Bucharest, Splaiul Independenţei 313, 060042 Bucharest, Romania. [3]"Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie Nr. 13, 050711 Bucharest, Romania. [4]Research Center in Geometry, Topology and Algebra, Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, 010014 Bucharest, Romania.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 April 2021 Accepted: 2 September 2022 Published online: 09 September 2022

References

1. Alencar, H., do Carmo, M.: Hypersurfaces with constant mean curvature in spheres. Proc. Am. Math. Soc. **120**(4), 1223–1229 (1994)
2. Alías, L.J., Meléndez, J.: Integral inequalities for compact hypersurfaces with constant scalar curvature in the Euclidean sphere. Mediterr. J. Math. **17**(2), Paper No. 61, 14 pp. (2020)
3. Bansal, P., Shahid, M.H., Lee, J.W.: ζ-Ricci soliton on real hypersurfaces of nearly Kaehler 6-sphere with SSMC. Mediterr. J. Math. **18**(3), 93 (2021)
4. Blair, D.E., Ludden, G.D., Yano, K.: Hypersurfaces of odd-dimensional spheres. J. Differ. Geom. **5**, 479–486 (1971)
5. Chaubey, S.K., De, U.C., Suh, Y.J.: Kenmotsu manifolds satisfying the Fischer-Marsden equation. J. Korean Math. Soc. **58**(3), 597–607 (2021)
6. Chen, B.-Y.: Minimal hypersurfaces in an *m*-sphere. Proc. Am. Math. Soc. **29**, 375–380 (1971)
7. Cheng, Q.-M.: Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature. J. Lond. Math. Soc. (2) **64**(3), 755–768 (2001)
8. Chern, S.S., do Carmo, M., Kobayashi, S.: Minimal submanifolds of a sphere with second fundamental form of constant length. In: Functional Analysis and Related Fields, pp. 59–75. Springer, Berlin (1970)
9. De, U.C., Mandal, K.: The Fischer-Marsden conjecture on almost Kenmotsu manifolds. Quaest. Math. (2020). https://doi.org/10.2989/16073606.2018.1533499
10. Deshmukh, S.: Jacobi-type vector fields on Ricci solitons. Bull. Math. Soc. Sci. Math. Roum. **55**(103) No. 1, 41–50 (2012)
11. Deshmukh, S.: First nonzero eigenvalue of a minimal hypersurface in the unit sphere. Ann. Mat. Pura Appl. **191**(3), 529–537 (2012)
12. Deshmukh, S.: A note on hypersurfaces in a sphere. Monatshefte Math. **174**(3), 413–426 (2014)
13. do Carmo, M.P., Warner, F.W.: Rigidity and convexity of hypersurfaces in spheres. J. Differ. Geom. **4**, 133–144 (1970)
14. Duggal, K.L., Sharma, R.: Symmetries of Spacetimes and Riemannian Manifolds. Springer, Berlin (1999)
15. Erkekoglu, F., García-Río, E., Kupeli, D.N., Ünal, B.: Characterizing specific Riemannian manifolds by differential equations. Acta Appl. Math. **76**(2), 195–219 (2003)
16. Fischer, A.E., Marsden, J.E.: Manifolds of Riemannian metrics with prescribed scalar curvature. Bull. Am. Math. Soc. **80**(3), 479–484 (1974)
17. García-Río, E., Kupeli, D.N., Ünal, B.: Some conditions for Riemannian manifolds to be isometric with Euclidean spheres. J. Differ. Equ. **194**(2), 287–299 (2003)
18. Hasanis, T., Vlachos, T.: A pinching theorem for minimal hypersurfaces in a sphere. Arch. Math. (Basel) **75**(6), 469–471 (2000)
19. Hou, Z.H.: Hypersurfaces in a sphere with constant mean curvature. Proc. Am. Math. Soc. **125**(4), 1193–1196 (1997)
20. Jagy, W.C.: Minimal hypersurfaces foliated by spheres. Mich. Math. J. **38**(2), 255–270 (1991)
21. Lawson, H.B. Jr.: Local rigidity theorems for minimal hypersurfaces. Ann. Math. (2) **89**, 187–197 (1969)
22. Min, S.-H., Seo, K.: Characterizations of a Clifford hypersurface in a unit sphere via Simons' integral inequalities. Monatshefte Math. **181**(2), 437–450 (2016)
23. Nomizu, K., Smyth, B.: On the Gauss mapping for hypersurfaces of constant mean curvature in the sphere. Comment. Math. Helv. **44**, 484–490 (1969)
24. Okumura, M.: Certain hypersurfaces of an odd dimensional sphere. Tohoku Math. J. (2) **19**, 381–395 (1967)
25. Patra, D.S., Ghosh, A.: The Fischer-Marsden conjecture and contact geometry. Period. Math. Hung. **76**, 207–216 (2018)
26. Peng, C.K., Terng, C.-L.: The scalar curvature of minimal hypersurfaces in spheres. Math. Ann. **266**(1), 105–113 (1983)
27. Perdomo, O.M.: Spectrum of the Laplacian and the Jacobi operator on rotational CMC hypersurfaces of spheres. Pac. J. Math. **308**(2), 419–433 (2020)
28. Scherfner, M., Weiss, S., Yau, S.T.: A review of the Chern conjecture for isoparametric hypersurfaces in spheres. In: Adv. Lect. Math. (ALM), vol. 21, pp. 175–187. Int. Press, Somerville (2012)
29. Seo, K.: Characterizations of a Clifford hypersurface in a unit sphere. In: Hermitian-Grassmannian Submanifolds. Springer Proc. Math. Stat., vol. 203, pp. 145–153. Springer, Singapore (2017)
30. Shen, C.L.: A global pinching theorem of minimal hypersurfaces in the sphere. Proc. Am. Math. Soc. **105**(1), 192–198 (1989)
31. Suh, Y.J., Yang, H.Y.: The scalar curvature of minimal hypersurfaces in a unit sphere. Commun. Contemp. Math. **9**(2), 183–200 (2007)
32. Tanno, S., Takahashi, T.: Some hypersurfaces of a sphere. Tohoku Math. J. (2) **22**, 212–219 (1970)
33. Venkatesha, V., Naik, D.M., Kumara, H.A.: Real hypersurfaces of complex space forms satisfying Fischer-Marsden equation. Ann. Univ. Ferrara (2021). https://doi.org/10.1007/s11565-021-00361-x
34. Wang, Q., Xia, C.: Rigidity theorems for closed hypersurfaces in a unit sphere. J. Geom. Phys. **55**(3), 227–240 (2005)

35. Wei, G.: J. Simons' type integral formula for hypersurfaces in a unit sphere. J. Math. Anal. Appl. **340**(2), 1371–1379 (2008)
36. Wei, S.-M., Xu, H.-W.: Scalar curvature of minimal hypersurfaces in a sphere. Math. Res. Lett. **14**(3), 423–432 (2007)
37. Xu, H.W., Xu, Z.Y.: The second pinching theorem for hypersurfaces with constant mean curvature in a sphere. Math. Ann. **356**, 869–883 (2013)
38. Yang, H.C., Cheng, Q.M.: An estimate of the pinching constant of minimal hypersurfaces with constant scalar curvature in the unit sphere. Manuscr. Math. **84**(1), 89–100 (1994)
39. Yau, S.T.: Seminar on Differential Geometry. Annals of Mathematics Studies, vol. 102, pp. 669–706. Princeton University Press, Princeton (1982)
40. Zhu, P.: Hypersurfaces in spheres with finite total curvature. Results Math. **74**(4), Paper No. 153, 13 pp. (2019)