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Some inequalities and majorization for products of τ-measurable operators

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Abstract

In this paper, for a semi-finite von Neumann algebra $\mathcal{M}$, we study the Young, Hölder and Heinz means inequalities and extend results for τ-measurable operators. We obtain some refinements of the those inequalities for τ-measurable operators. We have also presented several inequalities in the sense of majorization.

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1 Introduction and preliminaries

Purpose of this paper is to formulate some of the norm matrix inequalities at the concept of von Neumann algebras. The inequalities include forms of the arithmetic-geometric mean, the Cauchy-Schwarz, the Heinz mean inequality, the Young inequality, and some inequalities related to the Bourin question for τ-measurable operators. Among many other studies, we focus more on the papers [4, 12] devoted to extensions to unitary invariant norms on spaces of matrices. We also improve and present a new method for obtaining the inequalities mentioned in [10, 11]. A number of new inequalities based on [20] are presented. We obtain the case of equality for most of these inequalities by the method used in [9, 19].

In this section, we set up some notations and certain terminologies and give their basic properties. Let $\mathcal{H}$ be an infinite dimensional Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators in $\mathcal{H}$. In what follows, $\mathcal{N}$ is a von Neumann algebra on $\mathcal{H}$, that is a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ closed in the weak operator topology. The identity in $\mathcal{N}$ is denoted by 1. A von Neumann algebra is said to be σ-finite if it admits at most countably many orthogonal projections. We are only interested in semi-finite von Neumann algebras, that is, those which admit a faithful normal semi-finite trace τ. We fix a couple $(\mathcal{M},\tau)$ for semi-finite von Neumann algebra $\mathcal{M}$ with semi-finite trace τ. The cone of positive operators, the identity, and the projection lattice in $\mathcal{M}$ are denoted by $\mathcal{M}^+$, 1 and $\mathcal{P}(\mathcal{M})$, respectively.

An (unbounded) operator x with domain $\mathcal{D}(x) \subseteq \mathcal{H}$ is densely defined if $\mathcal{D}(x)$ is dense in $\mathcal{H}$. The operator x is called closed whenever its graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$. If $x : \mathcal{D}(x) \to \mathcal{H}$ is a closed densely defined linear operator, then it can be shown that the operator x^*x is self-adjoint and positive. The modulus $|x|$ of x is defined by $|x| = (x^*x)^{\frac{1}{2}}$. It is well known that for every self-adjoint operator a, there exists a unique spectral measure $e^a : \mathcal{B}(\mathbb{R}) \to \mathbf{B}(\mathcal{H})$ such that

$$a = \int_{\mathbb{R}} \lambda \, de^a(\lambda)$$

as a spectral decomposition of a.

Now, we are ready to introduce the non-commutative L_p-spaces. A linear operator $x : \mathcal{D}(x) \to \mathcal{H}$ is called affiliated with $\mathcal{M}$, if $ux = xu$ for all unitary $u \in \mathcal{M}'$. Note that if $x \in B(\mathcal{H})$, then x is affiliated with $\mathcal{M}$ if and only if $x \in \mathcal{M}$.

A closed and densely defined linear operator $x : \mathcal{D}(x) \to \mathcal{H}$ is said to be τ-measurable if x affiliated with $\mathcal{M}$, and there exists $\lambda \geq 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$. In fact, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of orthogonal projections in $\mathcal{M}$ such that $p_n(\mathcal{H}) \subseteq \mathcal{D}(x)$ for all $n, p_n \uparrow 1$ and $\tau(1 - p_n) \downarrow 0$ as $n \to \infty$. The collection of all τ-measurable operators is denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a complex $*$-algebra with unit element 1. The von Neumann algebra $\mathcal{M}$ is a $*$-subalgebra of $L_0(\mathcal{M})$.

The measure topology on $L_0(\mathcal{M})$ is defined by fundamental systems of neighborhoods around zero is given by

$$\nu(\epsilon, \delta) = \big\{x \in L_0(\mathcal{M}) : \exists p \in \mathcal{P}(\mathcal{M}) \text{ s.t. } \|px\| < \epsilon \text{ and } \tau(1 - p) < \delta\big\},$$

where ϵ and δ run over all strictly positive numbers. It is known [8] that $\mathcal{M}$ is dense in $L_0(\mathcal{M})$. In fact, if $x = u|x| \in L_0(\mathcal{M})$ and $|x| = \int_0^{\infty} \lambda \, de^{|x|}(\lambda)$, then the sequences $\{u \int_0^n \lambda \, de^{|x|}(\lambda)\}_{n=0}^{\infty}$ in $\mathcal{M}$ tends to x as $n \to \infty$ in the measure topology.

Let x be a τ-measurable operator and $t > 0$. The tth singular value of x (or generalized s-numbers) is the number denoted by $\mu_t(x)$ and for each $t \in \mathbb{R}_0^+$ (set of nonnegative real numbers) is defined by

$$\mu_t(x) = \inf\big\{\|xe\| : e \in \mathcal{P}(\mathcal{M}), \tau(1 - e) \leq t\big\}.$$

The notation of generalized s-numbers for τ-measurable operators was carefully developed by T. Fack and H. Kosaki [8]. For every $x \in \mathcal{M}$, $\mu_t(x)$ is nonincreasing and right continuous. As well as we have

$$\mu_t\big(|x|\big) = \mu_t\big(x^*\big) = \mu_t(x).$$

In the following theorem, we collect some other known [7, 8] facts on s-numbers that we will use later.

Theorem 1.1 *Let x, y be τ-measurable operators and $a, b \in \mathcal{M}$. Then for $t \in \mathbb{R}_0^+$ and $\alpha \in \mathbb{C}$,*

(1) $\mu_t(\alpha x) = |\alpha|\mu_t(x)$.
(2) $\mu_t(x^r) = \mu_t(x)^r$, *for any positive real number r and positive τ-measurable operator x.*
(3) $\mu_t(xy) = \mu_t((xy)^*) = \mu_t(y^*x^*)$ *and besides that $\mu_t(xy) = \mu_t(yx)$ for hermitian operators x and y.*
(4) $\mu_t(|xy^*|) = \mu_t(|x|.|y|)$.
(5) $\mu_t(axb) \leq \|a\|\mu_t(x)\|b\|$, *if $x \leq y$, then $\mu_t(x) \leq \mu_t(y)$.*

(6) $|\mu_t(x) - \mu_t(y)| \leq \|x - y\|$.
(7) $\mu_{s+t}(x + y) \leq \mu_s(x) + \mu_t(y)$, *for* $s \geq 0$.
(8) *if* $p \in \mathcal{P}(\mathcal{M})$, *then* $\mu_t(xp) = 0$ *for each* $t \geq \tau(p)$.
(9) $\tau(|x|) = \int_0^\infty \mu_t(x)\,dt$.
(10) *let* f *be a bounded continuous increasing function on* $[0,\infty)$ *with* $f(0) = 0$. *If* $x \in \mathcal{M}^+$, *then*

$$\mu_t\bigl(f(x)\bigr) = f\bigl(\mu_t(x)\bigr),$$

and

$$\tau\bigl(f(x)\bigr) = \int_0^{\tau(1)} f\bigl(\mu_t(x)\bigr)\,dt. \tag{1}$$

(11) *let* f *be continuous increasing function on* $[0,\infty)$ *such that* $f(e^t)$ *is convex, then*

$$\int_0^s f\bigl(\mu_t(xy)\bigr) \leq \int_0^s f\bigl(\mu_t(x)\mu_t(y)\bigr)\,dt, \quad \textit{for all} \quad s > 0.$$

(12) $\int_0^s f(\mu_t(x+y))\,dt \leq \int_0^s f(\mu_t(x) + \mu_t(y))\,dt$ *for any convex continuous increasing function* f *on* $\mathbb{R}^+$.
(13) $\int_0^s \mu_t(xy)^r\,dt \leq \int_0^s \mu_t(x)^r \mu_t(y)^r\,dt$, *for all* $s \in \mathbb{R}_0^+$ *and positive* τ*-measurable operator* x, y.
(14) *for positive real numbers* r, p, q *such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ *and* x, y *be positive* τ*-measurable operators, we have*

$$\frac{1}{r}\int_0^s \mu_t(xy)^r\,dt \leq \frac{1}{p}\int_0^s \mu_t(x)^p\,dt + \frac{1}{q}\int_0^s \mu_t(y)^q\,dt.$$

The next result was stated in [8, Proposition 3.2].

