Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Generalized Lommel–Wright function and its geometric properties

Hanaa M. Zayed[1*] and Khaled Mehrez[2]

*Correspondence:
hanaa@science.menofia.edu.eg
[1]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin Elkom 32511, Egypt
Full list of author information is available at the end of the article

Abstract

The normalization of the combination of generalized Lommel–Wright function $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)$ ($m \in \mathbb{N}$, $\kappa_3 > 0$ and $\kappa_1, \kappa_2 \in \mathbb{C}$) defined by $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) := \Gamma^m(\kappa_1+1)\Gamma(\kappa_1+\kappa_2+1)2^{2\kappa_1+\kappa_2}z^{1-(\kappa_2/2)-\kappa_1}\mathcal{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(\sqrt{z})$, where $\mathcal{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) := (1-2\kappa_1-\kappa_2)J^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) + z(J^{\kappa_3,m}_{\kappa_1,\kappa_2}(z))'$ and

$$J^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) = \left(\frac{z}{2}\right)^{2\kappa_1+\kappa_2}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma^m(n+\kappa_1+1)\Gamma(n\kappa_3+\kappa_1+\kappa_2+1)}\left(\frac{z}{2}\right)^{2n},$$

was previously introduced and some of its geometric properties have been considered. In this paper, we report conditions for $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)$ to be starlike and convex of order α, $0 \le \alpha < 1$, inside the open unit disk using some technical manipulations of the gamma and digamma functions, as well as inequality for the digamma function that has been proved (Guo and Qi in Proc. Am. Math. Soc. 141(3):1007–1015, 2013). In addition, a method presented by Lorch (J. Approx. Theory 40(2):115–120 1984) and further developed by Laforgia (Math. Compet. 42(166):597–600 1984) is applied to establish firstly sharp inequalities for the shifted factorial that will be used to obtain the order of starlikeness and convexity. We compare then the obtained orders of starlikeness and convexity with some important consequences in the literature as well as the results proposed by all techniques to demonstrate the accuracy of our approach. Ultimately, a lemma by (Fejér in Acta Litt. Sci. 8:89–115 1936) is used to prove that the modified form of the function $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)$ defined by $\mathcal{I}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) = \mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) * z/(1+z)$ is in the class of starlike and convex functions, respectively. Further work regarding the function $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)$ is underway and will be presented in a forthcoming paper.

MSC: 30C45; 33C50

Keywords: Analytic; Univalent; Starlike; Convex; Generalized Lommel–Wright functions

1 Introduction and preliminaries

In the literature, there has been a growing interest in special functions that have applications in different fields of mathematical analysis, functional analysis, geometry, and physics. Special functions are an old subject, but due to their essential position in mathematics, they continue to play an essential role, for instance, in combinatorics, applied mathematics, and engineering. More recently, further progress has been made towards the geometric properties for the normalized form of some special functions such as univalence, starlikeness, convexity, and close to convexity inside the open unit disk. Regarding treatises on this investigation, we refer, e.g., to [7–9, 12, 13] for generalized Bessel function, to [3] for hyper-Bessel functions, to [25, 32, 34] for generalized Struve function, to [31] for Lommel function, to [24, 28, 29] for hypergeometric function, to [33] for generalized Lommel–Wright function, and [23] for Fox–Wright function. In addition, the radii of starlikeness and the convexity of Bessel and its q-analog, Struve and Lommel functions, were investigated by several authors (see [1, 2, 4–6, 10–12]). These results would be fruitful to enrich the understanding of the geometrical properties of such functions as tools in such applications of geometric function theory.

Over the past few years considerable attention has been given to the role played by generalized Lommel–Wright function in concrete problems in physics, mechanics, engineering, and astronomy. In continuation of [33], we report conditions for $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)$ to be starlike and convex of order α, $0 \leq \alpha < 1$, inside the open unit disk using some technical manipulations of the gamma and digamma functions as well as an inequality for the digamma function that has been proved [18]. In addition, a method presented by Lorch [22], and further developed by Laforgia [21], is applied to establish firstly sharp inequalities for the shifted factorial that will be used to obtain the order starlikeness and convexity. We compare then the obtained orders of starlikeness and convexity with some important consequences in the literature as well as the results proposed by all techniques to demonstrate the accuracy of our approach. Ultimately, a lemma by [16] is used to prove that the modified form of the function $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)$ defined by $\mathcal{I}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z) = \mathfrak{J}_{\kappa_1,\kappa_2}^{\lambda_3,m}(z) * z/(1+z)$ is in the class of starlike and convex functions, respectively.

Throughout this paper, let $\mathcal{H}$ indicate the family of all functions that are analytic in $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}$ the subfamily of $\mathcal{H}$ consisting of functions of the form

$$f(z) = \sum_{n=1}^{\infty} A_n z^n, \quad A_1 = 1, z \in \mathbb{U}, \tag{1.1}$$

and by $\mathcal{S}$ the subfamily of $\mathcal{A}$ which are univalent in $\mathbb{U}$. If $g \in \mathcal{A}$ is given by

$$g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad B_1 = 1, z \in \mathbb{U},$$

then the *convolution* of the functions f and g is given by

$$(f * g)(z) := \sum_{n=1}^{\infty} A_n B_n z^n, \quad A_1 B_1 = 1, z \in \mathbb{U}. \tag{1.2}$$

The above definition of convolution arises from the formula (see [15])

$$(f * g)\bigl(r^2 e^{i\theta}\bigr) = \frac{1}{2\pi} \int_0^{2\pi} f\bigl(re^{i(\theta - t)}\bigr) g\bigl(re^{it}\bigr)\, dt \quad (r < 1).$$

Let us recall now the subclasses of the class of analytic functions which are considered the cornerstone of the univalent function theory such as the subclasses of starlike and convex functions. These classes admit geometrical and analytical characteristics, which do not pass in the status of those functions that are utilized in the ordinary analysis. The interested reader is referred for further information to [15, 17, 27], whereas general aspects are found in [19]. Traditionally, a domain $D \subset \mathbb{C}$ is called *starlike with respect to an interior point* z_0 if the line segment joining z_0 to any other point in D lies entirely in D. In particular, if $z_0 = 0$, then the domain D is called *starlike domain*. A function $f(z) \in \mathcal{S}$ is called *starlike with respect to the origin (or starlike)*, denoted by $\mathcal{S}^*$, if $f(\mathbb{U})$ is a starlike domain. The well-known analytical characterization of starlikeness is given by the following theorem.

Theorem A *Let $f \in \mathcal{S}$. Then f is starlike if and only if*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}.$$

Further, if the line segment joining any two points of $D \subset \mathbb{C}$ lies entirely in D, then the domain is convex. A function $f(z) \in \mathcal{S}$ *is called convex, denoted by* $\mathcal{K}$, if $f(\mathbb{U})$ is a convex domain. The following theorem gives an analytic description of the convex functions.

Theorem B *Let $f \in \mathcal{S}$. Then f is convex if and only if*

$$1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{U}.$$

For instance, $f(z) = z/(1-z)^2 \in \mathcal{S}^*$ because of

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{1+z}{1-z}\right) > 0,$$

while $f(z) = -\log(1-z) \in \mathcal{K}$ since

$$1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) = 1 + \operatorname{Re}\left(\frac{z}{1-z}\right) > 0.$$

It is noteworthy to mention that the classes $\mathcal{S}^*$ and $\mathcal{K}$ have a particular interest if more restrictions are enjoined, it gives us various types of classes of functions. Moreover, thanks to the positivity of $\operatorname{Re}(zf'(z)/f(z))$ and $1 + \operatorname{Re}(zf''(z)/f'(z))$ for $\mathcal{S}^*$ and $\mathcal{K}$, respectively, it allows us to study several families of conformal transformation with other motivating geometric properties. On the other hand, $f \in \mathcal{A}$ is *starlike functions of order* α, $0 \le \alpha \le 1$, if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{for all } z \in \mathbb{U},$$

and is in the class of *convex functions of order* α, denoted by $\mathcal{K}(\alpha)$, if and only if

$$1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \quad \text{for all } z \in \mathbb{U}.$$

It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{K}(\alpha) \subset \mathcal{K}(0) = \mathcal{K}$, and $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$.

In [14], Oteiza et al. defined the *generalized Lommel–Wright function* $J_{\lambda_2,\kappa_1}^{\kappa_3,m}(z)$ as

$$\begin{aligned} J_{\kappa_1,\kappa_2}^{\kappa_3,m}(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n\kappa_3+\kappa_1+\kappa_2+1)\Gamma^m(n+\kappa_1+1)} \left(\frac{z}{2}\right)^{2n+2\kappa_1+\kappa_2} \\ &= \left(\frac{z}{2}\right)^{2\kappa_1+\kappa_2} {}_1\Psi_{m+1}\left[\begin{matrix} (1,1) \\ (\kappa_1+1,1),\ldots,(\kappa_1+1,1),(\kappa_1+\kappa_2+1,\kappa_3) \end{matrix}\middle| -\frac{z^2}{4}\right], \end{aligned} \tag{1.3}$$

for $\kappa_1,\kappa_2 \in \mathbb{C}$, $m \in \mathbb{N} := \{1,2,\ldots\}$, and $\kappa_3 > 0$. Noting that ${}_p\Psi_q$ stands the *Fox–Wright function* which is defined by

$${}_p\Psi_q\left[\begin{matrix} (\mathrm{a}_1,\mathrm{A}_1),\ldots,(\mathrm{a}_p,\mathrm{A}_p) \\ (\mathrm{b}_1,\mathrm{B}_1),\ldots,(\mathrm{b}_q,\mathrm{B}_q) \end{matrix}\middle| z\right] =: {}_p\Psi_q\left[\begin{matrix} (\mathbf{a}_p,\mathbf{A}_p) \\ (\mathbf{b}_q,\mathbf{B}_q) \end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \psi_n \frac{z^n}{n!}, \tag{1.4}$$

with

$$\psi_n = \frac{\Gamma(\mathrm{a}_1+\mathrm{A}_1 n)\ldots\Gamma(\mathrm{a}_p+\mathrm{A}_p n)}{\Gamma(\mathrm{b}_1+\mathrm{B}_1 n)\ldots\Gamma(\mathrm{b}_q+\mathrm{B}_q n)},$$

for $\mathrm{A}_i, \mathrm{B}_j \in \mathbb{R}^+$ $(i = 1,\ldots,p,\ j = 1,\ldots,q)$ and $\mathrm{a}_i, \mathrm{b}_j \in \mathbb{C}$. It is worthy to note that (1.4) converges absolutely in the entire complex z-plane when $\Delta := \sum_{j=1}^{q} \mathrm{B}_j - \sum_{i=1}^{p} \mathrm{A}_i > -1$, while if $\Delta = -1$, it is absolutely convergent for $|z| < \rho$ and $|z| = \rho$ under the restriction $\mathrm{Re}(\sigma) > 1/2$, where

$$\rho = \left(\prod_{i=1}^{p} \mathrm{A}_i^{-\mathrm{A}_i}\right)\left(\prod_{j=1}^{q} \mathrm{B}_j^{-\mathrm{B}_j}\right), \qquad \sigma = \sum_{j=1}^{q} \mathrm{b}_j - \sum_{i=1}^{p} \mathrm{a}_i + \frac{p-q}{2}.$$

For more details concerning the Fox–Wright functions, we refer to [20] and the references therein.

