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Weighted dynamic Hardy-type inequalities involving many functions on arbitrary time scales

Ahmed A. El-Deeb[1*], Karim A. Mohamed[1], Dumitru Baleanu[2,3] and Haytham M. Rezk[1]

*Correspondence:
ahmedeldeeb@azhar.edu.eg
[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, 11884, Cairo, Egypt
Full list of author information is available at the end of the article

Abstract

The objective of this paper is to prove some new dynamic inequalities of Hardy type on time scales which generalize and improve some recent results given in the literature. Further, we derive some new weighted Hardy dynamic inequalities involving many functions on time scales. As special cases, we get continuous and discrete inequalities.

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Keywords: Delta derivative; Hardy's inequality; Hölder's inequality; Time scales

1 Introduction

In [1], Hardy showed that if $\alpha > 1$ and $\Psi(\zeta) \geq 0$ over the interval $(0,\infty)$ such that $\int_0^\infty \Psi^\alpha(\zeta)\,d\zeta < \infty$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x \Psi(\zeta)\,d\zeta\right)^\alpha dx \leq \left(\frac{\alpha}{\alpha-1}\right)^\alpha \int_0^\infty \Psi^\alpha(x)\,dx, \tag{1}$$

where the constant $(\alpha/(\alpha-1))^\alpha$ is sharp. In [2], Hardy obtained that if $\alpha > 1$ and $m > 1$, then

$$\int_0^\infty \frac{1}{x^m}\left(\int_0^x \Psi(\zeta)\,d\zeta\right)^\alpha dx \leq \left(\frac{\alpha}{m-1}\right)^\alpha \int_0^\infty \frac{1}{x^{m-\alpha}}\Psi^\alpha(x)\,dx. \tag{2}$$

In [3], Levinson proved that if $\alpha > 1$, $\Psi(x) \geq 0$, $f(x) > 0$ is an absolutely continuous function and

$$\frac{\alpha}{\alpha-1} + \frac{f'(x)}{f(x)} \geq \frac{1}{\beta} > 0, \quad \text{for all } x > 0,$$

then

$$\int_0^\infty \left(\frac{1}{xf(x)}\int_0^x f(\zeta)\Psi(\zeta)\,d\zeta\right)^\alpha dx \leq \beta^\alpha \int_0^\infty \Psi^\alpha(x)\,dx. \tag{3}$$

In [4], S. Hussan et al. proved that for any $\mathrm{i} = 1, 2, \ldots, n$, $n \in \mathbb{N}$, $f_\mathrm{i}(\zeta) \geq 0$ and integrable function on $(0, \infty)$ and $\hat{w}$, u_i, z_i are absolutely continuous functions with z'_i essentially bounded and positive, if u_i is increasing and

$$1 + \frac{u_\mathrm{i}(\zeta)\hat{w}'(\zeta)}{(1-2m)u'_\mathrm{i}(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\lambda_\mathrm{i}} > 0, \quad \text{for } m > \frac{1}{2},$$
$$1 + \frac{u_\mathrm{i}(\zeta)\hat{w}'(\zeta)}{(1-2m)u'_\mathrm{i}(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\delta_\mathrm{i}} > 0, \quad \text{for } m < \frac{1}{2},$$

then

$$\sum_{\mathrm{i}=1}^{n} \int_0^\infty \hat{w}(\zeta) R_\mathrm{i}(\zeta) R_{\mathrm{i}+1}(\zeta)\, d\zeta \leq \sum_{\mathrm{i}=1}^{n} \left(\frac{2\beta_\mathrm{i}}{|2m-1|}\right)^2 \int_0^\infty \hat{w}(\zeta) g_\mathrm{i}(\zeta)\, d\zeta, \tag{4}$$

where

$$R_\mathrm{i}(\zeta) = \begin{cases} \frac{\sqrt{u'_\mathrm{i}(\zeta)}}{u_\mathrm{i}^m(\zeta)} \int_0^\zeta \frac{u_\mathrm{i}(x) z'_\mathrm{i}(x)}{z_\mathrm{i}(x)} f_\mathrm{i}(x)\, dx, & m > \frac{1}{2}, \\ \frac{\sqrt{u'_\mathrm{i}(\zeta)}}{u_\mathrm{i}^m(\zeta)} \int_\zeta^\infty \frac{u_\mathrm{i}(x) z'_\mathrm{i}(x)}{z_\mathrm{i}(x)} f_\mathrm{i}(x)\, dx, & m < \frac{1}{2}, \end{cases}$$
$$g_\mathrm{i}(\zeta) = \frac{[u_\mathrm{i}(\zeta)]^{4-2m}[z'_\mathrm{i}(\zeta)]^2 f_\mathrm{i}^2(\zeta)}{z_\mathrm{i}^2(\zeta) u'_\mathrm{i}(\zeta)} \quad \text{and} \quad \beta_\mathrm{i} = \max_{1 \leq \mathrm{i} \leq n}(\lambda_\mathrm{i}, \delta_\mathrm{i}).$$

Also in the same paper [4], the authors proved that for any $\mathrm{i} = 1, 2, \ldots, n$, $n \geq \kappa - 1$, $n, \kappa \in \mathbb{N}$, if $\alpha_\mathrm{i} > 1$, $\delta_\mathrm{i} = \alpha_\mathrm{i}/(\kappa\alpha_\mathrm{i} - 1)$ and

$$1 + \frac{u_\mathrm{i}(\zeta)\hat{w}'(\zeta)}{(1-\kappa\alpha_\mathrm{i} m)u'_\mathrm{i}(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\lambda_\mathrm{i}} > 0, \quad \text{for } m > \frac{1}{\kappa\alpha_\mathrm{i}},$$
$$1 + \frac{u_\mathrm{i}(\zeta)\hat{w}'(\zeta)}{(1-\kappa\alpha_\mathrm{i} m)u'_\mathrm{i}(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\delta_\mathrm{i}} > 0, \quad \text{for } m < \frac{1}{\kappa\alpha_\mathrm{i}},$$

then

$$\sum_{\mathrm{i}=1}^{n-\kappa+2} \int_0^\infty \hat{w}(\zeta) \left[\prod_{j=\mathrm{i}}^{\mathrm{i}+\kappa-1} R_j^{\alpha_j}(\zeta)\right] d\zeta \leq \sum_{\mathrm{i}=1}^{n} \left(\frac{\kappa\alpha_\mathrm{i}\beta_\mathrm{i}}{|\kappa\alpha_\mathrm{i} m - 1|}\right)^{\kappa\alpha_\mathrm{i}} \int_0^\infty \hat{w}(\zeta) g_\mathrm{i}(\zeta)\, d\zeta, \tag{5}$$

where

$$R_\mathrm{i}(\zeta) = \begin{cases} \frac{\sqrt[\kappa\alpha_\mathrm{i}]{u'_\mathrm{i}(\zeta)}}{u_\mathrm{i}^m(\zeta)} \int_0^\zeta \frac{u_\mathrm{i}(x) z'_\mathrm{i}(x)}{z_\mathrm{i}(x)} f_\mathrm{i}(x)\, dx, & m > \frac{1}{\kappa\alpha_\mathrm{i}}, \\ \frac{\sqrt[\kappa\alpha_\mathrm{i}]{u'_\mathrm{i}(\zeta)}}{u_\mathrm{i}^m(\zeta)} \int_\zeta^\infty \frac{u_\mathrm{i}(x) z'_\mathrm{i}(x)}{z_\mathrm{i}(x)} f_\mathrm{i}(x)\, dx, & m < \frac{1}{\kappa\alpha_\mathrm{i}}, \end{cases}$$
$$g_\mathrm{i}(\zeta) = \frac{[u_\mathrm{i}(\zeta)]^{\kappa\alpha_\mathrm{i}(2-m)}[z'_\mathrm{i}(\zeta)]^{\kappa\alpha_\mathrm{i}} f_\mathrm{i}^{\kappa\alpha_\mathrm{i}}(\zeta)}{z_\mathrm{i}^{\kappa\alpha_\mathrm{i}}(\zeta)[u'_\mathrm{i}(\zeta)]^{k\alpha_\mathrm{i}-1}} \quad \text{and} \quad \beta_\mathrm{i} = \max_{1 \leq \mathrm{i} \leq n}(\lambda_\mathrm{i}, \delta_\mathrm{i}).$$

The main aim is to establish some dynamic inequalities where the involved functions are defined on the $\mathbb{T}$ domain. These results involve the classical discrete and continuous inequalities. For more details, we point the reader to the books [5, 6]. In [7], Řehak found the time scale version of Hardy's inequality. Especially, Řehak derived that if $\alpha > 1$ and $\Psi(\zeta) \geq 0$ are such that $\int_a^\infty \Psi^\alpha(\zeta)\Delta\zeta < \infty$ then

$$\int_a^\infty \left(\frac{1}{\sigma(\zeta)-a}\int_a^{\sigma(\zeta)} \Psi(x)\Delta x\right)^\alpha \Delta\zeta \leq \left(\frac{\alpha}{\alpha-1}\right)^\alpha \int_a^\infty \Psi^\alpha(\zeta)\Delta\zeta.$$

In addition, if $\mu(\zeta)/\zeta \to 0$ as $\zeta \to \infty$, then the constant $(\alpha/(\alpha-1))^\alpha$ is sharp.

In [8], the authors showed that if $f(\zeta) > 0$, $\Psi(\zeta) \geq 0$ and $f^\Delta(\zeta) \leq 0$ on $[0,\infty)_{\mathbb{T}}$, $\alpha > 1$ and there exist constants $\kappa, \beta > 0$ such that $\zeta/\sigma(\zeta) \geq 1/\kappa$ and

$$\frac{\alpha}{\alpha-1} + \frac{\kappa^\alpha \Phi(\zeta)}{\Phi^\sigma(\zeta)}\frac{\zeta f^\Delta(\zeta)}{f^\sigma(\zeta)} \geq \frac{1}{\beta}, \quad \text{for } \zeta \in [0,\infty)_{\mathbb{T}},$$

then

$$\int_0^\infty \frac{1}{\zeta^\alpha}\left(\Phi^\sigma(\zeta)\right)^\alpha \Delta\zeta \leq \left(\beta\kappa^\alpha\right)^\alpha \int_0^\infty \left(\frac{f(\zeta)\Psi(\zeta)}{f^\sigma(\zeta)}\right)^\alpha \Delta\zeta,$$

where

$$\Phi(\zeta) = \frac{1}{f(\zeta)}\int_0^\zeta f(x)\Psi(x)\Delta x, \quad \zeta \in [0,\infty)_{\mathbb{T}}.$$

The purpose of this manuscript is to establish some new Hardy-type inequalities on time scales $\mathbb{T}$ involving many functions which generalize and improve some results in [4]. The following is the format of the paper: In Sect. 2, we begin with some background information about the delta derivative on $\mathbb{T}$. Our main findings are obtained in Sect. 3.

2 Basic principles

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(\zeta) = \inf\{s \in \mathbb{T} : s > \zeta\}$ and define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(\zeta) = \sup\{s \in \mathbb{T} : s < \zeta\}$, respectively, where $\sup\emptyset = \inf\mathbb{T}$.

A point $\zeta \in \mathbb{T}$ is called right-dense if $\sigma(\zeta) = \zeta$, left-dense if $\rho(\zeta) = \zeta$, right-scattered if $\sigma(\zeta) > \zeta$, and left-scattered if $\rho(\zeta) < \zeta$ If $\sup\mathbb{T}$ is finite and left-scattered, then $\mathbb{T}^k = \mathbb{T}\backslash\{\sup\mathbb{T}\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is a right-dense continuous (rd-continuous) if f is continuous at right-dense points and its left-hand limits are finite at left-dense points in $\mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a real-valued function on $\mathbb{T}$. Then for $\zeta \in \mathbb{T}^k$, we define $f^\Delta(\zeta)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood u of ζ such that, for all $s \in u$, we have

$$\left|\left[f\left(\sigma(\zeta)\right) - f(s)\right] - f^\Delta(\zeta)[\zeta - s]\right| \leq \varepsilon\left|\sigma(\zeta) - s\right|.$$

In this case, we say that f is delta differentiable on $\mathbb{T}^k$ provided $f(\zeta)$ exists for all $\zeta \in \mathbb{T}^k$. If $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $\zeta \in \mathbb{T}$, then

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \quad \text{where } f^\sigma(\zeta) = (f \circ \sigma)(\zeta) = f\left(\sigma(\zeta)\right). \tag{6}$$

For $a, b \in \mathbb{T}$ and a delta differentiable function f, the Cauchy integral of f^{Δ} is defined by $\int_a^b f^{\Delta}(\zeta)\Delta\zeta = f(b) - f(a)$. The integration by parts formula on $\mathbb{T}$ is given by

$$\int_a^b \Psi(\zeta)\Phi^{\Delta}(\zeta)\Delta\zeta = (\Psi\Phi)(b) - (\Psi\Phi)(a) - \int_a^b \Psi^{\Delta}(\zeta)\Phi^{\sigma}(\zeta)\Delta\zeta. \tag{7}$$

Lemma 1 (Leibniz rule [9]) *If f, f^{Δ} are continuous and $u, v : \mathbb{T} \longrightarrow \mathbb{T}$ are delta differentiable functions and $f^{\Delta}(\zeta, s)$ mean the delta derivative of $f(\zeta, s)$ with respect to ζ, then*

$$\begin{aligned}&\left(\int_{u(\zeta)}^{v(\zeta)} f(\zeta,s)\Delta s\right)^{\Delta} \\ &\quad = \int_{u(\zeta)}^{v(\zeta)} f^{\Delta}(\zeta,s)\Delta s + v^{\Delta}(\zeta)f\bigl(\sigma(\zeta), v(\zeta)\bigr) - u^{\Delta}(\zeta)f\bigl(\sigma(\zeta), u(\zeta)\bigr).\end{aligned} \tag{8}$$

Lemma 2 (Chain rule [10]) *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^k$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists a point c in the real interval $[\zeta, \sigma(\zeta)]$ with*

$$(f \circ g)^{\Delta}(\zeta) = f^{\backprime}\bigl(g(c)\bigr)g^{\Delta}(\zeta). \tag{9}$$

Lemma 3 (Hölder's inequality [10]) *Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$, we have*

$$\int_a^b \bigl|f(\zeta)g(\zeta)\bigr|\Delta\zeta \le \left(\int_a^b \bigl|f(\zeta)\bigr|^{\alpha}\Delta\zeta\right)^{\frac{1}{\alpha}} \left(\int_a^b \bigl|g(\zeta)\bigr|^{\delta}\Delta\zeta\right)^{\frac{1}{\delta}}, \tag{10}$$

where $\alpha > 1$ and $\delta = \alpha/(\alpha - 1)$.