Proposition 1.2 *For* τ*-measurable operator* x, *the following conditions are equivalent*:
1 $\tau(p^x(s,\infty)) < \infty$ *for all* $s > 0$,
2 $\lim_{t\to\infty} \mu_t(x) = 0$,
3 *there exists a sequence of bounded operators* $x_n \in L_1(\mathcal{M})$ *such that* $x_n \to x$ *in the measure topology.*

With any of these three characterizations, we say that x is τ-compact operator. These operators form a complete bilateral ideal in $L_0(\mathcal{M})$ that we will denote by $\mathcal{K}(L_0(\mathcal{M}))$. It is known that for every positive operator $x \in \mathcal{K}(L_0(\mathcal{M}))$,

$$\sigma(x) = \overline{\bigl\{\mu_t(x) : t > 0\bigr\}}.$$

Note that a τ-compact operator is not necessarily bounded.

For $0 < p < \infty$, $L_p(\mathcal{M},\tau)$ is defined as the set of all τ-measurable operators x such that

$$\|x\|_p = \tau\bigl(|x|^p\bigr)^{\frac{1}{p}} < \infty. \tag{2}$$

Moreover, we put $L_\infty(\mathcal{M},\tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty$ the usual operator norm [2, 5, 8, 21]. For simplicity from now on $L_p(\mathcal{M},\tau)$ will denoted by $L_p(\mathcal{M})$. In this paper, we establish an analogue for majorization type of the Young and Heinz inequalities in the setting of operators affiliated to semi-finite von Neumann algebras. By using these results, we generalize the $\|\cdot\|_p$ type of those inequalities for τ- measurable operators in Theorems 2.25 and 3.3.

2 Young and Cauchy-Schwarz inequalities

The Young inequality is a well-known inequality, valid for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ is usually stated as

$$\alpha\beta \le \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q, \tag{3}$$

for any positive real numbers α and β, equality holds if and only if $\alpha^p = \beta^q$. Several generalisations of the Young inequality where α and β are replaced by Hilbert space operators or by singular values, norms, or traces of operators are known. For more references and further discussion on the subject of the Young inequality for matrices and operators, we refer the reader to [1, 6, 9, 20]. In particular, we remark that it was the fundamental paper by T. Ando [1], which initiated the study of the Young inequality for the singular values of $n \times n$ matrices. The present paper adds to these results by formulating new Young-type norm inequalities in $L_p(\mathcal{M})$. Farenick and Manjegani [9] proved the case of equality in the Young inequality for operators in a semi-finite von Neumann algebra.

Proposition 2.1 ([9]) *Let x, y be operators in $\mathcal{M}$.*

$$\mu_t\big(|xy^*|\big) \le \mu_t\left(\frac{1}{p}|x|^p + \frac{1}{q}|y|^q\right), \quad \text{for all } t > 0. \tag{4}$$

Moreover, if $x, y \in L_1(\mathcal{M})$ and bounded, then equality holds if and only if

$$|x|^p = |y|^q.$$

In the above proposition, x and y are bounded operators playing important role in the proof of the equality case in (4). In [19, Theorem 3.3], the authors extended (4) to singular values of τ-measurable operators. Since τ-measurable operators are not necessarily bounded with finite trace, the same proof for the case of equality in last proposition does not work. Still, we do not know for operators in $L_p(\mathcal{M})$: Does $\mu_t(|xy^*|) = \mu_t(\frac{1}{p}|x|^p + \frac{1}{q}|y|^q)$ for all $t > 0$ imply that $|x|^p = |y|^q$?

Let $1 \le p < \infty$, an operator $x \in \mathcal{M}$ is said to be locally integrable if there exists $\delta > 0$ such that

$$\int_0^\delta \mu_t(x)^p\, dt < \infty.$$

The set containing all these operators is denoted by $\mathfrak{L}^p_{\mathrm{loc}}(\mathcal{M})$. Note that, in particular, all bounded operators $a \in \mathcal{M}$ are of this class. Moreover,

$$\int_0^\delta \mu_t(x)^p\, dt \ge \mu_\delta(x)^{p-1} \int_0^\delta \mu_t(x)\, dt$$

implies that $\mathfrak{L}^p_{\mathrm{loc}}(\mathcal{M}) \subset \mathfrak{L}^1_{\mathrm{loc}}(\mathcal{M})$ for each $p \geq 1$. In [19], Larotonda proved the following theorem that is answer to the above question for τ-compact operators.

Theorem 2.2 ([19]) *Let $a, b \in \mathcal{K}(L_0(\mathcal{M}))^+$ with $ab \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$. If*

$$\mu_t(|ab|) = \mu_t\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \quad \textit{for all} \quad t > 0,$$

then $a^p = b^q$. If $p = q = 2$, it suffices to assume that $ab \in \mathfrak{L}^1_{\mathrm{loc}}(\mathcal{M})$.

According to part (4) of Theorem 1.1, we have the following corollary.

Corollary 2.3 *Let $x, y \in \mathcal{K}(L_0(\mathcal{M}))$ with $xy \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$. If*

$$\mu_t(|xy^*|) = \mu_t\left(\frac{1}{p}|x|^p + \frac{1}{q}|y|^q\right) \quad \textit{for all} \quad t > 0,$$

then $|x|^p = |y|^q$.

This powerful version of the Young inequality for singular values is important in that all results on τ-measurable operators, such as norms inequalities and majorization, can be deduced from it. So, it makes sense to find more general cases of this inequality. However, in considering the young inequality, it is not known whether the above theorem has a formulation in which it is true that

$$\mu_t(|azb|) \leq \mu_t\left(\frac{1}{p}|az|^p + \frac{1}{q}|bz|^q\right),$$

when a, b are positive τ-measurable operators, and $z \in L_0(\mathcal{M})$. However, this topic exceeds the scope of this paper.

As a result of [19, Theorem 3.3], for every $x, y \in L_1(\mathcal{M})$, we have

$$\|xy^*\|_1 \leq \left\|\frac{1}{p}|x|^p + \frac{1}{q}|y|^q\right\|_1 \leq \frac{1}{p}\||x|^p\|_1 + \frac{1}{q}\||y|^q\|_1. \tag{5}$$

If x and y are bounded operators or $xy \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, then equality holds if and only if $|x|^p = |y|^q$ using Proposition 2.1 or Theorem 2.2. More general, using similar argument in [20, Corollary 2.5] and part (4) in Theorem 1.1, we have Hölder and Young inequalities within the $\|\cdot\|_1$ in $L_1(\mathcal{M})$.

Corollary 2.4 *If $x, y \in L_1(\mathcal{M})$, then for all positive real numbers p, q, and r with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,*

$$\|xy^*\|_1^r \leq \left(\||x|^p\|_1\right)^{\frac{r}{p}}\left(\||y|^q\|_1\right)^{\frac{r}{q}} \leq \frac{r}{p}\||x|^p\|_1 + \frac{r}{q}\||y|^q\|_1.$$

Moreover, if x and y are bounded operators or $xy \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, then equality holds if and only if $|x|^p = |y|^q$.

Theorem 2.5 (Cases of Equality in Tracial Hölder and Young Inequalities) *Assume that a, b are positive bounded operators in $L_p(\mathcal{M})$ and $p > 1$. The following statements are equivalent.*

1 $\|ab\|_1 = \|a^p\|_1^{1/p}\|b^q\|_1^{1/q}$;
2 $\||ab|\|_1 = \|a^p\|_1^{1/p}\|b^q\|_1^{1/q}$;
3 $\|ab\|_1 = \frac{1}{p}\|a^p\|_1 + \frac{1}{q}\|b^q\|_1$;
4 $\||ab|\|_1 = \frac{1}{p}\|a^p\|_1 + \frac{1}{q}\|b^q\|_1$;
5 $b^q = a^p$.

Proof Using similar method used in the proof of [9, Theorem 3.6]. □

In the rest of this section, first, we demonstrate an extension of refinement of the Young inequality for τ-measurable operators [16, 17]. Kittaneh and Manasrah gave a refinement of the Young inequality as follows in [17]:

For $m = 1, 2, \dots$,

$$\left(a^\nu b^{1-\nu}\right)^m + \left(\min\{\nu, 1-\nu\}\right)^m \left(\sqrt{a^m} - \sqrt{b^m}\right)^2 \leq \left(\nu a + (1-\nu)b\right)^m, \tag{6}$$

where a, b are positive real numbers, and $0 \leq \nu \leq 1$.

We generalize (6) for positive τ-measurable operators in the following theorem.