We processed to insert some special cases of the *generalized Lommel–Wright function.* The *Bessel–Maitland function* introduced by Pathak [26] as

$$J_{\kappa_1,\kappa_2}^{\kappa_3}(z) := J_{\kappa_1,\kappa_2}^{\kappa_3,1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n\kappa_3+\kappa_1+\kappa_2+1)\Gamma(n+\kappa_1+1)} \left(\frac{z}{2}\right)^{2n+2\kappa_1+\kappa_2},$$

is obtained by taking $m = 1$ in (1.3) for $\kappa_3 > 0$ and $\kappa_1,\kappa_2 \in \mathbb{C}$. Putting $\kappa_1 = 1/2$ and $m = \kappa_3 = 1$ in (1.3), we obtain the *Struve function* defined by

$$H_{\kappa_2}(z) := J_{1/2,\kappa_2}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\kappa_2+3/2)\Gamma(n+3/2)} \left(\frac{z}{2}\right)^{\kappa_2+2n}, \quad \kappa_2 \in \mathbb{C}.$$

For $\kappa_1 = 0$ and $m = \kappa_3 = 1$ in (1.3), we have the *Bessel function* defined by

$$J_{\kappa_2}(z) := J_{0,\kappa_2}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\kappa_2+1)} \left(\frac{z}{2}\right)^{\kappa_2+2n},$$

where $z \neq 0$, $z, \kappa_2 \in \mathbb{C}$, and $\mathrm{Re}\,\kappa_2 > -1$.

We shall need the following definition.

Definition 1 The normalization *of the combination of generalized Lommel–Wright function is defined by*

$$\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) := \Gamma^m(\kappa_1+1)\Gamma(\kappa_1+\kappa_2+1)2^{2\kappa_1+\kappa_2}z^{1-(\kappa_2/2)-\kappa_1}\mathcal{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(\sqrt{z}),$$

where $\mathcal{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) := (1-2\kappa_1-\kappa_2)J^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) + z(J^{\kappa_3,m}_{\kappa_1,\kappa_2}(z))'$ with $m \in \mathbb{N}$, $\kappa_1 \in \mathbb{C} \setminus \mathbb{Z}^-$, $\mathbb{Z}^- := \{-1,-2,-3,\dots\}$ and $\kappa_2, \kappa_3 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Clearly, $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)$ can be written as

$$\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) = z + \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{4^n[(\kappa_1+1)_n]^m(\kappa_1+\kappa_2+1)_{n\kappa_3}} z^{n+1}, \tag{1.5}$$

where $(\lambda)_n$ stands for the *Pochhammer symbol* given by

$$(\lambda)_n := \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

The next technical lemmas will be helpful to obtain the main results.

Lemma 1 ([18]) *For $x \in (0,\infty)$ and $k \in \mathbb{N}$, the following inequalities hold*:

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}, \tag{1.6}$$

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}. \tag{1.7}$$

Lemma 2 ([16]) *Suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers such that $A_1 = 1$. If $nA_n - (n+1)A_{n+1} \geq 0$ and $nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} \geq 0$ for all $n \in \mathbb{N}$. Then $f(z) = \sum_{n=1}^{\infty} A_n z^n$ is starlike in $\mathbb{U}$.*

Lemma 3 *Suppose that $a > 0$, $\kappa_3 \geq 1$, $4\kappa_3^2 \geq (\kappa_3 - 1)a$ and $\gamma \geq \max\{\gamma_1, \gamma_2\}$, where γ_1 is the greatest root of the quadratic equation*

$$-\kappa_3\gamma^2 - \gamma\kappa_3(2a+1) + 5(a+1)\kappa_3^2 = 0,$$

whilst γ_2 is the greatest root of the equation

$$(1-3\kappa_3)\gamma^2 + \gamma(1-3\kappa_3)(2a+1) + (a+1)\big(a - \kappa_3 a + 6\kappa_3^2\big) = 0,$$

then the following inequality

$$\frac{\Gamma(a+n\kappa_3)}{\Gamma(a+1)} \geq (a+\gamma)^{(n-1)\kappa_3}, \tag{1.8}$$

holds for all $n \geq 3$.

Proof We have to start by defining the functions $f(a)$ and $g(a)$ as follows:

$$f(a) := \frac{\Gamma(a+n\kappa_3)(a+\gamma)^{(1-n)\kappa_3}}{\Gamma(a+1)},$$

and

$$g(a) := \frac{f(a+1)}{f(a)} = \left(\frac{a+n\kappa_3}{a+1}\right)\left(\frac{a+\gamma+1}{a+\gamma}\right)^{(1-n)\kappa_3}.$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial a}g(a) &= \left(\frac{a+n\kappa_3}{a+1}\right)\left(\frac{a+\gamma+1}{a+\gamma}\right)^{(1-n)\kappa_3} \\ &\quad\times \frac{(1-n\kappa_3)(a+\gamma)(a+1+\gamma)-(1-n)\kappa_3(a+n\kappa_3)(a+1)}{(a+n\kappa_3)(a+1)(a+\gamma)(a+1+\gamma)}.\end{aligned}$$

Consider the function $\chi : [3,\infty) \to \mathbb{R}$ defined by

$$\chi(n) := (1-n\kappa_3)(a+\gamma)(a+1+\gamma) - (1-n)\kappa_3(a+n\kappa_3)(a+1).$$

We proceed to establish for which values of a, γ, and κ_3, the function $\chi(n)$ is increasing on $[3,\infty)$. This means that $\chi(n) \geq \chi(3)$ for all $n \geq 3$. On the other hand, if $\chi(3) \geq 0$, this implies to $\chi(n) \geq 0$, which leads to $a \mapsto g(a)$ is increasing for $n \geq 3$, $\kappa_3 \geq 1$, and $\gamma \geq \max\{\gamma_1, \gamma_2\}$. Now,

$$\begin{aligned}\chi'(n) &= -\kappa_3(a+\gamma)(a+1+\gamma) + \kappa_3(a+n\kappa_3)(a+1) + (n-1)\kappa_3^2(a+1) \\ &\geq -\kappa_3(a+\gamma)(a+\gamma+1) + \kappa_3(a+3\kappa_3)(a+1) + 2\kappa_3^2(a+1) \\ &= 5(a+1)\kappa_3^2 + \kappa_3\bigl[-(a+\gamma)(a+1+\gamma) + a(a+1)\bigr] \\ &= -\kappa_3\gamma^2 - \gamma\kappa_3(2a+1) + 5(a+1)\kappa_3^2.\end{aligned}$$

The last expression is positive if

$$\gamma \geq \frac{-(2a+1)+\sqrt{(2a+1)^2+20\kappa_3(a+1)}}{2} := \gamma_1, \quad \gamma_1 \geq 0.$$

It is worth mentioning that $\chi(3) \geq 0$ if

$$(1-3\kappa_3)\gamma^2 + \gamma(1-3\kappa_3)(2a+1) + (a+1)\bigl(a-\kappa_3 a + 6\kappa_3^2\bigr) \geq 0,$$

which holds if

$$\gamma \geq \frac{-(1-3\kappa_3)(2a+1) - \sqrt{(3\kappa_3-1)[24\kappa_3^2+(8a^2+8a+3)\kappa_3-1]}}{2(1-3\kappa_3)} := \gamma_2,$$

where $\gamma_2 \geq 0$ if

$$(3\kappa_3-1)\bigl[24\kappa_3^2 + \bigl(8a^2+8a+3\bigr)\kappa_3 - 1\bigr] > (1-3\kappa_3)^2(2a+1)^2,$$

which implies that

$$18\kappa_3^3 + 18a\kappa_3^3 - 3a^2\kappa_3^2 - 9a\kappa_3^2 - 6\kappa_3^2 + 4a^2\kappa_3 + 4a\kappa_3 - a - a^2$$

$$= 18\kappa_3^2(a+1) - 3a^2\kappa_3(a+3) + 4a\kappa_3(a+1) - a(a+1) - 6\kappa_3^2$$
$$= (a+1)\big[18\kappa_3^3 - 3a^2\kappa_3 + 4a\kappa_3 - a - 6\kappa_3^2\big] > 0.$$

If $\kappa_3 \geq 1$, then the term between the brackets is positive if $4\kappa_3^2 \geq (\kappa_3 - 1)a$. In addition, by making use of the asymptotic expansion of the ratio of gamma function $\Gamma(z+\alpha)/\Gamma(z+\beta)$, that is,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{z^2} + O\left(\frac{1}{z^2}\right)\right],$$

we obtain

$$\begin{aligned}
&\lim_{a\to+\infty} (a+\gamma)^{(1-n)\kappa_3}\frac{\Gamma(a+n\kappa_3)}{\Gamma(a+1)} \\
&\quad= \lim_{a\to+\infty}\left[a^{1-n\kappa_3}\frac{\Gamma(a+n\kappa_3)}{\Gamma(a+1)}\right]\left[a^{\kappa_3-1}\left(1+\frac{\gamma}{a}\right)^{(1-n)\kappa_3}\right] \\
&\quad= \lim_{a\to+\infty}\left[a^{\kappa_3-1}\left(1+\frac{\gamma}{a}\right)^{(1-n)\kappa_3}\right] \\
&\quad= \begin{cases} 1, & \text{if } \kappa_3 = 1, \\ +\infty, & \text{if } \kappa_3 > 1, \end{cases}
\end{aligned}$$

for $a > 0$, $\kappa_3 \geq 1$, $4\kappa_3 \geq (\kappa_3 - 1)a$, and $\gamma \geq \max\{\gamma_1, \gamma_2\}$, where γ_1 and γ_2 are given above and $\lim_{a\to\infty} g(a) = 1$. Bearing in mind that the function g is increasing for all $a > 0$, $\kappa_3 \geq 1$, $4\kappa_3 \geq (\kappa_3 - 1)a$, and $\gamma \geq \max\{\gamma_1, \gamma_2\}$, then $g(a) \leq 1$. Moreover, $f(a+1) \leq f(a)$, which leads to $(a)_{n\kappa_3} \geq a(a+\gamma)^{(n-1)\kappa_3}$ for all $a > 0$, $\kappa_3 \geq 1$, and $\gamma \geq \max\{\gamma_1, \gamma_2\}$. This completes the proof. □

2 Main results

Our first two theorems of this section contain some interesting and applicable results involving the order of starlikeness and the order of convexity inside $\mathbb{U}$ using some technical manipulations of the gamma and digamma functions which improve slightly the results given in [33].

