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Graph convergence with an application for system of variational inclusions and fixed-point problems

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Abstract

This paper aims at proposing an iterative algorithm for finding an element in the intersection of the solutions set of a system of variational inclusions and the fixed-points set of a total uniformly L-Lipschitzian mapping. Applying the concepts of graph convergence and the resolvent operator associated with an $\widehat{H}$-accretive mapping, a new equivalence relationship between graph convergence and resolvent-operator convergence of a sequence of $\widehat{H}$-accretive mappings is established. As an application of the obtained equivalence relationship, the strong convergence of the sequence generated by our proposed iterative algorithm to a common point of the above two sets is proved under some suitable hypotheses imposed on the parameters and mappings. At the same time, the notion of $H(\cdot,\cdot)$-accretive mapping that appeared in the literature, where $H(\cdot,\cdot)$ is an α, β-generalized accretive mapping, is also investigated and analyzed. We show that the notions $H(\cdot,\cdot)$-accretive and $\widehat{H}$-accretive operators are actually the same, and point out some comments on the results concerning them that are available in the literature.

MSC: 47H05; 47H09; 47J20; 47J22; 47J25; 49J40

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1 Introduction

During the last six decades, variational inequality theory, originally introduced for the study of partial differential equations by Hartman and Stampacchia [1], has been recognized as a strong tool in the mathematical study of many nonlinear problems of physics and mechanics, as the complexity of the boundary conditions and the diversity of the constitutive equations lead to variational formulations of inequality type. As many nonlinear problems arising in optimization, operations research, structural analysis, and engineering sciences can be transformed into variational inequality problems (see, e.g., [2, 3]), since the appearance of this theory, there has been an increasing interest in extending and generalizing variational inequalities in many different directions using novel and innovative techniques, see, for example, [4, 5] and the references therein. Without doubt, one of the most important and well-known generalizations of variational inequalities is variational inclusions and thanks to their wide applications in the optimization and control, economics and transportation equilibrium, engineering science, etc., the study of different classes of variational inclusion problems continues to attract the interest of many researchers. For more related details, we refer the readers to [6–19] and the references therein.

It is important to emphasize that two important problems in the theory of the variational inequalities/icnlusions are the existence of solutions and approximation of solutions by the iterative algorithms. This has been one of the main motivations of researchers to develop alternative methods to study iterative algorithms for approximating solutions of various kinds of variational inequality/inclusion problems in the setting of Hilbert and Banach spaces. Among the methods that have appeared in the literature, the resolvent-operator technique is interesting and important, and plays a crucial role in computing approximate solutions of different classes of variational inequality/inclusion problems and their generalizations. More information along with relevant commentaries can be found in [6, 11–14, 17, 18, 20–29] and the references therein.

Monotone operators and accretive mappings have gained impetus, due to their wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations, integral equations, mathematical economics, optimal control, and so forth. Due to the importance and their many diverse applications in a huge variety of scientific fields, considerable attention has been paid to the development and generalization of monotone and accretive operators in the framework of different spaces. By the same token, the introduction of the notion of generalized m-accretive mapping as an extension of maximal monotone operators and m-accretive mapping along with a definition of its resolvent operator in a Banach space setting was first made by Huang and Fang [16] in 2001. Afterwards, many authors have shown interest in extending maximal monotone operators and generalized m-accretive mappings, and further generalizations of them have appeared in the literature. For instance, Huang and Fang [15], Ding and Lou [30] and Lee et al. [21], Fang and Huang [12], Xia and Huang [31], Fang and Huang [11], Fang et al. [14], Kazmi and Khan [24] and Peng and Zhu [25], Verma [27, 32], Verma [33], and Lan et al. [17] introduced and studied the notions of η-monotone operators, η-subdifferential operators, H-monotone operators, general H-monotone operators, H-accretive (to avoid confusion, throughout the paper we call it $\widehat{H}$-accretive) mappings, (H,η)-monotone operators, P-η-accretive (also referred to as (H,η)-accretive) mappings, A-monotone operators, (A,η)-monotone operators, and (A,η)-accretive (also referred to as A-maximal m-relaxed η-accretive) mappings, respectively. Motivated by these advances, in 2008, Sun et al. [34] introduced the class of M-monotone operators as a generalization of maximal monotone and H-monotone operators. With inspiration and motivation from the work of Sun et al. [34], in the same year, Zou and Huang [35] succeeded in introducing the notion of $H(\cdot,\cdot)$-accretive mappings in a Banach-space setting as a generalization of generalized m-accretive, H-monotone, $\widehat{H}$-accretive, and M-monotone operators.

The notion of graph convergence has attracted many researchers since 1984 after the pioneering work of Attouch [36]. It is worthwhile to stress that the attention of the author in [36] was limited to maximal monotone operators. In later years, considerable research efforts have been made to generalize and study the concept of graph convergence for generalized monotone operators and generalized accretive mappings available in the literature. For instance, Li and Huang [22] introduced the notion of graph convergence concerned with $H(\cdot,\cdot)$-accretive operators in Banach spaces and proved some equivalence theorems between graph convergence and resolvent-operator convergence of a sequence of $H(\cdot,\cdot)$-accretive mappings. For a detailed description of the concept of graph convergence for other generalizations of generalized monotone (accretive) operators existing in the literature, we refer the interested reader to [6, 22, 23, 28, 36] and the references therein. Using the properties of graph convergence of $H(\cdot,\cdot)$-accretive operators introduced by Li and Huang [22], recently, Tang and Wang [26] constructed a perturbed iterative algorithm for solving a system of ill-posed variational inclusions involving $H(\cdot,\cdot)$-accretive operators. At the same time, they proved that under some suitable conditions, the sequence generated by their proposed iterative algorithm is strongly convergent to the unique solution of the system of variational inclusions considered in [26].