Theorem 2.6 *Let a, b be positive operators in $L_1(\mathcal{M})$. Then for $m = 1, 2, \dots$,*

$$\left\|a^\nu b^{1-\nu}\right\|_1^m + \min\{\nu, 1-\nu\}^m \left(\|a\|_1^{\frac{m}{2}} - \|b\|_1^{\frac{m}{2}}\right)^2 \leq \left\|\nu a + (1-\nu)b\right\|_1^m. \tag{7}$$

Proof By Corollary 2.4, we have

$$\left\|a^\nu b^{1-\nu}\right\|_1 \leq \|a\|_1^\nu . \|b\|_1^{1-\nu}.$$

Thus by (6),

$$\begin{aligned} &\left(\left\|a^\nu b^{1-\nu}\right\|_1\right)^m + \left(\min\{\nu, 1-\nu\}\right)^m \left(\|a\|_1^{\frac{m}{2}} - \|b\|_1^{\frac{m}{2}}\right)^2 \\ &\quad \leq \left(\|a\|_1^\nu \|b\|_1^{1-\nu}\right)^m + \left(\min\{\nu, 1-\nu\}\right)^m \left(\|a\|_1^{\frac{m}{2}} - \|b\|_1^{\frac{m}{2}}\right)^2 \\ &\quad \leq \left(\nu\|a\|_1 + (1-\nu)\|b\|_1\right)^m. \end{aligned}$$

□

Remark 2.7 A result similar to Theorem 2.6 has been proved in [22, Theorem 3.8].

The following result is another extension of refinement of the Young inequality for τ-measurable operators. Let us first recall the inequality introduced by Xing-ka in [15].

Lemma 2.8 ([15, Lemma 1.1]) *Suppose that a and b are non-negative real numbers.*

1. *if $0 \leq \nu \leq \frac{1}{2}$, then*

$$\left[(\nu a)^\nu b^{1-\nu}\right]^2 + \nu^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2. \tag{8}$$

2. *if $\frac{1}{2} \leq \nu \leq 1$, then*

$$\left[a^\nu \left((1-\nu)b\right)^{1-\nu}\right]^2 + (1-\nu)^2(a-b)^2 \leq \nu^2 a^2 + (1-\nu)^2 b^2. \tag{9}$$

Lemma 2.9 (Cases of Equality in (8) and (9)) *Equality holds in* (8) *if and only if* $b = \nu a$, *and equality holds in* (9) *if and only if* $a = (1-\nu)b$.

Proof As we see in the proof of Lemma 2.8,

$$\nu^2 a^2 + (1-\nu)^2 b^2 - \nu^2 (a-b)^2 = b\big[2\nu(\nu a) + (1-2\nu)b\big] \geq b(\nu a)^{2\nu} b^{1-2\nu} = \big[(\nu a)^{\nu} b^{1-\nu}\big]^2.$$

Thus, equality holds in (8) if and only if

$$b\big[2\nu(\nu a) + (1-2\nu)b\big] = b(\nu a)^{2\nu} b^{1-2\nu}.$$

This relationship exists if and only if $b = \nu a$. Using the similar method, we can prove the case of equality in (9). □

Theorem 2.10 *Let* $a, b \in L_2(\mathcal{M})$ *be positive operator.*

1. *If* $0 \leq \nu \leq \frac{1}{2}$, *then*

$$\tau\Big(\big[(\nu a)^{\nu} b^{1-\nu}\big]^2\Big) + \nu^2\Big(\sqrt{\tau\big(a^2\big)} - \sqrt{\tau\big(b^2\big)}\Big)^2 \leq \tau\big(\nu^2 a^2 + (1-\nu)^2 b^2\big), \tag{10}$$

2. *If* $\frac{1}{2} \leq \nu \leq 1$, *then*

$$\begin{aligned} &\tau\Big(\big\{a^{\nu}\big[(1-\nu)b\big]^{1-\nu}\big\}^2\Big) + (1-\nu)^2\Big(\sqrt{\tau\big(a^2\big)} - \sqrt{\tau\big(b^2\big)}\Big)^2 \\ &\quad \leq \tau\big(\nu^2 a^2 + (1-\nu)^2 b^2\big). \end{aligned} \tag{11}$$

Moreover, if a and b are bounded operators or $ab \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, *then equality holds in* (10) *if and only if* $b = \nu a$, *and equality holds in* (11) *if and only if* $a = (1-\nu)b$.

Proof According to Lemma 2.8, we have

$$\big[\big(\nu\mu_t(a)\big)^{\nu}\mu_t(b)^{1-\nu}\big]^2 + \nu^2\big(\mu_t(a) - \mu_t(b)\big)^2 \leq \nu^2\mu_t(a)^2 + (1-\nu)^2\mu_t(b)^2, \quad \forall t > 0.$$

Therefore,

$$\begin{aligned} \tau\big(\nu^2 a^2 + (1-\nu)^2 b^2\big) &= \nu^2\tau\big(a^2\big) + (1-\nu)^2\tau\big(b^2\big) \\ &= \int_0^{\infty}\big[\nu^2\mu_t\big(a^2\big) + (1-\nu)^2\mu_t\big(b^2\big)\big]\,dt \\ &\geq \int_0^{\infty}\big[\big(\nu\mu_t(a)\big)^{\nu}\mu_t(b)^{1-\nu}\big]^2\,dt \\ &\quad + \nu^2\int_0^{\infty}\big[\mu_t(a)^2 + \mu_t(b)^2 - 2\mu_t(a)\mu_t(b)\big]\,dt \\ &= \int_0^{\infty}\big[\big(\nu\mu_t(a)\big)^{2\nu}\mu_t(b)^{2(1-\nu)}\big]\,dt \\ &\quad + \nu^2\Big[\tau\big(a^2\big) + \tau\big(b^2\big) - 2\int_0^{s}\mu_t(a)\mu_t(b)\,dt\Big] \\ &= I. \end{aligned}$$

Now, the combination of Theorem 1.1 part (11) and the Hölder inequality gives us

$$\begin{aligned} I &\geq \int_0^\infty \mu_t\big((\nu a)^{2\nu} b^{2(1-\nu)}\big)\,dt \\ &\quad + \nu^2\Big[\tau\big(a^2\big) + \tau\big(b^2\big) - 2\Big(\int_0^s \mu_t\big(a^2\big)\,dt\Big)^{\frac{1}{2}}\Big(\int_0^s \mu_t\big(b^2\big)\,dt\Big)^{\frac{1}{2}}\Big] \\ &= \tau\big(\big[(\nu a)^\nu b^{(1-\nu)}\big]^2\big) + \nu^2\big[\tau\big(a^2\big) + \tau\big(b^2\big) - 2\tau\big(a^2\big)^{\frac{1}{2}}\tau\big(b^2\big)^{\frac{1}{2}}\big] \\ &= \tau\big(\big[(\nu a)^\nu b^{(1-\nu)}\big]^2\big) + \nu^2\big[\sqrt{\tau\big(a^2\big)} - \sqrt{\tau\big(b^2\big)}\big]^2. \end{aligned}$$

Thus,

$$\tau\big(\big[(\nu a)^\nu b^{1-\nu}\big]^2\big) + \nu^2\big(\sqrt{\tau\big(a^2\big)} - \sqrt{\tau\big(b^2\big)}\big)^2 \leq \tau\big(\nu^2 a^2 + (1-\nu)^2 b^2\big).$$

Now, if $\frac{1}{2} \leq \nu \leq 1$, using the similar argument, we have the second inequality. The case of equality follows by the similar argument used in the proof of [9, Theorem 3.4]. □

Corollary 2.11 *Let a, b be positive τ-measurable operators. If $0 \leq \nu \leq \frac{1}{2}$, then*

$$\big\|(\nu a)^\nu b^{1-\nu}\big\|_2^2 + \nu^2\big(\|a\|_2 - \|b\|_2\big)^2 \leq \nu^2\|a\|_2^2 + (1-\nu)^2\|b\|_2^2,$$

and if $\frac{1}{2} \leq \nu \leq 1$, then

$$\big\|a^\nu\big[(1-\nu)b\big]^{1-\nu}\big\|_2^2 + (1-\nu)^2\big(\|a\|_2 - \|b\|_2\big)^2 \leq \nu^2\|a\|_2^2 + (1-\nu)^2\|b\|_2^2.$$

The next theorems give a converse of the Young inequality and its refinement for τ-measurable operators.