Theorem 1 *Let $\kappa_1, \kappa_2 \geq 0$ such that $\kappa_1 + \kappa_2 \geq \frac{1}{2}$. Also, assume that $\kappa_3, m \in \mathbb{N}$ and*

$$0 \leq \alpha \leq 1 - \frac{3(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)e}{4\Gamma(\kappa_1+\kappa_2+\kappa_3+1) - 3(e-1)(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)} =: \widetilde{\alpha}_{\max}, \tag{2.1}$$

then $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z) \in \mathcal{S}^*(\alpha)$.

Proof To prove that $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z) \in \mathcal{S}^*(\alpha)$ for all $z \in \mathbb{U}$, it is sufficient to show that

$$\left|\frac{z(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z))'}{\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)} - 1\right| < 1 - \alpha,$$

for $z \in \mathbb{U}$. Using the maximum modulus theorem of an analytic function as well as the well-known inequality $|z_1 + z_2| \leq |z_1| + |z_2|$, we get

$$\begin{aligned}
\left|\left(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\right)' - \frac{\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)}{z}\right| &= \left|\sum_{n=1}^{\infty} \frac{(-1)^n n(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} z^n\right| \\
&= \left|\sum_{n=1}^{\infty} \frac{(-1)^n (2n^2+n)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} e^{in\theta}\right| \\
&< \sum_{n=1}^{\infty} \frac{n(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} \\
&= \sum_{n=1}^{\infty} \frac{\Gamma^m(\kappa_1+1)\Gamma(\kappa_1+\kappa_2+1)\Gamma(2n+2)\Gamma(n+1)}{4^n\Gamma(\kappa_1+\kappa_2+n\kappa_3+1)\Gamma^m(\kappa_1+n+1)\Gamma(2n+1)\Gamma(n)},
\end{aligned}$$

for $\theta \in \mathbb{R}$ and $z \in \mathbb{U}$. Using the fact that the gamma function satisfies $\Gamma(z+1) = z\Gamma(z)$, we get

$$\Gamma\left(z+\frac{1}{2}\right) = \frac{1 \cdot 3 \cdots (2n-1)}{\sqrt{\pi}},$$

and so

$$(2)_{2n} = 4^n (1)_n \left(\frac{3}{2}\right)_n, \quad n \in \mathbb{N}.$$

Now,

$$\begin{aligned}
&\left|\left(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\right)' - \frac{\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)}{z}\right| \\
&\quad < \frac{\Gamma^m(\kappa_1+1)\Gamma(\kappa_1+\kappa_2+1)}{\Gamma(3/2)} \\
&\qquad \times \sum_{n=1}^{\infty} \frac{[\Gamma(n+1)]^2\Gamma(n+3/2)}{\Gamma(\kappa_1+\kappa_2+1+n\kappa_3)\Gamma^m(\kappa_1+n+1)\Gamma(2n+1)\Gamma(n)}.
\end{aligned} \tag{2.2}$$

Suppose that

$$\mathrm{F}(n) := \frac{[\Gamma(n+1)]^2\Gamma(n+3/2)}{\Gamma(\kappa_1+\kappa_2+n\kappa_3+1)\Gamma^m(\kappa_1+n+1)\Gamma(2n+1)}, \quad n \in \mathbb{N}. \tag{2.3}$$

Differentiating (2.3) logarithmically with respect to n, we find

$$\begin{aligned}
\mathrm{F}'(n) &= \big[2\psi(n+1) + \psi(n+3/2) - \kappa_3\psi(\kappa_1+\kappa_2+1+\kappa_3 n) \\
&\quad - m\psi(\kappa_1+n+1) - 2\psi(2n+1)\big]\mathrm{F}(n) \\
&= \big[2\psi(n+1) - 2\psi(2n+1) + \psi(n+3/2) - \kappa_3\psi(\kappa_1+\kappa_2+1+\kappa_3 n) \\
&\quad - m\psi(\kappa_1+n+1)\big]\mathrm{F}(n),
\end{aligned} \tag{2.4}$$

here, ψ stands for the digamma function defined by

$$\psi(z) = \frac{\partial}{\partial z}\big[\log \Gamma(z)\big] = \frac{\Gamma'(z)}{\Gamma(z)}.$$

By using the fact that the digamma function is increasing on $(0,\infty)$ and $\psi(z) \geq 0$ for all $z \geq x^*$, where $x^* \simeq 1.461632144\ldots$ is the abscissa of the minimum of the gamma function, and with the help of (2.4), we deduce that the sequence $\{\mathrm{F}(n)\}_{n\geq 1}$ is decreasing. Then we get

$$\begin{aligned}\left|\left(\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)\right)' - \frac{\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)}{z}\right| &< \frac{3(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)}{4\Gamma(\kappa_1+\kappa_2+\kappa_3+1)}\sum_{n=1}^{\infty}\frac{1}{\Gamma(n)} \\ &= \frac{3e(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)}{4\Gamma(\kappa_1+\kappa_2+\kappa_3+1)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|\frac{\mathfrak{J}^{\lambda_3,m}_{\kappa_1,\kappa_2}(z)}{z}\right| &= \left|1+\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}z^n\right| \\ &\geq 1-\left|\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}z^n\right| \\ &\geq 1-\left|\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}e^{in\theta}\right| \\ &> 1-\frac{\Gamma^m(\kappa_1+1)\Gamma(\kappa_1+\kappa_2+1)}{\Gamma(3/2)} \\ &\quad\times\sum_{n=1}^{\infty}\frac{\Gamma(n+3/2)[\Gamma(n+1)]^2}{\Gamma(\kappa_1+\kappa_2+1+n\kappa_3)\Gamma^m(\kappa_1+n+1)\Gamma(2n+1)\Gamma(n+1)} \\ &\geq 1-\frac{3(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)}{4\Gamma(\kappa_1+\kappa_2+\kappa_3+1)}\sum_{n=1}^{\infty}\frac{1}{\Gamma(n+1)} \\ &= \frac{4\Gamma(\kappa_1+\kappa_2+\kappa_3+1)-3(e-1)(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)}{4\Gamma(\kappa_1+\kappa_2+\kappa_3+1)},\end{aligned}$$

for $\theta \in \mathbb{R}$ and $z \in \mathbb{U}$. Putting everything together, we see that

$$\left|\frac{z(\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z))'}{\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)}-1\right| < \frac{3(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)e}{4\Gamma(\kappa_1+\kappa_2+\kappa_3+1)-3(e-1)(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)}, \quad z \in \mathbb{U},$$

and then we conclude that $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) \in \mathcal{S}^*(\alpha)$. □

Theorem 2 *Suppose that* $\kappa_1 \geq 0, \kappa_2 \geq 0, \kappa_3, m \in \mathbb{N}$, *and*

$$0 \leq \alpha \leq 1-\frac{3(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)e}{2\Gamma(\kappa_1+\kappa_2+\kappa_3+1)-3(e-1)(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)} =: \widehat{\alpha}_{\max}, \qquad (2.5)$$

then $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) \in \mathcal{K}(\alpha)$.

Proof To prove that $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z) \in \mathcal{K}(\alpha)$ for all $z \in \mathbb{U}$, it is sufficient to show that

$$\left|\frac{z(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z))''}{(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z))'} - 1\right| < 1 - \alpha,$$

for $z \in \mathbb{U}$. As in Theorem 1, we shall base the proof on the maximum modulus theorem of an analytic function to get

$$\begin{aligned}
\left|z\big(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\big)''\right| &= \left|\sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} z^n\right| \\
&< \sum_{n=1}^{\infty} \frac{n(n+1)(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} \\
&= \sum_{n=1}^{\infty} \frac{n(n+1)(2n+1)\Gamma(\kappa_1+\kappa_2+1)\Gamma^m(\kappa_1+1)}{4^n\Gamma(\kappa_1+\kappa_2+n\kappa_3+1)\Gamma^m(\kappa_1+n+1)} \frac{(2)_n(2)_{2n}(1)_n}{(2)_n(2)_{2n}(1)_n} \\
&= \frac{\Gamma(\kappa_1+\kappa_2+1)\Gamma^m(\kappa_1+1)}{\Gamma(3/2)} \\
&\quad \times \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(n+2)\Gamma(n+3/2)}{\Gamma(\kappa_1+\kappa_2+1+n\kappa_3)\Gamma^m(\kappa_1+n+1)\Gamma(2n+1)\Gamma(n)},
\end{aligned}$$

for $z \in \mathbb{U}$. Using the increasing property of the digamma functions, it is easy to observe that

$$\mathrm{G}(n) = \frac{\Gamma(n+1)\Gamma(n+2)\Gamma(n+3/2)}{\Gamma(\kappa_1+\kappa_2+1+n\kappa_3)\Gamma^m(\kappa_1+n+1)\Gamma(2n+1)}, \quad n \in \mathbb{N},$$

is a strictly decreasing function of n. Thus, we get

$$\left|z\big(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\big)''\right| < \frac{3(\kappa_1+1)^{-m} e\Gamma(\kappa_1+\kappa_2+1)}{2\Gamma(\kappa_1+\kappa_2+\kappa_3+1)}.$$