On the other hand, the theory of fixed points that the starting point of its study dates back to the beginning of the 1920s with the pioneering work of Polish mathematician Stefan Banach [37], is a very attractive subject, which has recently drawn much attention from the communities of physics, engineering, mathematics, etc. The existence of a strong connection between the variational inequality problems and the fixed-point problems motivated many investigators to study the problem of finding common elements of the set of solutions of variational inequalities/inclusions and the set of fixed points of given operators. For more details and information, the reader is referred to [4, 38–46] and the references therein.

In addition, after the emergence of the notion of nonexpansive mapping in the 1960s, the number of works dedicated to study fixed-point theory for nonexpansive mappings in the setting of different spaces has grown rapidly and has influenced several branches of mathematics. This is mainly because there is a very close relation between the classes of monotone and accretive operators, which arise naturally in the theory of differential equations, and the class of nonexpansive mappings. Due to its many diverse applications in the theory of fixed points, the interest in extending and generalizing the notion of nonexpansive mapping has increased rapidly over the past forty years. One of the first attempts in this direction was carried out by Goebel and Kirk [47] in 1972 who introduced a class of generalized nonexpansive mappings, the so-called asymptotically nonexpansive mappings. In 2005, Sahu [48] succeeded in introducing the concept of nearly asymptotically nonexpansive mapping as a generalization of the notion of asymptotically nonexpansive mapping. One year later, another class of generalized nonexpansive mappings, the so-called total asymptotically nonexpansive mappings, which is essentially more general than the classes of nearly asymptotically nonexpansive mappings and asymptotically nonexpansive mappings, was introduced and studied by Alber et al. [49]. The efforts in this direction have continued and in a successfully attempt by Kiziltunc and Purtas [50], the class of total uniformly L-Lipschitzian mappings was introduced as a unifying framework for the classes of generalized nonexpansive mappings existing in the literature. To find more information about different classes of generalized nonexpansive mappings and relevant commentaries, we refer the reader to [20, 47–51] and the references therein.

Motivated and inspired by the excellent work mentioned above, this paper pursues two purposes. The first objective is to prove the existence of a unique solution for a system of variational inclusions (SVI) involving $\widehat{H}$-accretive mappings under some suitable mild conditions. With the goal of finding a common point lying in the solutions set of the SVI and the set of fixed points of a total uniformly L-Lipschitzian mapping, an iterative algorithm is constructed. Employing the notions of graph convergence and the resolvent operator associated with an $\widehat{H}$-accretive mapping, a new equivalence relationship between the graph convergence of a sequence of $\widehat{H}$-accretive mappings and their associated resolvent operators, respectively, to a given $\widehat{H}$-accretive mapping and its associated resolvent operator under some appropriate conditions is established. As an application of the obtained equivalence relationship, we prove the strong convergence of the sequence generated by our proposed iterative algorithm to a common point of the set of fixed points of the total uniformly L-Lipschitzian mapping and the set of solutions of the SVI. The second goal of this paper is to investigate and analyze the notion of $H(\cdot,\cdot)$-accretive mapping that appeared in [26], where $H(\cdot,\cdot)$ is an α,β-generalized accretive mapping, and to point out some comments concerning it. We prove that under the assumptions imposed on the $H(\cdot,\cdot)$-accretive mapping considered in [26], every $H(\cdot,\cdot)$-accretive mapping is actually an $\widehat{H}$-accretive mapping and is not a new one. All the results derived by the authors in [26] are reviewed and some remarks regarding them are stated. We show that our results improve and generalize the corresponding results of [26] and recent related works.

2 Notation and preliminaries

In this section, we briefly present the notation and some preliminary material to be used later in this paper. First, we make clear that all linear spaces used in this paper are assumed to be real. Unless it is stated otherwise, in this paper we denote by X a real Banach space with norm $\|\cdot\|$, we denote by X^* its topological dual, and $\langle\cdot,\cdot\rangle$ will represent the duality pairing of X and X^*. We denote by S_X and S_{X^*}, respectively, the unit sphere in X and X^*. For a given set-valued mapping $M : X \rightrightarrows X$, the set Graph($M$) defined by

$$\mathrm{Graph}(M) := \bigl\{(u,v) \in X \times X : v \in M(u)\bigr\},$$

is called the *graph* of M.

Let us recall that a normed space X is called strictly convex if S_X is strictly convex, that is, the inequality $\|x+y\| < 2$ holds for all $x,y \in S_X$ with $x \neq y$. It is said to be smooth if for every $x \in S_X$ there is exactly one $x^* \in S_{X^*}$ such that $x^*(x) = 1$. Equivalently, a normed space X is said to be *smooth* provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x,y \in S_X$. It is known that if a Banach space X is reflexive, then X is strictly convex if and only if X^* is smooth and X is smooth if and only if X^* is strictly convex.