Theorem 2.12 ([20, Corollary 3.7]) *Let a, b be positive invertible operators in $\mathcal{M}$. Then for $\nu > 1$,*

$$\big\|\nu a + (1-\nu)b\big\|_1 \leq \big\|a^\nu b^{1-\nu}\big\|_1. \tag{12}$$

Moreover, if a and b bounded operators in $L_1(\mathcal{M})$ or $ab \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, then equality holds if and only if $a = b$.

Theorem 2.13 ([20, Theorem 3.20]) *Let a, b be positive invertible operator in $\mathcal{M}$ and $\nu > 1$. Then*

$$\begin{aligned} &\big\|\nu a + (1-\nu)b\big\|_1 + \min\{1, \nu - 1\}\Big(\sqrt{\|a\|_1} - \sqrt{\|b\|_1}\Big)^2 \\ &\quad \leq \big\|a^\nu b^{1-\nu}\big\|_1 \leq \big\|\nu a + (1-\nu)b\big\|_1 + \max\{1, \nu - 1\}\Big(\sqrt{\|a\|_1} - \sqrt{\|a\|_1}\Big)^2. \end{aligned}$$

Moreover, if a and b bounded operators in $L_1(\mathcal{M})$, then equality holds if and only if $\mu_t(a) = \mu_t(b)$ for all $t > 0$.

Concerning all results discussed so far, we can prove the following theorems for τ-measurable operators.

Theorem 2.14 *Let $a, b \in L_1(\mathcal{M})$ be positive invertible operators. Then for $\nu > 1$,*

$$\left\| \nu a + (1-\nu)b \right\|_1 \leq \left\| a^\nu b^{1-\nu} \right\|_1. \tag{13}$$

Moreover, if a and b are bounded operators or $ab \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, then equality holds if and only if $a = b$.

Theorem 2.15 *Let $a, b \in L_1(\mathcal{M})$ be positive invertible operators. Then for $\nu > 1$,*

$$\begin{aligned} &\left\| \nu a + (1-\nu)b \right\|_1 + \min\{1, \nu - 1\}\left(\sqrt{\|a\|_1} - \sqrt{\|b\|_1}\right)^2 \\ &\quad \leq \left\| a^\nu b^{1-\nu} \right\|_1 \leq \left\| \nu a + (1-\nu)b \right\|_1 + \max\{1, \nu - 1\}\left(\sqrt{\|a\|_1} - \sqrt{\|a\|_1}\right)^2. \end{aligned}$$

Moreover, if a and b are bounded operators or $ab \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, then equality holds if and only if $\mu_t(a) = \mu_t(b)$ for all $t > 0$.

Hu and Xue [14] obtained another improvement of reverses of the scalar Young type inequalities for non-negative real numbers a and b in the following form.

If $0 \leq \nu \leq \frac{1}{2}$, then

$$\nu^2 a^2 + (1-\nu)^2 b^2 + r_0 a\left(\sqrt{(1-\nu)b} - \sqrt{a}\right)^2 \leq (1-\nu)^2(a-b)^2 + a^{2\nu}\left[(1-\nu)b\right]^2, \tag{14}$$

where $r_0 = \min\{2\nu, 1 - 2\nu\}$.

If $\frac{1}{2} \leq \nu \leq 1$, then

$$\nu^2 a^2 + (1-\nu)^2 b^2 + r_0 b(\sqrt{b} - \sqrt{\nu a})^2 \leq \nu^2(a-b)^2 + (\nu a)^{2\nu} b^{2-2\nu}, \tag{15}$$

where $r_0 = \min\{2\nu - 1, 2 - 2\nu\}$.

Let a, b be positive τ-measurable operators. Then, we have the same inequalities for singular values if we replace the positive real numbers a and b in the above equation with $\mu_t(a)$ and $\mu_t(b)$, respectively. We are interested in proving some versions of those inequalities and the case of equality for trace and norm of τ-measurable operators, but it is unclear for us.

In the following, we use the method of Bhatia and Davis [4] to extend some inequalities for τ-measurable operators. Zhou, Wang, and Wu established the Schwarz inequality for τ-measurable operators [24]. Using the similar argument in the proof of [24, Theorem 1], we prove the following theorem that is an important key feature in the next results.

Theorem 2.16 *Let x, y are bounded τ-measurable operators such that xy is self-adjoint. Then for every $s > 0$,*

$$\int_0^s \mu_t(xy)\,dt \leq \int_0^s \mu_t(yx)\,dt.$$

Proof This theorem is a special case of [23, Lemma 3.1] when put $p = 1$. Since xy is a self-adjoint τ-measurable operator, for every positive integer n and $t > 0$, part (2) of Theorem 1.1 implies that

$$\mu_t(xy)^{2n} = \mu_t\big(|xy|\big)^{2n} = \mu_t\big(|xy|^2\big)^n.$$

Let f be an increasing function on $[0,\infty)$ with $f(0) = 0$ and $t \to f(e^t)$ is convex. Then,

$$\begin{aligned}\int_0^s f\big(\mu_t\big((xy)^{2n}\big)\big)\,dt &= \int_0^s f\big(\mu_t\big(\big((xy)^*(xy)\big)^n\big)\big)\,dt\\ &\le \int_0^s f\big(\mu_t(yx)^{2n-1}\|x\|\cdot\|y\|\big)\,dt, \qquad \text{[Theorem 1.1(11)]}\end{aligned}$$

particularly if $f(t) = t^{\frac{1}{2n-1}}$, then

$$\int_0^s \mu_t(xy)^{\frac{2n}{2n-1}}\,dt \le \int_0^s \mu_t(yx)\big(\|x\|\cdot\|y\|\big)^{\frac{1}{2n-1}}\,dt = \big(\|x\|\cdot\|y\|\big)^{\frac{1}{2n-1}}\int_0^s \mu_t(yx)\,dt.$$

Taking the $\lim_{n\to\infty}\inf$ of both sides, by the Fatou lemma, we get

$$\begin{aligned}\int_0^s \mu_t(xy)\,dt &= \int_0^s \liminf \mu_t(xy)^{\frac{2n}{2n-1}}\,dt\\ &\le \liminf \int_0^s \mu_t(xy)^{\frac{2n}{2n-1}}\,dt\\ &\le \liminf \int_0^s \mu_t(yx)\big(\|x\|\cdot\|y\|\big)^{\frac{1}{2n-1}}\,dt\\ &= \liminf\big(\|x\|\cdot\|y\|\big)^{\frac{1}{2n-1}}\int_0^s \mu_t(yx)\,dt\\ &= \int_0^s \mu_t(yx)\,dt,\end{aligned}$$

which proves the result. □

Corollary 2.17 *Let x, y are bounded τ-measurable operators such that xy is self-adjoint. Then, for every $r > 0$,*

$$\big\||xy|^r\big\|_p \le \big\||yx|^r\big\|_p, \quad \textit{for every } p > 0.$$

Proof By taking $f(t) = \frac{pr}{2n-1}$ in the proof of Theorem 2.16. □

Corollary 2.18 ([24]) *Let x, y, z be bounded τ-measurable operators and $r > 0$. Then*

$$\big\||x^*zy|^r\big\|_1^2 \le \big\||xx^*z|^r\big\|_1 \cdot \big\||zyy^*|^r\big\|_1. \tag{16}$$

Corollary 2.19 ([24]) *Let a, b, z are bounded τ-measurable operators such that a and b are positive. Then for $0 \le \nu \le 1$,*

$$\big\|a^\nu z b^{1-\nu}\big\|_1 \le \|az\|_1^\nu \|zb\|_1^{1-\nu}.$$

In the following, we are going to present some new versions of majorization and norm type Young's inequality.

Let a, b be positive τ-measureable operators. We say that a is submajorized (weakly majorized) by b in symbol $a \prec_w b$ [13], if $\int_0^s \mu_t(a)\,dt \leq \int_0^s \mu_t(b)\,dt$ for all $s > 0$. Moreover, a is said to be majorized by b and is indicated by $a \prec b$, if $a \prec_w y$ and $\int_0^\infty \mu_t(a)\,dt = \int_0^\infty \mu_t(b)\,dt$.