Lemma 4 ([11]) *If $C_1, C_2, \dots, C_n$ are reals and $C_{n+1} = C_1$, then*

$$\sum_{r=1}^{n-\kappa+2} C_r C_{r+1} \cdots C_{r+\kappa-1} \le \sum_{r=1}^{n} (C_r)^{\kappa}, \quad \text{where } n \ge \kappa - 1. \tag{11}$$

Lemma 5 ([11]) *If $C_1, C_2, \dots, C_n$ are reals and $C_{n+1} = C_1$, for $\kappa \ge 1$, then*

$$\left(\sum_{r=1}^{n} C_r\right)^{\kappa} \le n^{\kappa-1}\sum_{r=1}^{n} (C_r)^{\kappa}. \tag{12}$$

3 Main results

Throughout this section, any time scale $\mathbb{T}$ is unbounded above with $a, b \in \mathbb{T}$. We will make the assumption that the functions $\hat{w}$, u_{i}, z_{i} in the statements of the theorems are rd-continuous, nonnegative and increasing, and $f_{\mathrm{i}}(\zeta) > 0$ is an integrable function.

Theorem 6 *For any $1 \le \mathrm{i} \le n$, $n \ge \kappa - 1$ and $n, \kappa \in \mathbb{N}$, if there exist constants $\lambda_{\mathrm{i}} > 0$, $\delta_{\mathrm{i}} > 0$ such that*

$$1 - \frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{\alpha m}\hat{w}^{\Delta}(\zeta)}{(\alpha m - 1)[u_{\mathrm{i}}(\zeta)]^{\alpha m-1}u_{\mathrm{i}}^{\Delta}(\zeta)\hat{w}^{\sigma}(\zeta)} \ge \frac{1}{\lambda_{\mathrm{i}}} > 0, \quad \text{for } m > \frac{1}{2}, \tag{13}$$

$$1-\frac{u_{\mathrm{i}}(\zeta)\hat{w}^{\Delta}(\zeta)}{(\alpha m-1)u_{\mathrm{i}}^{\Delta}(\zeta)\hat{w}^{\sigma}(\zeta)}\geq\frac{1}{\delta_{\mathrm{i}}}>0,\quad \textit{for } m<\frac{1}{2}, \tag{14}$$

then

$$\sum_{\mathrm{i}=1}^{n}\int_{0}^{\infty}\hat{w}^{\sigma}(\zeta)R_{\mathrm{i}}^{\frac{\alpha}{2}}(\zeta)R_{\mathrm{i}+1}^{\frac{\alpha}{2}}(\zeta)\Delta\zeta\leq\sum_{\mathrm{i}=1}^{n}\left(\frac{\alpha\beta_{\mathrm{i}}}{|\alpha m-1|}\right)^{\alpha}\int_{0}^{\infty}\hat{w}^{\sigma}(\zeta)g_{\mathrm{i}}(\zeta)\Delta\zeta, \tag{15}$$

where

$$R_{\mathrm{i}}(\zeta)=\begin{cases}\frac{\sqrt[\alpha]{u_{\mathrm{i}}^{\Delta}(\zeta)}}{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m}}\int_{0}^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\\ \quad \textit{and}\quad \int_{0}^{\infty}\frac{u_{\mathrm{i}}^{\Delta}(x)}{(u_{\mathrm{i}}^{\sigma}(x))^{\alpha m}}\Delta\varkappa<\infty, & \textit{for } m>\frac{1}{2},\alpha\geq 2,\\ \frac{\sqrt[\alpha]{u_{\mathrm{i}}^{\Delta}(\zeta)}}{u_{\mathrm{i}}^{m}(\zeta)}\int_{\sigma(\zeta)}^{\infty}\frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x, & \textit{for } m<\frac{1}{2},1\leq\alpha\leq 2,\end{cases}$$

$$g_{\mathrm{i}}(\zeta)=\begin{cases}\frac{u_{\mathrm{i}}^{\alpha(2-\alpha m)}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m\alpha(\alpha-1)}[z_{\mathrm{i}}^{\Delta}(\zeta)]^{\alpha}f_{\mathrm{i}}^{\alpha}(\zeta)\hat{w}^{\alpha}(\zeta)}{z_{\mathrm{i}}^{\alpha}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{\alpha-1}(\hat{w}^{\sigma}(\zeta))^{\alpha}}, & \textit{for } m>\frac{1}{2},\alpha\geq 2,\\ \frac{u_{\mathrm{i}}^{\alpha(2-m)}(\zeta)[z_{\mathrm{i}}^{\Delta}(\zeta)]^{\alpha}f_{\mathrm{i}}^{\alpha}(\zeta)\hat{w}^{\alpha}(\zeta)}{z_{\mathrm{i}}^{\alpha}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{\alpha-1}(\hat{w}^{\sigma}(\zeta))^{\alpha}}, & \textit{for } m<\frac{1}{2},1\leq\alpha\leq 2,\end{cases}$$

and $\beta_{\mathrm{i}}=\max_{1\leq\mathrm{i}\leq n}(\lambda_{\mathrm{i}},\delta_{\mathrm{i}})$, $u_{\mathrm{i}}(\infty)=\infty$.

Proof First, let us define for $m>\frac{1}{2}$, $\alpha\geq 2$, and $0<a<b<\infty$,

$$R_{\mathrm{i}a}(\zeta)=\frac{\sqrt[\alpha]{u_{\mathrm{i}}^{\Delta}(\zeta)}}{(u_{\mathrm{i}}^{\sigma}(\zeta))^{m}}\int_{a}^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x,\quad 1\leq\mathrm{i}\leq n,$$

with $R_{\mathrm{i}0}(\zeta)=R_{\mathrm{i}}(\zeta)$. Using (11) with $\kappa=2$ for $C_{\mathrm{i}}=R_{\mathrm{i}}^{\frac{\alpha}{2}}(\zeta)$, we get

$$\sum_{\mathrm{i}=1}^{n}R_{\mathrm{i}a}^{\frac{\alpha}{2}}(\zeta)R_{(\mathrm{i}+1)a}^{\frac{\alpha}{2}}(\zeta)\leq\sum_{\mathrm{i}=1}^{n}R_{\mathrm{i}a}^{\alpha}(\zeta). \tag{16}$$

Multiplying (16) by $\hat{w}^{\sigma}(\zeta)$ and integrating from a to b, we have

$$\sum_{\mathrm{i}=1}^{n}\int_{a}^{b}\hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\frac{\alpha}{2}}(\zeta)R_{(\mathrm{i}+1)a}^{\frac{\alpha}{2}}(\zeta)\Delta\zeta\leq\sum_{\mathrm{i}=1}^{n}\int_{a}^{b}\hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\alpha}(\zeta)\Delta\zeta. \tag{17}$$

Now,

$$\begin{aligned}J&=\int_{a}^{b}\hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\alpha}(\zeta)\Delta\zeta\\ &=\int_{a}^{b}\hat{w}^{\sigma}(\zeta)\left(\frac{\sqrt[\alpha]{u_{\mathrm{i}}^{\Delta}(\zeta)}}{(u_{\mathrm{i}}^{\sigma}(\zeta))^{m}}\int_{a}^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\alpha}\Delta\zeta\\ &=\int_{a}^{b}\frac{u_{\mathrm{i}}^{\Delta}(\zeta)}{(u_{\mathrm{i}}^{\sigma}(\zeta))^{\alpha m}}\left(\sqrt[\alpha]{\hat{w}^{\sigma}(\zeta)}\int_{a}^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\alpha}\Delta\zeta.\end{aligned} \tag{18}$$

Integrating (18) by parts using formula (7) with

$$\Phi^{\Delta}(\zeta)=\frac{u_{\mathrm{i}}^{\Delta}(\zeta)}{(u_{\mathrm{i}}^{\sigma}(\zeta))^{\alpha m}},\qquad \Psi^{\sigma}(\zeta)=\left(\sqrt[\alpha]{\hat{w}^{\sigma}(\zeta)}\int_{a}^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\alpha},$$

we obtain

$$\begin{aligned} J &= \left[\Phi(\zeta)\Psi(\zeta)\right]_a^b + \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^\Delta \Delta\zeta \\ &= \left[-\Psi(\zeta)\int_\zeta^\infty \frac{u_i^\Delta(x)}{(u_i^\sigma(x))^{\alpha m}}\Delta x\right]_a^b + \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^\Delta \Delta\zeta \\ &= \left[-\Psi(b)\int_b^\infty \frac{u_i^\Delta(x)}{(u_i^\sigma(x))^{\alpha m}}\Delta x\right] + \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^\Delta \Delta\zeta \\ &\le \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^\Delta \Delta\zeta, \end{aligned} \tag{19}$$

where $\Phi(\zeta) = -\int_\zeta^\infty \frac{u_i^\Delta(x)}{(u_i^\sigma(x))^{\alpha m}}\Delta \varkappa$ and $(\Psi(\zeta))^\Delta > 0$. From (9), since $u_i^\Delta(\zeta) \ge 0$ and $c \in [\zeta, \sigma(\zeta)]$, we have

$$\begin{aligned} \left[u_i^{1-\alpha m}(\zeta)\right]^\Delta &= (1-\alpha m)u_i^{-\alpha m}(c)u_i^\Delta(\zeta) \\ &= (1-\alpha m)\frac{u_i^\Delta(\zeta)}{u_i^{\alpha m}(c)} \\ &\le (1-\alpha m)\frac{u_i^\Delta(\zeta)}{(u_i^\sigma(\zeta))^{\alpha m}}. \end{aligned} \tag{20}$$

Therefore, integrating (20) from ζ to ∞ with respect to x, we have

$$-\Phi(\zeta) \le \frac{1}{\alpha m - 1}u_i^{1-\alpha m}(\zeta). \tag{21}$$

Combining (21) and (19), we get

$$J \le \frac{1}{\alpha m - 1}\int_a^b u_i^{1-\alpha m}(\zeta)\left(\Psi(\zeta)\right)^\Delta \Delta\zeta. \tag{22}$$

Now, by applying (6) to $\Psi(\zeta) = \hat{w}(\zeta)\hat{Y}^\alpha(\zeta)$ and using (9), we obtain

$$\begin{aligned} \left(\Psi(\zeta)\right)^\Delta &= \hat{w}^\Delta(\zeta)\left[\hat{Y}^\sigma(\zeta)\right]^\alpha + \hat{w}(\zeta)\left[\hat{Y}^\alpha(\zeta)\right]^\Delta \\ &= \hat{w}^\Delta(\zeta)\left[\hat{Y}^\sigma(\zeta)\right]^\alpha + \alpha\hat{w}(\zeta)\hat{Y}^{\alpha-1}(c)\hat{Y}^\Delta(\zeta) \\ &\le \hat{w}^\Delta(\zeta)\left[\hat{Y}^\sigma(\zeta)\right]^\alpha + \alpha\hat{w}(\zeta)\frac{u_i(\zeta)z_i^\Delta(\zeta)}{z_i(\zeta)}f_i(\zeta)\left[\hat{Y}^\sigma(\zeta)\right]^{\alpha-1}, \end{aligned} \tag{23}$$

where

$$\hat{Y}(\zeta) = \int_a^\zeta \frac{u_i(x)z_i^\Delta(x)}{z_i(x)}f_i(x)\Delta x.$$

Substituting (23) into (22), we get

$$J \le \frac{1}{\alpha m - 1}\int_a^b u_i^{1-\alpha m}(\zeta)\hat{w}^\Delta(\zeta)\left(\int_a^{\sigma(\zeta)} \frac{u_i(x)z_i^\Delta(x)}{z_i(x)}f_i(x)\Delta x\right)^\alpha \Delta\zeta$$

$$
\begin{aligned}
&+ \frac{\alpha}{\alpha m-1}\int_a^b u_{\mathrm{i}}^{2-\alpha m}(\zeta)\hat{w}(\zeta)\frac{z_{\mathrm{i}}^{\Delta}(\zeta)}{z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta) \\
&\times \left(\int_a^{\sigma(\zeta)} \frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\alpha-1}\Delta\zeta \\
={}& \frac{1}{\alpha m-1}\int_a^b \frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{\alpha m}\hat{w}^{\Delta}(\zeta)}{u_{\mathrm{i}}^{\alpha m-1}(\zeta)u_{\mathrm{i}}^{\Delta}(\zeta)}R_{\mathrm{i}a}^{\alpha}(\zeta)\Delta\zeta \\
&+ \frac{\alpha}{\alpha m-1}\int_a^b \frac{u_{\mathrm{i}}^{2-\alpha m}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m(\alpha-1)}z_{\mathrm{i}}^{\Delta}(\zeta)f_{\mathrm{i}}(\zeta)\hat{w}(\zeta)}{z_{\mathrm{i}}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{1-\frac{1}{\alpha}}}R_{\mathrm{i}a}^{\alpha-1}(\zeta)\Delta\zeta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_a^b \hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\alpha}(\zeta)\left(1-\frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{\alpha m}\hat{w}^{\Delta}(\zeta)}{(\alpha m-1)u_{\mathrm{i}}^{\alpha m-1}(\zeta)u_{\mathrm{i}}^{\Delta}(\zeta)\hat{w}^{\sigma}(\zeta)}\right)\Delta\zeta \\
&\quad\le \frac{\alpha}{\alpha m-1}\int_a^b \frac{u_{\mathrm{i}}^{2-\alpha m}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m(\alpha-1)}z_{\mathrm{i}}^{\Delta}(\zeta)f_{\mathrm{i}}(\zeta)\hat{w}(\zeta)}{z_{\mathrm{i}}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{1-\frac{1}{\alpha}}}R_{\mathrm{i}a}^{\alpha-1}(\zeta)\Delta\zeta.
\end{aligned}
\tag{24}
$$