With each $x \in X$, we associate the set

$$J(x) = \bigl\{x^* \in X^* : \bigl\langle x, x^*\bigr\rangle = \|x\|^2 = \bigl\|x^*\bigr\|^2\bigr\}.$$

The operator $J : X \rightrightarrows X^*$ is called the *normalized duality mapping* of X. We observe immediately that if $X = \mathcal{H}$, a Hilbert space, then J is the identity mapping on $\mathcal{H}$. At the same time, from the Hahn–Banach theorem, it follows that $J(x)$ is nonempty for each $x \in X$. In general, the normalized duality mapping is set-valued. However, it is single-valued in a smooth Banach space.

Definition 2.1 For a given real smooth Banach space X, an operator $T : X \to X$ is said to be

(i) accretive if

$$\langle T(x) - T(y), J(x-y)\rangle \geq 0, \quad \forall x, y \in X;$$

(ii) strictly accretive if T is accretive and equality holds if and only if $x = y$;
(iii) r-strongly accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J(x-y)\rangle \geq r\|x-y\|^2, \quad \forall x, y \in X;$$

(iv) α-relaxed accretive if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), J(x-y)\rangle \geq -\alpha\|x-y\|^2, \quad \forall x, y \in X;$$

(v) γ-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$\|T(x) - T(y)\| \leq \gamma\|x-y\|, \quad \forall x, y \in X.$$

Definition 2.2 For a given real smooth Banach space X, a set-valued operator $M : X \rightrightarrows X$ is said to be

(i) accretive if

$$\langle u - v, J(x-y)\rangle \geq 0, \quad \forall (x,u), (y,v) \in \mathrm{Graph}(M);$$

(ii) m-accretive if M is accretive and $(I + \lambda M)(X) = X$, for all $\lambda > 0$, where I denotes the identity mapping on X.

Example 2.3 ([52]) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and let for $1 < p < \infty$, $L^p(\Omega)$ be the space of all the Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega |f|^p\,dx < \infty$. Suppose further that T is a maximal monotone graph in $\mathbb{R}$. With appropriate domains,

(i) the operator $A_1 u := -\Delta u + T(u)$ with homogeneous Neuman boundary condition, and the operator $A_2 u := -\Delta u$, $-\frac{\partial u}{\partial n} \in T(u)$ on $\partial\Omega$, where Δ denotes the Laplacian, are accretive on $L^p(\Omega)$. Meanwhile,
(ii) the operator $A_3 u := -\sum(\frac{\partial}{\partial x_i})(|\frac{\partial u}{\partial x_i}|^{r-1}\frac{\partial}{\partial x_i})$ is accretive for $r \geq 1$.

Example 2.4 ([52]) The operator $-\Delta$, where Δ denotes the Laplacian, is an m-accretive operator.

Remark 2.5 As was pointed out in [52], the interest and importance of the concept of accretive mappings, which was introduced and studied independently by Browder [53] and Kato [54], stems from the fact that many physically significant problems can be modeled in terms of an initial-valued problem of the form

$$\begin{cases} \frac{du}{dt} = -Au, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where A is either an accretive or strongly accretive mapping on an appropriate Banach space. Typical examples of such evolution equations are found in models involving the heat, wave or Schrödinger equation (see, e.g., Browder [53]). An early fundamental result in the theory of accretive mappings, due to Browder [55], states that the initial-value problem (2.1) is solvable if A is locally Lipschitzian and accretive on X. Utilizing the existence result for the equation (2.1), Browder [53] proved that if A is locally Lipschitzian and accretive on X, then A is m-accretive. Obviously, a consequence of this is that the equation $x + Tx = f$, for a given $f \in X$, where $T := I - A$, has a solution. Martin [56, 57] proved that the equation (2.1) is solvable if A is continuous and accretive on X, and using this result, he further established that if A is continuous and accretive, then A is m-accretive. In [52], the author verified that if $A : X \to X$ is a Lipschitz and strongly accretive mapping, then A is surjective. Consequently, for each $f \in X$, the equation $Ax = f$ has a solution in X.

We note that M is an m-accretive mapping if and only if M is accretive and there is no other accretive mapping whose graph contains strictly Graph(M). The m-accretivity is to be understood in terms of inclusion of graphs. If $M : X \rightrightarrows X$ is an m-accretive mapping, then adding anything to its graph, so as to obtain the graph of a new set-valued mapping, destroys the accretivity. In fact, the extended mapping is no longer accretive. In other words, for every pair $(x, u) \in X \times X \setminus \mathrm{Graph}(M)$ there exists $(y, v) \in \mathrm{Graph}(M)$ such that $\langle u - v, J(x - y)\rangle < 0$. Thanks to the above-mentioned arguments, a necessary and sufficient condition for set-valued mapping $M : X \rightrightarrows X$ to be m-accretive is that the property

$$\big\langle u - v, J(x - y)\big\rangle \geq 0, \quad \forall (y, v) \in \mathrm{Graph}(M)$$

is equivalent to $u \in M(x)$. The above characterization of m-accretive mappings provides us with a useful and manageable way for recognizing that an element u belongs to $M(x)$.

Definition 2.6 Given a smooth Banach space X and a mapping $\widehat{H} : X \to X$, the set-valued mapping $M : X \rightrightarrows X$ is said to be $\widehat{H}$-accretive if M is accretive and $(\widehat{H} + \lambda M)(X) = X$ holds for all $\lambda > 0$.