Proposition 2.20 ([18]) *Let f be a continuous increasing function on $\mathbb{R}^+$ such that $f(0) = 0$ and $t \longrightarrow f(e^t)$ is convex, then for positive operators a and b in $L_0(\mathcal{M})$ and for $s \in \mathbb{R}_0^+$,*

$$\int_0^s f\big(\mu_t\big(|ab|^r\big)\big)\,dt \leq \int_0^s f\big(\mu_t\big(a^r b^r\big)\big)\,dt.$$

Theorem 2.21 *Let a, b, z are bounded τ-measurable operators such that a and b are positive. Then for $p > 0$,*

$$\big|a^{\frac{1}{2}} z b^{\frac{1}{2}}\big|^p \prec_w \frac{1}{2}\big(\big|z^* a\big|^p + \big|bz^*\big|^p\big). \tag{17}$$

Proof Suppose that $s \in \mathbb{R}_0^+$, we have for all $t > 0$,

$$\begin{aligned}
\int_0^s \mu_t\big(\big|a^{\frac{1}{2}} z b^{\frac{1}{2}}\big|^p\big)\,dt &= \int_0^s \mu_t\big(\big(a^{\frac{1}{2}} z b^{\frac{1}{2}}\big)^*\big(a^{\frac{1}{2}} z b^{\frac{1}{2}}\big)\big)^{\frac{p}{2}}\,dt \quad \text{[Theorem 1.1(2)]} \\
&= \int_0^s \mu_t\big(b^{\frac{1}{2}} z^* a^{\frac{1}{2}} a^{\frac{1}{2}} z b^{\frac{1}{2}}\big)^{\frac{p}{2}}\,dt \\
&\leq \int_0^s \mu_t\big(bz^* az\big)^{\frac{p}{2}}\,dt \quad \text{[Theorem 2.16]} \\
&= \int_0^s \mu_t\big(bz^*\big(z^* a\big)^*\big)^{\frac{p}{2}}\,dt \\
&= \int_0^s \mu_t\big(\big|bz^*\big| \cdot \big|z^* a\big|\big)^{\frac{p}{2}}\,dt \quad \text{[Theorem 1.1(4)]} \\
&\leq \int_0^s \mu_t\big(\big|bz^*\big|^{\frac{p}{2}} \cdot \big|z^* a\big|^{\frac{p}{2}}\big)\,dt \quad \text{[Proposition 2.20]} \\
&\leq \frac{1}{2} \int_0^s \mu_t\big(\big|bz^*\big|^p + \big|z^* a\big|^p\big)\,dt, \quad \text{[Proposition 2.1]}
\end{aligned}$$

which proves the result. □

Corollary 2.22 *Let a, b be positive bounded operators in $L_p(\mathcal{M})$, and z be a bounded operator in $L_0(\mathcal{M})$. Then*

$$\big\|\big(a^{\frac{1}{2}} z b^{\frac{1}{2}}\big)^p\big\|_1 \leq \frac{1}{2}\big(\big\|\big|z^* a\big|^p + \big|bz^*\big|^p\big\|_1\big), \tag{18}$$

and

$$\big\|a^{\frac{1}{2}} z b^{\frac{1}{2}}\big\|_p^p \leq \frac{1}{2}\big(\big\||z^* a|\big\|_p^p + \big\||bz^*|\big\|_p^p\big). \tag{19}$$

Moreover, the equality holds if $|bz^*| = |z^* a|$.

Proof From the argument in the proof of the above theorem, for all $s \in \mathbb{R}_0^+$ and $t > 0$, we have

$$\int_0^s \mu_t\big(\big|a^{\frac{1}{2}} z b^{\frac{1}{2}}\big|^p\big)\,dt \le \int_0^s \mu_t\big(bz^* az\big)^{\frac{p}{2}}\,dt.$$

Setting $f(t) = t^{\frac{p}{2}}$ in Proposition 2.20 implies

$$\begin{aligned}\int_0^s \mu_t\big(\big|a^{\frac{1}{2}} z b^{\frac{1}{2}}\big|^p\big)\,dt &\le \int_0^s \mu_t\big(\big|bz^*\big|^{\frac{p}{2}} \cdot \big|z^* a\big|^{\frac{p}{2}}\big)\,dt \\ &\le \frac{1}{2}\int_0^s \mu_t\big(\big|bz^*\big|^p + \big|z^* a\big|^p\big)\,dt, \\ &\le \frac{1}{2}\bigg(\int_0^s \mu_t\big(\big|bz^*\big|\big)^p\,dt + \int_0^s \mu_t\big(\big|z^* a\big|\big)^p\,dt\bigg) \\ &= \frac{1}{2}\bigg(\int_0^s \mu_t\big(|zb|\big)^p\,dt + \int_0^s \mu_t\big(|az|\big)^p\,dt\bigg)\end{aligned}$$

by [19, Theorem 3.3] and parts (2) and (12) of Theorem 1.1. Letting $s \to \infty$, then we obtain the first and second inequalities.

If the equality holds, then from the above argument, we have

$$\int_0^\infty \mu_t\big(\big|bz^*\big|^{\frac{p}{2}} \cdot \big|z^* a\big|^{\frac{p}{2}}\big)\,dt = \int_0^\infty \frac{1}{2}\mu_t\big(\big|bz^*\big|^p + \big|z^* a\big|^p\big)\,dt. \tag{20}$$

By the Young inequality for singular values,

$$\mu_t\big(\big|bz^*\big|^{\frac{p}{2}} \cdot \big|z^* a\big|^{\frac{p}{2}}\big) \le \frac{1}{2}\mu_t\big(\big|bz^*\big|^p + \big|z^* a\big|^p\big), \tag{21}$$

for every $t > 0$. Therefore, (20) shows, when coupled with (21), that

$$\mu_t\big(\big|bz^*\big|^{\frac{p}{2}} \cdot \big|z^* a\big|^{\frac{p}{2}}\big) = \frac{1}{2}\mu_t\big(\big|bz^*\big|^p + \big|z^* a\big|^p\big),$$

for almost all $t > 0$. However, for $x \in L_0(\mathcal{M})$ as the nonincreasing function $\mu_t(x)$ are right continuous, $\mu_t(|bz^*|^{\frac{p}{2}} \cdot |z^* a|^{\frac{p}{2}}) = \frac{1}{2}\mu_t(|bz^*|^p + |z^* a|^p)$ for all $t > 0$ which implies $|bz^*| = |z^* a|$ by Corollary 2.3. □

Corollary 2.23 *Let a, b be positive bounded operators in $L_p(\mathcal{M})$, and z be bounded operator in $L_0(\mathcal{M})$. Then, for $p > 0$,*

$$\big|a^{\frac{1}{2}} z b^{\frac{1}{2}}\big|^p \prec \frac{1}{2}\big(\big|z^* a\big|^p + \big|bz^*\big|^p\big), \tag{22}$$

if $|z^* a| = |bz^*|$.

Theorem 2.24 *Let a, b be positive bounded operators in $L_p(\mathcal{M})$, and z be a bounded operator in $L_0(\mathcal{M})$. Then, for $p > 0$,*

$$\big\|a^{\frac{1}{2}} z b^{\frac{1}{2}}\big\|_p \le \|az\|_p^{\frac{1}{2}} \|zb\|_p^{\frac{1}{2}} \le \frac{1}{2}\big(\|az\|_p + \|zb\|_p\big). \tag{23}$$

Proof Suppose that $s \in \mathbb{R}_0^+$, we have for all $t > 0$

$$\begin{aligned}
\int_0^s \mu_t\big(\big|a^{\frac{1}{2}}zb^{\frac{1}{2}}\big|^p\big)\,dt &= \int_0^s \mu_t\big(\big(a^{\frac{1}{2}}zb^{\frac{1}{2}}\big)^*\big(a^{\frac{1}{2}}zb^{\frac{1}{2}}\big)\big)^{\frac{p}{2}}\,dt \quad \text{[Theorem 1.1(2)]}\\
&= \int_0^s \mu_t\big(b^{\frac{1}{2}}z^*a^{\frac{1}{2}}a^{\frac{1}{2}}zb^{\frac{1}{2}}\big)^{\frac{p}{2}}\,dt\\
&= \int_0^s \mu_t\big(bz^*az\big)^{\frac{p}{2}}\,dt \quad \text{[Theorem 2.16]}\\
&= \int_0^s \mu_t\big(bz^*\big(z^*a\big)^*\big)^{\frac{p}{2}}\,dt\\
&= \int_0^s \mu_t\big(\big|bz^*\big|\cdot\big|z^*a\big|\big)^{\frac{p}{2}}\,dt \quad \text{[Theorem 1.1(4)]}\\
&\le \int_0^s \mu_t\big(\big|bz^*\big|^{\frac{p}{2}}\cdot\big|z^*a\big|^{\frac{p}{2}}\big)\,dt \quad \text{[Proposition 2.20]}\\
&\le \int_0^s \mu_t\big(\big|bz^*\big|^{\frac{p}{2}}\big)\cdot\mu_t\big(\big|z^*a\big|^{\frac{p}{2}}\big)\,dt \quad \text{[Theorem 1.1(13)]}\\
&= \int_0^s \mu_t\big(|zb|\big)^{\frac{p}{2}}\cdot\mu_t\big(|az|\big)^{\frac{p}{2}}\,dt\\
&\le \left(\int_0^s \mu_t\big(|zb|^p\big)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^s \mu_t\big(|az|^p\big)\,dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Letting $s \to \infty$, we obtain

$$\big\|a^{\frac{1}{2}}zb^{\frac{1}{2}}\big\|_p \le \|zb\|_p^{\frac{1}{2}}.\|az\|_p^{\frac{1}{2}} \le \frac{1}{2}\big(\|az\|_p + \|zb\|_p\big). \qquad \square$$