From (13) and (24), we have

$$
\begin{aligned}
&\int_a^b \hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\alpha}(\zeta)\Delta\zeta \\
&\quad\le \frac{\alpha\lambda_{\mathrm{i}}}{\alpha m-1}\int_a^b \frac{u_{\mathrm{i}}^{2-\alpha m}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m(\alpha-1)}z_{\mathrm{i}}^{\Delta}(\zeta)f_{\mathrm{i}}(\zeta)\hat{w}(\zeta)}{z_{\mathrm{i}}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{1-\frac{1}{\alpha}}}R_{\mathrm{i}a}^{\alpha-1}(\zeta)\Delta\zeta \\
&\quad= \frac{\alpha\lambda_{\mathrm{i}}}{\alpha m-1}\int_a^b \left[\hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\alpha}(\zeta)\right]^{\frac{\alpha-1}{\alpha}}\frac{u_{\mathrm{i}}^{2-\alpha m}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m(\alpha-1)}z_{\mathrm{i}}^{\Delta}(\zeta)f_{\mathrm{i}}(\zeta)\hat{w}(\zeta)}{z_{\mathrm{i}}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{1-\frac{1}{\alpha}}(\hat{w}^{\sigma}(\zeta))^{1-\frac{1}{\alpha}}}\Delta\zeta.
\end{aligned}
$$

Applying Hölder's inequality with α and $\alpha/(\alpha-1)$, we have

$$
\begin{aligned}
&\int_a^b \hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\alpha}(\zeta)\Delta\zeta \\
&\quad\le \left(\frac{\alpha\lambda_{\mathrm{i}}}{\alpha m-1}\right)^{\alpha}\int_a^b \frac{u_{\mathrm{i}}^{\alpha(2-\alpha m)}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m\alpha(\alpha-1)}[z_{\mathrm{i}}^{\Delta}(\zeta)]^{\alpha}f_{\mathrm{i}}^{\alpha}(\zeta)\hat{w}^{\alpha}(\zeta)}{z_{\mathrm{i}}^{\alpha}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{\alpha-1}(\hat{w}^{\sigma}(\zeta))^{\alpha-1}}\Delta\zeta \\
&\quad\le \left(\frac{\alpha\lambda_{\mathrm{i}}}{\alpha m-1}\right)^{\alpha}\int_a^{\infty} \frac{u_{\mathrm{i}}^{\alpha(2-\alpha m)}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m\alpha(\alpha-1)}[z_{\mathrm{i}}^{\Delta}(\zeta)]^{\alpha}f_{\mathrm{i}}^{\alpha}(\zeta)\hat{w}^{\alpha}(\zeta)}{z_{\mathrm{i}}^{\alpha}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{\alpha-1}(\hat{w}^{\sigma}(\zeta))^{\alpha-1}}\Delta\zeta.
\end{aligned}
\tag{25}
$$

By letting $a \to 0$, $b \to \infty$ and from (25), (17), we have

$$
\sum_{\mathrm{i}=1}^{n}\int_0^{\infty}\hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\frac{\alpha}{2}}(\zeta)R_{(\mathrm{i}+1)a}^{\frac{\alpha}{2}}(\zeta)\Delta\zeta \le \sum_{\mathrm{i}=1}^{n}\left(\frac{\alpha\beta_{\mathrm{i}}}{\alpha m-1}\right)^{\alpha}\int_0^{\infty}\hat{w}^{\sigma}(\zeta)g_{\mathrm{i}}(\zeta)\Delta\zeta. \tag{26}
$$

Second, let us define for $m < \frac{1}{2}$, $1 \le \alpha \le 2$, and $0 < a < b < \infty$,

$$
R_{\mathrm{i}b}(\zeta) = \frac{\sqrt[\alpha]{u_{\mathrm{i}}^{\Delta}(\zeta)}}{u_{\mathrm{i}}^{m}(\zeta)}\int_{\sigma(\zeta)}^{b}\frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x, \quad 1 \le \mathrm{i} \le n,
$$

with $R_{\mathrm{i}\infty}(\zeta) = R_{\mathrm{i}}(\zeta)$. Following the same steps as in the proof of (26), we obtain

$$\sum_{\mathrm{i}=1}^{n} \int_0^\infty \hat{w}^\sigma(\zeta) R_{\mathrm{i}}^{\frac{\alpha}{2}}(\zeta) R_{\mathrm{i}+1}^{\frac{\alpha}{2}}(\zeta) \Delta\zeta \le \sum_{\mathrm{i}=1}^{n} \left(\frac{\alpha\beta_{\mathrm{i}}}{1-\alpha m} \right)^\alpha \int_0^\infty \hat{w}^\sigma(\zeta) g_{\mathrm{i}}(\zeta) \Delta\zeta. \tag{27}$$

Inequalities (26) and (27) are equivalent to (15). □

In Theorem 6, if we take $\mathbb{T} = \mathbb{N}$, then we have $\sigma(s) = s+1$ and obtain the next corollary.

Corollary 7 *Let $\{u(s)\}_{s=1}^\infty$, $\{\hat{w}(s)\}_{s=1}^\infty$, and $\{z(s)\}_{s=1}^\infty$ be increasing and nonnegative sequences. For any $1 \le \mathrm{i} \le n$, $n \ge \kappa - 1$, $n, \kappa \in \mathbb{N}$, and*

$$1 - \frac{[u_{\mathrm{i}}(s+1)]^{\alpha m} \Delta\hat{w}(s)}{(\alpha m - 1)[u_{\mathrm{i}}(s)]^{\alpha m - 1} \hat{w}(s+1) \Delta u_{\mathrm{i}}(s)} \ge \frac{1}{\lambda_{\mathrm{i}}} > 0, \quad \text{for } m > \frac{1}{2},$$
$$1 - \frac{u_{\mathrm{i}}(s) \Delta\hat{w}(s)}{(\alpha m - 1)\hat{w}(s+1)\Delta u_{\mathrm{i}}(s)} \ge \frac{1}{\delta_{\mathrm{i}}} > 0, \quad \text{for } m < \frac{1}{2},$$

we have

$$\sum_{\mathrm{i}=1}^{n} \left(\sum_{s=1}^{\infty} \hat{w}(s+1) R_{\mathrm{i}}^{\frac{\alpha}{2}}(s) R_{\mathrm{i}+1}^{\frac{\alpha}{2}}(s) \right) \le \sum_{\mathrm{i}=1}^{n} \left(\frac{\alpha\beta_{\mathrm{i}}}{|1-\alpha m|} \right)^\alpha \sum_{s=1}^{\infty} \hat{w}(s+1) g_{\mathrm{i}}(s),$$

where

$$R_{\mathrm{i}}(s) = \begin{cases} \frac{\sqrt[\alpha]{\Delta u_{\mathrm{i}}(s)}}{[u_{\mathrm{i}}(s+1)]^m} \sum_{r=1}^{s} \frac{u_{\mathrm{i}}(r)\Delta z_{\mathrm{i}}(r)}{z_{\mathrm{i}}(r)} f_{\mathrm{i}}(r), & \text{for } m > \frac{1}{2}, \alpha \ge 2, \\ \frac{\sqrt[\alpha]{\Delta u_{\mathrm{i}}(s)}}{u_{\mathrm{i}}^m(s)} \sum_{r=s+1}^{\infty} \frac{u_{\mathrm{i}}(r)\Delta z_{\mathrm{i}}(r)}{z_{\mathrm{i}}(r)} f_{\mathrm{i}}(r), & \text{for } m < \frac{1}{2}, 1 \le \alpha \le 2, \end{cases}$$

$$g_{\mathrm{i}}(s) = \begin{cases} \frac{u_{\mathrm{i}}^{\alpha(2-\alpha m)}(s)[u_{\mathrm{i}}(s+1)]^{m\alpha(\alpha-1)}[\Delta z_{\mathrm{i}}(s)]^\alpha f_{\mathrm{i}}^\alpha(s)\hat{w}^\alpha(s)}{z_{\mathrm{i}}^\alpha(s)(\Delta u_{\mathrm{i}}(s))^{\alpha-1}(\hat{w}(s+1))^\alpha}, & \text{for } m > \frac{1}{2}, \alpha \ge 2, \\ \frac{u_{\mathrm{i}}^{\alpha(2-\alpha m)}(s)[\Delta z_{\mathrm{i}}(s)]^\alpha f_{\mathrm{i}}^\alpha(s)\hat{w}^\alpha(s)}{z_{\mathrm{i}}^\alpha(s)(\Delta u_{\mathrm{i}}(s))^{\alpha-1}(\hat{w}(s+1))^\alpha}, & \text{for } m < \frac{1}{2}, 1 \le \alpha \le 2, \end{cases}$$

with $\Delta y(s) = y(s+1) - y(s)$, $\beta_{\mathrm{i}} = \max_{1 \le \mathrm{i} \le n}(\lambda_{\mathrm{i}}, \delta_{\mathrm{i}})$, and $u_{\mathrm{i}}(\infty) = \infty$.

Remark 8 If we put $\mathbb{T} = \mathbb{R}$ and $\alpha = 2$, in Theorem 6, then (15) reduces to (4).

The next corollary follows from Theorem 6 by taking $u_{\mathrm{i}}(\zeta) = z_{\mathrm{i}}(\zeta) = \zeta$, $f_{\mathrm{i}}(\zeta) = f_{\mathrm{i}+1}(\zeta)$, $m = 1$, and $\alpha = 2$.

Corollary 9 *For any $1 \le \mathrm{i} \le n$, $n \ge \kappa - 1$ and $\kappa \in \mathbb{N}$, if there exist $\lambda_{\mathrm{i}} > 0$ such that*

$$1 - \frac{\sigma^2(\zeta)\hat{w}^\Delta(\zeta)}{\zeta \hat{w}^\sigma(\zeta)} \ge \frac{1}{\lambda_{\mathrm{i}}} > 0,$$

then

$$\sum_{\mathrm{i}=1}^{n} \int_0^\infty \hat{w}^\sigma(\zeta) \left[\frac{1}{\sigma(\zeta)} \int_0^{\sigma(\zeta)} f_{\mathrm{i}}(x) \Delta x \right]^2 \Delta\zeta \le \sum_{\mathrm{i}=1}^{n} (2\lambda_{\mathrm{i}})^2 \int_0^\infty \hat{w}^\sigma(\zeta) g_{\mathrm{i}}(\zeta) \Delta\zeta, \tag{28}$$

where

$$g_i(\zeta) = \frac{\sigma^2(\zeta) f_i^2(\zeta) \hat{w}^2(\zeta)}{\zeta^2 [\hat{w}^\sigma(\zeta)]^2}.$$

Remark 10 Letting $\mathbb{T} = \mathbb{R}$ in Corollary 9, we have that $\sigma(\zeta) = \zeta$ and

$$1 - \frac{\zeta \hat{w}'(\zeta)}{\hat{w}(\zeta)} \geq \frac{1}{\lambda_i} > 0.$$

Then

$$\sum_{i=1}^{n} \int_0^\infty \hat{w}(\zeta) \left[\frac{1}{\zeta} \int_0^\zeta f_i(x)\, dx \right]^2 d\zeta \leq \sum_{i=1}^{n} (2\lambda_i)^2 \int_0^\infty \hat{w}(\zeta) f_i^2(\zeta)\, d\zeta,$$

which agrees with [4, Corollary 1].

Theorem 11 *For any* $1 \leq i \leq n$, $n \geq \kappa - 1$, *and* $n, \kappa \in \mathbb{N}$, *if* $\alpha_i > 1$, $\delta_i = \alpha_i/(\kappa\alpha_i - 1)$ *and there exist* $\lambda_i > 0$, $\delta_i > 0$ *such that*

$$1 - \frac{[u_i^\sigma(\zeta)]^{\kappa\alpha_i m} \hat{w}^\Delta(\zeta)}{(\kappa\alpha_i m - 1) u_i^{\kappa\alpha_i m - 1}(\zeta) u_i^\Delta(\zeta) \hat{w}^\sigma(\zeta)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{\kappa\alpha_i}, \tag{29}$$

$$1 - \frac{u_i(\zeta) \hat{w}^\Delta(\zeta)}{(\kappa\alpha_i m - 1) u_i^\Delta(\zeta) \hat{w}^\sigma(\zeta)} \geq \frac{1}{\delta_i} > 0, \quad \text{for } m < \frac{1}{\kappa\alpha_i}, \tag{30}$$

then

$$\begin{aligned} &\sum_{i=1}^{n-\kappa+2} \int_0^\infty \hat{w}^\sigma(\zeta) \left(\prod_{j=i}^{i+\kappa-1} R_j^{\alpha_j}(\zeta) \right) \Delta\zeta \\ &\quad \leq \sum_{i=1}^{n} \left(\frac{\kappa\alpha_i \beta_i}{|\kappa\alpha_i m - 1|} \right)^{\kappa\alpha_i} \int_0^\infty \hat{w}^\sigma(\zeta) g_i(\zeta) \Delta\zeta, \end{aligned} \tag{31}$$

where

$$R_i(\zeta) = \begin{cases} \frac{\sqrt[\kappa\alpha_i]{u_i^\Delta(\zeta)}}{[u_i^\sigma(\zeta)]^m} \int_0^{\sigma(\zeta)} \frac{u_i(x) z_i^\Delta(x)}{z_i(x)} f_i(x) \Delta x & \\ \quad \textit{and} \quad \int_0^\infty \frac{u_i^\Delta(x)}{(u_i^\sigma(x))^{\kappa\alpha_i m}} \Delta\varkappa < \infty, & \text{for } m > \frac{1}{\kappa\alpha_i}, \\ \frac{\sqrt[\kappa\alpha_i]{u_i^\Delta(\zeta)}}{u_i^m(\zeta)} \int_{\sigma(\zeta)}^\infty \frac{u_i(x) z_i^\Delta(x)}{z_i(x)} f_i(x) \Delta x, & \text{for } m < \frac{1}{\kappa\alpha_i}, \end{cases}$$

$$g_i(\zeta) = \begin{cases} \frac{u_i^{\kappa\alpha_i(2-\kappa\alpha_i m)}(\zeta) [u_i^\sigma(\zeta)]^{\kappa\alpha_i m(\kappa\alpha_i - 1)} (z_i^\Delta(\zeta))^{\kappa\alpha_i} f_i^{\kappa\alpha_i}(\zeta) \hat{w}^{\kappa\alpha_i}(\zeta)}{z_i^{\kappa\alpha_i}(\zeta) (u_i^\Delta(\zeta))^{\kappa\alpha_i - 1} (\hat{w}^\sigma(\zeta))^{\kappa\alpha_i}}, & \text{for } m > \frac{1}{\kappa\alpha_i}, \\ \frac{u_i^{\kappa\alpha_i(2-m)}(\zeta) (z_i^\Delta(\zeta))^{\kappa\alpha_i} f_i^{\kappa\alpha_i}(\zeta) \hat{w}^{\kappa\alpha_i}(\zeta)}{z_i^{\kappa\alpha_i}(\zeta) (u_i^\Delta(\zeta))^{\kappa\alpha_i - 1} (\hat{w}^\sigma(\zeta))^{\kappa\alpha_i}}, & \text{for } m < \frac{1}{\kappa\alpha_i}, \end{cases}$$

and $\beta_i = \max_{1 \leq i \leq n}(\lambda_i, \delta_i)$, $u_i(\infty) = \infty$.