Further computations yield

$$\begin{aligned}
\left|\big(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\big)'\right| &\geq 1 - \left|\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} z^n\right| \\
&> 1 - \sum_{n=1}^{\infty} \frac{(n+1)(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} \\
&= 1 - \frac{\Gamma(\kappa_1+\kappa_2+1)\Gamma^m(\kappa_1+1)}{\Gamma(3/2)} \\
&\quad \times \sum_{n=1}^{\infty} \frac{\Gamma(n+2)\Gamma(n+3/2)\Gamma(n+1)}{\Gamma(\kappa_1+\kappa_2+1+n\kappa_3)\Gamma^m(\kappa_1+n+1)\Gamma(2n+1)\Gamma(n+1)} \\
&\geq 1 - \frac{3(\kappa_1+2)^{-m}\Gamma(\kappa_1+\kappa_2+1)}{2\Gamma(\kappa_1+\kappa_2+\kappa_3+1)} \sum_{n=1}^{\infty} \frac{1}{\Gamma(n+1)} \\
&= \frac{2\Gamma(\kappa_1+\kappa_2+\kappa_3+1) - 3(e-1)(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)}{2\Gamma(\kappa_1+\kappa_2+\kappa_3+1)}.
\end{aligned}$$

Combining everything together to get

$$\left|\frac{z(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z))''}{(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z))'}-1\right|<\frac{3(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)e}{2\Gamma(\kappa_1+\kappa_2+\kappa_3+1)-3(e-1)(\kappa_1+1)^{-m}\Gamma(\kappa_1+\kappa_2+1)},\quad z\in\mathbb{U},$$

and from the above inequality, we conclude that $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\in\mathcal{K}(\alpha)$. □

Remark 1 It is worth noting that special cases will follow if we set $\kappa_1=0$, $\kappa_3=m=1$, and $\kappa_1=1/2$, $\kappa_3=m=1$, respectively, in Theorems 1 and 2.

In the following results, that is, Theorems 3 and 4, the starlikeness and convexity with its order have been evaluated where the leading concept of the proofs comes from Lemma 1.

Theorem 3 *Assume that $\kappa_1,\kappa_2,\kappa_3$ are positive numbers, $m\in\mathbb{N}$ such that*

$$m\ln(\kappa_1+2)+\kappa_3\ln(\kappa_1+\kappa_2+1+\kappa_3)-\frac{m}{\kappa_1+2}-\frac{\kappa_3}{\kappa_1+\kappa_2+1+\kappa_3}\geq\frac{5}{3},$$

and

$$0\leq\alpha\leq 1-\frac{\Gamma(\kappa_1+\kappa_2+1)}{(\kappa_1+1)^m\Gamma(\kappa_1+\kappa_2+1+\kappa_3)-\Gamma(\kappa_1+\kappa_2+1)}=:\widetilde{\delta}_{\max},\tag{2.6}$$

then $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\in\mathcal{S}^*(\alpha)$.

Proof From Theorem 1, we have

$$\begin{aligned}\left|\left(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\right)'-\frac{\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)}{z}\right|&<\frac{\Gamma^m(\kappa_1+1)\Gamma(\kappa_1+\kappa_2+1)}{4}\\&\quad\times\sum_{n=1}^{\infty}\frac{n(2n+1)}{4^{n-1}\Gamma^m(\kappa_1+n+1)\Gamma(\kappa_1+\kappa_2+1+n\kappa_3)},\end{aligned}$$

for $z\in\mathbb{U}$. Letting

$$\mathrm{D}_1(x)=\frac{x(2x+1)}{\Gamma^m(\kappa_1+x+1)\Gamma(\kappa_1+\kappa_2+1+\kappa_3x)},\quad x\geq 1.\tag{2.7}$$

Hence,

$$\frac{\mathrm{D}_1'(x)}{\mathrm{D}_1(x)}=\frac{1}{x}+\frac{2}{2x+1}-m\psi(\kappa_1+x+1)-\kappa_3\psi(\kappa_1+\kappa_2+1+\kappa_3x):=\mathrm{D}_2(x)$$

and

$$\mathrm{D}_2'(x)=-\frac{1}{x^2}-\frac{4}{(2x+1)^2}-m\frac{\partial}{\partial x}\psi(\kappa_1+x+1)-\kappa_3^2\frac{\partial}{\partial x}\psi(\kappa_1+\kappa_2+1+\kappa_3x).$$

From Lemma 1, we have

$$\mathrm{D}_2'(x)<-\frac{1}{x^2}-\frac{4}{(2x+1)^2}-m\left(\frac{1}{\kappa_1+x+1}+\frac{1}{2(\kappa_1+x+1)^2}\right)$$

$$- \kappa_3^2\left(\frac{1}{\kappa_1+\kappa_2+1+\kappa_3 x}+\frac{1}{2(\kappa_1+\kappa_2+1+\kappa_3 x)^2}\right)<0,$$

under the given hypotheses, which leads to $\mathrm{D}_2(x)$ is a strictly decreasing function on $[1,\infty)$ with $\mathrm{D}_2(1)<0$ to conclude that $\mathrm{D}_2(x)<0$ for all $x\geq 1$. Consequently, $\mathrm{D}_1'(x)<0$ under the given hypotheses, that is, $\mathrm{D}_1(x)$ is a strictly decreasing function on $[1,\infty)$ and

$$\begin{aligned}\left|\left(\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\right)'-\frac{\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)}{z}\right| &< \frac{3\Gamma^m(\kappa_1+1)\Gamma(\kappa_1+\kappa_2+1)}{4\Gamma^m(\kappa_1+2)\Gamma(\kappa_1+\kappa_2+1+\kappa_3)}\sum_{n=1}^{\infty}\frac{1}{4^{n-1}}\\ &= \frac{\Gamma(\kappa_1+\kappa_2+1)}{(\kappa_1+1)^m\Gamma(\kappa_1+\kappa_2+1+\kappa_3)}.\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}\left|\frac{\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)}{z}\right| &\geq 1-\frac{3\Gamma(\kappa_1+\kappa_2+1)}{4(\kappa_1+1)^m\Gamma(\kappa_1+\kappa_2+\kappa_3+1)}\sum_{n=1}^{\infty}\frac{1}{4^{n-1}}\\ &= \frac{(\kappa_1+1)^m\Gamma(\kappa_1+\kappa_2+\kappa_3+1)-\Gamma(\kappa_1+\kappa_2+1)}{(\kappa_1+1)^m\Gamma(\kappa_1+\kappa_2+1+\kappa_3)},\end{aligned}$$

for $z\in\mathbb{U}$, which ultimates our proof. □

Using arguments similar to Theorem 3, we get the following result regarding the order of convexity by using (1.6) and (1.7).

Theorem 4 *Assume that* $\kappa_1,\kappa_2,\kappa_3$ *are positive numbers,* $m\in\mathbb{N}$ *such that*

$$m\ln(\kappa_1+2)+\kappa_3\ln(\kappa_1+\kappa_2+1+\kappa_3)-\frac{m}{\kappa_1+2}-\frac{\kappa_3}{\kappa_1+\kappa_2+1+\kappa_3}\geq\frac{13}{6},$$

and

$$0\leq\alpha\leq 1-\frac{2\Gamma(\kappa_1+\kappa_2+1)}{(\kappa_1+1)^m\Gamma(\kappa_1+\kappa_2+1+\kappa_3)-2\Gamma(\kappa_1+\kappa_2+1)}=:\widehat{\delta}_{\max}, \tag{2.8}$$

then $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)\in\mathcal{K}(\alpha)$.

In the next two theorems, we are going with other results including the order of starlikeness and the order of convexity that are evaluated using the sharp inequalities for the shifted factorial, which improve slightly the results given in [33].

Theorem 5 *Suppose that*

$$0\leq\alpha\leq 1-\frac{\mathrm{R}_1}{\mathrm{R}_2}=:\widetilde{\zeta}_{\max},$$

where

$$\begin{aligned}\mathrm{R}_1 &= 48(\kappa_1+2)^m\Gamma(\kappa_1+\kappa_2+2)\Gamma(\kappa_1+\kappa_2+1+2\kappa_3)\\ &\quad\times\left[(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\right]\end{aligned}$$

$$+40\Gamma(\kappa_1+\kappa_2+2)\Gamma(\kappa_1+\kappa_2+1+\kappa_3)\big[(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\big]$$
$$+21(\kappa_1+2)^m(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}\prod_{\ell=1}^{2}\Gamma(\kappa_1+\kappa_2+1+\ell\kappa_3),$$

and

$$\begin{aligned}\mathrm{R}_2 = {} & 64(\kappa_1+\kappa_2+1)\prod_{\ell=1}^{2}\Gamma(\kappa_1+\kappa_2+1+\ell\kappa_3)\prod_{\ell=1}^{2}(\kappa_1+\ell)^m \\ & \times\big[(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\big]-48(\kappa_1+2)^m\Gamma(\kappa_1+\kappa_2+1+2\kappa_3) \\ & \times\Gamma(\kappa_1+\kappa_2+2)\big[(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\big] \\ & -20\Gamma(\kappa_1+\kappa_2+2)\Gamma(\kappa_1+\kappa_2+1+\kappa_3)\big[(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\big] \\ & -7(\kappa_1+2)^m(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}\prod_{\ell=1}^{2}\Gamma(\kappa_1+\kappa_2+1+\ell\kappa_3),\end{aligned}$$

with $\kappa_1>-1$, $\kappa_2\geq 0$, $\kappa_3\geq 1$, $\gamma\geq\max\{\gamma_1,\gamma_2\}$, *where* γ_1 *and* γ_2 *are given in Lemma* 3 *and* $\mathrm{R}_2>0$, *then* $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}\in\mathcal{S}^*(\alpha)$.