Theorem 2.25 *Let a, b be positive bounded operators in $L_p(\mathcal{M})$, and z be a bounded operator in $L_0(\mathcal{M})$. Then, for $0 \le \nu \le 1$,*

$$\big\|a^\nu zb^{1-\nu}\big\|_p \le \|az\|_p^\nu\cdot\|zb\|_p^{1-\nu} \le \nu\|az\|_p + (1-\nu)\|zb\|_p. \tag{24}$$

Moreover, $\|a^\nu zb^{1-\nu}\|_p = \nu\|az\|_p + (1-\nu)\|zb\|_p$ if $\|az\|_p = \|zb\|_p$.

Proof The first part is [22, Lemma 2.1], and the second part of inequality is obtained from the classical Young inequality. Since some partial proof are needed for the case of equality, we write the proof with our notations. The inequality is trivial statement for $\nu = 0, 1$. We will use induction, for all indices $\nu = \frac{k}{2^n}$, $k = 0, 1, \ldots, 2^n$. The general case then followed by continuity. Theorem 2.24 shows the result for $\nu = \frac{1}{2}$. Suppose that inequality (24) is valid for all dyadic rational numbers with denominator $\frac{k}{2^{n-1}}$. Let $\nu = \frac{2k+1}{2^n}$ be any dyadic rational. Then $\nu = \eta + \omega$, where $\omega = \frac{1}{2^n}$ and $\eta = \frac{2k}{2^n}$. Two such rational numbers are η and $\lambda = \eta + 2\omega = \nu + \omega$. Let $s \in \mathbb{R}_0^+$, then for all $t > 0$, we have

$$\begin{aligned}
\int_0^s \mu_t\big(\big|a^\nu zb^{1-\nu}\big|\big)^p\,dt &= \int_0^s \mu_t\big(\big|a^{\eta+\omega}zb^{1-\lambda+\omega}\big|^p\big)\,dt\\
&= \int_0^s \mu_t\big(\big|a^\omega\big(a^\eta zb^{1-\lambda}\big)b^\omega\big|^p\big)\,dt\\
&= \int_0^s \mu_t\big(\big|a^\omega\big(a^\eta zb^{1-\lambda}\big)b^\omega\big|^2\big)^{\frac{p}{2}}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^s \mu_t\big(\big(b^\omega b^{1-\lambda} z^* a^\eta a^\omega\big)\big(a^\omega a^\eta z b^{1-\lambda} b^\omega\big)\big)^{\frac{p}{2}}\, dt \\
&= \int_0^s \mu_t\big(\big(a^\omega a^\omega a^\eta z b^{1-\lambda}\big)\big(b^\omega b^\omega b^{1-\lambda} z^* a^\eta\big)\big)^{\frac{p}{2}}\, dt \quad \text{[Theorem 2.16]} \\
&= \int_0^s \mu_t\big(\big(a^\lambda z b^{1-\lambda}\big)\big(b^{1-\eta} z^* a^\eta\big)\big)^{\frac{p}{2}}\, dt \\
&= \int_0^s \mu_t\big(\big(a^\lambda z b^{1-\lambda}\big)\big(a^\eta z b^{1-\eta}\big)^*\big)^{\frac{p}{2}}\, dt \\
&\leq \int_0^s \mu_t\big(\big|a^\lambda z b^{1-\lambda}\big|^{\frac{p}{2}}\big|a^\eta z b^{1-\eta}\big|^{\frac{p}{2}}\big)\, dt, \quad \text{[Proposition 2.20]}
\end{aligned}
$$

where it has been obtained using (16) and replacing $\lambda = \eta + 2\omega$. Now, using induction hypothesis and the Hölder inequality implies

$$
\begin{aligned}
\int_0^s \mu_t\big(\big|a^\nu z b^{1-\nu}\big|\big)^p\, dt &\leq \bigg(\int_0^s \mu_t\big(\big|a^\lambda z b^{1-\lambda}\big|^p\big)\, dt\bigg)^{\frac{1}{2}} \\
&\quad \times \bigg(\int_0^s \mu_t\big(\big|a^\eta z b^{1-\eta}\big|^p\big)\, dt\bigg)^{\frac{1}{2}}\, dt \\
&\leq \bigg(\int_0^s \mu_t\big(|az|^p\big)\, dt\bigg)^{\frac{\lambda}{2}} \bigg(\int_0^\infty \mu_t\big(|zb|^p\big)\, dt\bigg)^{\frac{1-\lambda}{2}} \\
&\quad \times \bigg(\int_0^s \mu_t\big(|az|^p\big)\, dt\bigg)^{\frac{\eta}{2}} \bigg(\int_0^s \mu_t\big(|zb|^p\big)\, dt\bigg)^{\frac{1-\eta}{2}} \\
&= \bigg(\int_0^s \mu_t\big(|az|^p\big)\, dt\bigg)^{\nu} \bigg(\int_0^s \mu_t\big(|zb|^p\big)\, dt\bigg)^{1-\nu}.
\end{aligned}
$$

The general case then followed by continuity, and the proof is complete. $s \to \infty$ implies the first part of inequality in (24). If the equality holds, then the above argument gives us case of equality in the Hölder inequality, which implies $\|az\|_p = \|zb\|_p$. □

Corollary 2.26 *Let a, b be non-negative bounded operators in $L_p(\mathcal{M})$, and z be a bounded operator in $L_p(\mathcal{M})$. Then, for $0 \leq \nu \leq 1$, we have*

$$
\big\|a^\nu z b^\nu\big\|_p \leq \|z\|_p^{1-\nu} \|azb\|_p^\nu.
$$

Proof It suffices to prove this when a is strictly positive (invertible); the general case follows from this by continuity. Using (24), we obtain

$$
\big\|a^\nu z b^\nu\big\|_p = \big\|\big(a^{-1}\big)^{1-\nu} azb^{1-(1-\nu)}\big\|_p \leq \|z\|_p^{1-\nu} \|azb\|_p^\nu. \qquad \square
$$

Here are some other types of the Young inequality. Note that these inequalities can also be expressed for the generalized s-numbers.

Theorem 2.27 *Let x, y be operators in $L_0(\mathcal{M})$. Then for $p, q, r \in \mathbb{R}^+$ that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,*

$$
\frac{1}{r}\big|xy^*\big|^r \prec_w \frac{1}{p}|x|^p + \frac{1}{q}|y|^q. \tag{25}
$$

Moreover, if $x, y \in L_1(\mathcal{M})$ are bounded operators or $xy \in \mathfrak{L}^2_{\text{loc}}(\mathcal{M})$, then

$$\frac{1}{r}\left|xy^*\right|^r \prec \frac{1}{p}|x|^p + \frac{1}{q}|y|^q, \tag{26}$$

if and only if $|x|^p = |y|^q$.