Proof Let us define for $m > \frac{1}{\kappa\alpha_i}$ and $0 < a < b < \infty$,

$$R_{ia}(\zeta) = \frac{\sqrt[\kappa\alpha_i]{u_i^\Delta(\zeta)}}{[u_i^\sigma(\zeta)]^m} \int_a^{\sigma(\zeta)} \frac{u_i(x) z_i^\Delta(x)}{z_i(x)} f_i(x) \Delta x, \quad 1 \leq i \leq n, \tag{32}$$

with $R_{\mathrm{i}0}(\zeta) = R_{\mathrm{i}}(\zeta)$. Using (11) with $C_{\mathrm{i}} = R^{\alpha_{\mathrm{i}}}_{\mathrm{i}a}(\zeta)$, we get

$$\sum_{\mathrm{i}=1}^{n-\kappa+2} R^{\alpha_{\mathrm{i}}}_{\mathrm{i}a}(\zeta)R^{\alpha_{\mathrm{i}+1}}_{(\mathrm{i}+1)a}(\zeta)\cdots R^{\alpha_{\mathrm{i}+\kappa-1}}_{(\mathrm{i}+\kappa-1)a}(\zeta) \le \sum_{\mathrm{i}=1}^{n} R^{\kappa\alpha_{\mathrm{i}}}_{\mathrm{i}a}(\zeta). \tag{33}$$

Multiplying (33) by $\hat{w}^{\sigma}(\zeta)$ and integrating from a to b, we have

$$\sum_{\mathrm{i}=1}^{n-\kappa+2} \int_a^b \hat{w}^{\sigma}(\zeta)\left(\prod_{j=\mathrm{i}}^{\mathrm{i}+\kappa-1} R^{\alpha_j}_{ja}(\zeta)\right)\Delta\zeta \le \sum_{\mathrm{i}=1}^{n}\int_a^b \hat{w}^{\sigma}(\zeta)R^{\kappa\alpha_{\mathrm{i}}}_{\mathrm{i}a}(\zeta)\Delta\zeta. \tag{34}$$

Now,

$$\begin{aligned}
\mathfrak{I} &= \int_a^b \hat{w}^{\sigma}(\zeta)R^{\kappa\alpha_{\mathrm{i}}}_{\mathrm{i}a}(\zeta)\Delta\zeta \\
&= \int_a^b \hat{w}^{\sigma}(\zeta)\left(\frac{\sqrt[\kappa\alpha_{\mathrm{i}}]{u^{\Delta}_{\mathrm{i}}(\zeta)}}{[u^{\sigma}_{\mathrm{i}}(\zeta)]^{m}}\int_a^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z^{\Delta}_{\mathrm{i}}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\kappa\alpha_{\mathrm{i}}}\Delta\zeta \\
&= \int_a^b \frac{u^{\Delta}_{\mathrm{i}}(\zeta)}{[u^{\sigma}_{\mathrm{i}}(\zeta)]^{\kappa\alpha_{\mathrm{i}}m}}\left(\sqrt[\kappa\alpha_{\mathrm{i}}]{\hat{w}^{\sigma}(\zeta)}\int_a^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z^{\Delta}_{\mathrm{i}}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\kappa\alpha_{\mathrm{i}}}\Delta\zeta.
\end{aligned} \tag{35}$$

Integrating (35) by parts using formula (7) with

$$\Phi^{\Delta}(\zeta) = \frac{u^{\Delta}_{\mathrm{i}}(\zeta)}{[u^{\sigma}_{\mathrm{i}}(\zeta)]^{\kappa\alpha_{\mathrm{i}}m}}, \qquad \Psi^{\sigma}(\zeta) = \left(\sqrt[\kappa\alpha_{\mathrm{i}}]{\hat{w}^{\sigma}(\zeta)}\int_a^{\sigma(\zeta)}\frac{u_{\mathrm{i}}(x)z^{\Delta}_{\mathrm{i}}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\kappa\alpha_{\mathrm{i}}},$$

we obtain

$$\begin{aligned}
J &= \left[\Phi(\zeta)\Psi(\zeta)\right]_a^b + \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^{\Delta}\Delta\zeta \\
&= \left[-\Psi(\zeta)\int_{\zeta}^{\infty}\frac{u^{\Delta}_{\mathrm{i}}(x)}{[u^{\sigma}_{\mathrm{i}}(x)]^{\kappa\alpha_{\mathrm{i}}m}}\Delta x\right]_a^b + \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^{\Delta}\Delta\zeta \\
&= \left[-\Psi(b)\int_{b}^{\infty}\frac{u^{\Delta}_{\mathrm{i}}(x)}{[u^{\sigma}_{\mathrm{i}}(x)]^{\kappa\alpha_{\mathrm{i}}m}}\Delta x\right] + \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^{\Delta}\Delta\zeta \\
&\le \int_a^b \left(-\Phi(\zeta)\right)\left(\Psi(\zeta)\right)^{\Delta}\Delta\zeta,
\end{aligned} \tag{36}$$

where $\Phi(\zeta) = -\int_{\zeta}^{\infty}\frac{u^{\Delta}_{\mathrm{i}}(x)}{[u^{\sigma}_{\mathrm{i}}(x)]^{\kappa\alpha_{\mathrm{i}}m}}\Delta\varkappa$ and $(\Psi(\zeta))^{\Delta} > 0$. From (9), since $u^{\Delta}_{\mathrm{i}}(\zeta) \ge 0$ and $c \in [\zeta,\sigma(\zeta)]$, we have

$$\begin{aligned}
\left[u^{1-\kappa\alpha_{\mathrm{i}}m}_{\mathrm{i}}(\zeta)\right]^{\Delta} &= (1-\kappa\alpha_{\mathrm{i}}m)u^{-\kappa\alpha_{\mathrm{i}}m}_{\mathrm{i}}(c)u^{\Delta}_{\mathrm{i}}(\zeta) \\
&= (1-\kappa\alpha_{\mathrm{i}}m)\frac{u^{\Delta}_{\mathrm{i}}(\zeta)}{u^{\kappa\alpha_{\mathrm{i}}m}_{\mathrm{i}}(c)} \\
&\le (1-\kappa\alpha_{\mathrm{i}}m)\frac{u^{\Delta}_{\mathrm{i}}(\zeta)}{[u^{\sigma}_{\mathrm{i}}(\zeta)]^{\kappa\alpha_{\mathrm{i}}m}}.
\end{aligned} \tag{37}$$

Therefore, integrating (37) from ζ to ∞ with respect to x, we have

$$-\Phi(\zeta) \leq \frac{1}{\kappa\alpha_{\mathrm{i}} m - 1} u_{\mathrm{i}}^{1-\kappa\alpha_{\mathrm{i}} m}(\zeta). \tag{38}$$

Combining (38) and (36), we have

$$J \leq \frac{1}{\kappa\alpha_{\mathrm{i}} m - 1} \int_a^b u_{\mathrm{i}}^{1-\kappa\alpha_{\mathrm{i}} m}(\zeta)\big(\Psi(\zeta)\big)^{\Delta}\Delta\zeta. \tag{39}$$

Now, by applying (6) to $\Psi(\zeta) = \hat{w}(\zeta)\hat{Y}^{\kappa\alpha_{\mathrm{i}}}(\zeta)$ and using (9), we obtain

$$\begin{aligned}\big(\Psi(\zeta)\big)^{\Delta} &= \hat{w}^{\Delta}(\zeta)\big[\hat{Y}^{\sigma}(\zeta)\big]^{\kappa\alpha_{\mathrm{i}}} + \hat{w}(\zeta)\big[\hat{Y}^{\kappa\alpha_{\mathrm{i}}}(\zeta)\big]^{\Delta} \\ &= \hat{w}^{\Delta}(\zeta)\big[\hat{Y}^{\sigma}(\zeta)\big]^{\kappa\alpha_{\mathrm{i}}} + \kappa\alpha_{\mathrm{i}}\hat{w}(\zeta)\hat{Y}^{\kappa\alpha_{\mathrm{i}}-1}(c)\hat{Y}^{\Delta}(\zeta) \\ &\leq \hat{w}^{\Delta}(\zeta)\big[\hat{Y}^{\sigma}(\zeta)\big]^{\kappa\alpha_{\mathrm{i}}} + \kappa\alpha_{\mathrm{i}}\hat{w}(\zeta)\frac{u_{\mathrm{i}}(\zeta)z_{\mathrm{i}}^{\Delta}(\zeta)}{z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta)\big[\hat{Y}^{\sigma}(\zeta)\big]^{\kappa\alpha_{\mathrm{i}}-1},\end{aligned} \tag{40}$$

where

$$\hat{Y}(\zeta) = \int_a^{\zeta} \frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x.$$

Substituting (40) into (39) and using (32), we get

$$\begin{aligned}J &\leq \frac{1}{\kappa\alpha_{\mathrm{i}} m - 1}\int_a^b u_{\mathrm{i}}^{1-\kappa\alpha_{\mathrm{i}} m}(\zeta)\hat{w}^{\Delta}(\zeta)\bigg(\int_a^{\sigma(\zeta)} \frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\bigg)^{\kappa\alpha_{\mathrm{i}}}\Delta\zeta \\ &\quad + \frac{\kappa\alpha_{\mathrm{i}}}{\kappa\alpha_{\mathrm{i}} m - 1}\int_a^b \frac{u_{\mathrm{i}}^{2-\kappa\alpha_{\mathrm{i}} m}(\zeta)\hat{w}(\zeta)z_{\mathrm{i}}^{\Delta}(\zeta)}{z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta) \\ &\quad \times \bigg(\int_a^{\sigma(\zeta)} \frac{u_{\mathrm{i}}(x)z_{\mathrm{i}}^{\Delta}(x)}{z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\bigg)^{\kappa\alpha_{\mathrm{i}}-1}\Delta\zeta \\ &= \frac{1}{\kappa\alpha_{\mathrm{i}} m - 1}\int_a^b \frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{\kappa\alpha_{\mathrm{i}} m}\hat{w}^{\Delta}(\zeta)}{u_{\mathrm{i}}^{\kappa\alpha_{\mathrm{i}} m-1}(\zeta)u_{\mathrm{i}}^{\Delta}(\zeta)}R_{\mathrm{i}a}^{\kappa\alpha_{\mathrm{i}}}(\zeta)\Delta\zeta \\ &\quad + \frac{\kappa\alpha_{\mathrm{i}}}{\kappa\alpha_{\mathrm{i}} m - 1}\int_a^b \frac{u_{\mathrm{i}}^{2-\kappa\alpha_{\mathrm{i}} m}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m(\kappa\alpha_{\mathrm{i}}-1)}z_{\mathrm{i}}^{\Delta}(\zeta)f_{\mathrm{i}}(\zeta)\hat{w}(\zeta)}{z_{\mathrm{i}}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{1-\frac{1}{\kappa\alpha_{\mathrm{i}}}}}R_{\mathrm{i}a}^{\kappa\alpha_{\mathrm{i}}-1}(\zeta)\Delta\zeta.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_a^b \hat{w}^{\sigma}(\zeta)R_{\mathrm{i}a}^{\kappa\alpha_{\mathrm{i}}}(\zeta)\bigg(1 - \frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{\kappa\alpha_{\mathrm{i}} m}\hat{w}^{\Delta}(\zeta)}{(\kappa\alpha_{\mathrm{i}} m - 1)u_{\mathrm{i}}^{\kappa\alpha_{\mathrm{i}} m-1}(\zeta)u_{\mathrm{i}}^{\Delta}(\zeta)\hat{w}^{\sigma}(\zeta)}\bigg)\Delta\zeta \\ &\quad \leq \frac{\kappa\alpha_{\mathrm{i}}}{\kappa\alpha_{\mathrm{i}} m - 1}\int_a^b \frac{u_{\mathrm{i}}^{2-\kappa\alpha_{\mathrm{i}} m}(\zeta)[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m(\kappa\alpha_{\mathrm{i}}-1)}z_{\mathrm{i}}^{\Delta}(\zeta)f_{\mathrm{i}}(\zeta)\hat{w}(\zeta)}{z_{\mathrm{i}}(\zeta)(u_{\mathrm{i}}^{\Delta}(\zeta))^{1-\frac{1}{\kappa\alpha_{\mathrm{i}}}}}R_{\mathrm{i}a}^{\kappa\alpha_{\mathrm{i}}-1}(\zeta)\Delta\zeta.\end{aligned} \tag{41}$$