It should be remarked that Fang and Huang [11] were the first to introduce the class of $\widehat{H}$-accretive mappings on q-uniformly smooth Banach spaces for some real constant $q > 1$. We recall that for a given real constant $q > 1$, X is called *q-uniformly smooth* if there exits a constant $C > 0$ such that $\rho_X(\tau) \leq C\tau^q$, for all $\tau \in \mathbb{R}^+$, where the function $\rho_X : \mathbb{R}^+ \to \mathbb{R}^+$ is given by formula

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\| - 1 : x, y \in S_X\right\}.$$

At the same time, it should be pointed out that if $\widehat{H} = I$, then Definition 2.6 reduces to the definition of an m-accretive operator, and if $X = \mathcal{H}$ and $\widehat{H} = I$, then Definition 2.6 becomes just the definition of a maximal monotone operator.

The following example shows that for a given mapping $\widehat{H} : X \to X$, an m-accretive mapping may not be $\widehat{H}$-accretive.

Example 2.7 Let $M_2(\mathbb{C})$ be the space of all 2×2 matrices with complex entries. Then,

$$M_2(\mathbb{C}) = \left\{ \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ x_3 + iy_3 & x_4 + iy_4 \end{pmatrix} \Big| x_k, y_k \in \mathbb{R}, k = 1,2,3,4 \right\}$$

is a Hilbert space together with the inner product $\langle A, B\rangle := \mathrm{tr}(AB^*)$, for all $A, B \in M_2(\mathbb{C})$, where tr denotes the trace, that is, the sum of the diagonal entries, and B^* denotes the Hermitian conjugate (or adjoint) of the matrix B, that is, $B^* = \overline{B^t}$, the complex conjugate of the transpose B. The inner product defined above induces a norm on $M_2(\mathbb{C})$ as follows:

$$\|A\| = \sqrt{\sum_{i=1}^{4} \left(x_k^2 + y_k^2\right)}, \quad \text{for all } A = \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ x_3 + iy_3 & x_4 + iy_4 \end{pmatrix} \in M_2(\mathbb{C}).$$

Thereby, the Hilbert space $(M_2(\mathbb{C}), \|\cdot\|)$ is a 2-uniformly smooth Banach space. For any

$$A = \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ x_3 + iy_3 & x_4 + iy_4 \end{pmatrix} \in M_2(\mathbb{C}),$$

we have

$$\begin{aligned} A &= \frac{y_1 + y_4 - i(x_1 + x_4)}{2} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} + \frac{y_2 + y_3 - i(x_2 + x_3)}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &\quad + \frac{x_3 - x_2 + i(y_3 - y_2)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{y_1 - y_4 - i(x_1 - x_4)}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ &= \frac{y_1 + y_4 - i(x_1 + x_4)}{2} \mu_1 + \frac{y_2 + y_3 - i(x_2 + x_3)}{2} \mu_2 \\ &\quad + \frac{x_3 - x_2 + i(y_3 - y_2)}{2} \mu_3 + \frac{y_1 + y_4 - i(x_1 + x_4)}{2} \mu_4, \end{aligned}$$

where

$$\mu_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \mu_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mu_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Therefore, the set $\{\mu_k : k = 1,2,3,4\}$ spans the Hilbert space $M_2(\mathbb{C})$. Letting $\theta_k := \frac{1}{\sqrt{2}}\mu_k$ for $k = 1,2,3,4$, it is easy to see that the set $\mathfrak{B}$ consisting of the rescaled 2×2 matrices θ_k ($k = 1,2,3,4$) is also a spanner of $M_2(\mathbb{C})$. At the same time, $\|\theta_k\| = 1$, for $k = 1,2,3,4$ and $\langle \theta_k, \theta_j\rangle = 0$ for $1 \leq k, j \leq 4$, that is, the set $\mathfrak{B}$ is orthonormal. Accordingly, the set $\mathfrak{B} = \{\theta_k : k = 1,2,3,4\}$ is an orthonormal basis for the Banach space $M_2(\mathbb{C})$. Let the mappings $M, \widehat{H} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be defined, respectively, by $M(A) = \gamma A + \alpha_1\theta_1 + \alpha_3\theta_3$ and $\widehat{H}(A) = -\gamma A + \alpha_2\theta_2 + \alpha_4\theta_4$, for all $A \in M_2(\mathbb{C})$, where γ is an arbitrary positive real constant and α_k ($k = 1,2,3,4$) are arbitrary real constants. Then, for all $A, B \in M_2(\mathbb{C})$, it yields

$$\begin{aligned} \big\langle M(A) - M(B), J(A - B)\big\rangle &= \big\langle M(A) - M(B), A - B\big\rangle \\ &= \gamma \langle A - B, A - B\rangle \\ &= \gamma \|A - B\|^2 \geq 0, \end{aligned}$$

i.e., M is an accretive mapping. By virtue of the fact that for any $A \in M_2(\mathbb{C})$ and $\lambda > 0$, we have

$$\begin{aligned}(I + \lambda M)(A) &= A + \gamma\lambda A + \lambda\alpha_1\theta_1 + \lambda\alpha_3\theta_3 \\ &= (1 + \gamma\lambda)A + \lambda(\alpha_1\theta_1 + \alpha_3\theta_3),\end{aligned}$$