Proof For all $s \in \mathbb{R}_0^+$ and $t > 0$, we have

$$\begin{aligned}
\int_0^s \mu_t\left(\left|xy^*\right|^r\right) dt &= \int_0^s \mu_t\left(\left|xy^*\right|^2\right)^{\frac{r}{2}} dt \\
&= \int_0^s \mu_t\left(yx^*xy^*\right)^{\frac{r}{2}} dt \\
&= \int_0^s \mu_t\left(|x|^2|y|^2\right)^{\frac{r}{2}} dt \quad \text{[24, Theorem 1]} \\
&\le \int_0^s \mu_t\left(|x|^r|y|^r\right) dt \quad \text{[Proposition 2.20]} \\
&\le \int_0^s \mu_t\left(\frac{r}{p}|x|^p + \frac{r}{q}|y|^q\right) dt, \quad \text{[Proposition 2.1]}
\end{aligned}$$

which proves (25). If x, y in $L_1(\mathcal{M})$, then (26) holds if

$$\tau\left(\left|xy^*\right|^r\right) = \tau\left(\frac{r}{p}|x|^p + \frac{r}{q}|y|^q\right).$$

As we see in the proof of Corollary 2.22, this equality holds if and only if

$$\mu_t\left(|x|^r|y|^r\right) = \mu_t\left(\frac{r}{p}|x|^p + \frac{r}{q}|y|^q\right),$$

which implies the result by applying Corollary 2.3. □

Theorem 2.28 *Let a, b be positive bounded operators, and z be a bounded operator in $L_0(\mathcal{M})$. Then for $p, q, r \in \mathbb{R}^+$ that $\frac{1}{p} + \frac{1}{q} = 1$,*

$$|azb|^r \prec_w \frac{1}{p}\left|b^2z^*\right|^{\frac{pr}{2}} + \frac{1}{q}\left|z^*a^2\right|^{\frac{qr}{2}} \tag{27}$$

Moreover, $|azb|^r \prec \frac{1}{p}|b^2z^*|^{\frac{pr}{2}} + \frac{1}{q}|z^*a^2|^{\frac{qr}{2}}$ if

$$\left|zb^2\right|^p = \left|a^2z\right|^q.$$

Proof For all $s \in \mathbb{R}_0^+$, in accordance with the previous proof process, we have

$$\begin{aligned}
\int_0^s \mu_t\left(|azb|^r\right) dt &= \int_0^s \mu_t\left((azb)^*(azb)\right)^{\frac{r}{2}} dt \\
&= \int_0^s \mu_t\left(bz^*aazb\right)^{\frac{r}{2}} dt
\end{aligned}$$

$$\begin{aligned} &= \int_0^s \mu_t\bigl(bbz^*aaz\bigr)^{\frac{r}{2}}\,dt \quad \text{[Theorem 2.16]} \\ &= \int_0^s \mu_t\bigl(\bigl|b^2z^*\bigr| \cdot \bigl|z^*a^2\bigr|\bigr)^{\frac{r}{2}}\,dt \quad \text{[Theorem 1.1(4)]} \\ &\le \int_0^s \mu_t\bigl(\bigl|b^2z^*\bigr|^{\frac{r}{2}} \cdot \bigl|z^*a^2\bigr|^{\frac{r}{2}}\bigr)\,dt \quad \text{[Proposition 2.20]} \\ &\le \int_0^s \mu_t\biggl(\frac{1}{p}\bigl|b^2z^*\bigr|^{\frac{pr}{2}} + \frac{1}{q}\bigl|z^*a^2\bigr|^{\frac{qr}{2}}\biggr)\,dt. \end{aligned}$$

The similar argument used in the proof of Theorem 2.27 implies the majorization case. □

Corollary 2.29 *Let a, b be positive bounded operators in $L_1(\mathcal{M})$, and z be a bounded operator in $L_0(\mathcal{M})$. Then for $p,q,r \in \mathbb{R}^+$ that $\frac{1}{p} + \frac{1}{q} = 1$,*

1. $\|(azb)^r\|_1 \le \|\frac{1}{p}|z^*a^2|^{\frac{pr}{2}} + \frac{1}{q}|b^2z^*|^{\frac{qr}{2}}\|_1$.
2. $\|(azb)^r\|_1^2 \le \|(a^2z^*)^r\|_1\|(z^*b^2)^r\|_1 \le \frac{1}{p}\|(a^2z^*)^r\|_1^p + \frac{1}{q}\|(z^*b^2)^r\|_1^q$.
3. $\|(azb)^r\|_1 \le \|(a^2z^*)^{\frac{r}{2}}\|_p\|(z^*b^2)^{\frac{r}{2}}\|_q \le \frac{1}{p}\|(a^2z^*)^{\frac{r}{2}}\|_p^p + \frac{1}{q}\|(z^*b^2)^{\frac{r}{2}}\|_q^q$.

Moreover, the equality holds if $|a^2z^*|^p = |z^*b^2|^q$.

Theorem 2.30 *Let a, b be positive bounded operators, and z be a bounded operator in $L_0(\mathcal{M})$. Then for $p,q,r \in \mathbb{R}^+$ that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,*

$$\frac{1}{r}\bigl|az^2b\bigr|^r \prec_w \frac{1}{p}\bigl|bz^*\bigr|^p + \frac{1}{q}|az|^q. \tag{28}$$

Moreover, $\frac{1}{r}|az^2b|^r \prec \frac{1}{p}|bz^*|^p + \frac{1}{q}|az|^q$ if

$$\bigl|bz^*\bigr|^p = |az|^q.$$

Proof For all $s \in \mathbb{R}_0^+$ and $t > 0$, we have

$$\begin{aligned} \int_0^s \mu_t\bigl(\bigl|az^2b\bigr|^r\bigr)\,dt &= \int_0^s \mu_t\bigl(\bigl(az^2b\bigr)^*\bigl(az^2b\bigr)\bigr)^{\frac{r}{2}}\,dt \\ &= \int_0^s \mu_t\bigl(b\bigl(z^2\bigr)^*aaz^2b\bigr)^{\frac{r}{2}}\,dt \\ &= \int_0^s \mu_t\bigl(zbbz^*z^*aaz\bigr)^{\frac{r}{2}}\,dt \quad \text{[Theorem 2.16]} \\ &= \int_0^s \mu_t\bigl(\bigl|bz^*\bigr|^2 \cdot |az|^2\bigr)^{\frac{r}{2}}\,dt \quad \text{[Theorem 1.1(4)]} \\ &\le \int_0^s \mu_t\bigl(\bigl|bz^*\bigr|^r \cdot |az|^r\bigr)\,dt \quad \text{[Proposition 2.20]} \\ &\le \int_0^s \mu_t\biggl(\frac{r}{p}\bigl|bz^*\bigr|^p + \frac{r}{q}|az|^q\biggr)\,dt, \end{aligned}$$

which implies the result. The similar argument in the proof of Theorem 2.27 implies the majorization case. □

Corollary 2.31 *Let a, b be positive bounded τ-measurable operators, and z be a bounded operator in $L_0(\mathcal{M})$. Then for $p,q,r \in \mathbb{R}^+$ that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$,*

1 $\frac{1}{r}\|(az^2b)^r\|_1 \le \|\frac{1}{p}|bz^*|^p + \frac{1}{q}|az|^q\|_1$.

2 $\|(az^2b)^r\|_1^2 \le \|(zb)^r\|_1\|(az)^r\|_1 \le \frac{1}{p}\|(zb)^r\|_1^p + \frac{1}{q}\|(az)^r\|_1^q$.

3 $\|(az^2b)^r\|_r^r \le \|zb\|_p^r\|az\|_q^r \le \frac{r}{p}\|zb\|_p^p + \frac{r}{q}\|az\|_q^q$.

Moreover, equality holds if $|az|^q = |zb|^p$.

3 Heinz mean

The purpose of this section is to prove the Heinz means inequality for τ-measurable operators and generalizations of that results. Heinz means, introduced in [3], are means that interpolate in a certain way between the arithmetic and geometric mean. For every positive real numbers a, b and $0 \le \nu \le 1$, the Heinz mean is defined as

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu}b^\nu}{2}. \tag{29}$$

The function H_ν is symmetric about the point $\nu = \frac{1}{2}$. It is easy to see that

$$H_{\frac{1}{2}}(a,b) = \sqrt{ab} \le H_\nu(a,b) \le \frac{a+b}{2} = H_1(a,b). \tag{30}$$

Let $A,B,X \in M_n(\mathbb{C})$ such that A and B are positive. It was shown in [4, 12] that

$$\big\| \big|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\big| \big\| \le \big\| \big|AX + XB\big| \big\|,$$

and

$$\big\| \big|A^\nu XB^{1-\nu} + B^\nu X^* A^{1-\nu}\big| \big\| \le \big\| |AX| \big\| + \big\| |XB| \big\|,$$

for $\nu \in [0,1]$ and for every invariant norm. Now, we present the Heinz mean inequalities for τ-measurable operators.

Theorem 3.1 *Let a, b be positive operators in $L_1(\mathcal{M})$ and let $0 \le \nu \le 1$, then*

$$2\big|a^{\frac{1}{2}}b^{\frac{1}{2}}\big| \prec_w \big|a^{\frac{\nu}{2}}b^{\frac{1-\nu}{2}}\big|^2 + \big|a^{\frac{1-\nu}{2}}b^{\frac{\nu}{2}}\big|^2, \tag{31}$$

and

$$2\big\|a^{\frac{1}{2}}b^{\frac{1}{2}}\big\|_1 \le \big\|a^\nu b^{1-\nu} + a^{1-\nu}b^\nu\big\|_1 \le \|a+b\|_1. \tag{32}$$

Moreover, if a and b are bounded or $ab \in \mathfrak{L}^2_{\text{loc}}(\mathcal{M})$, then equality holds in (32), and, therefore, we have majorization relation ($\prec$) in (31) if and only if $a = b$.