Proof To begin with, we note that if $f\in\mathcal{A}$ satisfies $\sum_{n=2}^{\infty}(n-\alpha)|A_n|\leq 1-\alpha$, then $f\in\mathcal{S}^*(\alpha)$ (see [30, Theorem 1]). Therefore, according to (1.5), it is sufficient to show that

$$\mathrm{H}_1:=\sum_{n=2}^{\infty}(n-\alpha)\left|\frac{(-1)^{n-1}(2n-1)}{4^{n-1}(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_{n-1}]^m}\right|\leq 1-\alpha.$$

Since $\kappa_1>-1$, $\kappa_2\geq 0$, and $\kappa_3\geq 1$, we have

$$\begin{aligned}\mathrm{H}_1 &= \sum_{n=1}^{\infty}\frac{(2n+1)(n+1-\alpha)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} \\ &= \frac{3(1-\alpha)}{4(\kappa_1+\kappa_2+1)_{\kappa_3}(\kappa_1+1)^m}+\frac{5(3-\alpha)}{16(\kappa_1+\kappa_2+1)_{2\kappa_3}(\kappa_1+1)^m(\kappa_1+2)^m} \\ &\quad+\sum_{n=3}^{\infty}\frac{n(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} \\ &\quad+(1-\alpha)\sum_{n=3}^{\infty}\frac{2n+1}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}.\end{aligned}$$

Using the fact that

$$n(2n+1)\leq(21/64)\cdot 4^n,\qquad 2n+1\leq(7/64)\cdot 4^n,\quad n\geq 3,$$

which can be verified using the concept of mathematical induction and

$$(\kappa_1+\kappa_2+1)(\kappa_1+\kappa_2+1+\gamma)^{(n-1)\kappa_3}\leq(\kappa_1+\kappa_2+1)_{n\kappa_3},$$

for all $\kappa_1 > -1$, $\kappa_2 \geq 0$, $\kappa_3 \geq 1$, and $\gamma \geq \max\{\gamma_1, \gamma_2\}$ that follows from Lemma 3, we obtain

$$\begin{aligned}
\mathrm{H}_1 &\leq \frac{3(1-\alpha)}{4(\kappa_1+\kappa_2+1)_{\kappa_3}(\kappa_1+1)^m} + \frac{5(3-\alpha)}{16(\kappa_1+\kappa_2+1)_{2\kappa_3}(\kappa_1+1)^m(\kappa_1+2)^m} \\
&\quad + \frac{21}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\sum_{n=3}^{\infty}\frac{1}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3(n-1)}(\kappa_1+1+\gamma)^{m(n-1)}} \\
&\quad + \frac{7(1-\alpha)}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\sum_{n=3}^{\infty}\frac{1}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3(n-1)}(\kappa_1+1+\gamma)^{m(n-1)}} \\
&= \frac{3(1-\alpha)}{4(\kappa_1+\kappa_2+1)_{\kappa_3}(\kappa_1+1)^m} + \frac{5(3-\alpha)}{16(\kappa_1+\kappa_2+1)_{2\kappa_3}(\kappa_1+1)^m(\kappa_1+2)^m} \\
&\quad + \frac{21}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\cdot\frac{(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1} \\
&\quad + \frac{7(1-\alpha)}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\cdot\frac{(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1} \\
&\leq 1-\alpha.
\end{aligned}$$

Thus, we conclude that $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m} \in \mathcal{S}^*(\alpha)$, as required. □

Theorem 6 *Suppose that*

$$0 \leq \alpha \leq 1 - \frac{\mathrm{T}_1}{\mathrm{T}_2} =: \widehat{\zeta}_{\max},$$

where

$$\begin{aligned}
\mathrm{T}_1 &= 96(\kappa_1+2)^m\Gamma(\kappa_1+\kappa_2+2)\Gamma(\kappa_1+\kappa_2+1+2\kappa_3) \\
&\quad \times \prod_{\ell=1}^{2}\bigl[\ell(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr] \\
&\quad + 120\Gamma(\kappa_1+\kappa_2+2)\Gamma(\kappa_1+\kappa_2+1+\kappa_3)\prod_{\ell=1}^{2}\bigl[\ell(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr] \\
&\quad + 84(\kappa_1+2)^m(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}\prod_{\ell=1}^{2}\Gamma(\kappa_1+\kappa_2+1+\ell\kappa_3) \\
&\quad \times \bigl[2(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr],
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{T}_2 &= 64(\kappa_1+\kappa_2+1)\prod_{\ell=1}^{2}\Gamma(\kappa_1+\kappa_2+1+\ell\kappa_3)\prod_{\ell=1}^{2}(\kappa_1+\ell)^m \\
&\quad \times \prod_{\ell=1}^{2}\bigl[\ell(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr] \\
&\quad - 96(\kappa_1+2)^m\Gamma(\kappa_1+\kappa_2+1+2\kappa_3)\Gamma(\kappa_1+\kappa_2+2)
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{\ell=1}^{2}\bigl[\ell(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr] \\
&-60\Gamma(\kappa_1+\kappa_2+1+\kappa_3)\Gamma(\kappa_1+\kappa_2+2)\prod_{\ell=1}^{2}\bigl[\ell(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr] \\
&-21(\kappa_1+2)^m(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}\prod_{\ell=1}^{2}\Gamma(\kappa_1+\kappa_2+1+\ell\kappa_3) \\
&\times\bigl[2(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr] \\
&-14(\kappa_1+2)^m(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m} \\
&\times\bigl[(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1\bigr]\prod_{\ell=1}^{2}\Gamma(\kappa_1+\kappa_2+1+\ell\kappa_3),
\end{aligned}
$$

with $\kappa_1>-1$, $\kappa_2\geq 0$, $\kappa_3\geq 1$, $\gamma\geq\max\{\gamma_1,\gamma_2\}$, *where* γ_1 *and* γ_2 *are given in Lemma* 3 *and* $\mathrm{T}_2>0$, *then* $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}\in\mathcal{K}(\alpha)$.

Proof Using the Alexander duality relation and according to [30, Corollary on p. 110], it suffices to show that

$$
\mathrm{H}_2:=\sum_{n=2}^{\infty}n(n-\alpha)\left|\frac{(-1)^{n-1}(2n-1)}{4^{n-1}(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_{n-1}]^m}\right|\leq 1-\alpha. \tag{2.9}
$$

Since $\kappa_1>-1$, $\kappa_2\geq 0$, and $\kappa_3\geq 1$, we have

$$
\begin{aligned}
\mathrm{H}_2 &= \frac{6(2-\alpha)}{4(\kappa_1+\kappa_2+1)_{\kappa_3}(\kappa_1+1)^m}+\frac{15(3-\alpha)}{16(\kappa_1+\kappa_2+1)_{2\kappa_3}(\kappa_1+1)^m(\kappa_1+2)^m} \\
&+\sum_{n=3}^{\infty}\frac{n^2(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}+(2-\alpha)\sum_{n=3}^{\infty}\frac{n(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} \\
&+(1-\alpha)\sum_{n=3}^{\infty}\frac{(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}.
\end{aligned}
$$

Recalling the fact that

$$
\begin{aligned}
&n^2(2n+1)\leq(63/64)\cdot 4^n, \qquad n(2n+1)\leq(21/64)\cdot 4^n, \\
&2n+1\leq(7/8)\cdot 2^n, \quad n\geq 3,
\end{aligned}
$$

and Lemma 3, it follows that

$$
\begin{aligned}
\mathrm{H}_2 &\leq \frac{6(2-\alpha)}{4(\kappa_1+\kappa_2+1)_{\kappa_3}(\kappa_1+1)^m}+\frac{15(3-\alpha)}{16(\kappa_1+\kappa_2+1)_{2\kappa_3}(\kappa_1+1)^m(\kappa_1+2)^m} \\
&+\frac{63}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\sum_{n=3}^{\infty}\frac{1}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3(n-1)}(\kappa_1+1+\gamma)^{m(n-1)}} \\
&+\frac{21(2-\alpha)}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\sum_{n=3}^{\infty}\frac{1}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3(n-1)}(\kappa_1+1+\gamma)^{m(n-1)}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{7(1-\alpha)}{16(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\sum_{n=3}^{\infty}\frac{1}{2^{n-1}(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3(n-1)}(\kappa_1+1+\gamma)^{m(n-1)}}\\
&=\frac{6(2-\alpha)}{4(\kappa_1+\kappa_2+1)_{\kappa_3}(\kappa_1+1)^m}+\frac{15(3-\alpha)}{16(\kappa_1+\kappa_2+1)_{2\kappa_3}(\kappa_1+1)^m(\kappa_1+2)^m}\\
&+\frac{63}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\cdot\frac{(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1}\\
&+\frac{21(2-\alpha)}{64(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\cdot\frac{(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}}{(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1}\\
&+\frac{7(1-\alpha)}{32(\kappa_1+\kappa_2+1)(\kappa_1+1)^m}\cdot\frac{(\kappa_1+\kappa_2+1+\gamma)^{-\kappa_3}(\kappa_1+1+\gamma)^{-m}}{2(\kappa_1+\kappa_2+1+\gamma)^{\kappa_3}(\kappa_1+1+\gamma)^m-1}\\
&\leq 1-\alpha.
\end{aligned}
$$

This proves the claim that $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}\in\mathcal{K}(\alpha)$. □

Remark 2 1. Theorem 1, Theorem 3, and Theorem 5 assign sufficient conditions for starlikeness of $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}$. As it appears in Table 1, the first one gives better result than the second and the third ones for suitable choices of the parameters.

2. Due to Theorem 2, Theorem 4, and Theorem 6, which assign sufficient conditions for convexity of $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}$, it is important to observe as that Theorem 2 sometimes gives a better estimation than the others, while occasionally Theorem 4 is the best. See Table 2.

In the remainder of this section, we shall use Lemma 2 to prove $\mathcal{I}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)=\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)*z/(1+z)$ is in the class of starlike and convex functions, respectively.

Theorem 7 *If* $\kappa_1\geq(\sqrt{13}-3)/2\simeq 0.302776\ldots$ *and* $\kappa_2,\kappa_3,m\in\mathbb{N}$, *then* $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)*z/(1+z)$ *is starlike in* $\mathbb{U}$.