where I is the identity mapping on $M_2(\mathbb{C})$, we conclude that $(I + \lambda M)(M_2(\mathbb{C})) = M_2(\mathbb{C})$, for every real constant $\lambda > 0$, that is, the mapping $I + \lambda M$ is surjective for every positive real constant λ. Hence, M is an m-accretive mapping. Since for any $A \in M_2(\mathbb{C})$, we have

$$(\widehat{H} + M)(A) = \sum_{k=1}^{4} \alpha_k\theta_k = \frac{\sqrt{2}}{2}\begin{pmatrix} (\alpha_1 + \alpha_4)i & -\alpha_3 + i\alpha_2 \\ \alpha_3 + i\alpha_2 & -(\alpha_1 + \alpha_4)i \end{pmatrix},$$

it follows that

$$\big\|(\widehat{H} + M)(A)\big\| = \left\|\sum_{k=1}^{4} \alpha_k\theta_k\right\| = \sqrt{2\big((\alpha_1 + \alpha_4)^2 + \alpha_3^2 + \alpha_4^2\big)} > 0.$$

This implies that $\mathbf{0} \notin (\widehat{H} + M)(H_2(\mathbb{C}))$, i.e., $\widehat{H} + M$ is not surjective. Thus, the mapping M is not $\widehat{H}$-accretive.

Example 2.8 Let $H_2(\mathbb{C})$ be the set of all Hermitian matrices with complex entries. We recall that a square matrix A is said to be Hermitian (or self-adjoint) if it is equal to its own Hermitian conjugate, i.e., $A^* = \overline{A^t} = A$. In the light of the definition of a Hermitian 2×2 matrix, the condition $A^* = A$ implies that the 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is Hermitian if and only if $a, d \in \mathbb{R}$ and $b = \bar{c}$. Therefore,

$$H_2(\mathbb{C}) = \left\{ \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \middle| x, y, z, w \in \mathbb{R} \right\}.$$

Then, $H_2(\mathbb{C})$ is a subspace of $M_2(\mathbb{C})$, the space of all 2×2 matrices with complex entries, with respect to the operations of addition and scalar multiplication defined on $M_2(\mathbb{C})$, when $M_2(\mathbb{C})$ is considered as a real vector space. In other words, $H_2(\mathbb{C})$ together with the mentioned operators is a vector space over $\mathbb{R}$. We introduce the scalar product on $H_2(\mathbb{C})$ as $\langle A, B\rangle := \frac{1}{2}\operatorname{tr}(AB)$, for all $A, B \in H_2(\mathbb{C})$. By an easy check, we observe that $(H_2(\mathbb{C}), \langle\cdot,\cdot\rangle)$ is an inner product space. The inner product defined above induces a norm on $H_2(\mathbb{C})$ as follows:

$$\begin{aligned}\|A\| &= \sqrt{\langle A, A\rangle} = \sqrt{\frac{1}{2}\operatorname{tr}(AA)} \\ &= \left\{\frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} x^2 + y^2 + z^2 & (z + w)(x - iy) \\ (z + w)(x + iy) & x^2 + y^2 + w^2 \end{pmatrix}\right)\right\}^{\frac{1}{2}} \\ &= \sqrt{x^2 + y^2 + \frac{1}{2}\big(z^2 + w^2\big)}, \quad \forall A \in H_2(\mathbb{C}).\end{aligned}$$

Taking into account that every finite-dimensional normed space is a Banach space, it follows that $(H_2(\mathbb{C}), \|\cdot\|)$ is a Hilbert space and so it is a 2-uniformly smooth Banach space.

Let the mappings $\widehat{H}_1, \widehat{H}_2, M : H_2(\mathbb{C}) \to H_2(\mathbb{C})$ be defined, respectively, by

$$\widehat{H}_1(A) = \begin{pmatrix} |\sqrt{3}\sin z - \sqrt{6}\cos z| - \alpha z^l & -2|x| - x^k + 1 - i(\frac{3^y-1}{3^y+1} - y^k) \\ 2|x| - x^k + 1 + i(\frac{3^y-1}{3^y+1} - y^k) & (\frac{1}{2})^{|w|-2} - \beta w^p \end{pmatrix},$$

$$\widehat{H}_2(A) = \begin{pmatrix} \cos(\gamma z + \theta) & \frac{e^x}{e^x+1} - \frac{2i}{1+y^2} \\ \frac{e^x}{e^x+1} + \frac{2i}{1+y^2} & \sin(\varsigma w + \xi) \end{pmatrix} \quad \text{and} \quad M(A) = \begin{pmatrix} \alpha z^l & x^k - iy^k \\ x^k + iy^k & \beta w^p \end{pmatrix},$$

for all $A = \left(\begin{smallmatrix} z & x-iy \\ x+iy & w \end{smallmatrix}\right) \in H_2(\mathbb{C})$, where α and β are arbitrary positive real constants, γ and ς are arbitrary nonzero real constants, θ and ξ are arbitrary real constants, and k, l, and p are arbitrary but fixed odd natural numbers.

