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Global error bounds of the extended vertical linear complementarity problems for Dashnic–Zusmanovich matrices and Dashnic–Zusmanovich-B matrices

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Abstract

Global error bounds of the extended vertical linear complementarity problems for Dashnic–Zusmanovich (*DZ*) matrices and Dashnic–Zusmanovich-B (*DZ*-*B*) matrices are presented, respectively. The obtained error bounds are sharper than those of Zhang et al. (Comput. Optim. Appl. 42(3):335–352, 2009) in some cases. Some numerical examples are given to illustrate the obtained results.

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1 Introduction

The extended vertical linear complementarity problem (EVLCP) is to find a vector $x \in R^n$ such that

$$r(x) := \min(M_0x + q_0, M_1x + q_1, \dots, M_kx + q_k),$$

or to prove that there is no such vector x, where the min operator works componentwise for both vectors and matrices. It is denoted by EVLCP(M, q), where

$$q = (q_0, q_1, \dots, q_k), \qquad q_l \in R^n, \quad l = 0, 1, \dots, k,$$

is a block vector and

$$M = (M_0, M_1, \dots, M_k), \qquad M_l \in R^{n\times n}, \quad l = 0, 1, \dots, k,$$

is a block matrix. When $k = 1$, $M_0 = I$, $q_0 = 0$, the EVLCP(M, q) comes back to linear complementarity problems (LCP), and when $M_0 = I$, $q_0 = 0$, the EVLCP(M, q) reduces to vertical linear complementarity problems (VLCP) [2]. The extended vertical linear complementarity problems are widely used in optimization theory, control theory, neural network model, convergence analysis, sensitive analysis, verification of the solutions, and so on, see [1, 3–7].

Many scholars are interested in the research on the error of the solution for the EVLCP(M,q) including the LCP case. Various results on the solution and its error bounds for the EVLCP(M,q) have appeared recently, see [8–13]. For example, Gowda and Sznajder [14] extended the sufficient and necessary condition for the existence and uniqueness of the solution from the LCP to the EVLCP. Afterwards, Sznajder and Gowda [15] provided some equivalent forms for the condition above. Xiu and Zhang [16] extended the error bound for the LCP given by [6] to the EVLCP. Zhang et al. [1] extended the error bound of the LCP given by Chen and Xiang [8] to the general EVLCP by the row rearrangement technique and provided some computable error bounds for two types of special block matrices. However, these error bounds generally can not be calculated accurately because they involve computing the inverse of matrices. In order to overcome this shortcoming, in this paper, we continue to explore the extended vertical linear complementarity problems, and we propose new error bounds for the other types of special block matrices, named *DZ* matrices [17, 18] and *DZ-B* matrices [19], only relying on the elements of such matrices. The obtained results extend the corresponding results in [1]. The validity of new error bounds is theoretically guaranteed, and numerical examples show the validity of the new results.

The remainder of this paper is organized as follows. In Sect. 2, we recall some related definitions, theorems, lemmas, and notations, which will be used in the proof of this paper. In Sect. 3, we prove that each block in any row rearrangement of the block matrix $M = (M_0, M_1, \dots, M_k)$ is a *DZ* matrix if each matrix M_l $(l = 0, 1, \dots, k)$ is a *DZ* matrix, propose a computable error bound of the EVLCP(M,q) with each matrix in M being a *DZ* matrix, and present numerical examples to show the effectiveness of the new error bound. In Sect. 4, we prove that each block in any row rearrangement of the block matrix $M = (M_0, M_1, \dots, M_k)$ is a *DZ-B* matrix if each matrix M_l $(l = 0, 1, \dots, k)$ is a *DZ-B* matrix, provide a calculable error bound of the EVLCP(M,q) with each matrix in M being a *DZ-B* matrix, and use numerical examples to indicate the validity of the new error bound. Some conclusions are summarized in Sect. 5.

2 Preliminaries

In this section, we recall some theorems, definitions, lemmas, and notations. Given a matrix $A = (a_{ij}) \in R^{n\times n}$, $i, j \in N = \{1, 2, \dots, n\}$, denote

$$r_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad \forall i \in N.$$

The first one is the existence and uniqueness condition of the solution for the EVLCP(M,q) given by Gowda and Sznajder [14].

Theorem 1 ([14]) *For any block vector q, the* EVLCP(M,q) *has a unique solution if and only if the block matrix $M = (M_0, M_1, \dots, M_k)$ has the W-property, i.e.,*

$$\min(M_0x, M_1x, \dots, M_kx) \le 0 \le \max(M_0x, M_1x, \dots, M_kx) \quad \Rightarrow \quad x = 0, \tag{1}$$

where the min and max operators work componentwise for both vectors and matrices.

Using the row rearrangement technique, Zhang et al. [1] presented a sufficient and necessary condition for the block matrix M with the row W-property and proposed a global error bound for the $EVLCP(M,q)$.

Definition 1 ([1]) The block matrix $M' = (M'_0, M'_1, \dots, M'_k)$ is called a row rearrangement of $M = (M_0, M_1, \dots, M_k)$ if, for any $i \in N$,

$$\left(M_j^{'}\right)_{i.} = (M_{j_i})_{i.} \in \left\{(M_0)_{i.}, (M_1)_{i.}, \dots, (M_k)_{i.}\right\} = \left\{\left(M_0^{'}\right)_{i.}, \left(M_1^{'}\right)_{i.}, \dots, \left(M_k^{'}\right)_{i.}\right\}, \tag{2}$$

where $A_{i.}$ means the ith row of a given matrix A. This is also true for the block vectors q and $q^{'}$. Denote by $R(M)$ and $R(q)$ the set of all row rearrangements of M and q, respectively.

Lemma 1 ([1]) *The block matrix $M = (M_0, M_1, \dots, M_k)$ has the row W-property if and only if $(I-D)M_j^{'} + DM_l^{'}$ is nonsingular for any two blocks $M_j^{'}$ and $M_l^{'}$ of $M^{'} \in R(M)$ and for any $D = \operatorname{diag}(d_i)$ with $d_i \in [0,1]$ $(i \in N)$.*

Theorem 2 ([1]) *Let x^* be the solution of the* EVLCP(M,q). *If the block matrix $M = (M_0, M_1, \dots, M_k)$ has the row W-property, then for any $x \in R^n$,*

$$\left\|x - x^*\right\| \le \alpha(M) \cdot \left\|r(x)\right\|, \tag{3}$$

where

$$\alpha(M) := \max_{M^{'} \in R(M)} \max_{j<l \in \{0,1,\dots,k\}} \max_{d \in [0,1]^n} \left\|\left[(I-D)M_j^{'} + DM_l^{'}\right]^{-1}\right\|,$$

$D = \operatorname{diag}(d_i)$ is defined as Lemma 1, and $M_j^{'}$, $M_l^{'}$ are any two blocks in $M^{'} \in R(M)$.

Obviously, the upper bound in (3) for $\|x - x^*\|$ cannot be calculated easily because it is difficult to compute $\|[(I-D)M_j^{'} + DM_l^{'}]^{-1}\|$ precisely in general. Hence, some computable upper bounds for $\alpha(M)$ are provided under various matrix norms by using the structural properties of matrices M_j, $j \in \{1,2,\dots,k\}$. When all M_j are strictly diagonally dominant (SDD) matrices, Zhang et al. [1] gave a calculable upper bound for $\alpha(M)$ under the infinity norm (denoted by $\alpha_\infty(M)$) as follows. Here a matrix $A = (a_{ij}) \in C^{n\times n}$ is called an SDD matrix if $|a_{ii}| > \sum_{j\neq i}^n |a_{ij}|$ for each $i \in N$.

Theorem 3 ([1]) *If $M_0, M_1, \dots, M_k$ are SDD, and for each $i \in N$, $(M_j)_{ii}(M_l)_{ii} > 0$ for any $j < l \in \{0,1,\dots,k\}$, then $M = (M_0, M_1, \dots, M_k)$ has the row W-property and*

$$\alpha_\infty(M) \le \frac{1}{\min_{i\in N}\{(\min(\tilde{M}_0 e, \tilde{M}_1 e, \dots, \tilde{M}_k e))_i\}},$$

where $\tilde{M}_i$ is the comparison matrix of M_i, i.e., $(\tilde{M}_i)_{\tau\tau} = |(M_i)_{\tau\tau}|$, $(\tilde{M}_i)_{\tau j} = -|(M_i)_{\tau j}|$ for $\tau \neq j$, $(M_i)_{\tau j}$ is the element in the τth row and the jth column of M_i, and $(\tilde{M}_i)_{\tau j}$ is the element in the τth row and the jth column of $\tilde{M}_i$.