Table 1 Comparison of the order of starlikeness given by Theorems 1, 3 and 5

κ_1	κ_2	κ_3	m	$\widetilde{\alpha}_{\max}$ Theorem 1	$\widetilde{\delta}_{\max}$ Theorem 3	$\widetilde{\zeta}_{\max}$ Theorem 5 with min γ
3	2	1	2	0.97847446275028873123	0.98947368421052631578	0.99208830838480806639
7	3	1	2	0.99709879234283707259	0.998577524893314366999	0.99998335359501052005
4	3	2	2	0.99886657109218533610	0.99944413563090605892	0.99958307190258777911
9	6	3	2	0.99999583595437582265	0.99999795751216812126	0.99999846813358621261
5	5	3	4	0.99999908328632086372	0.99999955034657038088	0.99999966275993086008
12	10	5	3	0.99999999990421235635	0.99999999995301559358	0.99999999996476169518
14	13	8	5	0.99999999999999999717	0.99999999999999999861	0.99999999999999999896

Table 2 Comparison of the order of convexity given by Theorems 2, 4 and 6

κ_1	κ_2	κ_3	m	$\widehat{\alpha}_{\max}$ Theorem 2	$\widehat{\delta}_{\max}$ Theorem 4	$\widehat{\zeta}_{\max}$ Theorem 6 with min γ
3	2	1	3	0.98930995964631139740	0.99476439790575916230	0.99607281772648946631
7	3	1	4	0.99990949804677554452	0.99995560882496559684	0.99996670645961244179
4	3	2	3	0.99954682324769373114	0.99977772838408535230	0.99983329678044753707
9	6	3	2	0.99999167188683056198	0.999995915015992712389	0.99999969362732106008
5	5	3	4	0.99999816657157930414	0.99999910069273638516	0.9999993255195994225
12	10	5	3	0.99999999980842471269	0.99999999990603118715	0.99999999995364882391
14	13	8	5	0.99999999999999999434	0.99999999999999999722	0.99999999999999999792

Proof From (1.5) and bearing in mind that $z/(1+z)$ can be expressed as

$$\frac{z}{1+z} = z + \sum_{n=1}^{\infty}(-1)^n z^{n+1},$$

we have

$$\mathfrak{I}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z) * \frac{z}{1+z} = \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{U}, \tag{2.10}$$

where

$$A_n = \frac{2n-1}{4^{n-1}[(\kappa_1+1)_{n-1}]^m(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}} \quad \text{for all } n \in \mathbb{N},$$

Thanks to Lemma 2, we shall prove that $nA_n \geq (n+1)A_{n+1}$ and $nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} \geq 0$ for all $n \in \mathbb{N}$, where $A_1 = 1$, $A_n > 0$ for all $n \geq 2$. Bearing in mind that

$$\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3} \leq (\kappa_1+\kappa_2+1)_{n\kappa_3}, \tag{2.11}$$

for $\kappa_1 \geq (\sqrt{13}-3)/2$ and $\kappa_2, \kappa_3, m \in \mathbb{N}$, it is easy to observe that

$$\begin{aligned} &nA_n - (n+1)A_{n+1} \\ &\quad = \frac{n(2n-1)}{4^{n-1}(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_{n-1}]^m} - \frac{(n+1)(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m} \\ &\quad \geq \frac{1}{4^n(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_{n-1}]^m} \\ &\qquad \times \left[4n(2n-1) - \frac{(n+1)(2n+1)(\kappa_1+n)^{-m}}{\kappa_1+\kappa_2+1+(n-1)\kappa_3}\right], \end{aligned} \tag{2.12}$$

that is,

$$nA_n - (n+1)A_{n+1} \geq \frac{\mathrm{U}(n)}{4^n(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_n]^m(\kappa_1+\kappa_2+1+(n-1)\kappa_3)},$$

where

$$\mathrm{U}(n) := 4n(2n-1)\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)(\kappa_1+n)^m - (n+1)(2n+1).$$

Since $\kappa_1 \geq (\sqrt{13}-3)/2$ and $\kappa_2, \kappa_3, m \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{U}(n) &:= 4n(2n-1)\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)(\kappa_1+n)^m - (n+1)(2n+1) \\ &\geq 4n(2n-1)\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)(\kappa_1+n) - (n+1)(2n+1) \\ &= 8\kappa_3 n^4 + (8\kappa_1\kappa_3 + 8\kappa_1 + 8\kappa_2 + 8 - 12\kappa_3)n^3 \\ &\quad + \big(8\kappa_1^2 + 4\kappa_1 + 8\kappa_1\kappa_2 - 12\kappa_1\kappa_3 - 4\kappa_2 + 4\kappa_3 - 6\big)n^2 \\ &\quad + \big(-4\kappa_1^2 - 4\kappa_1\kappa_2 - 4\kappa_1 + 4\kappa_1\kappa_3 - 3\big)n - 1 =: \widetilde{\mathrm{U}}(n). \end{aligned}$$

It is worth mentioning that $(n+1)A_{n+1} \leq nA_n$ if $\widetilde{\mathrm{U}}(n) \geq 0$ for all $n \in \mathbb{N}$. Noting that

$$1 - 4\kappa_1^2 - 8\kappa_1 - 4\kappa_2 - 4\kappa_1\kappa_2 \leq -1,$$

which holds for $\kappa_1 \geq (\sqrt{13}-3)/2$ and $\kappa_2, \kappa_3, m \in \mathbb{N}$, it follows that

$$\begin{aligned}
\widetilde{\mathrm{U}}(n) \geq{}& 8\kappa_3 n^4 + (8\kappa_1\kappa_3 + 8\kappa_1 + 8\kappa_2 + 8 - 12\kappa_3)n^3 \\
&+ \bigl(8\kappa_1^2 + 4\kappa_1 + 8\kappa_1\kappa_2 - 12\kappa_1\kappa_3 - 4\kappa_2 + 4\kappa_3 - 6\bigr)n^2 \\
&+ \bigl(-4\kappa_1^2 - 4\kappa_1\kappa_2 - 4\kappa_1 + 4\kappa_1\kappa_3 - 3\bigr)n + 1 - 4\kappa_1^2 - 8\kappa_1 - 4\kappa_2 - 4\kappa_1\kappa_2 \\
={}& (n-1)\bigl[8\kappa_3 n^3 + (8 + 8\kappa_1 - 4\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2)n^2 \\
&+ \bigl(2 + 12\kappa_1 + 8\kappa_1^2 - 4\kappa_1\kappa_3 + 4\kappa_2 + 8\kappa_1\kappa_2\bigr)n \\
&- 1 + 8\kappa_1 + 4\kappa_1^2 + 4\kappa_2 + 4\kappa_1\kappa_2\bigr].
\end{aligned}$$

Furthermore, since $\kappa_1 \geq (\sqrt{13}-3)/2$ and $\kappa_2, \kappa_3, m \in \mathbb{N}$, we have

$$-1 + 8\kappa_1 + 4\kappa_1^2 + 4\kappa_2 + 4\kappa_1\kappa_2 \geq -10 - 20\kappa_1 - 8\kappa_1^2 - 12\kappa_2 - 8\kappa_1\kappa_2 - 4\kappa_3 - 4\kappa_1\kappa_3,$$

and so

$$\begin{aligned}
\widetilde{\mathrm{U}}(n) \geq{}& (n-1)\bigl[8\kappa_3 n^3 + (8 + 8\kappa_1 - 4\kappa_3 + 8\kappa_3 + 8\kappa_2)n^2 \\
&+ \bigl(2 + 12\kappa_1 + 8\kappa_1^2 - 4\kappa_1\kappa_3 + 4\kappa_2 + 8\kappa_1\kappa_2\bigr)n \\
&- 10 - 20\kappa_1 - 8\kappa_1^2 - 12\kappa_2 - 8\kappa_1\kappa_2 - 4\kappa_3 - 4\kappa_1\kappa_3\bigr] \\
={}& (n-1)^2\bigl[8\kappa_3 n^2 + (8 + 4\kappa_3 + 8\kappa_1 + 8\kappa_1\kappa_3 + 8\kappa_2)n \\
&+ 10 + 20\kappa_1 + 8\kappa_1^2 + 4\kappa_3 + 4\kappa_1\kappa_3 + 12\kappa_2 + 8\kappa_1\kappa_2\bigr].
\end{aligned}$$

Continuing in this manner we get

$$\begin{aligned}
\widetilde{\mathrm{U}}(n) &\geq (n-1)^3[8\kappa_3 n + 8 + 8\kappa_1 + 12\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2] \\
&\geq 8\kappa_3(n-1)^3 \geq 0, \quad \kappa_3, n \in \mathbb{N}.
\end{aligned}$$

It remains to show that $nA_n + (n+2)A_{n+2} \geq 2(n+1)A_{n+1}$ for all $n \in \mathbb{N}$. Since $A_{n+2} > 0$ for all $n \in \mathbb{N}$, we easily get

$$\begin{aligned}
&nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} \\
&\quad > nA_n - 2(n+1)A_{n+1} \\
&\quad = \frac{1}{4^{n-1}}\left[\frac{n(2n-1)}{(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_{n-1}]^m} - \frac{2(n+1)(2n+1)}{4(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}\right].
\end{aligned}$$

Using (2.11), we have

$$nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2}$$

$$\geq \frac{1}{4^{n-1}(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_{n-1}]^m} \times \left[4n(2n-1) - \frac{2(n+1)(2n+1)(\kappa_1+n)^{-m}}{\kappa_1+\kappa_2+1+(n-1)\kappa_3}\right], \tag{2.13}$$

and so,

$$\begin{aligned} &nA_n - 2(n+1)A_{n+1} + (n+2)A_{n+2} \\ &\quad \geq \frac{\mathrm{V}(n)}{4^n(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_n]^m(\kappa_1+\kappa_2+1+(n-1)\kappa_3)}, \end{aligned}$$

where

$$\mathrm{V}(n) := 4n(2n-1)\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)(\kappa_1+n)^m - 2(n+1)(2n+1).$$

Since $\kappa_1 \geq (\sqrt{13}-3)/2$ and $\kappa_2, \kappa_3, m \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{V}(n) &\geq 4n(2n-1)(\kappa_1+n)\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big) - 2(n+1)(2n+1) \\ &= 8\kappa_3 n^4 + (8\kappa_1\kappa_3 + 8\kappa_1 + 8\kappa_2 + 8 - 12\kappa_3)n^3 \\ &\quad + \big(8\kappa_1^2 + 4\kappa_1 + 8\kappa_1\kappa_2 - 12\kappa_1\kappa_3 - 4\kappa_2 + 4\kappa_3 - 8\big)n^2 \\ &\quad + \big(-4\kappa_1^2 - 4\kappa_1\kappa_2 - 4\kappa_1 + 4\kappa_1\kappa_3 - 6\big)n - 2 =: \widetilde{\mathrm{V}}(n). \end{aligned}$$

Again, since $\kappa_1 \geq (\sqrt{13}-3)/2$ and $\kappa_2, \kappa_3, m \in \mathbb{N}$, we have

$$\begin{aligned} \widetilde{\mathrm{V}}(n) &\geq (n-1)\big[8\kappa_3 n^3 + (8 + 8\kappa_1 - 4\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2)n^2 \\ &\quad + \big(12\kappa_1 + 8\kappa_1^2 - 4\kappa_1\kappa_3 + 4\kappa_2 + 8\kappa_1\kappa_2\big)n \\ &\quad - 6 + 8\kappa_1 + 4\kappa_1^2 + 4\kappa_2 + 4\kappa_1\kappa_2\big] \\ &\geq (n-1)^2\big[8\kappa_3 n^2 + (8 + 8\kappa_1 + 4\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2)n + 8 + 20\kappa_1 + 8\kappa_1^2 \\ &\quad + 4\kappa_3 + 4\kappa_1\kappa_3 + 12\kappa_2 + 8\kappa_1\kappa_2\big] \\ &\geq (n-1)^3[8\kappa_3 n + 8 + 8\kappa_1 + 12\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2] \\ &\geq 8\kappa_3(n-1)^4 \geq 0 \quad \text{for } \kappa_3, n \in \mathbb{N}, \end{aligned}$$

which ends the proof. □

Theorem 8 *Suppose that* $\kappa_1 \geq 0$ *and* $\kappa_2, \kappa_3, m \in \mathbb{N}$. *Then* $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z) * z/(1+z)$ *is starlike in* $\mathbb{U}$.