Then, for any $A = \left(\begin{smallmatrix} z_1 & x_1-iy_1 \\ x_1+iy_1 & w_1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} z_2 & x_2-iy_2 \\ x_2+iy_2 & w_2 \end{smallmatrix}\right) \in H_2(\mathbb{C})$, it yields

$$\begin{aligned}
&\big\langle M(A) - M(B), J(A-B)\big\rangle \\
&\quad = \big\langle M(A) - M(B), A - B\big\rangle \\
&\quad = \left\langle \begin{pmatrix} \alpha(z_1^l - z_2^l) & x_1^k - x_2^k - i(y_1^k - y_2^k) \\ x_1^k - x_2^k + i(y_1^k - y_2^k) & \beta(w_1^p - w_2^p) \end{pmatrix}, \right. \\
&\qquad \left. \begin{pmatrix} z_1 - z_2 & x_1 - x_2 - i(y_1 - y_2) \\ x_1 - x_2 + i(y_1 - y_2) & w_1 - w_2 \end{pmatrix} \right\rangle \\
&\quad = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} \Theta_{11}(x_1,x_2,y_1,y_2,z_1,z_2) & \Theta_{12}(x_1,x_2,y_1,y_2,z_1,z_2) \\ \Theta_{21}(x_1,x_2,y_1,y_2,z_1,z_2) & \Theta_{22}(x_1,x_2,y_1,y_2,z_1,z_2) \end{pmatrix} \right) \\
&\quad = \frac{\alpha\varrho}{2}|z_1 z_2|(z_1 - z_2)\big(z_1^k - z_2^k\big) + \frac{\beta\varsigma}{2}e^{w_1 w_2}(w_1 - w_2)\big(\sqrt[n]{w_1} - \sqrt[n]{w_2}\big) \\
&\qquad + (x_1 - x_2)\big(x_1^m - x_2^m\big) + (y_1 - y_2)\big(y_1^m - y_2^m\big) \\
&\quad = \frac{\alpha}{2}\big(z_1^l - z_2^l\big)(z_1 - z_2) + (x_1 - x_2)\big(x_1^k - x_2^k\big) \\
&\qquad + (y_1 - y_2)\big(y_1^k - y_2^k\big) + \frac{\beta}{2}\big(w_1^p - w_2^p\big)(w_1 - w_2) \\
&\quad = \frac{\alpha}{2}(z_1 - z_2)^l \sum_{j=1}^{l} z_1^{l-j} z_2^{j-1} + (x_1 - x_2)^2 \sum_{t=1}^{k} x_1^{k-t} x_2^{t-1} \\
&\qquad + (y_1 - y_2)^2 \sum_{s=1}^{k} y_1^{k-s} y_2^{s-1} + \frac{\beta}{2}(w_1 - w_2)^2 \sum_{r=1}^{p} w_1^{p-r} w_2^{r-1},
\end{aligned}$$

where

$$\begin{aligned}
&\Theta_{11}(x_1,x_2,y_1,y_2,z_1,z_2) \\
&\quad = \alpha\big(z_1^l - z_2^l\big)(z_1 - z_2) + (x_1 - x_2)\big(x_1^k - x_2^k\big) \\
&\qquad + (y_1 - y_2)\big(y_1^k - y_2^k\big) + i\big(x_1^k - x_2^k\big)(y_1 - y_2) - i(x_1 - x_2)\big(y_1^k - y_2^k\big), \\
&\Theta_{12}(x_1,x_2,y_1,y_2,z_1,z_2) \\
&\quad = \alpha\big(x_1 - x_2 - i(y_1 - y_2)\big)\big(z_1^l - z_2^l\big) \\
&\qquad + \big(x_1^k - x_2^k - i\big(y_1^k - y_2^k\big)\big)(w_1 - w_2),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{21}(x_1,x_2,y_1,y_2,z_1,z_2)\\
&\quad= \bigl(x_1^k - x_2^k + i\bigl(y_1^k - y_2^k\bigr)\bigr)(z_1 - z_2)\\
&\qquad + \beta\bigl(w_1^p - w_2^p\bigr)\bigl(x_1 - x_2 + i(y_1 - y_2)\bigr),\\
&\Theta_{22}(x_1,x_2,y_1,y_2,z_1,z_2)\\
&\quad= (x_1 - x_2)\bigl(x_1^k - x_2^k\bigr) + (y_1 - y_2)\bigl(y_1^k - y_2^k\bigr)\\
&\qquad - i(y_1 - y_2)\bigl(x_1^k - x_2^k\bigr) + i(x_1 - x_2)\bigl(y_1^k - y_2^k\bigr) + \beta\bigl(w_1^p - w_2^p\bigr)(w_1 - w_2).
\end{aligned}$$

Taking into account the fact that k, l, and p are odd natural numbers, it can be easily observed that $\sum_{j=1}^{l} z_1^{l-j} z_2^{j-1} \geq 0$, $\sum_{t=1}^{k} x_1^{k-t} x_2^{t-1} \geq 0$, $\sum_{s=1}^{k} y_1^{k-s} y_2^{s-1} \geq 0$ and $\sum_{r=1}^{p} w_1^{p-r} w_2^{r-1} \geq 0$. Since $\alpha, \beta > 0$, the preceding relation implies that

$$\bigl\langle M(A) - M(B), J(A - B)\bigr\rangle \geq 0, \quad \forall A, B \in H_2(\mathbb{C}),$$

that is, M is an accretive mapping. Let us define now the functions $f, g, h, \varphi : \mathbb{R} \to \mathbb{R}$, for all $\nu \in \mathbb{R}$, respectively, as

$$\begin{aligned}
&f(\nu) := |\sqrt{3}\sin\nu - \sqrt{6}\cos\nu|,\\
&g(\nu) := 2|\nu| + 1,\\
&h(\nu) := \frac{3^\nu - 1}{3^\nu + 1} \quad \text{and} \quad \varphi(\nu) := \left(\frac{1}{2}\right)^{|\nu|-2}.
\end{aligned}$$