Theorem 4 ([1]) *If $M_0, M_1, \dots, M_k$ are matrices with positive diagonal entries, and the spectral radius $\rho(\max(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \dots, \Lambda_k^{-1}|B_k|)) < 1$, then $M = (M_0, M_1, \dots, M_k)$ has* *the row W-property and*

$$\alpha_\infty(M) \le \left\| \left[I - \max_{i=0,1,\dots,k} \left(\Lambda_i^{-1} |B_i| \right) \right]^{-1} \max_{i=0,1,\dots,k} \left(\Lambda_i^{-1} \right) \right\|_\infty,$$

where Λ_i *is the diagonal part of* M_i, $B_i = \Lambda_i - M_i$ *for* $i = 0, 1, \dots, k$.

3 A global error bound for the EVLCP of Dashnic–Zusmanovich matrices

Dashnic–Zusmanovich matrix, as a subclass of the class of nonsingular P-matrix, was introduced by Dashnic and Zusmanovich [17] to upper bound for the infinity norm of its inverse matrix, whose definition and related conclusion are listed as follows.

Definition 2 ([17]) A matrix $A = (a_{ij}) \in C^{n\times n}$ is called a Dashnic–Zusmanovich (DZ) matrix if there exists an index $i \in N$ such that for any $j \in N$, $j \ne i$,

$$|a_{ii}| \left(|a_{jj}| - r_j(A) + |a_{ji}| \right) > r_i(A)|a_{ji}|.$$

Theorem 5 ([18]) *Let* $A = (a_{ij}) \in C^{n\times n}$ *be a DZ matrix. Then*

$$\left\| A^{-1} \right\|_\infty \le \max \left\{ \max_{j\in N, j\ne i} \alpha_1(A), \max_{j\in N, j\ne i} \alpha_2(A) \right\},$$

where

$$\alpha_1(A) = \frac{|a_{ji}| + |a_{ii}|}{(|a_{jj}| - r_j(A) + |a_{ji}|)|a_{ii}| - |a_{ji}|r_i(A)},$$
$$\alpha_2(A) = \frac{|a_{jj}| - r_j(A) + |a_{ji}| + r_i(A)}{(|a_{jj}| - r_j(A) + |a_{ji}|)|a_{ii}| - |a_{ji}|r_i(A)}.$$

Next, we will propose an upper bound for $\alpha_\infty(M)$ with each M_l ($l = 0, 1, \dots, k$) being a DZ matrix. Before that, some useful propositions are provided below.

Proposition 1 *Let* $A = (a_{ij}) \in C^{n\times n}$ *and* $B = (b_{ij}) \in C^{n\times n}$ *be all DZ matrices with positive diagonal elements. If there exists an index* $i \in N$ *such that, for any* $j \in N$, $j \ne i$, $a_{ij}b_{ij} \ge 0$ (*or* $a_{ji}b_{ji} \ge 0$), *and*

$$|a_{ii}| \left(|a_{jj}| - r_j(A) + |a_{ji}| \right) > r_i(A)|a_{ji}|, \qquad |b_{ii}| \left(|b_{jj}| - r_j(B) + |b_{ji}| \right) > r_i(B)|b_{ji}|,$$
$$|a_{ii}| \left(|b_{jj}| - r_j(B) + |b_{ji}| \right) > r_i(A)|b_{ji}|, \qquad |b_{ii}| \left(|a_{jj}| - r_j(A) + |a_{ji}| \right) > r_i(B)|a_{ji}|,$$

then $(I - D)A + DB$ *is also a DZ matrix for any* $D = \operatorname{diag}(d_t)$ *with* $d_t \in [0, 1]$ ($t \in N$).

Proof Both A and B are DZ matrices, and there exists $i \in N$ such that, for any $j \in N$, $j \ne i$,

$$|a_{ii}| \left(|a_{jj}| - r_j(A) + |a_{ji}| \right) > r_i(A)|a_{ji}|, \qquad |b_{ii}| \left(|b_{jj}| - r_j(B) + |b_{ji}| \right) > r_i(B)|b_{ji}|.$$

Note that $d_i \in [0, 1]$, then $1 - d_i \ge 0$ and $d_i \ge 0$, and they are not equal to 0 at the same time. Let $(I - D)A + DB = C = (c_{ij})$, then

$$|c_{ii}| = (1 - d_i)|a_{ii}| + d_i|b_{ii}|, \qquad |c_{ij}| = (1 - d_i)|a_{ij}| + d_i|b_{ij}|,$$

$$r_i(C) = \sum_{j\neq i}^{n} \bigl((1-d_i)|a_{ij}| + d_i|b_{ij}|\bigr) = (1-d_i)r_i(A) + d_i r_i(B).$$

Hence, we get

$$\begin{aligned}
&|c_{jj}| - r_j(C) + |c_{ji}| \\
&\quad = (1-d_j)|a_{jj}| + d_j|b_{jj}| - \bigl((1-d_j)r_j(A) + d_j r_j(B)\bigr) + (1-d_j)|a_{ji}| + d_j|b_{ji}| \\
&\quad = (1-d_j)\bigl[|a_{jj}| - r_j(A)|a_{ji}|\bigr] + d_j\bigl[|b_{jj}| - r_j(B) + |b_{ji}|\bigr],
\end{aligned}$$

and

$$\begin{aligned}
&|c_{ii}|\bigl(|c_{jj}| - r_j(C) + |c_{ji}|\bigr) \\
&\quad = \bigl[(1-d_i)|a_{ii}| + d_i|b_{ii}|\bigr] \times \bigl[(1-d_j)\bigl(|a_{jj}| - r_j(A)|a_{ji}|\bigr) + d_j\bigl(|b_{jj}| - r_j(B) + |b_{ji}|\bigr)\bigr] \\
&\quad = (1-d_i)(1-d_j)\bigl(|a_{jj}| - r_j(A)|a_{ji}|\bigr)|a_{ii}| + (1-d_i)d_j\bigl(|b_{jj}| - r_j(B) + |b_{ji}|\bigr)|a_{ii}| \\
&\qquad + d_i(1-d_j)\bigl(|a_{jj}| - r_j(A)|a_{ji}|\bigr)|b_{ii}| + d_i d_j\bigl(|b_{jj}| - r_j(B) + |b_{ji}|\bigr)|b_{ii}| \\
&\quad > (1-d_i)(1-d_j)r_i(A)|a_{ji}| + (1-d_i)d_j r_i(A)|b_{ji}| \\
&\qquad + d_i(1-d_j)r_i(B)|a_{ji}| + d_i d_j r_i(B)|b_{ji}| \\
&\quad = \bigl[(1-d_i)r_i(A) + d_i r_i(B)\bigr]\bigl[(1-d_j)|a_{ji}| + d_j|b_{ji}|\bigr] \\
&\quad = r_i(C)|c_{ji}|,
\end{aligned}$$

that is,

$$|c_{ii}|\bigl(|c_{jj}| - r_j(C) + |c_{ji}|\bigr) > r_i(C)|c_{ji}|.$$

From Definition 2, the conclusion follows. □

Based on Proposition 1, Lemma 1, and the fact that a DZ matrix is nonsingular, the block matrix composed of DZ matrices has the row W-property.

Proposition 2 *If $M_0, M_1, \dots, M_k$ are DZ matrices and each M_{l_i} ($l_i = 0, 1, \dots, k$) satisfies the hypotheses of Proposition 1, then each $M_l^{'}$ in $M^{'} \in R(M)$ is a DZ matrix, and consequently, $M = (M_0, M_1, \dots, M_k)$ has the row W-property.*

Proof By Definition 1, for the ith row $(M_l^{'})_i$ of $M_l^{'}$, $i \in N$, there exists $l_i \in \{0, 1, \dots, k\}$ such that $(M_l^{'})_i = (M_{l_i})_i$, i.e., $(M_{l_i})_{ii} = (M_l^{'})_{ii}$, $r_i(M_{l_i}) = r_i(M_l^{'})$. Since M_{l_i} is a DZ matrix, then

$$\bigl|(M_{l_i})_{ii}\bigr|\bigl(\bigl|(M_{l_i})_{jj}\bigr| - r_j(M_{l_i}) + \bigl|(M_{l_i})_{ji}\bigr|\bigr) > r_i(M_{l_i})\bigl|(M_{l_i})_{ji}\bigr|, \quad j \neq i, j \in N,$$

that is,

$$\bigl|(M_l^{'})_{ii}\bigr|\bigl(\bigl|(M_l^{'})_{jj}\bigr| - r_j(M_l^{'}) + \bigl|(M_l^{'})_{ji}\bigr|\bigr) > r_i(M_l^{'})\bigl|(M_l^{'})_{ji}\bigr|, \quad j \neq i, j \in N.$$

By Definition 2, $M_l^{'}$ is a DZ matrix.