Proof Using the same method used in lemma 3.1 of [20], we get for every $s \in \mathbb{R}_0^+$

$$\begin{aligned}\int_0^s \mu_t\big|a^{\frac{1}{2}}b^{\frac{1}{2}}\big|\,dt &= \int_0^s \mu_t\big(a^{\frac{1}{2}}b^{\frac{1}{2}}\big)\,dt \\ &\le \int_0^s \mu_t\big(\big|a^{\frac{\nu}{2}}b^{\frac{1-\nu}{2}}\big| \cdot \big|a^{\frac{1-\nu}{2}}b^{\frac{\nu}{2}}\big|\big)\,dt \quad [20, \text{Lemma 3.1}]\end{aligned}$$

$$\leq \int_0^s \mu_t\left(\frac{1}{2}\left|a^{\frac{\nu}{2}}b^{\frac{1-\nu}{2}}\right|^2 + \frac{1}{2}\left|a^{\frac{1-\nu}{2}}b^{\frac{\nu}{2}}\right|^2\right) dt, \quad \text{[Prop. 2.1]}$$

which proves (31). According to the above argument, we have

$$\begin{aligned}
\left\|a^{\frac{1}{2}}b^{\frac{1}{2}}\right\| &= \int_0^\infty \mu_t\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right) dt \\
&= \frac{1}{2}\int_0^\infty \mu_t\left(b^{\frac{1-\nu}{2}}a^{\frac{\nu}{2}}a^{\frac{\nu}{2}}b^{\frac{1-\nu}{2}} + b^{\frac{\nu}{2}}a^{\frac{1-\nu}{2}}a^{\frac{1-\nu}{2}}b^{\frac{\nu}{2}}\right) dt \\
&\leq \frac{1}{2}\int_0^\infty \left(\mu_t\left(b^{\frac{1-\nu}{2}}a^{\frac{\nu}{2}}a^{\frac{\nu}{2}}b^{\frac{1-\nu}{2}}\right) + \mu\left(b^{\frac{\nu}{2}}a^{\frac{1-\nu}{2}}a^{\frac{1-\nu}{2}}b^{\frac{\nu}{2}}\right)\right) dt \\
&\quad \text{[By part (6) of Theorem 1.1]} \\
&\leq \frac{1}{2}\int_0^\infty \left(\mu_t\left(a^\nu b^{1-\nu}\right) + \mu\left(a^{1-\nu}b^\nu\right)\right) dt \quad \text{[Theorem 2.16]} \\
&\leq \frac{1}{2}\int_0^\infty \left(\mu_t\left(\nu a + (1-\nu)b\right) + \mu\left(\nu b + (1-\nu)a\right)\right) dt \quad \text{[Prop. 2.1]} \\
&\leq \frac{1}{2}\int_0^\infty \left(\mu_t(a) + \mu(b)\right) dt. \quad \text{[By part (6) of Theorem 1.1]}
\end{aligned}$$

Thus, (32) is proved. The case of equality will be derived by the same method applied in [20, Theorem 3.3]. □

The following lemma provides another version of inequality (32) when $\nu > 1$.

Lemma 3.2 ([20]) *Let a, b be positive invertible operators in $L_1(\mathcal{M})$. Then for $\nu > 1$,*

$$\left\|a^{\frac{1}{2}}b^{\frac{1}{2}}\right\|_1 \leq \frac{1}{2}\|a+b\|_1 \leq \frac{1}{2}\left\|a^\nu b^{1-\nu} + a^{1-\nu}b^\nu\right\|_1. \tag{33}$$

Moreover, if a and b are bounded or $ab \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, then equality holds in each part if and only if $a = b$.

In the following, we present another general form of the Heinz mean inequality. Note that Han and Shao [11], proved that if x, y be τ-measurable operators and $0 \leq \nu \leq 1$, then

$$\mu_t\left(x^\nu y^{1-\nu} + x^{1-\nu}y^\nu\right) \leq \mu_t(x+y).$$

So by definition of $\|.\|_p$, for a, b be positive τ-measurable operators in $L_p(\mathcal{M})$, we have

$$\left\|a^\nu b^{1-\nu} + a^{1-\nu}b^\nu\right\|_p \leq \|a+b\|_p.$$

Using Theorem 2.25, the following inequalities can also be obtained.

Theorem 3.3 *Let a, b be positive bounded operators in $L_p(\mathcal{M})$ and $z \in L_0(\mathcal{M})$ bounded. Then for $0 \leq \nu \leq 1$,*

$$\left\|a^\nu z b^{1-\nu} + b^\nu z^* a^{1-\nu}\right\|_p \leq \|az\|_p + \|zb\|_p. \tag{34}$$

If a and b are bounded operators, then equality holds if $\|az\|_p = \|zb\|_p$.

Proof By the Minkowski inequality and Theorem 2.25, we have

$$\begin{aligned}\left\|a^{\nu}zb^{1-\nu}+b^{\nu}z^{*}a^{1-\nu}\right\|_{p} &\leq \left\|a^{\nu}zb^{1-\nu}\right\|_{p}+\left\|b^{\nu}z^{*}a^{1-\nu}\right\|_{p}\\ &\leq \|az\|_{p}^{\nu}\|zb\|_{p}^{1-\nu}+\left\|bz^{*}\right\|_{p}^{\nu}\left\|z^{*}a\right\|_{p}^{1-\nu}\\ &= \|az\|_{p}^{\nu}\|zb\|_{p}^{1-\nu}+\|zb\|_{p}^{\nu}\|az\|_{p}^{1-\nu}\\ &\quad \big(\text{since } \left\|x^{*}\right\|=\|x\| \text{ for every } x\in L_{0}(\mathcal{M})\big)\\ &\leq \|az\|_{p}+\|zb\|_{p}.\end{aligned}$$

If equality holds in (34), then we have

$$\left\|a^{\nu}zb^{1-\nu}\right\|_{p}=\|az\|_{p}^{\nu}\|zb\|_{p}^{1-\nu}=\nu\|az\|_{p}+(1-\nu)\|zb\|_{p}$$

and

$$\left\|b^{\nu}z^{*}a^{1-\nu}\right\|_{p}=\left\|bz^{*}\right\|_{p}^{\nu}\|za\|_{p}^{1-\nu}=\nu\left\|bz^{*}\right\|_{p}+(1-\nu)\|az\|_{p}.$$

Theorem 2.25 now shows that $\|az\|_p = \|zb\|_p$. □

Corollary 3.4 *Let a, b be positive operators in $L_p(\mathcal{M})$. Then for $0 \leq \nu \leq 1$,*

$$\left\|a^{\nu}b^{1-\nu}+b^{\nu}a^{1-\nu}\right\|_{p}\leq\|a\|_{p}+\|b\|_{p}.$$

If a and b are bounded operators or $ab \in \mathfrak{L}^2_{\mathrm{loc}}(\mathcal{M})$, then equality holds if $a = b$.

Corollary 3.5 *Let a, b be positive bounded operators in $L_1(\mathcal{M})$, $z \in L_0(\mathcal{M})$ bounded and $0 \leq \nu \leq 1$. Then*

$$\left\|a^{\nu}zb^{1-\nu}+b^{\nu}za^{1-\nu}\right\|_{1}\leq\|a+b\|_{1}\|z\|.$$

Proof We have

$$\begin{aligned}\left\|a^{\nu}zb^{1-\nu}+b^{\nu}za^{1-\nu}\right\|_{1} &\leq \|az\|_{1}+\|zb\|_{1}\\ &\leq \|a\|_{1}\|z\|+\|b\|_{1}\|z\|\\ &= \big(\|a\|_{1}+\|b\|_{1}\big)\|z\|\\ &= \|a+b\|_{1}\|z\|.\end{aligned}$$

□

It is not clear to us as to whether or not the following inequality for singular values of τ-measurable operators is true

$$\mu_{t}\big(a^{\nu}zb^{1-\nu}+b^{\nu}za^{1-\nu}\big)\leq\mu_{t}(az+zb),\quad \forall t>0.$$

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Authors' contributions
The main idea of this paper was proposed by corresponding author SMM, and ZM prepared the manuscripts initially and preformed all the steps and proofs in this research. All authors read and approved the final manuscripts.

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