From (41) and (29), we have

$$\begin{aligned}
&\int_a^b \hat{w}^\sigma(\zeta) R_{\mathrm{i}a}^{\kappa\alpha_\mathrm{i}}(\zeta)\Delta\zeta \\
&\quad\leq \frac{\kappa\alpha_\mathrm{i}\lambda_\mathrm{i}}{\kappa\alpha_\mathrm{i} m-1}\int_a^b \frac{u_\mathrm{i}^{2-\kappa\alpha_\mathrm{i} m}(\zeta)[u_\mathrm{i}^\sigma(\zeta)]^{m(\kappa\alpha_\mathrm{i}-1)} z_\mathrm{i}^\Delta(\zeta) f_\mathrm{i}(\zeta)\hat{w}(\zeta)}{z_\mathrm{i}(\zeta)(u_\mathrm{i}^\Delta(\zeta))^{1-\frac{1}{\kappa\alpha_\mathrm{i}}}} R_{\mathrm{i}a}^{\kappa\alpha_\mathrm{i}-1}(\zeta)\Delta\zeta \\
&\quad= \frac{\kappa\alpha_\mathrm{i}\lambda_\mathrm{i}}{\kappa\alpha_\mathrm{i} m-1}\int_a^b \left(\hat{w}^\sigma(\zeta) R_{\mathrm{i}a}^{\kappa\alpha_\mathrm{i}}(\zeta)\right)^{\frac{\kappa\alpha_\mathrm{i}-1}{\kappa\alpha_\mathrm{i}}} \\
&\qquad\times \frac{u_\mathrm{i}^{2-\kappa\alpha_\mathrm{i} m}(\zeta)[u_\mathrm{i}^\sigma(\zeta)]^{m(\kappa\alpha_\mathrm{i}-1)} z_\mathrm{i}^\Delta(\zeta) f_\mathrm{i}(\zeta)\hat{w}(\zeta)}{z_\mathrm{i}(\zeta)(u_\mathrm{i}^\Delta(\zeta))^{1-\frac{1}{\kappa\alpha_\mathrm{i}}}(\hat{w}^\sigma(\zeta))^{1-\frac{1}{\kappa\alpha_\mathrm{i}}}}\Delta\zeta.
\end{aligned}$$

Applying Hölder's inequality with $\kappa\alpha_\mathrm{i}$ and $\kappa\delta_\mathrm{i} = \kappa\alpha_\mathrm{i}/(\kappa\alpha_\mathrm{i}-1)$, we have

$$\begin{aligned}
&\int_a^b \hat{w}^\sigma(\zeta) R_{\mathrm{i}a}^{\kappa\alpha_\mathrm{i}}(\zeta)\Delta\zeta \\
&\quad\leq \left(\frac{\kappa\alpha_\mathrm{i}\lambda_\mathrm{i}}{\kappa\alpha_\mathrm{i} m-1}\right)^{\kappa\alpha_\mathrm{i}} \\
&\qquad\times \int_a^b \frac{u_\mathrm{i}^{\kappa\alpha_\mathrm{i}(2-\kappa\alpha_\mathrm{i} m)}(\zeta)[u_\mathrm{i}^\sigma(\zeta)]^{\kappa\alpha_\mathrm{i} m(\kappa\alpha_\mathrm{i}-1)}(z_\mathrm{i}^\Delta(\zeta))^{\kappa\alpha_\mathrm{i}} f_\mathrm{i}^{\kappa\alpha_\mathrm{i}}(\zeta)\hat{w}^{\kappa\alpha_\mathrm{i}}(\zeta)}{z_\mathrm{i}^{\kappa\alpha_\mathrm{i}}(\zeta)(u_\mathrm{i}^\Delta(\zeta))^{\kappa\alpha_\mathrm{i}-1}(\hat{w}^\sigma(\zeta))^{\kappa\alpha_\mathrm{i}-1}}\Delta\zeta \\
&\quad= \left(\frac{\kappa\alpha_\mathrm{i}\lambda_\mathrm{i}}{\kappa\alpha_\mathrm{i} m-1}\right)^{\kappa\alpha_\mathrm{i}} \int_a^b \hat{w}^\sigma(\zeta) g_\mathrm{i}(\zeta)\Delta\zeta.
\end{aligned}\tag{42}$$

From (42) and (34), we have

$$\begin{aligned}
&\sum_{\mathrm{i}=1}^{n-\kappa+2}\int_a^b \hat{w}^\sigma(\zeta)\left(\prod_{j=\mathrm{i}}^{\mathrm{i}+\kappa-1} R_j^{\alpha_j}(\zeta)\right)\Delta\zeta \\
&\quad\leq \sum_{\mathrm{i}=1}^{n}\left(\frac{\kappa\alpha_\mathrm{i}\beta_\mathrm{i}}{\kappa\alpha_\mathrm{i} m-1}\right)^{\kappa\alpha_\mathrm{i}} \int_a^b \hat{w}^\sigma(\zeta) g_\mathrm{i}(\zeta)\Delta\zeta \\
&\quad\leq \sum_{\mathrm{i}=1}^{n}\left(\frac{\kappa\alpha_\mathrm{i}\beta_\mathrm{i}}{\kappa\alpha_\mathrm{i} m-1}\right)^{\kappa\alpha_\mathrm{i}} \int_a^\infty \hat{w}^\sigma(\zeta) g_\mathrm{i}(\zeta)\Delta\zeta.
\end{aligned}\tag{43}$$

Let us define for $m < 1/\kappa\alpha_\mathrm{i}$ and $0 < a < b < \infty$,

$$R_{\mathrm{i}b}(\zeta) = \frac{\sqrt[\kappa\alpha_\mathrm{i}]{u_\mathrm{i}^\Delta(\zeta)}}{u_\mathrm{i}^m(\zeta)}\int_{\sigma(\zeta)}^b \frac{u_\mathrm{i}(x) z_\mathrm{i}^\Delta(x)}{z_\mathrm{i}(x)} f_\mathrm{i}(x)\Delta x, \quad 1 \leq \mathrm{i} \leq n,$$

with $R_{\mathrm{i}\infty}(\zeta) = R_\mathrm{i}(\zeta)$. Following the same steps as in the proof of (43), we obtain

$$\begin{aligned}
&\sum_{\mathrm{i}=1}^{n-\kappa+2}\int_a^b \hat{w}^\sigma(\zeta)\left(\prod_{j=\mathrm{i}}^{\mathrm{i}+\kappa-1} R_j^{\alpha_j}(\zeta)\right)\Delta\zeta \\
&\quad\leq \sum_{\mathrm{i}=1}^{n}\left(\frac{\kappa\alpha_\mathrm{i}\beta_\mathrm{i}}{1-\kappa\alpha_\mathrm{i} m}\right)^{\kappa\alpha_\mathrm{i}} \int_a^b \hat{w}^\sigma(\zeta) g_\mathrm{i}(\zeta)\Delta\zeta
\end{aligned}\tag{44}$$

$$\leq \sum_{i=1}^{n}\left(\frac{\kappa\alpha_i\beta_i}{1-\kappa\alpha_i m}\right)^{\kappa\alpha_i}\int_a^{\infty}\hat{w}^{\sigma}(\zeta)g_i(\zeta)\Delta\zeta.$$

By letting $a\to 0$ and $b\to\infty$ in (43) and (44), we get (31). □

In Theorem 11, if we take $\mathbb{T}=\mathbb{N}$, then we have the following corollary.

Corollary 12 *Let $\{u(s)\}_{s=1}^{\infty}$, $\{\hat{w}(s)\}_{s=1}^{\infty}$, and $\{z(s)\}_{s=1}^{\infty}$ be increasing and nonnegative sequences. Then for any $1\leq i\leq n$, $n>\kappa-1$, $\alpha_i>1$, $\kappa\in\mathbb{N}$, $\delta_i=\alpha_i/(\kappa\alpha_i-1)$ and*

$$\begin{aligned}
&1-\frac{[u_i(s+1)]^{\kappa\alpha_i m}\Delta\hat{w}(s)}{(\kappa\alpha_i m-1)[u_i(s)]^{\kappa\alpha_i m-1}\hat{w}(s+1)\Delta u_i(s)}\geq\frac{1}{\lambda_i}>0,\quad \textit{for } m>\frac{1}{2},\\
&1-\frac{u_i(s)\Delta\hat{w}(s)}{(\kappa\alpha_i m-1)\hat{w}(s+1)\Delta u_i(s)}\geq\frac{1}{\lambda_i}>0,\quad \textit{for } m>\frac{1}{2},
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{i=1}^{n-\kappa+2}\left(\sum_{s=1}^{\infty}\hat{w}(s+1)\left[\prod_{j=i}^{i+\kappa-1}R_j^{\alpha_j}(s)\right]\right)\\
&\quad\leq\sum_{i=1}^{n}\left(\frac{\kappa\alpha_i\beta_i}{|1-\kappa\alpha_i m|}\right)^{\kappa\alpha_i}\sum_{s=1}^{\infty}\hat{w}(s+1)g_i(s),\\
&R_i(s)=\begin{cases}\frac{\sqrt[\kappa\alpha_i]{\Delta u_i(s)}}{[u_i(s+1)]^m}\sum_{r=1}^{s}\frac{u_i(r)\Delta z_i(r)}{z_i(r)}f_i(r), & \textit{for } m>\frac{1}{\kappa\alpha_i},\\ \frac{\sqrt[\kappa\alpha_i]{\Delta u_i(s)}}{u_i^m(s)}\sum_{r=s+1}^{\infty}\frac{u_i(r)\Delta z_i(r)}{z_i(r)}f_i(r), & \textit{for } m<\frac{1}{\kappa\alpha_i},\end{cases}\\
&g_i(s)=\begin{cases}\frac{u_i^{\kappa\alpha_i(2-\kappa\alpha_i m)}(s)[u_i(s+1)]^{\kappa\alpha_i m(\kappa\alpha_i-1)}[\Delta z_i(s)]^{\kappa\alpha_i}f_i^{\kappa\alpha_i}(s)\hat{w}^{\kappa\alpha_i}(s)}{z_i^{\kappa\alpha_i}(s)(\Delta u_i(s))^{\kappa\alpha_i-1}(\hat{w}(s+1))^{\kappa\alpha_i}}, & \textit{for } m>\frac{1}{\kappa\alpha_i},\\ \frac{u_i^{\kappa\alpha_i(2-m)}(s)[\Delta z_i(s)]^{\kappa\alpha_i}f_i^{\kappa\alpha_i}(s)\hat{w}^{\kappa\alpha_i}(s)}{z_i^{\kappa\alpha_i}(s)(\Delta u_i(s))^{\kappa\alpha_i-1}(\hat{w}(s+1))^{\kappa\alpha_i}}, & \textit{for } m<\frac{1}{\kappa\alpha_i},\end{cases}
\end{aligned}$$

and $\beta_i=\max_{1\leq i\leq n}(\lambda_i,\delta_i)$, $u_i(\infty)=\infty$.

Remark 13 If we put $\mathbb{T}=\mathbb{R}$ in Theorem 11, then (31) reduces to (5).

The next corollary follows from Theorem 11 by taking $u_i(\zeta)=z_i(\zeta)=\zeta, f_i(\zeta)\to\zeta^{m-1}h_i$, $h_i=h_{i+1}$, $\alpha_i=\alpha_{i+1}$ and $\kappa=2$.

Corollary 14 *For any $1\leq i\leq n$, $n>\kappa-1$, $\kappa\in\mathbb{N}$, if h_i are rd-continuous functions and there exist $\lambda_i>0$ such that*

$$1-\frac{\sigma^{2\alpha_i m}(\zeta)\hat{w}^{\Delta}(\zeta)}{(2\alpha_i m-1)\zeta^{2\alpha_i m-1}\hat{w}^{\sigma}(\zeta)}\geq\frac{1}{\lambda_i}>0,$$

then

$$\begin{aligned}
&\sum_{i=1}^{n}\int_0^{\infty}\hat{w}^{\sigma}(\zeta)\left[\frac{1}{\sigma^m(\zeta)}\int_0^{\sigma(\zeta)}x^{m-1}h_i(x)\Delta x\right]^{2\alpha_i}\Delta\zeta\\
&\quad\leq\sum_{i=1}^{n}\left(\frac{2\alpha_i\lambda_i}{2\alpha_i m-1}\right)^{2\alpha_i}\int_0^{\infty}\hat{w}^{\sigma}(\zeta)g_i(\zeta)\Delta\zeta,
\end{aligned}$$

where

$$g_i(\zeta) = \frac{\sigma^{2\alpha_i m(2\alpha_i-1)}(\zeta)\zeta^{2\alpha_i m(1-\alpha_i)}h_i^{2\alpha_i}(\zeta)\hat{w}^{2\alpha_i}(\zeta)}{[\hat{w}^\sigma(\zeta)]^{2\alpha_i}}, \quad \textit{for } m > \frac{1}{2\alpha_i}.$$

Remark 15 Letting $\mathbb{T} = \mathbb{R}$ in Corollary 14, we have that $\sigma(\zeta) = \zeta$ and

$$1 - \frac{\zeta\hat{w}'(\zeta)}{(2\alpha_i m - 1)\hat{w}(\zeta)} \geq \frac{1}{\lambda_i} > 0.$$

Then

$$\begin{aligned} &\sum_{i=1}^{n}\int_0^\infty \hat{w}(\zeta)\left[\frac{1}{\zeta^m}\int_0^\zeta x^{m-1}h_i(x)\,dx\right]^{2\alpha_i} d\zeta \\ &\quad\leq \sum_{i=1}^{n}\left(\frac{2\alpha_i\lambda_i}{2\alpha_i m - 1}\right)^{2\alpha_i}\int_0^\infty \hat{w}(\zeta)h_i^{2\alpha_i}(\zeta)\,d\zeta, \end{aligned}$$

which agrees with [4, Corollary 2].