Let $M_j^{'}$, $M_l^{'}$ be any two blocks in $M^{'} \in R(M)$, then $M_j^{'}$, $M_l^{'}$ are all DZ matrices. Based on Proposition 1, $(I - D)M_j^{'} + DM_l^{'}$ is a DZ matrix for any $D = \operatorname{diag}(d_i)$, $d_i \in [0,1]$ $(i \in N)$. Thus $(I - D)M_j^{'} + DM_l^{'}$ is nonsingular. By Lemma 1, the block matrix $M = (M_0, M_1, \ldots, M_k)$ has the row W-property. □

Theorem 6 *Let $M = (M_0, M_1, \ldots, M_k)$, and let each M_p $(p = 0, 1, 2, \ldots, k)$ be a DZ matrix with positive diagonal elements and satisfy the hypotheses of Proposition 1. Then*

$$\alpha(M)_\infty \le 2\max\left\{\frac{\max\{\varphi_{\max}^p\}}{\min\{\beta_{\min}^p\}}, \frac{\max\{\omega_{\max}^p\}}{\min\{\beta_{\min}^p\}}\right\},$$

where

$$\varphi_{\max}^p = \max_{\tau \ne i \in N}\{\varphi_{\tau i}^p\}, \qquad \varphi_{\tau i}^p = \left|(M_p)_{\tau i}\right| + \left|(M_p)_{ii}\right|, \qquad \beta_{\min}^p = \min_{\tau \ne i \in N}\{\beta_{i\tau}^p\},$$

$$\omega_{\max}^p = \max_{\tau \ne i \in N}\{\omega_{i\tau}^p\}, \qquad \omega_{i\tau}^p = \left|(M_p)_{\tau\tau}\right| - r_\tau(M_p) + \left|(M_p)_{\tau i}\right| + r_i(M_p),$$

$$\beta_{i\tau}^p = \left|(M_p)_{ii}\right|\left(\left|(M_p)_{\tau\tau}\right| - r_\tau(M_p) + \left|(M_p)_{\tau i}\right|\right) - r_i(M_p)\left|(M_p)_{\tau i}\right|.$$

Proof For any two blocks $M_j^{'}, M_l^{'}$ in $M^{'} \in R(M)$ and any $D = \operatorname{diag}(d_i)$ with $d_i \in [0,1]$ $(i \in N)$, let $M_D = (m_{ij}) = (I - D)M_j^{'} + DM_l^{'}$. By Propositions 1 and 2, it holds that $M_j^{'}$, $M_l^{'}$, and M_D are all DZ matrices with positive diagonal elements. By Theorem 5, we have that

$$\left\|M_D^{-1}\right\|_\infty \le \max\left\{\max_{\tau \in N, \tau \ne i}\alpha_1(M_D), \max_{\tau \in N, \tau \ne i}\alpha_2(M_D)\right\}$$

holds for each matrix $D = \operatorname{diag}(d_i)$ with $d_i \in [0,1]$ $(i \in N)$, where

$$\alpha_1(M_D) = \frac{|m_{\tau i}| + |m_{ii}|}{(|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}|)|m_{ii}| - |m_{\tau i}|r_i(M_D)},$$

$$\alpha_2(A) = \frac{|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}| + r_i(M_D)}{(|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}|)|m_{ii}| - |m_{\tau i}|r_i(M_D)}.$$

Hence, it holds that

$$\begin{aligned}
|m_{\tau i}| + |m_{ii}| &= (1 - d_\tau)\left|\left(M_j^{'}\right)_{\tau i}\right| + d_\tau\left|\left(M_l^{'}\right)_{\tau i}\right| + (1 - d_i)\left|\left(M_j^{'}\right)_{ii}\right| + d_i\left|\left(M_l^{'}\right)_{ii}\right| \\
&< \left[\left|\left(M_j^{'}\right)_{\tau i}\right| + \left|\left(M_j^{'}\right)_{ii}\right|\right] + \left[\left|\left(M_l^{'}\right)_{\tau i}\right| + \left|\left(M_l^{'}\right)_{ii}\right|\right] \\
&= \left(\varphi_{\tau i}^j\right)^{'} + \left(\varphi_{\tau i}^l\right)^{'} \\
&= 2\max_{t=j,l}\left(\varphi_{\max}^t\right)^{'},
\end{aligned}$$

where

$$\left(\varphi_{\max}^t\right)^{'} = \max_{\tau \ne i \in N}\left\{\left(\varphi_{\tau i}^l\right)^{'}\right\}, \qquad \left(\varphi_{\tau i}^l\right)^{'} = \left|\left(M_t^{'}\right)_{\tau i}\right| + \left|\left(M_t^{'}\right)_{ii}\right|, \quad t = j, l.$$

Further, we get

$$\begin{aligned}
&\left(|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}|\right)|m_{ii}| - |m_{\tau i}|r_i(M_D) \\
&\quad = \left[(1 - d_\tau)\left(\left|\left(M_j^{'}\right)_{\tau\tau}\right| - r_\tau\left(M_j^{'}\right) + \left|\left(M_j^{'}\right)_{\tau i}\right|\right) + d_\tau\left(\left|\left(M_l^{'}\right)_{\tau\tau}\right| - r_\tau\left(M_l^{'}\right) + \left|\left(M_l^{'}\right)_{\tau i}\right|\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \big[(1-d_i)\big|(M_j')_{ii}\big| + d_i\big|(M_l')_{ii}\big|\big] \\
&- \big[(1-d_\tau)\big|(M_j')_{\tau i}\big| + d_\tau\big|(M_l')_{\tau i}\big|\big]\big[(1-d_i)r_i(M_j') + d_i r_i(M_l')\big] \\
={}& (1-d_\tau)(1-d_i)\big|(M_j')_{ii}\big|\big(\big|(M_j')_{\tau\tau}\big| - r_\tau(M_j') + \big|(M_j')_{\tau i}\big|\big) \\
&+ (1-d_\tau)d_i\big|(M_l')_{ii}\big|\big(\big|(M_j')_{\tau\tau}\big| - r_\tau(M_j') + \big|(M_j')_{\tau i}\big|\big) \\
&+ d_\tau(1-d_i)\big|(M_j')_{ii}\big|\big(\big|(M_l')_{\tau\tau}\big| - r_\tau(M_l') + \big|(M_l')_{\tau i}\big|\big) \\
&+ d_\tau d_i\big|(M_l')_{ii}\big|\big(\big|(M_l')_{\tau\tau}\big| - r_\tau(M_l') + \big|(M_l')_{\tau i}\big|\big) \\
&- (1-d_\tau)(1-d_i)r_i(M_j')\big|(M_j')_{\tau i}\big| - (1-d_\tau)d_i r_i(M_l')\big|(M_j')_{\tau i}\big| \\
&- d_\tau(1-d_i)r_i(M_j')\big|(M_l')_{\tau i}\big| - d_\tau d_i r_i(M_l')\big|(M_l')_{\tau i}\big| \\
>{}& (1-d_\tau)(1-d_i)\big|(M_j')_{ii}\big|\big(\big|(M_j')_{\tau\tau}\big| - r_\tau(M_j') + \big|(M_j')_{\tau i}\big|\big) \\
&+ (1-d_\tau)d_i r_i(M_l')\big|(M_j')_{\tau i}\big| + d_\tau(1-d_i)r_i(M_j')\big|(M_l')_{\tau i}\big| \\
&+ d_\tau d_i\big|(M_l')_{ii}\big|\big(\big|(M_l')_{\tau\tau}\big| - r_\tau(M_l') + \big|(M_l')_{\tau i}\big|\big) \\
&- (1-d_\tau)(1-d_i)r_i(M_j')\big|(M_j')_{\tau i}\big| - (1-d_\tau)d_i r_i(M_l')\big|(M_j')_{\tau i}\big| \\
&- d_\tau(1-d_i)r_i(M_j')\big|(M_l')_{\tau i}\big| - d_\tau d_i r_i(M_l')\big|(M_l')_{\tau i}\big| \\
={}& (1-d_\tau)(1-d_i)\big|(M_j')_{ii}\big|\big(\big|(M_j')_{\tau\tau}\big| - r_\tau(M_j') + \big|(M_j')_{\tau i}\big|\big) \\
&+ d_\tau d_i\big|(M_l')_{ii}\big|\big(\big|(M_l')_{\tau\tau}\big| - r_\tau(M_l') + \big|(M_l')_{\tau i}\big|\big) \\
&- (1-d_\tau)(1-d_i)r_i(M_j')\big|(M_j')_{\tau i}\big| - d_\tau d_i r_i(M_l')\big|(M_l')_{\tau i}\big| \\
={}& (1-d_\tau)(1-d_i)\big[\big|(M_j')_{ii}\big|\big(\big|(M_j')_{\tau\tau}\big| - r_\tau(M_j') + \big|(M_j')_{\tau i}\big|\big) - r_i(M_j')\big|(M_j')_{\tau i}\big|\big] \\
&+ d_\tau d_i\big[\big|(M_l')_{ii}\big|\big(\big|(M_l')_{\tau\tau}\big| - r_\tau(M_l') + \big|(M_l')_{\tau i}\big|\big) - r_i(M_l')\big|(M_l')_{\tau i}\big|\big] \\
={}& (1-d_\tau)(1-d_i)\big(\beta_{i\tau}^j\big)' + d_\tau d_i\big(\beta_{i\tau}^l\big)' \\
\geq{}& (1-d_\tau)(1-d_i)\big(\beta_{\min}^j\big)' + d_\tau d_i\big(\beta_{\min}^l\big)',
\end{aligned}
$$