Proof Under the hypotheses $\kappa_1 \geq 0$ and $\kappa_2, \kappa_3, m \in \mathbb{N}$, Lemma 2 and Theorem 7, we proceed to showing that $(n+1)A_{n+1} \leq nA_n$ and $nA_n + (n+2)A_{n+2} \geq 2(n+1)A_{n+1}$ for all $n \in \mathbb{N}$. At first, for $n = 1$, $\mathrm{U}(1) = 4(\kappa_1+1)^m(\kappa_1+\kappa_2+1) - 6 \geq 4(\kappa_1+1)(\kappa_1+2) - 6 > 0$ for $\kappa_1 \geq 0$, whilst for $n \geq 2$, we find

$$\frac{(n+1)(2n+1)}{4} < n(2n-1), \quad n \geq 2. \tag{2.14}$$

Further, it can be shown that

$$\Phi(n)=\frac{\Gamma(\kappa_1+\kappa_2+1)\Gamma^m(\kappa_1+1)}{\Gamma(\kappa_1+\kappa_2+1+(n-1)\kappa_3)\Gamma^m(\kappa_1+n)},$$

is a decreasing function with respect to n as follows:

$$\frac{\Phi'(n)}{\Phi(n)}=-m\psi(\kappa_1+n)-\kappa_3\psi\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big).$$

Since $\kappa_1\geq 0$ and $\kappa_2,\kappa_3,m\in\mathbb{N}$, we get $\psi(\kappa_1+n)\geq 0$ and $\psi(\kappa_1+\kappa_2+1+(n-1)\kappa_3)\geq 0$, these together with (2.12) lead to $nA_n\geq (n+1)A_{n+1}$, $n\geq 2$. A similar argument may be used to prove that $nA_n+(n+2)A_{n+2}\geq 2(n+1)A_{n+1}$. For $n=1,2$, it is easy to prove $nA_n+(n+2)A_{n+2}\geq 2(n+1)A_{n+1}$, whereas for $n\geq 3$, we have

$$\frac{(n+1)(2n+1)}{2}<n(2n-1),\qquad n\geq 3, \tag{2.15}$$

and since $\mathrm{V}(n)$ is a decreasing function with respect to $n, n\geq 3$, this would lead to $nA_n+(n+2)A_{n+2}\geq 2(n+1)A_{n+1}$ for $n\in\mathbb{N}$, which asserts our claim. □

Remark 3 It is important to note that Theorem 8 extends the range of validity for parameter κ_1 to $\kappa_1\geq 0$.

Theorem 9 *If* $\kappa_1,\kappa_2,\kappa_3,m\in\mathbb{N}$, *then*, $\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)*z/(1+z)$ *is convex in* $\mathbb{U}$.

Proof Using the classical Alexander theorem between the classes of starlike and convex functions, which asserts that $f(z)\in\mathcal{K}$ if and only if $zf'(z)\in\mathcal{S}^*$, it is sufficient to prove that $z(\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z))'*z/(1+z)$ is starlike in $\mathbb{U}$. We then have

$$z\big(\mathfrak{J}^{\kappa_3,m}_{\kappa_1,\kappa_2}(z)\big)'*\frac{z}{1+z}=\sum_{n=1}^{\infty}B_nz^n,\qquad z\in\mathbb{U},$$

where

$$B_n=\frac{n(2n-1)}{4^{n-1}[(\kappa_1+1)_{n-1}]^m(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}}\qquad\text{for all } n\geq 1.$$

To obtain the required result, we will use Lemma 2. It suffices to show that

$$(n+1)B_{n+1}\leq nB_n\quad\text{ and }\quad nB_n+(n+2)B_{n+2}\geq 2(n+1)B_{n+1}\quad\text{ for all } n\in\mathbb{N}.$$

We shall show that $(n+1)B_{n+1}\leq nB_n$ for all $n\in\mathbb{N}$ as follows:

$$\begin{aligned}
&nB_n-(n+1)B_{n+1}\\
&\quad=\frac{n^2(2n-1)}{4^{n-1}(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_{n-1}]^m}-\frac{(n+1)^2(2n+1)}{4^n(\kappa_1+\kappa_2+1)_{n\kappa_3}[(\kappa_1+1)_n]^m}\\
&\quad\geq\frac{\mathrm{W}(n)}{4^n(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}[(\kappa_1+1)_n]^m(\kappa_1+\kappa_2+1+(n-1)\kappa_3)},
\end{aligned}$$

where

$$\mathrm{W}(n) := 4n^2(2n-1)\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)(\kappa_1+n)^m-(n+1)^2(2n+1).$$

Since $\kappa_1,\kappa_2,\kappa_3,m\in\mathbb{N}$, we have

$$\begin{aligned}\mathrm{W}(n) &\geq 4n^2(2n-1)(\kappa_1+n)\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)-(n+1)^2(2n+1)\\ &= 8\kappa_3n^5+(8+8\kappa_1-12\kappa_3+8\kappa_1\kappa_3+8\kappa_2)n^4\\ &\quad+\big(-6+4\kappa_1+8\kappa_1^2+4\kappa_3-12\kappa_1\kappa_3-4\kappa_2+8\kappa_1\kappa_2\big)n^3\\ &\quad+\big(-5-4\kappa_1-4\kappa_1^2+4\kappa_1\kappa_3-4\kappa_1\kappa_2\big)n^2-4n-1=\widetilde{\mathrm{W}}(n).\end{aligned}$$

Obviously, $nB_n\geq(n+1)B_{n+1}$ if $\widetilde{\mathrm{W}}(n)\geq 0$ for all $n\in\mathbb{N}$. Bearing in mind that $\kappa_1,\kappa_3,\kappa_2\in\mathbb{N}$, it follows that

$$\begin{aligned}\widetilde{\mathrm{W}}(n) &\geq 8\kappa_3n^5+(8+8\kappa_1-12\kappa_3+8\kappa_1\kappa_3+8\kappa_2)n^4\\ &\quad+\big(-6+4\kappa_1+8\kappa_1^2+4\kappa_3-12\kappa_1\kappa_3-4\kappa_2+8\kappa_1\kappa_2\big)n^3\\ &\quad+\big(-5-4\kappa_1+8\kappa_1^2+4\kappa_3-12\kappa_1\kappa_3-4\kappa_2+8\kappa_1\kappa_2\big)n^2\\ &\quad-4n+7-8\kappa_1-4\kappa_1^2-4\kappa_2-4\kappa_1\kappa_2\\ &\geq(n-1)\big[8\kappa_3n^4+(8+8\kappa_1-4\kappa_3+8\kappa_1\kappa_3+8\kappa_2)n^3\\ &\quad+\big(2+12\kappa_1+8\kappa_1^2-4\kappa_1\kappa_3+4\kappa_2+8\kappa_1\kappa_2\big)n^2\\ &\quad+\big(-3+8\kappa_1+4\kappa_1^2+4\kappa_2+4\kappa_1\kappa_2\big)n-7+8\kappa_1+4\kappa_1^2+4\kappa_2+4\kappa_1\kappa_2\big]\\ &\geq(n-1)^2\big[8\kappa_3n^3+(8+8\kappa_1+4\kappa_3+8\kappa_1\kappa_3+8\kappa_2)n^2\\ &\quad+\big(2+12\kappa_1+8\kappa_1^2-4\kappa_1\kappa_3+4\kappa_2+8\kappa_1\kappa_2\big)n\\ &\quad+7+28\kappa_1+12\kappa_1^2+4\kappa_3+4\kappa_1\kappa_3+16\kappa_2+12\kappa_1\kappa_2\big]\\ &\geq(n-1)^3\big[8\kappa_3n^2+(8+8\kappa_1+12\kappa_3+8\kappa_1\kappa_3+8\kappa_2)n\\ &\quad+18+28\kappa_1+8\kappa_1^2+16\kappa_3+12\kappa_1\kappa_3+20\kappa_2+8\kappa_1\kappa_2\big]\\ &\geq(n-1)^4[8\kappa_3n+8+8\kappa_1+20\kappa_3+8\kappa_1\kappa_3+8\kappa_2]\\ &\geq 8\kappa_3(n-1)^5\geq 0.\end{aligned}$$

On the other hand, we prove that $nB_n+(n+2)B_{n+2}\geq 2(n+1)B_{n+1}$ for all $n\in\mathbb{N}$. We have

$$\begin{aligned}&nB_n-2(n+1)B_{n+1}+(n+2)B_{n+2}\\ &\quad\geq\frac{\mathrm{Y}(n)}{4^n[(\kappa_1+1)_n]^m(\kappa_1+\kappa_2+1)_{(n-1)\kappa_3}(\kappa_1+\kappa_2+1+(n-1)\kappa_3)},\end{aligned}$$

where

$$\begin{aligned}\mathrm{Y}(n) &:= 4n^2(2n-1)(\kappa_1+n)^m\big(\kappa_1+\kappa_2+1+(n-1)\kappa_3\big)-2(n+1)^2(2n+1)\\ &\geq 4n^2(2n-1)(\kappa_1+n)\big(\kappa_1+\kappa_2+1+(n-1)\lambda_3\big)-2(n+1)^2(2n+1)\end{aligned}$$

$$\begin{aligned}
&= 8\kappa_3 n^5 + (8 + 8\kappa_1 - 12\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2)n^4 + \bigl(-8 + 4\kappa_1 \\
&\quad + 8\kappa_1^2 + 4\kappa_3 - 12\kappa_1\kappa_3 - 4\kappa_2 + 8\kappa_1\kappa_2\bigr)n^3 \\
&\quad + \bigl(-10 - 4\kappa_1 - 4\kappa_1^2 + 4\kappa_1\kappa_3 - 4\kappa_1\kappa_2\bigr)n^2 - 8n - 2 = \widetilde{\Upsilon}(n).
\end{aligned}$$