Theorem 16 *For any* $1 \leq i \leq n$, $n > \kappa - 1$, $\kappa \in \mathbb{N}$ *and* $\frac{\zeta}{2}, \frac{\sigma(\zeta)}{2} \in \mathbb{T}$, *if* $\alpha_i > 1$, $\delta_i = \alpha_i/(3\alpha_i - 1)$, *and*

$$1 + \frac{[u_i^\sigma(\zeta)]^{3\alpha_i m}\hat{w}^\Delta(\zeta)}{(1 - 3\alpha_i m)u_i^{3\alpha_i m-1}(\zeta)u_i^\Delta(\zeta)\hat{w}^\sigma(\zeta)} \geq \frac{1}{\lambda_i} > 0, \quad \textit{for } m > \frac{1}{3\alpha_i}, \tag{45}$$

then

$$\begin{aligned} &\sum_{i=1}^{n-1}\int_0^x \hat{w}^\sigma(\zeta)\left[\Gamma_i^{\alpha_i}(\zeta)\Gamma_{i+1}^{\alpha_i+1}(\zeta)\Gamma_{i+2}^{\alpha_{i+2}}(\zeta)\right]\Delta\zeta \\ &\quad\leq \sum_{i=1}^{n}\left(\frac{3\alpha_i\lambda_i}{3\alpha_i m - 1}\right)^{3\alpha_i}\int_0^x \hat{w}^\sigma(\zeta)g_i(\zeta)\Delta\zeta, \end{aligned} \tag{46}$$

where

$$\begin{aligned} \Gamma_i(\zeta) &= \frac{\sqrt[3\alpha_i]{u_i^\Delta(\zeta)}}{[u_i^\sigma(\zeta)]^m}\int_{\frac{\sigma(\zeta)}{2}}^{\sigma(\zeta)}\frac{u_i(\eta)z_i^\Delta(\eta)}{z_i(\eta)}f_i(\eta)\Delta\eta \\ &\qquad \textit{and} \quad \int_0^\infty \frac{u_i^\Delta(x)}{(u_i^\sigma(x))^{\alpha_i m}}\Delta\varkappa < \infty, \\ g_i(\zeta) &= \left[\frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}z_i^\Delta(\zeta)}{u_i^{3\alpha_i m-2}(\zeta)\sqrt[3\delta_i]{u_i^\Delta(\zeta)}z_i(\zeta)}f_i(\zeta)\right. \\ &\qquad \left. - \frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}u_i(\frac{\zeta}{2})z_i^\Delta(\frac{\zeta}{2})}{2u_i^{3\alpha_i m-1}(\zeta)z_i(\frac{\zeta}{2})\sqrt[3\delta_i]{u_i^\Delta(\zeta)}}f_i\left(\frac{\zeta}{2}\right)\right]^{3\alpha_i}, \\ u_i(\infty) &= \infty. \end{aligned}$$

Proof Let us define for $m > \frac{1}{3\alpha_{\mathrm{i}}}$, $1 \leq \mathrm{i} \leq n$,

$$\Gamma_{\mathrm{i}}(\zeta) = \frac{\sqrt[3\alpha_{\mathrm{i}}]{u_{\mathrm{i}}^{\Delta}(\zeta)}}{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m}} \int_{\frac{\sigma(\zeta)}{2}}^{\sigma(\zeta)} \frac{u_{\mathrm{i}}(\eta) z_{\mathrm{i}}^{\Delta}(\eta)}{z_{\mathrm{i}}(\eta)} f_{\mathrm{i}}(\eta) \Delta \eta. \tag{47}$$

Using (11) with $\kappa = 3$, for $C_{\mathrm{i}} = \Gamma_{\mathrm{i}}^{\alpha_{\mathrm{i}}}(\zeta)$, we get

$$\sum_{\mathrm{i}=1}^{n-1} \Gamma_{\mathrm{i}}^{\alpha_{\mathrm{i}}}(\zeta) \Gamma_{\mathrm{i}+1}^{\alpha_{\mathrm{i}+1}}(\zeta) \Gamma_{\mathrm{i}+2}^{\alpha_{\mathrm{i}+2}}(\zeta) \leq \sum_{\mathrm{i}=1}^{n} \Gamma_{\mathrm{i}}^{3\alpha_{\mathrm{i}}}(\zeta). \tag{48}$$

Multiplying (48) by $\hat{w}^{\sigma}(\zeta)$ and integrating from 0 to $\varkappa$, we get

$$\sum_{\mathrm{i}=1}^{n-1} \int_0^{x} \hat{w}^{\sigma}(\zeta) \big[\Gamma_{\mathrm{i}}^{\alpha_{\mathrm{i}}}(\zeta) \Gamma_{\mathrm{i}+1}^{\alpha_{\mathrm{i}+1}}(\zeta) \Gamma_{\mathrm{i}+2}^{\alpha_{\mathrm{i}+2}}(\zeta)\big] \Delta \zeta \leq \sum_{\mathrm{i}=1}^{n} \int_0^{x} \hat{w}^{\sigma}(\zeta) \Gamma_{\mathrm{i}}^{3\alpha_{\mathrm{i}}}(\zeta) \Delta \zeta. \tag{49}$$

Now,

$$\begin{aligned} J &= \int_0^{x} \hat{w}^{\sigma}(\zeta) \Gamma_{\mathrm{i}}^{3\alpha_{\mathrm{i}}}(\zeta) \Delta \zeta \\ &= \int_0^{x} \hat{w}^{\sigma}(\zeta) \Bigg[\frac{\sqrt[3\alpha_{\mathrm{i}}]{u_{\mathrm{i}}^{\Delta}(\zeta)}}{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{m}} \int_{\frac{\sigma(\zeta)}{2}}^{\sigma(\zeta)} \frac{u_{\mathrm{i}}(\eta) z_{\mathrm{i}}^{\Delta}(\eta)}{z_{\mathrm{i}}(\eta)} f_{\mathrm{i}}(\eta) \Delta \eta \Bigg]^{3\alpha_{\mathrm{i}}} \Delta \zeta \\ &= \int_0^{x} \frac{u_{\mathrm{i}}^{\Delta}(\zeta)}{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{3\alpha_{\mathrm{i}} m}} \Bigg[\sqrt[3\alpha_{\mathrm{i}}]{\hat{w}^{\sigma}(\zeta)} \int_{\frac{\sigma(\zeta)}{2}}^{\sigma(\zeta)} \frac{u_{\mathrm{i}}(\eta) z_{\mathrm{i}}^{\Delta}(\eta)}{z_{\mathrm{i}}(\eta)} f_{\mathrm{i}}(\eta) \Delta \eta \Bigg]^{3\alpha_{\mathrm{i}}} \Delta \zeta. \end{aligned} \tag{50}$$

Integrating (50) by parts using formula (7) with

$$\Phi^{\Delta}(\zeta) = \frac{u_{\mathrm{i}}^{\Delta}(\zeta)}{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{3\alpha_{\mathrm{i}} m}}, \qquad \Psi^{\sigma}(\zeta) = \Bigg[\sqrt[3\alpha_{\mathrm{i}}]{\hat{w}^{\sigma}(\zeta)} \int_{\frac{\sigma(\zeta)}{2}}^{\sigma(\zeta)} \frac{u_{\mathrm{i}}(\varkappa) z_{\mathrm{i}}^{\Delta}(\varkappa)}{z_{\mathrm{i}}(\varkappa)} f_{\mathrm{i}}(\varkappa) \Delta \varkappa \Bigg]^{3\alpha_{\mathrm{i}}},$$

we obtain

$$\begin{aligned} J &= \big[\Phi(\zeta)\Psi(\zeta)\big]_0^{x} + \int_0^{x} \big(-\Phi(\zeta)\big)\big(\Psi(\zeta)\big)^{\Delta} \Delta \zeta \\ &= \Bigg[-\Psi(\zeta) \int_{\zeta}^{\infty} \frac{u_{\mathrm{i}}^{\Delta}(\eta)}{[u_{\mathrm{i}}^{\sigma}(\eta)]^{3\alpha_{\mathrm{i}} m}} \Delta \eta \Bigg]_0^{x} + \int_0^{x} \big(-\Phi(\zeta)\big)\big(\Psi(\zeta)\big)^{\Delta} \Delta \zeta \\ &= \Bigg[-\Psi(\varkappa) \int_{x}^{\infty} \frac{u_{\mathrm{i}}^{\Delta}(\eta)}{[u_{\mathrm{i}}^{\sigma}(\eta)]^{3\alpha_{\mathrm{i}} m}} \Delta \eta \Bigg] + \int_0^{x} \big(-\Phi(\zeta)\big)\big(\Psi(\zeta)\big)^{\Delta} \Delta \zeta \\ &\leq \int_0^{x} \big(-\Phi(\zeta)\big)\big(\Psi(\zeta)\big)^{\Delta} \Delta \zeta, \end{aligned} \tag{51}$$

where $\Phi(\zeta) = -\int_\zeta^\infty \frac{u_i^\Delta(\eta)}{[u_i^\sigma(\eta)]^{3\alpha_i m}}\Delta\eta$ and $(\Psi(\zeta))^\Delta > 0$. From (9), using $u_i^\Delta(\zeta) \geq 0$ and $c \in [\zeta, \sigma(\zeta)]$, we have

$$\begin{aligned}
\big[u_i^{1-3\alpha_i m}(\zeta)\big]^\Delta &= (1-3\alpha_i m)u_i^{-3\alpha_i m}(c)u_i^\Delta(\zeta) \\
&= (1-3\alpha_i m)\frac{u_i^\Delta(\zeta)}{u_i^{3\alpha_i m}(c)} \\
&\leq (1-3\alpha_i m)\frac{u_i^\Delta(\zeta)}{[u_i^\sigma(\zeta)]^{3\alpha_i m}}.
\end{aligned} \tag{52}$$

Therefore, integrating (52) from ζ to ∞ with respect to η, we have

$$-\Phi(\zeta) \leq \frac{1}{3\alpha_i m - 1}u_i^{1-3\alpha_i m}(\zeta). \tag{53}$$

Combining (53) and (51), we have

$$J \leq \frac{1}{3\alpha_i m - 1}\int_0^x u_i^{1-3\alpha_i m}(\zeta)\big(\Psi(\zeta)\big)^\Delta \Delta\zeta. \tag{54}$$

Now, by applying (6) to $\Psi(\zeta) = \hat{w}(\zeta)\hat{Y}^{3\alpha_i}(\zeta)$ and using (9), we obtain

$$\begin{aligned}
\big(\Psi(\zeta)\big)^\Delta &= \hat{w}^\Delta(\zeta)\big[\hat{Y}^\sigma(\zeta)\big]^{3\alpha_i} + \hat{w}(\zeta)\big[\hat{Y}^{3\alpha_i}(\zeta)\big]^\Delta \\
&= \hat{w}^\Delta(\zeta)\big[\hat{Y}^\sigma(\zeta)\big]^{3\alpha_i} + 3\alpha_i\hat{w}(\zeta)\hat{Y}^{(3\alpha_i-1)}(c)\hat{Y}^\Delta(\zeta) \\
&\leq \hat{w}^\Delta(\zeta)\big[\hat{Y}^\sigma(\zeta)\big]^{3\alpha_i} + 3\alpha_i\hat{w}^\sigma(\zeta)\big[\hat{Y}^\sigma(\zeta)\big]^{3\alpha_i-1}\hat{Y}^\Delta(\zeta).
\end{aligned} \tag{55}$$

From (55) and (54), as well as using (47), we have

$$\begin{aligned}
J &\leq \frac{1}{3\alpha_i m - 1}\int_0^x u_i^{1-3\alpha_i m}(\zeta)\hat{w}^\Delta(\zeta) \\
&\quad \times \left(\int_{\frac{\sigma(\zeta)}{2}}^{\sigma(\zeta)} \frac{u_i(\eta)z_i^\Delta(\eta)}{z_i(\eta)}f_i(\eta)\Delta\eta\right)^{3\alpha_i}\Delta\zeta \\
&\quad + \frac{3\alpha_i}{3\alpha_i m - 1}\int_0^x u_i^{1-3\alpha_i m}(\zeta)\hat{w}^\sigma(\zeta) \\
&\quad \times \left[\frac{u_i(\zeta)z_i^\Delta(\zeta)f_i(\zeta)}{z_i(\zeta)} - \frac{u_i(\frac{\zeta}{2})z_i^\Delta(\frac{\zeta}{2})f_i(\frac{\zeta}{2})}{2z_i(\frac{\zeta}{2})}\right] \\
&\quad \times \left(\int_{\frac{\sigma(\zeta)}{2}}^{\sigma(\zeta)} \frac{u_i(\eta)z_i^\Delta(\eta)}{z_i(\eta)}f_i(\eta)\Delta\eta\right)^{3\alpha_i-1}\Delta\zeta \\
&= \frac{1}{3\alpha_i m - 1}\int_0^x \frac{[u_i^\sigma(\zeta)]^{3\alpha_i m}\hat{w}^\Delta(\zeta)}{u_i^{3\alpha_i m-1}(\zeta)u_i^\Delta(\zeta)}\Gamma_i^{3\alpha_i}(\zeta)\Delta\zeta \\
&\quad + \frac{3\alpha_i}{3\alpha_i m - 1}\int_0^x \frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}\hat{w}^\sigma(\zeta)}{u_i^{3\alpha_i m-1}(\zeta)[u_i^\Delta(\zeta)]^{1-\frac{1}{3\alpha_i}}}\Gamma_i^{3\alpha_i-1}(\zeta) \\
&\quad \times \left[\frac{u_i(\zeta)z_i^\Delta(\zeta)f_i(\zeta)}{z_i(\zeta)} - \frac{u_i(\frac{\zeta}{2})z_i^\Delta(\frac{\zeta}{2})f_i(\frac{\zeta}{2})}{2z_i(\frac{\zeta}{2})}\right]\Delta\zeta.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^x \hat{w}^\sigma(\zeta)\Gamma_i^{3\alpha_i}(\zeta)\left(1-\frac{[u_i^\sigma(\zeta)]^{3\alpha_i m}\hat{w}^\Delta(\zeta)}{(3\alpha_i m-1)u_i^{3\alpha_i m-1}(\zeta)u_i^\Delta(\zeta)\hat{w}^\sigma(\zeta)}\right)\Delta\zeta \\
&\quad\le \frac{3\alpha_i}{3\alpha_i m-1}\int_0^x \frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}\hat{w}^\sigma(\zeta)}{u_i^{3\alpha_i m-1}(\zeta)[u_i^\Delta(\zeta)]^{1-\frac{1}{3\alpha_i}}}\Gamma_i^{3\alpha_i-1}(\zeta) \\
&\qquad\times\left[\frac{u_i(\zeta)z_i^\Delta(\zeta)}{z_i(\zeta)}f_i(\zeta)-\frac{u_i(\frac{\zeta}{2})z_i^\Delta(\frac{\zeta}{2})}{2z_i(\frac{\zeta}{2})}f_i\left(\frac{\zeta}{2}\right)\right]\Delta\zeta.
\end{aligned}\tag{56}$$