where

$$
\begin{aligned}
&\big(\beta_{\min}^j\big)' = \min_{\tau\neq i\in N}\big(\beta_{i\tau}^t\big)', \\
&\big(\beta_{i\tau}^t\big)' = \big|(M_t')_{ii}\big|\big(\big|(M_t')_{\tau\tau}\big| - r_\tau(M_t') + \big|(M_t')_{\tau i}\big|\big) - r_i(M_t')\big|(M_t')_{\tau i}\big|, \quad t = j, l.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\alpha_1(M_D) &= \frac{|m_{\tau i}| + |m_{ii}|}{(|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}|)|m_{ii}| - |m_{\tau i}|r_i(M_D)} \\
&\geq \frac{2\max_{t=j,l}(\varphi_{\max}^t)'}{(1-d_\tau)(1-d_i)(\beta_{\min}^j)' + d_\tau d_i(\beta_{\min}^l)'} \\
&\geq \frac{2\max_{t=j,l}(\varphi_{\max}^t)'}{\min\{(\beta_{\min}^j)', (\beta_{\min}^l)'\}}
\end{aligned}
$$

and

$$\begin{aligned}
&|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}| + r_i(M_D) \\
&\quad = \bigl[(1-d_\tau)\bigl|\bigl(M_j^{'}\bigr)_{\tau\tau}\bigr| + d_\tau\bigl|\bigl(M_l^{'}\bigr)_{\tau\tau}\bigr|\bigr] - \bigl[(1-d_\tau)r_\tau\bigl(M_j^{'}\bigr) + d_\tau r_\tau\bigl(M_l^{'}\bigr)\bigr] \\
&\qquad + \bigl[(1-d_\tau)\bigl|\bigl(M_j^{'}\bigr)_{\tau i}\bigr| + d_\tau\bigl|\bigl(M_l^{'}\bigr)_{\tau i}\bigr|\bigr] + \bigl[(1-d_i)r_i\bigl(M_j^{'}\bigr) + d_i r_i\bigl(M_l^{'}\bigr)\bigr] \\
&\quad = (1-d_\tau)\bigl[\bigl|\bigl(M_j^{'}\bigr)_{\tau\tau}\bigr| - r_\tau\bigl(M_j^{'}\bigr) + \bigl|\bigl(M_j^{'}\bigr)_{\tau i}\bigr|\bigr] \\
&\qquad + d_\tau\bigl[\bigl|\bigl(M_l^{'}\bigr)_{\tau\tau}\bigr| - r_\tau\bigl(M_l^{'}\bigr) + \bigl|\bigl(M_l^{'}\bigr)_{\tau i}\bigr|\bigr] + \bigl[(1-d_i)r_i\bigl(M_j^{'}\bigr) + d_i r_i\bigl(M_l^{'}\bigr)\bigr] \\
&\quad < \bigl[\bigl|\bigl(M_j^{'}\bigr)_{\tau\tau}\bigr| - r_\tau\bigl(M_j^{'}\bigr) + \bigl|\bigl(M_j^{'}\bigr)_{\tau i}\bigr| + r_i\bigl(M_j^{'}\bigr)\bigr] \\
&\qquad + \bigl[\bigl|\bigl(M_l^{'}\bigr)_{\tau\tau}\bigr| - r_\tau\bigl(M_l^{'}\bigr) + \bigl|\bigl(M_l^{'}\bigr)_{\tau i}\bigr| + r_i\bigl(M_l^{'}\bigr)\bigr] \\
&\quad = \bigl(\omega_{i\tau}^{j}\bigr)^{'} + \bigl(\omega_{i\tau}^{l}\bigr)^{'} \\
&\quad \le 2\max_{t=j,l}\bigl(\omega_{\max}^{t}\bigr)^{'},
\end{aligned}$$

where

$$\begin{aligned}
&\bigl(\omega_{\max}^{t}\bigr)^{'} = \max_{\tau\neq i\in N}\bigl(\omega_{i\tau}^{t}\bigr)^{'}, \\
&\bigl(\omega_{i\tau}^{t}\bigr)^{'} = \bigl|\bigl(M_t^{'}\bigr)_{\tau\tau}\bigr| - r_\tau\bigl(M_t^{'}\bigr) + \bigl|\bigl(M_t^{'}\bigr)_{\tau i}\bigr| + r_i\bigl(M_t^{'}\bigr), \quad t = j,l.
\end{aligned}$$

So, it holds that

$$\begin{aligned}
\alpha_2(A) &= \frac{|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}| + r_i(M_D)}{(|m_{\tau\tau}| - r_\tau(M_D) + |m_{\tau i}|)|m_{ii}| - |m_{\tau i}|r_i(M_D)} \\
&\le \frac{2\max_{t=j,l}(\omega_{\max}^{t})^{'}}{(1-d_\tau)(1-d_i)(\beta_{\min}^{j})^{'} + d_\tau d_i(\beta_{\min}^{l})^{'}} \\
&\le \frac{2\max_{t=j,l}(\omega_{\max}^{t})^{'}}{\min\{(\beta_{\min}^{j})^{'},(\beta_{\min}^{l})^{'}\}}
\end{aligned}$$

and

$$\bigl\|M_D^{-1}\bigr\|_\infty \le 2\max\left\{\frac{\max_{t=j,l}(\varphi_{\max}^{t})^{'}}{\min\{(\beta_{\min}^{j})^{'},(\beta_{\min}^{l})^{'}\}},\frac{\max_{t=j,l}(\omega_{\max}^{t})^{'}}{\min\{(\beta_{\min}^{j})^{'},(\beta_{\min}^{l})^{'}\}}\right\}.$$

By Definition 1, we can regard $M_j^{'}$, $M_l^{'}$ as two blocks in a row rearrangement of $M = (M_0, M_1, \dots, M_k)$, and thus for $t = j$ or $t = l$ and for $i \in N$, there exists $t_i \in \{0,1,\dots,k\}$ such that

$$\bigl(\varphi_{\tau i}^{t}\bigr)^{'} = \varphi_{\tau i}^{t_i}, \qquad \bigl(\omega_{i\tau}^{t}\bigr)^{'} = \omega_{i\tau}^{t_i}, \qquad \bigl(\beta_{i\tau}^{t}\bigr)^{'} = \beta_{i\tau}^{t_i},$$

this implies that

$$\max_{t=j,l}\bigl(\varphi_{\max}^{t}\bigr)^{'} = \max_{t=j,l}\Bigl\{\max_{\tau\neq i\in N}\bigl\{\bigl(\varphi_{\tau i}^{t}\bigr)^{'}\bigr\}\Bigr\} = \max_{t=j,l}\Bigl\{\max_{\tau\neq i\in N}\bigl\{\bigl(\varphi_{\tau i}^{t_i}\bigr)\bigr\}\Bigr\}$$

$$= \max_{\tau \neq i \in N} \left\{ \max_{t=j,l} \left\{ \left(\varphi_{\tau i}^{t_i} \right) \right\} \right\} \leq \max_{\tau \neq i \in N} \left\{ \max_{p=0,1,\ldots,k} \left\{ \left(\varphi_{\tau i}^{p} \right) \right\} \right\}$$
$$= \max_{p=0,1,\ldots,k} \left\{ \max_{\tau \neq i \in N} \left\{ \left(\varphi_{\tau i}^{p} \right) \right\} \right\} = \max_{p=0,1,\ldots,k} \left\{ \varphi_{\max}^{p} \right\}.$$