Then, for any $A = \left(\begin{smallmatrix} z & x-iy \\ x+iy & w \end{smallmatrix}\right) \in H_2(\mathbb{C})$, it yields

$$\begin{aligned}
(\widehat{H}_1 + M)(A) = (\widehat{H}_1 + M)\left(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}\right)
&= \begin{pmatrix} |\sqrt{3}\sin z - \sqrt{6}\cos z| & 2|x| + 1 - \frac{3^y-1}{3^y+1}i \\ 2|x| + 1 + \frac{3^y-1}{3^y+1}i & (\frac{1}{2})^{|w|-2} \end{pmatrix}\\
&= \begin{pmatrix} f(z) & g(x) - ih(y) \\ g(x) + ih(y) & \varphi(w) \end{pmatrix}.
\end{aligned}$$

It can be easily seen that $f(\mathbb{R}) = [0,3]$, $g(\mathbb{R}) = [1,+\infty)$, $h(\mathbb{R}) = (-1,1)$ and $\varphi(\mathbb{R}) = (0,4]$. These facts imply that $(\widehat{H}_1 + M)(H_2(\mathbb{C})) \neq H_2(\mathbb{C})$, i.e., $\widehat{H}_1 + M$ is not surjective and so M is not an $\widehat{H}_1$-accretive mapping. Now, let $\lambda > 0$ be an arbitrary real constant and assume that the functions $\widehat{f}, \widehat{g}, \widehat{h}, \widehat{\varphi} : \mathbb{R} \to \mathbb{R}$ are defined, respectively, by

$$\begin{aligned}
&\widehat{f}(\nu) := \cos(\gamma\nu + \theta) + \lambda\alpha\nu^l, \qquad \widehat{g}(\nu) := \frac{e^\nu}{e^\nu + 1} + \lambda\nu^k,\\
&\widehat{h}(\nu) := \frac{2}{1 + \nu^2} + \lambda x\nu^k \quad \text{and} \quad \widehat{\varphi}(\nu) := \sin(\varsigma\nu + \xi) + \lambda\beta\nu^p, \quad \forall \nu \in \mathbb{R}.
\end{aligned}$$

Then, for any $A = \left(\begin{smallmatrix} z & x-iy \\ x+iy & w \end{smallmatrix}\right) \in H_2(\mathbb{C})$, we obtain

$$\begin{aligned}
(\widehat{H}_2 + \lambda M)(A) &= (\widehat{H}_2 + \lambda M)\left(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}\right) \\
&= \begin{pmatrix} \cos(\gamma z + \theta) + \lambda \alpha z^l & \frac{e^x}{e^x+1} + \lambda x^k - i(\frac{2}{1+y^2} + \lambda y^k) \\ \frac{e^x}{e^x+1} + \lambda x^k + i(\frac{2}{1+y^2} + \lambda y^k) & \sin(\varsigma w + \xi) + \lambda \beta w^p \end{pmatrix} \\
&= \begin{pmatrix} \widehat{f}(z) & \widehat{g}(x) - i\widehat{h}(y) \\ \widehat{g}(x) + i\widehat{h}(y) & \widehat{\varphi}(w) \end{pmatrix}.
\end{aligned}$$

Relying on the fact that k, l, and p are odd natural numbers, it is easy to see that $\widehat{f}(\mathbb{R}) = \widehat{g}(\mathbb{R}) = \widehat{h}(\mathbb{R}) = \widehat{\varphi}(\mathbb{R}) = \mathbb{R}$. Consequently, $(\widehat{H}_2 + \lambda M)(H_2(\mathbb{C})) = H_2(\mathbb{C})$, that is, $\widehat{H}_2 + \lambda M$ is surjective. Taking into account the arbitrariness in the choice of $\lambda > 0$, we conclude that M is an $\widehat{H}_2$-accretive mapping.

As was pointed out, if $\widehat{H} = I$, then the definition of I-accretive mappings is that of m-accretive mappings. In fact, the class of $\widehat{H}$-accretive mappings has a close relation with that of m-accretive mappings. In the same way as the proofs of Theorems 2.1 and 2.2 in [11], we obtain the following assertions in a smooth Banach space setting.

Lemma 2.9 *Let X be a real smooth Banach space, $\widehat{H} : X \to X$ be a strictly accretive mapping, $M : X \rightrightarrows X$ be an $\widehat{H}$-accretive mapping, and let $x, u \in X$ be given points. If $\langle u - v, J(x - y)\rangle \geq 0$ holds, for all $(y, v) \in \mathrm{Graph}(M)$, then $u \in M(x)$, that is, M is an m-accretive mapping.*

Lemma 2.10 *Let X be a real smooth Banach space, $\widehat{H} : X \to X$ be a strictly accretive mapping and $M : X \rightrightarrows X$ be an $\widehat{H}$-accretive mapping. Then, the mapping $(\widehat{H} + \lambda M)^{-1}$ is single-valued for every constant $\lambda > 0$.*

It is worth noting that Lemma 2.10 allows us to define the resolvent operator $R^{\widehat{H}}_{M,\lambda}$ associated with $\widehat{H}$, M and an arbitrary real constant $\lambda > 0$ as follows.