From (56) and (45), we have

$$\begin{aligned}
&\int_0^x \hat{w}^\sigma(\zeta)\Gamma_i^{3\alpha_i}(\zeta)\Delta\zeta \\
&\quad\le \frac{3\alpha_i\lambda_i}{3\alpha_i m-1}\int_0^x \hat{w}^\sigma(\zeta)\Gamma_i^{3\alpha_i-1}(\zeta) \\
&\qquad\times\left[\frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}z_i^\Delta(\zeta)}{u_i^{3\alpha_i m-2}(\zeta)\sqrt[3\delta_i]{u_i^\Delta(\zeta)}z_i(\zeta)}f_i(\zeta)-\frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}u_i(\frac{\zeta}{2})z_i^\Delta(\frac{\zeta}{2})}{2u_i^{3\alpha_i m-1}(\zeta)z_i(\frac{\zeta}{2})\sqrt[3\delta_i]{u_i^\Delta(\zeta)}}f_i\left(\frac{\zeta}{2}\right)\right]\Delta\zeta \\
&\quad= \frac{3\alpha_i\lambda_i}{3\alpha_i m-1}\int_0^x \left(\hat{w}^\sigma(\zeta)\Gamma_i^{3\alpha_i}(\zeta)\right)^{\frac{3\alpha_i-1}{3\alpha_i}}\left[\hat{w}^\sigma(\zeta)\right]^{\frac{1}{3\alpha_i}} \\
&\qquad\times\left[\frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}z_i^\Delta(\zeta)}{u_i^{3\alpha_i m-2}(\zeta)\sqrt[3\delta_i]{u_i^\Delta(\zeta)}z_i(\zeta)}f_i(\zeta)-\frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}u_i(\frac{\zeta}{2})z_i^\Delta(\frac{\zeta}{2})}{2u_i^{3\alpha_i m-1}(\zeta)z_i(\frac{\zeta}{2})\sqrt[3\delta_i]{u_i^\Delta(\zeta)}}f_i\left(\frac{\zeta}{2}\right)\right]\Delta\zeta.
\end{aligned}$$

Applying Hölder's inequality with $3\alpha_i$ and $\alpha_i/(3\alpha_i-1)$, we have

$$\begin{aligned}
&\int_0^x \hat{w}^\sigma(\zeta)\Gamma_i^{3\alpha_i}(\zeta)\Delta\zeta \\
&\quad\le \left(\frac{3\alpha_i\lambda_i}{3\alpha_i m-1}\right)^{3\alpha_i}\int_0^x \hat{w}^\sigma(\zeta) \\
&\qquad\times\left[\frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}z_i^\Delta(\zeta)f_i(\zeta)}{u_i^{3\alpha_i m-2}(\zeta)\sqrt[3\delta_i]{u_i^\Delta(\zeta)}z_i(\zeta)}-\frac{[u_i^\sigma(\zeta)]^{(3\alpha_i-1)m}u_i(\frac{\zeta}{2})z_i^\Delta(\frac{\zeta}{2})f_i(\frac{\zeta}{2})}{2u_i^{3\alpha_i m-1}(\zeta)z_i(\frac{\zeta}{2})\sqrt[3\delta_i]{u_i^\Delta(\zeta)}}\right]^{3\alpha_i}\Delta\zeta \\
&\quad= \left(\frac{3\alpha_i\lambda_i}{3\alpha_i m-1}\right)^{3\alpha_i}\int_0^x \hat{w}^\sigma(\zeta)g_i(\zeta)\Delta\zeta.
\end{aligned}\tag{57}$$

From (57) and (49), we get (46). □

In Theorem 16, if we take $\mathbb{T}=\mathbb{N}$, then we obtain the following corollary.

Corollary 17 *For any* $\{u(s)\}_{s=1}^\infty$, $\{\hat{w}(s)\}_{s=1}^\infty$, *and* $\{z(s)\}_{s=1}^\infty$ *increasing and nonnegative sequences*, $1\le i\le n$, *if* $\alpha_i>1$, $\delta_i=\alpha_i/(3\alpha_i-1)$, *and*

$$1+\frac{[u_i(s+1)]^{3\alpha_i m}\Delta\hat{w}(s)}{(1-3\alpha_i m)u_i^{3\alpha_i m-1}(s)\hat{w}(s+1)\Delta u(s)}\ge\frac{1}{\lambda_i}>0,\quad \textit{for } m>\frac{1}{3\alpha_i},$$

then

$$\sum_{i=1}^{n-1}\left(\sum_{s=1}^{q-1}\hat{w}(s+1)\left[\Gamma_i^{\alpha_i}(s)\Gamma_{i+1}^{\alpha_i+1}(s)\Gamma_{i+2}^{\alpha_{i+2}}(s)\right]\right)$$
$$\leq \sum_{i=1}^{n}\left(\frac{3\alpha_i\lambda_i}{3\alpha_i m-1}\right)^{3\alpha_i}\sum_{s=1}^{q-1}\hat{w}(s+1)g_i(s),$$

where

$$\Gamma_i(s)=\frac{\sqrt[3\alpha_i]{\Delta u_i(s)}}{u_i^m(s+1)}\sum_{r=\frac{s+1}{2}}^{s}\frac{u_i(r)\Delta z_i(r)}{z_i(r)}f_i(r),\qquad \frac{s+1}{2},\frac{s}{2}\in\mathbb{N},$$
$$g_i(s)=\left[\frac{[u_i(s+1)]^{(3\alpha_i-1)m}\Delta z_i(s)}{u_i^{3\alpha_i m-2}(s)\sqrt[3\delta_i]{\Delta u_i(s)}z_i(s)}f_i(s)\right.$$
$$\left.-\frac{[u_i(s+1]^{(3\alpha_i-1)m}u_i(\frac{s}{2})\Delta z_i(\frac{s}{2})}{2u_i^{3\alpha_i m-1}(s)z_i(\frac{s}{2})\sqrt[3\delta_i]{\Delta u_i(s)}}f_i\left(\frac{s}{2}\right)\right]^{3\alpha_i},$$

and $u_i(\infty)=\infty$.

Remark 18 Clearly, for $\mathbb{T}=\mathbb{R}$, Theorem 16 reduces to [4, Theorem 3].

Theorem 19 *For any* $1\leq i\leq n$, *if* $\alpha>1$, $\kappa\geq 1$, $\delta=\alpha/(\kappa\alpha-1)$, *and there exist* $\lambda_i>0$, $m>0$ *such that*

$$1+\frac{u_i^\sigma(\zeta)\hat{w}^\Delta(\zeta)}{(1+\kappa\alpha m)u_i^\Delta(\zeta)\hat{w}(\zeta)}\geq\frac{1}{\lambda_i}>0, \tag{58}$$

then

$$\int_a^b\hat{w}^\sigma(\zeta)\left(\sum_{i=1}^{n}\Gamma_{ia}(\zeta)\right)^{\kappa\alpha}\Delta\zeta\leq\sum_{i=1}^{n}\left(\frac{-\kappa\alpha\lambda_i\sqrt[\kappa\delta]{n}}{1+\kappa\alpha m}\right)^{\kappa\alpha}\int_a^b\hat{w}(\zeta)g_i(\zeta)\Delta\zeta, \tag{59}$$

where

$$\Gamma_i(\zeta)=\left[u_i^\sigma(\zeta)\right]^m\sqrt[\kappa\alpha]{u_i^\Delta(\zeta)}\int_0^\zeta\frac{z_i^\Delta(x)}{u_i^\sigma(x)z_i(x)}f_i(x)\Delta x,\qquad \zeta\in[0,\infty)_{\mathbb{T}},$$

and

$$g_i(\zeta)=\frac{[u_i^\sigma(\zeta)]^{\kappa\alpha m}[\hat{w}^\sigma(\zeta)]^{\kappa\alpha}[z_i^\Delta(\zeta)]^{\kappa\alpha}}{[u_i^\Delta(\zeta)]^{\frac{\alpha}{\delta}}\hat{w}^{\kappa\alpha}(\zeta)z_i^{\kappa\alpha}(\zeta)}f_i^{\kappa\alpha}(\zeta).$$

Proof Let us define for $1\leq i\leq n$ and $0<a<b<\infty$,

$$\Gamma_{ia}(\zeta)=\left[u_i^\sigma(\zeta)\right]^m\sqrt[\kappa\alpha]{u_i^\Delta(\zeta)}\int_a^\zeta\frac{z_i^\Delta(x)}{u_i^\sigma(x)z_i(x)}f_i(x)\Delta x. \tag{60}$$

Using (12), for $C_{\mathrm{i}} = \Gamma_{\mathrm{i}}(\zeta)$ and $\kappa \to \kappa\alpha$, we get

$$\left(\sum_{\mathrm{i}=1}^{n} \Gamma_{\mathrm{i}a}(\zeta)\right)^{\kappa\alpha} \leq n^{\alpha\kappa-1} \sum_{\mathrm{i}=1}^{n} \Gamma_{\mathrm{i}a}^{\kappa\alpha}(\zeta). \tag{61}$$

Multiplying (61) by $\hat{w}^{\sigma}(\zeta)$ and integrating from a to b, we get

$$\int_a^b \hat{w}(\zeta)\left(\sum_{\mathrm{i}=1}^{n} \Gamma_{\mathrm{i}a}(\zeta)\right)^{\kappa\alpha} \Delta\zeta \leq n^{\alpha\kappa-1} \sum_{\mathrm{i}=1}^{n} \int_a^b \hat{w}(\zeta)\Gamma_{\mathrm{i}a}^{\kappa\alpha}(\zeta)\Delta\zeta. \tag{62}$$

Now,

$$\begin{aligned} J &= \int_a^b \hat{w}(\zeta)\Gamma_{\mathrm{i}a}^{\kappa\alpha}(\zeta)\Delta\zeta \\ &= \int_a^b \hat{w}(\zeta)\left[\left[u_{\mathrm{i}}^{\sigma}(\zeta)\right]^{m} \sqrt[\kappa\alpha]{u_{\mathrm{i}}^{\Delta}(\zeta)} \int_a^{\zeta} \frac{z_{\mathrm{i}}^{\Delta}(x)}{u_{\mathrm{i}}^{\sigma}(x)z_{\mathrm{i}}(x)} f_{\mathrm{i}}(x)\Delta x\right]^{\kappa\alpha} \Delta\zeta \\ &= \int_a^b \left[u_{\mathrm{i}}^{\sigma}(\zeta)\right]^{\kappa\alpha m} u_{\mathrm{i}}^{\Delta}(\zeta)\left[\sqrt[\kappa\alpha]{\hat{w}(\zeta)} \int_a^{\zeta} \frac{z_{\mathrm{i}}^{\Delta}(x)}{u_{\mathrm{i}}^{\sigma}(x)z_{\mathrm{i}}(x)} f_{\mathrm{i}}(x)\Delta x\right]^{\kappa\alpha} \Delta\zeta. \end{aligned} \tag{63}$$

Integrating (63) by parts using formula (7) with

$$\Phi^{\Delta}(\zeta) = \left[u_{\mathrm{i}}^{\sigma}(\zeta)\right]^{\kappa\alpha m} u_{\mathrm{i}}^{\Delta}(\zeta), \qquad \Psi(\zeta) = \left[\sqrt[\kappa\alpha]{\hat{w}(\zeta)} \int_a^{\zeta} \frac{z_{\mathrm{i}}^{\Delta}(x)}{u_{\mathrm{i}}^{\sigma}(x)z_{\mathrm{i}}(x)} f_{\mathrm{i}}(x)\Delta x\right]^{\kappa\alpha},$$

we obtain

$$\begin{aligned} J &= \left[\Phi(\zeta)\Psi(\zeta)\right]_a^b + \int_a^b \left(\Phi^{\sigma}(\zeta)\right)\left(-\Psi(\zeta)\right)^{\Delta}\Delta\zeta \\ &= \left[-\Psi(\zeta)\int_{\zeta}^{b}\left[u_{\mathrm{i}}^{\sigma}(x)\right]^{\kappa\alpha m} u_{\mathrm{i}}^{\Delta}(x)\Delta x\right]_a^b + \int_a^b \left(\Phi^{\sigma}(\zeta)\right)\left(-\Psi(\zeta)\right)^{\Delta}\Delta\zeta \\ &= \int_a^b \left(\Phi^{\sigma}(\zeta)\right)\left(-\Psi(\zeta)\right)^{\Delta}\Delta\zeta, \end{aligned} \tag{64}$$

where $\Phi(\zeta) = -\int_{\zeta}^{b}[u_{\mathrm{i}}^{\sigma}(\varkappa)]^{\kappa\alpha m} u_{\mathrm{i}}^{\Delta}(\varkappa)\Delta\varkappa$ and $(\Psi(\zeta))^{\Delta} > 0$. From (9), using $u_{\mathrm{i}}^{\Delta}(\zeta) \geq 0$ and $c \in [\zeta, \sigma(\zeta)]$, we have

$$\begin{aligned} \left[u_{\mathrm{i}}^{1+\kappa\alpha m}(\zeta)\right]^{\Delta} &= (1+\kappa\alpha m)u_{\mathrm{i}}^{\kappa\alpha m}(c)u_{\mathrm{i}}^{\Delta}(\zeta) \\ &\leq (1+\kappa\alpha m)\left[u_{\mathrm{i}}^{\sigma}(\zeta)\right]^{\kappa\alpha m} u_{\mathrm{i}}^{\Delta}(\zeta). \end{aligned}$$

This implies that

$$\frac{1}{1+\kappa\alpha m}\int_{\zeta}^{b}\left[u_{\mathrm{i}}^{1+\kappa\alpha m}(x)\right]^{\Delta}\Delta x \leq \int_{\zeta}^{b}\left[u_{\mathrm{i}}^{\sigma}(x)\right]^{\kappa\alpha m} u_{\mathrm{i}}^{\Delta}(x)\Delta x = -\Phi(\zeta).$$