Similarly, we get

$$\begin{aligned}
\max_{t=j,l} \left(\omega_{\max}^{t} \right)' &= \max_{t=j,l} \left\{ \max_{\tau \neq i \in N} \left\{ \left(\omega_{i\tau}^{t} \right)' \right\} \right\} = \max_{t=j,l} \left\{ \max_{\tau \neq i \in N} \left\{ \left(\omega_{i\tau}^{t_i} \right) \right\} \right\} \\
&= \max_{\tau \neq i \in N} \left\{ \max_{t=j,l} \left\{ \left(\omega_{i\tau}^{t_i} \right) \right\} \right\} \leq \max_{\tau \neq i \in N} \left\{ \max_{p=0,1,\ldots,k} \left\{ \left(\omega_{i\tau}^{p} \right) \right\} \right\} \\
&= \max_{p=0,1,\ldots,k} \left\{ \max_{\tau \neq i \in N} \left\{ \left(\omega_{i\tau}^{p} \right) \right\} \right\} = \max_{p=0,1,\ldots,k} \left\{ \omega_{\max}^{p} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\min_{\tau \neq i \in N} \left\{ \left(\beta_{\min}^{j} \right)', \left(\beta_{\min}^{l} \right)' \right\} &= \min_{t=j,l} \left\{ \min_{\tau \neq i \in N} \left\{ \beta_{\min}^{t} \right\} \right\} = \min_{t=j,l} \left\{ \min_{\tau \neq i \in N} \left\{ \beta_{i\tau}^{t_i} \right\} \right\} \\
&= \min_{\tau \neq i \in N} \left\{ \min_{t=j,l} \left\{ \beta_{i\tau}^{t_i} \right\} \right\} \geq \min_{\tau \neq i \in N} \left\{ \min_{p=0,1,\ldots,k} \left\{ \beta_{i\tau}^{p} \right\} \right\} \\
&= \min_{p=0,1,\ldots,k} \left\{ \min_{\tau \neq i \in N} \left\{ \beta_{i\tau}^{p} \right\} \right\} = \min_{p=0,1,\ldots,k} \left\{ \beta_{\min}^{p} \right\}.
\end{aligned}$$

Hence, for any two blocks M_j', M_l' in $M' \in R(M)$,

$$\left\| M_D^{-1} \right\|_\infty \leq 2 \max \left\{ \frac{\max\{\varphi_{\max}^{p}\}}{\min\{\beta_{\min}^{p}\}}, \frac{\max\{\omega_{\max}^{p}\}}{\min\{\beta_{\min}^{p}\}} \right\}, \quad p = 0, 1, \ldots, k.$$

By the arbitrariness of M_j' and M_l', the conclusion follows. □

We illustrate our results with the following two examples.

Example 1 Let $M = (M_0, M_1, M_2)$, where

$$M_0 = \begin{bmatrix} 2 & 0 & 0.2 & 0.2 \\ 0.3 & 3 & 1.2 & 1 \\ 0.5 & 0.1 & 3 & 2 \\ 0.3 & 0.1 & 1 & 2 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 2 & 0 & 0.8 & 0.1 \\ 0.5 & 2 & 0.3 & 0.2 \\ 1 & 0.5 & 3 & 0.5 \\ 0.2 & 0.5 & 1 & 3 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 2 & 0 & 0.2 & 0.8 \\ 1.7 & 3 & 0.85 & 0.39 \\ 0.12 & 0.5 & 3 & 0.1 \\ 0.2 & 0.3 & 0.1 & 3 \end{bmatrix}$$

are DZ matrices and SDD matrices. Thus $M = (M_0, M_1, M_2)$ has the row W-property. By Theorem 6, it holds that

$$\alpha_\infty(M) \leq 10.2509.$$

By Theorem 3 (Theorem 4.4 of [1]), we have

$$\alpha_\infty(M) \leq 16.6667.$$

Since

$$\rho\left(\max\left(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \Lambda_2^{-1}|B_2|\right)\right) = 0.9778 < 1,$$

then by Theorem 4 (Theorem 4.3 of [1]), we get

$$\alpha_\infty(M) \le 25.2523.$$

Example 2 Let $M = (M_0, M_1, M_2)$, where

$$M_0 = \begin{bmatrix} 2 & 0 & 0.2 & 0.2 \\ 0.3 & 3 & 1.2 & 1 \\ 0.5 & 0.1 & 3 & 2 \\ 1 & 0.1 & 1 & 2 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 2 & 0 & 0.8 & 0.1 \\ 0.5 & 2 & 0.3 & 0.2 \\ 1 & 0.5 & 3 & 0.5 \\ 0.2 & 0.5 & 1 & 3 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 2 & 0 & 0.2 & 0.8 \\ 1.7 & 3 & 0.85 & 0.39 \\ 0.12 & 0.5 & 3 & 0.1 \\ 0.2 & 0.3 & 0.1 & 3 \end{bmatrix}$$

are *DZ* matrices, but M_0 is not an *SDD* matrix, and

$$\rho\left(\max\left(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \Lambda_2^{-1}|B_2|\right)\right) = 1.0728 > 1.$$

Thus Theorems 3 and 4 cannot work for this case. However, by Theorem 6, it holds that

$$\alpha_\infty(M) \le 16.3429.$$

Examples 1 and 2 show that the bound in Theorem 6 is sharper than that in Theorems 3 and 4 in some cases.

4 A global error bound for the EVLCP of Dashnic–Zusmanovich-B matrices

In 2020, Zhou et al. [19] introduced error bounds of the linear complementarity problems of Dashnic–Zusmanovich-B matrices, whose definition is listed below.

Definition 3 ([19]) A matrix $M = (m_{ij}) \in R^{n\times n}$ is called a Dashnic–Zusmanovich-B (*DZ-B*) matrix if B^+ is a *DZ* matrix, where $M = B^+ + C$,

$$B^+ = (b_{ij}) = \begin{pmatrix} m_{11} - r_1^+ & \cdots & m_{1n} - r_1^+ \\ \vdots & \ddots & \vdots \\ m_{n1} - r_n^+ & \cdots & m_{nn} - r_n^+ \end{pmatrix}, \qquad C = \begin{pmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & \ddots & \vdots \\ r_n^+ & \cdots & r_n^+ \end{pmatrix}, \tag{4}$$

and $r_i^+ = \max\{0, m_{ij} | i \ne j\}$.

Next, we will present an upper bound for $\alpha_\infty(M)$ with each $M_l \in R^{n\times n}$ $(l = 0, 1, \ldots, k)$ being a *DZ-B* matrix. Before that, some useful results are presented as follows.

Proposition 3 *Let $A = (a_{ij}) \in R^{n\times n}$ and $B = (b_{ij}) \in R^{n\times n}$ be all DZ-B matrices of the form*

$$A = B_A^+ + C_A, \qquad B = B_B^+ + C_B,$$

where B_A^+, B_B^+, C_A, C_B are as (4), and B_A^+, B_B^+ are all DZ matrices and satisfy the hypotheses of Proposition 1. Then $(I - D)A + DB$ is also a DZ-B matrix for any $D = \operatorname{diag}(d_i)$ with $d_i \in [0,1]$ $(i \in N)$.

Proof Since both A and B are *DZ-B* matrices, then A and B can be split separately into

$$\begin{aligned}(I - D)A + DB &= (I - D)\bigl(B_A^+ + C_A\bigr) + D\bigl(B_B^+ + C_B\bigr)\\ &= \bigl[(I - D)B_A^+ + DB_B^+\bigr] + \bigl[(I - D)C_A + DC_B\bigr].\end{aligned}$$

By Proposition 1, we have that $(I - D)B_A^+ + DB_B^+$ is a *DZ* matrix, and $(I - D)B_A^+ + DB_B^+$ and $(I - D)C_A + DC_B$ satisfy formula (4). So, by Definition 3, $(I - D)A + DB$ is a *DZ-B* matrix. □

Based on Proposition 3, Lemma 1, and the fact that a *DZ-B* matrix is nonsingular, we can prove the following result.

Proposition 4 *If $M_0, M_1, \dots, M_k$ are all DZ-B matrices and each M_{l_i} $(l_i = 0, 1, \dots k)$ satisfies the hypotheses of Proposition 3, then each M_l' in $M' \in R(M)$ is a DZ-B matrix, and consequently, $M = (M_0, M_1, \dots, M_k)$ has the row W-property.*

Proof By Definition 1, for the ith row $(M_l')_i$ of M_l' $(i \in N)$, there exists $l_i \in \{0, 1, \dots, k\}$ such that $(M_l')_i = (M_{l_i})_i$ and M_{l_i} is a *DZ-B* matrix. So, it holds that

$$M_{l_i} = B_{l_i}^+ + C_{l_i}, \qquad (M_{l_i})_i = \bigl(B_{l_i}^+\bigr)_i + (C_{l_i})_i,$$

that is,

$$\bigl(M_l'\bigr)_i = \bigl(B_l^+\bigr)_i' + (C_l)_i', \qquad M_l' = \bigl(B_l^+\bigr)' + (C_l)'.$$

By Proposition 2, $(B_l^+)'$ is a *DZ* matrix, and $(B_l^+)'$ and $(C_l)'$ satisfy formula (4). So, M_l' is a *DZ-B* matrix.