For $\kappa_1, \kappa_3, \kappa_2 \in \mathbb{N}$, we find

$$\begin{aligned}
\widetilde{\Upsilon}(n) &\geq (n-1)\bigl[8\kappa_3 n^4 + (8 + 8\kappa_1 - 4\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2)n^3 \\
&\quad + \bigl(12\kappa_1 + 8\kappa_1^2 - 4\kappa_1\kappa_3 + 4\kappa_2 + 8\kappa_1\kappa_2\bigr)n^2 \\
&\quad + \bigl(-10 + 8\kappa_1 + 4\kappa_1^2 + 4\kappa_2 + 4\kappa_1\kappa_2\bigr)n - 18 + 8\kappa_1 + 4\kappa_1^2 + 4\kappa_2 + 4\kappa_1\kappa_2\bigr] \\
&\geq (n-1)^2\bigl[8\kappa_3 n^3 + (8 + 8\kappa_1 + 4\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2)n^2 \\
&\quad + \bigl(8 + 20\kappa_1 + 8\kappa_1^2 + 4\kappa_3 + 4\kappa_1\kappa_3 + 12\kappa_2 + 8\kappa_1\kappa_2\bigr)n \\
&\quad - 2 + 28\kappa_1 + 12\kappa_1^2 + 4\kappa_3 + 4\kappa_1\kappa_3 + 16\kappa_2 + 12\kappa_1\kappa_2\bigr] \\
&\geq (n-1)^3\bigl[8\kappa_3 n^2 + (8 + 8\kappa_1 + 12\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2)n \\
&\quad + 16 + 28\kappa_1 + 8\kappa_1^2 + 16\kappa_3 + 12\kappa_1\kappa_3 + 20\kappa_2 + 8\kappa_1\kappa_2\bigr] \\
&\geq (n-1)^4[8\kappa_3 n + 8 + 8\kappa_1 + 20\kappa_3 + 8\kappa_1\kappa_3 + 8\kappa_2] \\
&\geq 8\kappa_3(n-1)^5 \geq 0,
\end{aligned}$$

which completes the proof. We can propose another proof for the same result as outlined below. For $n = 1$, $\mathrm{W}(1) = 4(\kappa_1 + 1)^m(\kappa_1 + \kappa_2 + 1) - 12 \geq 4(\kappa_1 + 1)(\kappa_1 + 2) - 12 > 0$ for $\kappa_1 \geq 1$, whilst for $n \geq 2$, we find

$$\frac{(n+1)^2(2n+1)}{4} < n^2(2n-1), \quad n \geq 2,$$

and since $\mathrm{W}(n)$ is a decreasing function with respect to $n, n \geq 2$, we get $nB_n - (n+1)B_{n+1} \geq 0$, $n \in \mathbb{N}$. Secondly, it is easy to prove that $nB_n + (n+2)B_{n+2} \geq 2(n+1)B_{n+1}$ for $n = 1, 2$, whereas for $n \geq 3$, we have

$$\frac{(n+1)^2(2n+1)}{2} < n^2(2n-1), \quad n \geq 3,$$

and since $\Upsilon(n)$ is a decreasing function with respect to $n, n \geq 3$, it follows that $nB_n + (n+2)B_{n+2} \geq 2(n+1)B_{n+1}$ for $n \in \mathbb{N}$, and according to Lemma 2, we end the proof of the theorem. □

3 Conclusions

In the current paper, we have reported conditions for $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)$ to be starlike and convex of order α, $0 \leq \alpha < 1$, inside the open unit disk using some technical manipulations of the gamma and digamma functions as well as an inequality for the digamma function that has been proved in [18]. In addition, a method presented by Lorch [22] and further developed by Laforgia [21] has been applied to establish firstly sharp inequalities for the shifted factorial that would be used to obtain the order starlikeness and convexity. We then have compared the obtained orders of starlikeness and convexity with some important consequences in the literature as well as the results proposed by all techniques to demonstrate the accuracy of our approach. We conclude the paper showing that the modified form of the function $\mathfrak{J}_{\kappa_1,\kappa_2}^{\kappa_3,m}(z)$ is in the class of starlike and convex functions. Further investigations on this topic are now underway and will be reported in forthcoming papers.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
HMZ has a major contributor to writing the main manuscript text and analyzing the results, reviewing, and editing. KH contributed to analyzing all the results and making necessary improvements. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin Elkom 32511, Egypt. [2]Department of Mathematics, Kairouan Preparatory Institute for Engineering Studies, University of Kairouan, 3100 Kairouan, Tunisia.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 April 2022 Accepted: 29 August 2022 Published online: 07 September 2022

References

1. Aktaş, İ., Baricz, Á.: Bounds for radii of starlikeness of some q-Bessel functions. Results Math. **72**(1), 947–963 (2017)
2. Aktaş, İ., Baricz, Á., Orhan, H.: Bounds for radii of starlikeness and convexity of some special functions. Turk. J. Math. **42**(1), 211–226 (2018)
3. Aktaş, İ., Baricz, Á., Singh, S.: Geometric and monotonic properties of hyper-Bessel functions. Ramanujan J. **51**(2), 275–295 (2020)
4. Aktaş, İ., Baricz, Á., Yağmur, N.: Bounds for the radii of univalence of some special functions. Math. Inequal. Appl. **20**(3), 825–843 (2017)
5. Aktaş, İ., Orhan, H.: Bounds for radii of convexity of some q-Bessel functions. Bull. Korean Math. Soc. **57**(2), 355–369 (2020)
6. Ali, R.M., Lee, S.K., Mondal, S.R.: Starlikeness of a generalized Bessel function. Bull. Belg. Math. Soc. Simon Stevin **25**(4), 527–540 (2018)
7. Baricz, Á.: Bessel transforms and Hardy space of generalized Bessel functions. Mathematica **48(71)**(2), 127–136 (2006)
8. Baricz, Á.: Geometric properties of generalized Bessel functions of complex order. Mathematica **48(71)**(1), 13–18 (2006)
9. Baricz, Á.: Geometric properties of generalized Bessel functions. Publ. Math. (Debr.) **73**(1–2), 155–178 (2008)
10. Baricz, Á., Dimitrov, D.K., Orhan, H., Yağmur, N.: Radii of starlikeness of some special functions. Proc. Am. Math. Soc. **144**(8), 3355–3367 (2016)
11. Baricz, Á., Kupán, P.A., Szász, R.: The radius of starlikeness of normalized Bessel functions of the first kind. Proc. Am. Math. Soc. **142**(6), 2019–2025 (2014)
12. Baricz, Á., Ponnusamy, S.: Starlikeness and convexity of generalized Bessel functions. Integral Transforms Spec. Funct. **21**(9), 641–653 (2010)
13. Baricz, Á., Szász, R.: Close-to-convexity of some special functions and their derivatives. Bull. Malays. Math. Sci. Soc. **39**(1), 427–437 (2016)
14. de Oteiza, M.B.M., Kalla, S., Conde, S.: Un estudio sobre la función Lommel–Maitland. Rev. Téc. Fac. Ing., Univ. Zulia **9**(2), 33–40 (1986)
15. Duren, P.L.: Univalent Functions. Springer, New York (1983)
16. Fejér, L.: Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge. Acta Litt. Sci. **8**, 89–115 (1936)
17. Goodman, A.W.: Univalent Functions, Vols. 1–2. Mariner, Tampa (1983)

18. Guo, B.-N., Qi, F.: Refinements of lower bounds for polygamma functions. Proc. Am. Math. Soc. **141**(3), 1007–1015 (2013)
19. Hayman, W.K.: Multivalent Functions. Cambridge University Press, Cambridge (1958)
20. Kilbas, A.A., Saigo, M., Trujillo, J.J.: On the generalized Wright function. Fract. Calc. Appl. Anal. **5**(4), 437–460 (2002)
21. Laforgia, A.: Further inequalities for the gamma function. Math. Compet. **42**(166), 597–600 (1984)
22. Lorch, L.: Inequalities for ultraspherical polynomials and the gamma function. J. Approx. Theory **40**(2), 115–120 (1984)
23. Mehrez, K.: Some geometric properties of a class of functions related to the Fox–Wright functions. Banach J. Math. Anal. **14**, 1222–1240 (2020)
24. Miller, S.S., Mocanu, P.T.: Univalence of Gaussian and confluent hypergeometric functions. Proc. Am. Math. Soc. **110**(2), 333–342 (1990)
25. Orhan, H., Yağmur, N.: Geometric properties of generalized Struve functions. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. **63**(2), 229–244 (2017)
26. Pathak, R.S.: Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations. Proc. Natl. Acad. Sci., India **A–36**(1), 81–86 (1966)
27. Pommerenke, C.: Univalent Functions. Vandenhoeck & Ruprecht, Gottingen (1975)
28. Ponnusamy, S.: The Hardy space of hypergeometric functions. Complex Var. Theory Appl. **29**(1), 83–96 (1996)
29. Ponnusamy, S., Rønning, F.: Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a halfplane. Integral Transforms Spec. Funct. **8**, 121–138 (1999)
30. Silverman, H.: Univalent functions with negative coefficients. Proc. Am. Math. Soc. **51**(1), 109–116 (1975)
31. Yağmur, N.: Hardy space of Lommel functions. Bull. Korean Math. Soc. **52**, 1035–1046 (2015)
32. Yağmur, N., Orhan, H.: Hardy space of generalized Struve functions. Complex Var. Elliptic Equ. **59**(7), 929–936 (2014)
33. Zayed, H.M., Bulboaca, T.: On some geometric properties for the combination of generalized Lommel–Wright function. J. Inequal. Appl. **2021**, 158 (2021). https://doi.org/10.1186/s13660-021-02690-z
34. Zayed, H.M., Bulboaca, T., Morais, J.: The geometric characterizations for a combination of generalized Struve functions. Comput. Methods Funct. Theory (2021). https://doi.org/10.1007/s40315-021-00421-5