Therefore

$$\begin{aligned}\Phi(\zeta) &\leq \frac{-1}{1+\kappa\alpha m}\int_{\zeta}^{b}\left[u_{\mathrm{i}}^{1+\kappa\alpha m}(x)\right]^{\Delta}\Delta x\\ &= \frac{-1}{1+\kappa\alpha m}\left[u_{\mathrm{i}}^{1+\kappa\alpha m}(b)-u_{\mathrm{i}}^{1+\kappa\alpha m}(\zeta)\right]\\ &= \frac{1}{1+\kappa\alpha m}\left[u_{\mathrm{i}}^{1+\kappa\alpha m}(\zeta)-u_{\mathrm{i}}^{1+\kappa\alpha m}(b)\right]\\ &\leq \frac{1}{1+\kappa\alpha m}u_{\mathrm{i}}^{1+\kappa\alpha m}(\zeta).\end{aligned} \tag{65}$$

Substituting (65) into (64), we have

$$J \leq \frac{1}{1+\kappa\alpha m}\int_{a}^{b}\left[u_{\mathrm{i}}^{\sigma}(\zeta)\right]^{1+\kappa\alpha m}\left(-\Psi(\zeta)\right)^{\Delta}\Delta\zeta. \tag{66}$$

Now, by applying (6) to $\Psi(\zeta)=\hat{w}(\zeta)\hat{Y}^{\kappa\alpha}(\zeta)$ and using (9), we obtain

$$\begin{aligned}\left(\Psi(\zeta)\right)^{\Delta} &= \hat{w}^{\Delta}(\zeta)\hat{Y}^{\kappa\alpha}(\zeta)+\hat{w}^{\sigma}(\zeta)\left[\hat{Y}^{\kappa\alpha}(\zeta)\right]^{\Delta}\\ &= \hat{w}^{\Delta}(\zeta)\hat{Y}^{\kappa\alpha}(\zeta)+\kappa\alpha\hat{w}^{\sigma}(\zeta)\hat{Y}^{\kappa\alpha-1}(c)\hat{Y}^{\Delta}(\zeta)\\ &\geq \hat{w}^{\Delta}(\zeta)\hat{Y}^{\kappa\alpha}(\zeta)+\kappa\alpha\hat{w}^{\sigma}(\zeta)\left[\frac{z_{\mathrm{i}}^{\Delta}(\zeta)}{u_{\mathrm{i}}^{\sigma}(\zeta)z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta)\right]\hat{Y}^{\kappa\alpha-1}(\zeta).\end{aligned} \tag{67}$$

From (67) and (66), as well as using (60), we have

$$\begin{aligned}J &\leq \frac{-1}{1+\kappa\alpha m}\int_{a}^{b}\left[u_{\mathrm{i}}^{\sigma}(\zeta)\right]^{1+\kappa\alpha m}\hat{w}^{\Delta}(\zeta)\left(\int_{a}^{\zeta}\frac{z_{\mathrm{i}}^{\Delta}(x)}{u_{\mathrm{i}}^{\sigma}(x)z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\kappa\alpha}\Delta\zeta\\ &\quad+\frac{-\kappa\alpha}{1+\kappa\alpha m}\int_{a}^{b}\left[u_{\mathrm{i}}^{\sigma}(\zeta)\right]^{1+\kappa\alpha m}\hat{w}^{\sigma}(\zeta)\left(\frac{z_{\mathrm{i}}^{\Delta}(\zeta)}{u_{\mathrm{i}}^{\sigma}(\zeta)z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta)\right)\\ &\quad\times\left(\int_{a}^{\zeta}\frac{z_{\mathrm{i}}^{\Delta}(x)}{u_{\mathrm{i}}^{\sigma}(x)z_{\mathrm{i}}(x)}f_{\mathrm{i}}(x)\Delta x\right)^{\kappa\alpha-1}\Delta\zeta\\ &= \frac{-1}{1+\kappa\alpha m}\int_{a}^{b}\frac{u_{\mathrm{i}}^{\sigma}(\zeta)\hat{w}^{\Delta}(\zeta)}{u_{\mathrm{i}}^{\Delta}(\zeta)}\Gamma_{\mathrm{i}a}^{\kappa\alpha}(\zeta)\Delta\zeta\\ &\quad+\frac{-\kappa\alpha}{1+\kappa\alpha m}\int_{a}^{b}\frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{1+m}\hat{w}^{\sigma}(\zeta)}{[u_{\mathrm{i}}^{\Delta}(\zeta)]^{1-\frac{1}{\kappa\alpha}}}\Gamma_{\mathrm{i}a}^{\kappa\alpha-1}(\zeta)\left(\frac{z_{\mathrm{i}}^{\Delta}(\zeta)}{u_{\mathrm{i}}^{\sigma}(\zeta)z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta)\right)\Delta\zeta.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{a}^{b}\hat{w}(\zeta)\Gamma_{\mathrm{i}}^{\kappa\alpha}(\zeta)\left(1+\frac{u_{\mathrm{i}}^{\sigma}(\zeta)\hat{w}^{\Delta}(\zeta)}{(1+\kappa\alpha m)u_{\mathrm{i}}^{\Delta}(\zeta)\hat{w}(\zeta)}\right)\Delta\zeta\\ &\quad\leq \frac{-\kappa\alpha}{1+\kappa\alpha m}\int_{a}^{b}\frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{1+m}\hat{w}^{\sigma}(\zeta)}{[u_{\mathrm{i}}^{\Delta}(\zeta)]^{1-\frac{1}{\kappa\alpha}}}\Gamma_{\mathrm{i}a}^{\kappa\alpha-1}(\zeta)\left[\frac{z_{\mathrm{i}}^{\Delta}(\zeta)}{u_{\mathrm{i}}^{\sigma}(\zeta)z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta)\right]\Delta\zeta.\end{aligned} \tag{68}$$

From (68) and (58), we have

$$\begin{aligned}&\int_a^b \hat{w}(\zeta)\Gamma_{\mathrm{i}a}^{\kappa\alpha}(\zeta)\Delta\zeta\\&\quad\leq \frac{-\kappa\alpha\lambda_{\mathrm{i}}}{1+\kappa\alpha m}\int_a^b \big(\hat{w}(\zeta)\Gamma_{\mathrm{i}a}^{\kappa\alpha}(\zeta)\big)^{\frac{\kappa\alpha-1}{\kappa\alpha}}\frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^m\hat{w}^{\sigma}(\zeta)z_{\mathrm{i}}^{\Delta}(\zeta)}{[u_{\mathrm{i}}^{\Delta}(\zeta)]^{1-\frac{1}{\kappa\alpha}}\hat{w}^{\frac{\kappa\alpha-1}{\kappa\alpha}}(\zeta)z_{\mathrm{i}}(\zeta)}f_{\mathrm{i}}(\zeta)\Delta\zeta.\end{aligned}$$

Applying Hölder's inequality with $\kappa\alpha$ and $\alpha/(\kappa\alpha-1)$, we have

$$\begin{aligned}&\int_a^b \hat{w}(\zeta)\Gamma_{\mathrm{i}a}^{\kappa\alpha}(\zeta)\Delta\zeta\\&\quad\leq\left(\frac{-\kappa\alpha\lambda_{\mathrm{i}}}{1+\kappa\alpha m}\right)^{\kappa\alpha}\int_a^b \frac{[u_{\mathrm{i}}^{\sigma}(\zeta)]^{\kappa\alpha m}[\hat{w}^{\sigma}(\zeta)]^{\kappa\alpha}[z_{\mathrm{i}}^{\Delta}(\zeta)]^{\kappa\alpha}}{[u_{\mathrm{i}}^{\Delta}(\zeta)]^{\kappa\alpha-1}\hat{w}^{\kappa\alpha-1}(\zeta)z_{\mathrm{i}}^{\kappa\alpha}(\zeta)}\Gamma_{\mathrm{i}}^{\kappa\alpha}(\zeta)\Delta\zeta\\&\quad=\left(\frac{-\kappa\alpha\lambda_{\mathrm{i}}}{1+\kappa\alpha m}\right)^{\kappa\alpha}\int_a^b \hat{w}(\zeta)g_{\mathrm{i}}(\zeta)\Delta\zeta.\end{aligned}\tag{69}$$

From (69) and (62), we get (59). □

In Theorem 19, if we take $\mathbb{T}=\mathbb{N}$, then we obtain the following corollary.

Corollary 20 *For any* $\{u(s)\}_{s=1}^{\infty}$, $\{\hat{w}(s)\}_{s=1}^{\infty}$, *and* $\{z(s)\}_{s=1}^{\infty}$ *increasing and nonnegative sequences, if* $1\leq \mathrm{i}\leq n$, $\alpha>1$, $\kappa\geq 1$, $\delta=\alpha/(\kappa\alpha-1)$, *and*

$$1+\frac{u_{\mathrm{i}}(s+1)\Delta\hat{w}(s)}{(1+\kappa\alpha m)\hat{w}(s)\Delta u_{\mathrm{i}}(s)}\geq\frac{1}{\lambda_{\mathrm{i}}}>0,$$

then

$$\sum_{s=1}^{r-1}\hat{w}(s+1)\left(\sum_{\mathrm{i}=1}^{n}\Gamma_{\mathrm{i}a}(s)\right)^{\kappa\alpha}\leq\sum_{\mathrm{i}=1}^{n}\left(\frac{-\kappa\alpha\lambda_{\mathrm{i}}\sqrt[\kappa\delta]{n}}{1+\kappa\alpha m}\right)^{\kappa\alpha}\sum_{s=1}^{r-1}\hat{w}(s)g_{\mathrm{i}}(s),$$

where

$$\Gamma_{\mathrm{i}}(s)=\big[u_{\mathrm{i}}(s+1)\big]^m\sum_{q=1}^{s-1}\frac{\Delta z_{\mathrm{i}}(q)}{u_{\mathrm{i}}(q+1)z_{\mathrm{i}}(q)}f_{\mathrm{i}}(q)$$

and

$$g_{\mathrm{i}}(s)=\frac{[u_{\mathrm{i}}(s+1)]^{\kappa\alpha m}[\hat{w}(s+1)]^{\kappa\alpha}[\Delta z_{\mathrm{i}}(s)]^{\kappa\alpha}}{[\Delta u_{\mathrm{i}}(s)]^{\frac{\alpha}{\delta}}\hat{w}^{\kappa\alpha}(s)z_{\mathrm{i}}^{\kappa\alpha}(s)}f_{\mathrm{i}}^{\kappa\alpha}(s).$$

Remark 21 Clearly, for $\mathbb{T}=\mathbb{R}$, Theorem 16 reduces to [4, Theorem 4].

The next corollary follows from Theorem 19 by taking $u_{\mathrm{i}}(\zeta)=z_{\mathrm{i}}(\zeta)=\zeta, f_{\mathrm{i}}(\zeta)\to\zeta^{1-m}h_{\mathrm{i}}$.

Corollary 22 *For any* $1\leq\mathrm{i}\leq n$ *and* $\alpha>1$, *if* h_{i} *are rd-continuous functions and*

$$1+\frac{\sigma(\zeta)\hat{w}^{\Delta}(\zeta)}{(\kappa\alpha m+1)\hat{w}(\zeta)}\geq\frac{1}{\lambda_{\mathrm{i}}}>0,$$

then

$$\begin{aligned} &\int_a^b \hat{w}(\zeta)\left[\sum_{\mathrm{i}=1}^n \sigma^m(\zeta)\int_0^\zeta \frac{x^{2-m}}{\sigma(x)}h_{\mathrm{i}}(x)\Delta x\right]^{\kappa\alpha}\Delta\zeta \\ &\quad\le n^{\alpha\kappa-1}\sum_{\mathrm{i}=1}^n\left(\frac{-\kappa\alpha\lambda_{\mathrm{i}}}{1+\kappa\alpha m}\right)^{\kappa\alpha}\int_a^b \hat{w}(\zeta)g_{\mathrm{i}}(\zeta)\Delta\zeta, \end{aligned} \tag{70}$$

where

$$g_{\mathrm{i}}(\zeta) = \frac{[\sigma(\zeta)]^{\kappa\alpha m}[\hat{w}^\sigma(\zeta)]^{\kappa\alpha}\zeta^{\kappa\alpha-\kappa\alpha m}}{\hat{w}^{\kappa\alpha}(\zeta)\zeta^{\kappa\alpha}}h_{\mathrm{i}}^{\kappa\alpha}(\zeta).$$

Remark 23 Letting $\mathbb{T}=\mathbb{R}$ in Corollary 22, we have that $\sigma(\zeta)=\zeta$ and

$$1+\frac{\zeta\hat{w}'(\zeta)}{(\kappa\alpha m+1)\hat{w}(\zeta)} \ge \frac{1}{\lambda_{\mathrm{i}}} > 0.$$

Then

$$\begin{aligned} &\int_a^b \hat{w}(\zeta)\left[\sum_{\mathrm{i}=1}^n \zeta^m\int_0^\zeta \frac{1}{\zeta^{1+m}}h_{\mathrm{i}}(x)\,dx\right]^{\kappa\alpha}d\zeta \\ &\quad\le n^{\alpha\kappa-1}\sum_{\mathrm{i}=1}^n\left(\frac{-\kappa\alpha\lambda_{\mathrm{i}}}{1+\kappa\alpha m}\right)^{\kappa\alpha}\int_a^b \hat{w}(\zeta)h_{\mathrm{i}}^{\kappa\alpha}(\zeta)\,d\zeta, \end{aligned}$$

which agrees with [4, Corollary 3].

4 Conclusion

In this work, we explored some new generalized inequalities involving many functions of Hardy type on time scales by using delta calculus. Further, we also applied our inequalities to discrete and continuous calculus to obtain some new Hardy inequalities as special cases. In a future work, we will continue to generalize more dynamic inequalities by conformable delta fractional calculus on time scales.

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Author details
[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, 11884, Cairo, Egypt. [2]Department of Mathematics, Cankaya University, Ankara, 06530, Turkey. [3]Institute of Space Science, Magurele-Bucharest, Romania.

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