Let M_j', M_l' be any two blocks in $M' \in R(M)$, then M_j' and M_l' are all *DZ-B* matrices. By Proposition 3, $(I - D)M_j' + DM_l'$ is a *DZ-B* matrix for any $D = \operatorname{diag}(d_i)$, $d_i \in [0,1]$ $(i \in N)$. Thus $(I - D)M_j' + DM_l'$ is nonsingular. By Lemma 1, the block matrix $M = (M_0, M_1, \dots, M_k)$ has the row W-property. □

Theorem 7 *Let $M = (M_0, M_1, \dots, M_k)$, and let each M_p be a DZ-B matrix of the form $M_p = B_p^+ + C_p$ as (4), and let $B_p^+ = ((b^p)_{i\tau})$ satisfy the hypotheses of Proposition 3 for $p = 0, 1, \dots, k$. Then*

$$\alpha(M)_\infty \le 2(n-1)\max\left\{\frac{\max\{\chi_{\max}^p\}}{\min\{\lambda_{\min}^p\}}, \frac{\max\{\gamma_{\max}^p\}}{\min\{\lambda_{\min}^p\}}\right\},$$

where

$$\chi^p_{\max} = \max_{\tau\neq i\in N}\{\chi^p_{\tau i}\}, \qquad \chi^p_{\tau i} = \left|\left(b^p\right)_{\tau i}\right| + \left|\left(b^p\right)_{ii}\right|,$$
$$\gamma^p_{\max} = \max_{\tau\neq i\in N}\{\gamma^p_{i\tau}\}, \qquad \gamma^p_{i\tau} = \left|\left(b^p\right)_{\tau\tau}\right| - r_\tau\left(B^+_p\right) + \left|\left(b^p\right)_{\tau i}\right| + r_i\left(B^+_p\right),$$
$$\lambda^p_{\min} = \min_{\tau\neq i\in N}\{\lambda^p_{i\tau}\}, \qquad \lambda^p_{i\tau} = \left|\left(b^p\right)_{ii}\right|\left(\left|\left(b^p\right)_{\tau\tau}\right| - r_\tau\left(B^+_p\right) + \left|\left(b^p\right)_{\tau i}\right|\right) - r_i\left(B^+_p\right)\left|\left(b^p\right)_{\tau i}\right|.$$

Proof For any two blocks $M_j^{'}, M_l^{'}$ in $M^{'} \in R(M)$ and any $D = \operatorname{diag}(d_i)$ with $d_i \in [0,1]$ $(i\in N)$, denote $M_D = (m_{i\tau}) = (I-D)M_j^{'} + DM_l^{'}$. By Propositions 3 and 4, it holds that $M_j^{'}$, $M_l^{'}$, and M_D are all DZ-B matrices, then we can split $M_j^{'} = (B_j^+)^{'} + (C_j)^{'}$ and $M_l^{'} = (B_l^+)^{'} + (C_l)^{'}$ as (4). Let $(B_t^+)^{'} = ((b_t^{'})_{i\tau})$ for $t = j, l$. Since

$$\begin{aligned} M_D &= (I-D)M_j^{'} + DM_l^{'} \\ &= (I-D)\left[\left(B_j^+\right)^{'} + (C_j)^{'}\right] + D\left[\left(B_l^+\right)^{'} + (C_l)^{'}\right] \\ &= \left[(I-D)\left(B_j^+\right)^{'} + D\left(B_l^+\right)^{'}\right]\left[(I-D)(C_j)^{'} + D(C_l)^{'}\right], \end{aligned}$$

then $M_D = B_D^+ + C_D$, where

$$B_D^+ = (b_{i\tau}) = (I-D)\left(B_j^+\right)^{'} + D\left(B_l^+\right)^{'}, \qquad C_D = (I-D)(C_j)^{'} + D(C_l)^{'}.$$

So, both $(B_j^+)^{'}$ and $(B_l^+)^{'}$ are DZ matrices with positive diagonal elements, and B_D^+ is also a DZ matrix with positive diagonal entries by Proposition 1. Therefore, B_D^+ is a nonsingular matrix, and

$$M_D^{-1} = \left(B_D^+ + C_D\right)^{-1} = \left(B_D^+\left(I + \left(B_D^+\right)^{-1}C_D\right)\right)^{-1} = \left(I + \left(B_D^+\right)^{-1}C_D\right)^{-1}\left(B_D^+\right)^{-1},$$

that is,

$$\left\|M_D^{-1}\right\|_\infty \le \left\|\left(I + \left(B_D^+\right)^{-1}C_D\right)^{-1}\right\|_\infty \cdot \left\|\left(B_D^+\right)^{-1}\right\|_\infty \le (n-1)\left\|\left(B_D^+\right)^{-1}\right\|_\infty,$$

where the last equality holds because $\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty \le (n-1)$, see Theorem 2.2 in [9].

In fact, since $(B_j^+)^{'}$, $(B_l^+)^{'}$, and B_D^+ are all DZ Z-matrices with positive diagonal elements, by Theorem 5, we have that

$$\left\|\left(B_D^+\right)^{-1}\right\|_\infty \le \max\left\{\max_{\tau\in N,\tau\neq i}\alpha_1\left(B_D^+\right), \max_{\tau\in N,\tau\neq i}\alpha_2\left(B_D^+\right)\right\}$$

holds for each matrix $D = \operatorname{diag}(d_i)$ with $d_i \in [0,1]$ $(i\in N)$, where

$$\alpha_1\left(B_D^+\right) = \frac{|b_{\tau i}| + |b_{ii}|}{(|b_{\tau\tau}| - r_\tau(B_D^+) + |b_{\tau i}|)|b_{ii}| - |b_{\tau i}|r_i(B_D^+)},$$
$$\alpha_2\left(B_D^+\right) = \frac{|b_{\tau\tau}| - r_\tau(B_D^+) + |b_{\tau i}| + r_i(B_D^+)}{(|b_{\tau\tau}| - r_\tau(B_D^+) + |b_{\tau i}|)|b_{ii}| - |b_{\tau i}|r_i(B_D^+)}.$$

Hence, we get

$$\begin{aligned}|b_{\tau i}| + |b_{ii}| &= (1-d_\tau)\left|\left(b_j'\right)_{\tau i}\right| + d_\tau\left|\left(b_l'\right)_{\tau i}\right| + (1-d_i)\left|\left(b_j'\right)_{ii}\right| + d_i\left|\left(b_l'\right)_{ii}\right| \\ &< \left[\left|\left(b_j'\right)_{\tau i}\right| + \left|\left(b_j'\right)_{ii}\right|\right] + \left[\left|\left(b_l'\right)_{\tau i}\right| + \left|\left(b_l'\right)_{ii}\right|\right] \\ &= \left(\chi^j_{\tau i}\right)' + \left(\chi^l_{\tau i}\right)' = 2\max_{t=j,l}\left(\chi^t_{\max}\right)',\end{aligned}$$

where

$$\left(\chi^t_{\max}\right)' = \max_{\tau\neq i\in N}\left\{\left(\chi^l_{\tau i}\right)'\right\}, \qquad \left(\chi^l_{\tau i}\right)' = \left|\left(b_t'\right)_{\tau i}\right| + \left|\left(b_t'\right)_{ii}\right|, \quad t = j, l.$$

Therefore, we have

$$\begin{aligned}&\left(|b_{\tau\tau}| - r_\tau\left(B_D^+\right) + |b_{\tau i}|\right)|b_{ii}| - |b_{\tau i}|r_i\left(B_D^+\right) \\ &\quad= \left[(1-d_\tau)\left(\left|\left(b_j'\right)_{\tau\tau}\right| - r_\tau\left(B_j^+\right)' + \left|\left(b_j'\right)_{\tau i}\right|\right) + d_\tau\left(\left|\left(b_l'\right)_{\tau\tau}\right| - r_\tau\left(B_l^+\right)' + \left|\left(b_l'\right)_{\tau i}\right|\right)\right] \\ &\qquad\times\left[(1-d_i)\left|\left(b_j'\right)_{ii}\right| + d_i\left|\left(b_l'\right)_{ii}\right|\right] \\ &\qquad- \left[(1-d_\tau)\left|\left(b_j'\right)_{\tau i}\right| + d_\tau\left|\left(b_l'\right)_{\tau i}\right|\right]\left[(1-d_i)r_i\left(B_j^+\right)' + d_ir_i\left(B_l^+\right)'\right] \\ &\quad= (1-d_\tau)(1-d_i)\left|\left(b_j'\right)_{ii}\right|\left(\left|\left(b_j'\right)_{\tau\tau}\right| - r_\tau\left(B_j^+\right)' + \left|\left(b_j'\right)_{\tau i}\right|\right) \\ &\qquad+ (1-d_\tau)d_i\left|\left(b_l'\right)_{ii}\right|\left(\left|\left(b_j'\right)_{\tau\tau}\right| - r_\tau\left(B_j^+\right)' + \left|\left(b_j'\right)_{\tau i}\right|\right) \\ &\qquad+ d_\tau(1-d_i)\left|\left(b_j'\right)_{ii}\right|\left(\left|\left(b_l'\right)_{\tau\tau}\right| - r_\tau\left(B_l^+\right)' + \left|\left(b_l'\right)_{\tau i}\right|\right) \\ &\qquad+ d_\tau d_i\left|\left(b_l'\right)_{ii}\right|\left(\left|\left(b_l'\right)_{\tau\tau}\right| - r_\tau\left(B_l^+\right)' + \left|\left(b_l'\right)_{\tau i}\right|\right) \\ &\qquad- (1-d_\tau)(1-d_i)r_i\left(B_j^+\right)'\left|\left(b_j'\right)_{\tau i}\right| - (1-d_\tau)d_ir_i\left(B_l^+\right)'\left|\left(b_j'\right)_{\tau i}\right| \\ &\qquad- d_\tau(1-d_i)r_i\left(B_j^+\right)'\left|\left(b_l'\right)_{\tau i}\right| - d_\tau d_ir_i\left(B_l^+\right)'\left|\left(b_l'\right)_{\tau i}\right| \\ &\quad> (1-d_\tau)(1-d_i)\left|\left(b_j'\right)_{ii}\right|\left(\left|\left(b_j'\right)_{\tau\tau}\right| - r_\tau\left(B_j^+\right)' + \left|\left(b_j'\right)_{\tau i}\right|\right) \\ &\qquad+ (1-d_\tau)d_ir_i\left(B_l^+\right)'\left|\left(b_j'\right)_{\tau i}\right| + d_\tau(1-d_i)r_i\left(B_j^+\right)'\left|\left(b_l'\right)_{\tau i}\right| \\ &\qquad+ d_\tau d_i\left|\left(b_l'\right)_{ii}\right|\left(\left|\left(b_l'\right)_{\tau\tau}\right| - r_\tau\left(B_l^+\right)' + \left|\left(b_l'\right)_{\tau i}\right|\right) \\ &\qquad- (1-d_\tau)(1-d_i)r_i\left(B_j^+\right)'\left|\left(b_j'\right)_{\tau i}\right| - (1-d_\tau)d_ir_i\left(B_l^+\right)'\left|\left(b_j'\right)_{\tau i}\right| \\ &\qquad- d_\tau(1-d_i)r_i\left(B_j^+\right)'\left|\left(b_l'\right)_{\tau i}\right| - d_\tau d_ir_i\left(B_l^+\right)'\left|\left(b_l'\right)_{\tau i}\right| \\ &\quad= (1-d_\tau)(1-d_i)\left|\left(b_j'\right)_{ii}\right|\left(\left|\left(b_j'\right)_{\tau\tau}\right| - r_\tau\left(B_j^+\right)' + \left|\left(b_j'\right)_{\tau i}\right|\right) \\ &\qquad+ d_\tau d_i\left|\left(b_l'\right)_{ii}\right|\left(\left|\left(b_l'\right)_{\tau\tau}\right| - r_\tau\left(B_l^+\right)' + \left|\left(b_l'\right)_{\tau i}\right|\right) \\ &\qquad- (1-d_\tau)(1-d_i)r_i\left(B_j^+\right)'\left|\left(b_j'\right)_{\tau i}\right| - d_\tau d_ir_i\left(B_l^+\right)'\left|\left(b_l'\right)_{\tau i}\right| \\ &\quad= (1-d_\tau)(1-d_i)\left[\left|\left(b_j'\right)_{ii}\right|\left(\left|\left(b_j'\right)_{\tau\tau}\right| - r_\tau\left(B_j^+\right)' + \left|\left(b_j'\right)_{\tau i}\right|\right) - r_i\left(B_j^+\right)'\left|\left(b_j'\right)_{\tau i}\right|\right] \\ &\qquad+ d_\tau d_i\left[\left|\left(b_l'\right)_{ii}\right|\left(\left|\left(b_l'\right)_{\tau\tau}\right| - r_\tau\left(B_l^+\right)' + \left|\left(b_l'\right)_{\tau i}\right|\right) - r_i\left(B_l^+\right)'\left|\left(b_l'\right)_{\tau i}\right|\right] \\ &\quad= (1-d_\tau)(1-d_i)\left(\lambda^j_{i\tau}\right)' + d_\tau d_i\left(\lambda^l_{i\tau}\right)'\end{aligned}$$

$$\geq (1-d_\tau)(1-d_i)\big(\lambda_{\min}^{j}\big)' + d_\tau d_i\big(\lambda_{\min}^{l}\big)',$$

where

$$\begin{aligned}
&\big(\lambda_{\min}^{j}\big)' = \min_{\tau\neq i\in N}\big(\lambda_{i\tau}^{t}\big)',\\
&\big(\lambda_{i\tau}^{t}\big)' = \big|\big(b_t'\big)_{ii}\big|\big(\big|\big(b_t'\big)_{\tau\tau}\big| - r_\tau\big(B_t^+\big)' + \big|\big(b_t'\big)_{\tau i}\big|\big) - r_i\big(B_j^+\big)'\big|\big(b_t'\big)_{\tau i}\big|, \quad t=j,l.
\end{aligned}$$

Further, we get

$$\begin{aligned}
\alpha_1\big(B_D^+\big) &= \frac{|b_{\tau i}| + |b_{ii}|}{(|b_{\tau\tau}| - r_\tau(B_D^+) + |b_{\tau i}|)|b_{ii}| - |b_{\tau i}|r_i(B_D^+)}\\
&\geq \frac{2\max_{t=j,l}(\chi_{\max}^{t})'}{(1-d_\tau)(1-d_i)(\lambda_{\min}^{j})' + d_\tau d_i(\lambda_{\min}^{l})'}\\
&\geq \frac{2\max_{t=j,l}(\chi_{\max}^{t})'}{\min\{(\lambda_{\min}^{j})', (\lambda_{\min}^{l})'\}}
\end{aligned}$$

and

$$\begin{aligned}
&|b_{\tau\tau}| - r_\tau\big(B_D^+\big) + |b_{\tau i}| + r_i\big(B_D^+\big)\\
&\quad= \big[(1-d_\tau)\big|\big(b_j'\big)_{\tau\tau}\big| + d_\tau\big|\big(b_l'\big)_{\tau\tau}\big|\big] - \big[(1-d_\tau)r_\tau\big(B_j^+\big)' + d_\tau r_\tau\big(B_l^+\big)'\big]\\
&\qquad + \big[(1-d_\tau)\big|\big(b_j'\big)_{\tau i}\big| + d_\tau\big|\big(b_l'\big)_{\tau i}\big|\big] + \big[(1-d_i)r_i\big(B_j^+\big)' + d_i r_i\big(B_l^+\big)'\big]\\
&\quad= (1-d_\tau)\big[\big|\big(b_j'\big)_{\tau\tau}\big| - r_\tau\big(B_j^+\big)' + \big|\big(b_j'\big)_{\tau i}\big|\big]\\
&\qquad + d_\tau\big[\big|\big(b_l'\big)_{\tau\tau}\big| - r_\tau\big(B_l^+\big)' + \big|\big(b_l'\big)_{\tau i}\big|\big] + \big[(1-d_i)r_i\big(B_j^+\big)' + d_i r_i\big(B_l^+\big)'\big]\\
&\quad< \big[\big|\big(b_j'\big)_{\tau\tau}\big| - r_\tau\big(B_j^+\big)' + \big|\big(b_j'\big)_{\tau i}\big| + r_i\big(B_j^+\big)'\big]\\
&\qquad + \big[\big|\big(b_l'\big)_{\tau\tau}\big| - r_\tau\big(B_l^+\big)' + \big|\big(b_l'\big)_{\tau i}\big| + r_i\big(B_l^+\big)'\big]\\
&\quad= \big(\gamma_{i\tau}^{j}\big)' + \big(\gamma_{i\tau}^{l}\big)'\\
&\quad\leq 2\max_{t=j,l}\big(\gamma_{\max}^{t}\big)',
\end{aligned}$$

where

$$\begin{aligned}
&\big(\gamma_{\max}^{t}\big)' = \max_{\tau\neq i\in N}\big(\gamma_{i\tau}^{t}\big)',\\
&\big(\gamma_{i\tau}^{t}\big)' = \big|\big(b_t'\big)_{\tau\tau}\big| - r_\tau\big(B_t^+\big)' + \big|\big(b_t'\big)_{\tau i}\big| + r_i\big(B_t^+\big)', \quad t=j,l,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2\big(B_D^+\big) &= \frac{|b_{\tau\tau}| - r_\tau(B_D^+) + |b_{\tau i}| + r_i(B_D^+)}{(|b_{\tau\tau}| - r_\tau(B_D^+) + |b_{\tau i}|)|b_{ii}| - |b_{\tau i}|r_i(B_D^+)}\\
&\leq \frac{2\max_{t=j,l}(\gamma_{\max}^{t})'}{(1-d_\tau)(1-d_i)(\lambda_{\min}^{j})' + d_\tau d_i(\lambda_{\min}^{l})'}
\end{aligned}$$

$$\leq \frac{2\max_{t=j,l}(\gamma_{\max}^t)'}{\min\{(\lambda_{\min}^j)',(\lambda_{\min}^l)'\}}.$$

So, it holds that

$$\|M_D{}^{-1}\|_\infty \leq 2(n-1)\max\left\{\frac{\max_{t=j,l}(\chi_{\max}^t)'}{\min\{(\lambda_{\min}^j)',(\lambda_{\min}^l)'\}},\frac{\max_{t=j,l}(\gamma_{\max}^t)'}{\min\{(\lambda_{\min}^j)',(\lambda_{\min}^l)'\}}\right\}.$$

By Definition 1, we can regard M_j', M_l' as two blocks in a row rearrangement of $M = (M_0, M_1, \ldots, M_k)$, and thus for $t = j$ or $t = l$ and for $i \in N$, there exists $t_i \in \{0, 1, \ldots, k\}$ such that

$$(\chi_{\tau i}^t)' = \varphi_{\tau i}^{t_i}, \qquad (\gamma_{i\tau}^t)' = \gamma_{i\tau}^{t_i}, \qquad (\lambda_{i\tau}^t)' = \lambda_{i\tau}^{t_i},$$

this implies that

$$\begin{aligned}\max_{t=j,l}(\chi_{\max}^t)' &= \max_{t=j,l}\left\{\max_{\tau\neq i\in N}\left\{(\chi_{\tau i}^t)'\right\}\right\} = \max_{t=j,l}\left\{\max_{\tau\neq i\in N}\left\{(\chi_{\tau i}^{t_i})\right\}\right\}\\ &= \max_{\tau\neq i\in N}\left\{\max_{t=j,l}\left\{(\chi_{\tau i}^{t_i})\right\}\right\} \geq \max_{\tau\neq i\in N}\left\{\max_{p=0,1,\ldots,k}\left\{(\chi_{\tau i}^p)\right\}\right\}\\ &= \max_{p=0,1,\ldots,k}\left\{\max_{\tau\neq i\in N}\left\{(\chi_{\tau i}^p)\right\}\right\} = \max_{p=0,1,\ldots,k}\left\{\chi_{\max}^p\right\}.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\max_{t=j,l}(\gamma_{\max}^t)' &= \max_{t=j,l}\left\{\max_{\tau\neq i\in N}\left\{(\gamma_{i\tau}^t)'\right\}\right\} = \max_{t=j,l}\left\{\max_{\tau\neq i\in N}\left\{(\gamma_{i\tau}^{t_i})\right\}\right\}\\ &= \max_{\tau\neq i\in N}\left\{\max_{t=j,l}\left\{(\gamma_{i\tau}^{t_i})\right\}\right\} \leq \max_{\tau\neq i\in N}\left\{\max_{p=0,1,\ldots,k}\left\{(\gamma_{i\tau}^p)\right\}\right\}\\ &= \max_{p=0,1,\ldots,k}\left\{\max_{\tau\neq i\in N}\left\{(\gamma_{i\tau}^p)\right\}\right\} = \max_{p=0,1,\ldots,k}\left\{\gamma_{\max}^p\right\}\end{aligned}$$

and

$$\begin{aligned}\min_{\tau\neq i\in N}\left\{(\lambda_{\min}^j)',(\lambda_{\min}^l)'\right\} &= \min_{t=j,l}\left\{\min_{\tau\neq i\in N}\left\{\lambda_{\min}^t\right\}\right\} = \min_{t=j,l}\left\{\min_{\tau\neq i\in N}\left\{\lambda_{i\tau}^{t_i}\right\}\right\}\\ &= \min_{\tau\neq i\in N}\left\{\min_{t=j,l}\left\{\lambda_{i\tau}^{t_i}\right\}\right\} \geq \min_{\tau\neq i\in N}\left\{\min_{p=0,1,\ldots,k}\left\{\lambda_{i\tau}^p\right\}\right\}\\ &= \min_{p=0,1,\ldots,k}\left\{\min_{\tau\neq i\in N}\left\{\lambda_{i\tau}^p\right\}\right\} = \min_{p=0,1,\ldots,k}\left\{\lambda_{\min}^p\right\}.\end{aligned}$$

Thus, for any two blocks M_j',M_l' in $M' \in R(M)$, we have

$$\alpha(M)_\infty \leq 2(n-1)\max\left\{\frac{\max\{\chi_{\max}^p\}}{\min\{\lambda_{\min}^p\}},\frac{\max\{\gamma_{\max}^p\}}{\min\{\lambda_{\min}^p\}}\right\}, \quad p = 0, 1, \ldots, k.$$

By the arbitrariness of M_j' and M_l', the conclusion follows. □

We illustrate our results with the following two examples.

Example 3 Let $M = (M_0, M_1, M_2)$, and let each M_p be *DZ-B* matrices and *SDD* matrices of the form $M_p = B_p^+ + C_p$ $(p = 0, 1, 2)$ as (4), where

$$M_0 = \begin{bmatrix} 3.5 & 1.5 & 0.5 \\ 1 & 4 & -0.5 \\ -0.86 & 1.64 & 4.64 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 3.2 & 0.2 & -2 \\ 0.1 & 2.1 & 0.9 \\ -1.84 & 0.16 & 3.16 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 2.78 & 0.78 & -0.22 \\ 1.74 & 3.5 & 1.7 \\ -0.12 & 0.5 & 3.5 \end{bmatrix}, \qquad B_0^+ = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -1.5 \\ -2.5 & 0 & 3 \end{bmatrix},$$

$$B_1^+ = \begin{bmatrix} 3 & 0 & -2.2 \\ 0 & 2 & -1 \\ -2 & 0 & 3 \end{bmatrix}, \qquad B_2^+ = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1.76 & -0.04 \\ -0.62 & 0 & 3 \end{bmatrix}.$$

Thus $M = (M_0, M_1, M_2)$ has the row W-property. By Theorem 7, it holds that

$$\alpha_\infty(M) \le 14.6667.$$

By Theorem 3 (Theorem 4.4 of [1]), we have

$$\alpha_\infty(M) \le 16.6667.$$

Since

$$\rho\left(\max\left(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \Lambda_2^{-1}|B_2|\right)\right) = 0.9909 < 1,$$

then by Theorem 4 (Theorem 4.3 of [1]), we get

$$\alpha_\infty(M) \le 42.8184.$$

Example 4 Let $M = (M_0, M_1, M_2)$, and let M_p be *DZ-B* matrices of the form $M_p = B_p^+ + C_p$ $(p = 0, 1, 2)$ as (4), where

$$M_0 = \begin{bmatrix} 3.5 & 1.5 & 0.5 \\ 1 & 4 & -0.5 \\ -0.86 & 1.64 & 4.64 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 3.2 & 0.2 & -2 \\ 0.1 & 2.1 & 0.9 \\ -1.84 & 0.16 & 3.16 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 2.78 & 0.78 & -0.22 \\ 2 & 3.5 & 1.7 \\ -0.12 & 0.5 & 3.5 \end{bmatrix}, \qquad B_0^+ = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -1.5 \\ -2.5 & 0 & 3 \end{bmatrix},$$

$$B_1^+ = \begin{bmatrix} 3 & 0 & -2.2 \\ 0 & 2 & -1 \\ -2 & 0 & 3 \end{bmatrix}, \qquad B_2^+ = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1.5 & -0.3 \\ -0.62 & 0 & 3 \end{bmatrix}.$$

Thus $M = (M_0, M_1, M_2)$ has the row W-property. It is easy to check that M_2 is not *SDD* and

$$\rho\left(\max\left(\Lambda_0^{-1}|B_0|, \Lambda_1^{-1}|B_1|, \Lambda_2^{-1}|B_2|\right)\right) = 1.0347 > 1.$$

Hence, we cannot use these bounds in Theorems 3 and 4 to estimate $\alpha_\infty(M)$. But, by Theorem 7, we get

$$\alpha_\infty(M) \le 14.6667.$$

Examples 3 and 4 show that the bound in Theorem 7 is sharper than that in Theorems 3 and 4 in some cases.

5 Conclusions

In this paper, we present global error bounds for the extended vertical linear complementarity problems of DZ matrices and DZ-B matrices. These bounds are expressed in terms of elements of the matrices, so they can be checked easily. Numerical examples show the feasibility of new results. Finding computable global error bounds for the extended vertical linear complementarity problems of other matrices (S-SOB matrices, S-SOB-B matrices, weakly chained diagonally dominant B-matrices, SB-matrices, etc.) under some additional conditions is an interesting problem. It is worth studying in the future.

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Declarations

Competing interests
The authors declare no competing interests.

Author contribution
YXZ: original draft writing, review writing, and editing. DSS: conceptualization, supervision, and funding acquisition. All authors read and approved the final manuscript.

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