Definition 2.11 Let X be a real smooth Banach space, $\widehat{H} : X \to X$ be a strictly accretive mapping and $M : X \rightrightarrows X$ be an $\widehat{H}$-accretive mapping. The resolvent operator $R^{\widehat{H}}_{M,\lambda} : X \to X$ associated with $\widehat{H}$, M and an arbitrary positive real constant λ is defined by

$$R^{\widehat{H}}_{M,\lambda}(u) = (\widehat{H} + \lambda M)^{-1}(u), \quad \forall u \in X.$$

By a similar proof as in Theorem 2.3 of [11], we conclude the Lipschitz continuity of the resolvent operator $R^{\widehat{H}}_{M,\lambda}$ associated with $\widehat{H}$, M and $\lambda > 0$ and calculate its Lipschitz constant under some appropriate conditions as follows.

Lemma 2.12 *Let X be a real smooth Banach space, $\widehat{H} : X \to X$ be an r-strongly accretive mapping and $M : X \rightrightarrows X$ be an $\widehat{H}$-accretive mapping. Then, the resolvent operator $R^{\widehat{H}}_{M,\lambda} : X \to X$ is Lipschitz continuous with constant $\frac{1}{r}$, i.e.,*

$$\left\| R^{\widehat{H}}_{M,\lambda}(u) - R^{\widehat{H}}_{M,\lambda}(v) \right\| \leq \frac{1}{r} \|u - v\|, \quad \forall u, v \in X. \tag{2.2}$$

Remark 2.13 (i) It should be pointed out that Lemmas 2.1–2.3 improve, respectively, Theorems 2.1–2.3 in [11]. In fact, Theorems 2.1–2.3 in [11] have been presented in the framework of a q-uniformly smooth Banach space, whereas our results are given in a smooth Banach space setting.

(ii) There is a small mistake in the context of Theorem 2.3 of [11]. In fact, in the context of [11, Theorem 2.3], the inequality

$$\left\|R^{\widehat{H}}_{M,\lambda}(u)-R^{\widehat{H}}_{M,\lambda}(v)\right\| \leq \frac{1}{r\|u-v\|}, \quad \forall u,v\in X,$$

must be replaced by (2.2), as we have done in the context of Lemma 2.12.

3 System of variational inclusions: existence and uniqueness of solution and iterative algorithm

For given real Banach spaces X_1 and X_2, and the mappings $F:X_1\times X_2\to X_1$, $G:X_1\times X_2\to X_2$, $\widehat{H}_1:X_1\to X_1$, $\widehat{H}_2:X_2\to X_2$, $M:X_1\rightrightarrows X_1$, and $N:X_2\rightrightarrows X_2$, we consider the problem of finding $(a,b)\in X_1\times X_2$ such that

$$\begin{cases} 0\in F(a,b)+M(a),\\ 0\in G(a,b)+N(b), \end{cases} \tag{3.1}$$

which is called a system of variational inclusions (SVI) involving $\widehat{H}$-accretive mappings.

It is important to emphasize that by taking different choices of the operators F, G, $\widehat{H}_i$, M, N and the underlying spaces X_i $(i=1,2)$ in the SVI (3.1), one can easily obtain the problems studied in [12–14, 22, 29, 58] and the references therein.

The following conclusion, which tells us that SVI (3.1) is equivalent to a fixed-point problem, provides us with a characterization of the solution of the SVI (3.1).

Lemma 3.1 *Let X_1 and X_2 be two real smooth Banach spaces, and $\widehat{H}_1:X_1\to X_1$ and $\widehat{H}_2:X_2\to X_2$ be strictly accretive mappings. Suppose further that $M:X_1\rightrightarrows X_1$ is an $\widehat{H}_1$-accretive operator and $N:X_2\rightrightarrows X_2$ is an $\widehat{H}_2$-accretive operator. Then, the following statements are equivalent*:

(i) *$(a,b)\in X_1\times X_2$ is a solution of the* SVI (3.1);

(ii) *For any $\lambda,\rho>0$, (a,b) satisfies*

$$\begin{cases} a=R^{\widehat{H}_1}_{M,\lambda}[\widehat{H}_1(a)-\lambda F(a,b)],\\ b=R^{\widehat{H}_2}_{N,\rho}[\widehat{H}_2(b)-\rho G(a,b)]; \end{cases}$$

(iii) *For some $\lambda_0,\rho_0>0$, (a,b) satisfies*

$$\begin{cases} a=R^{\widehat{H}_1}_{M,\lambda_0}[\widehat{H}_1(a)-\lambda_0 F(a,b)],\\ b=R^{\widehat{H}_2}_{N,\rho_0}[\widehat{H}_2(b)-\rho_0 G(a,b)]. \end{cases}$$

Proof "(i) $\Rightarrow$ (ii)" Let us first assume that $(a,b) \in X_1 \times X_2$ is a solution of the SVI (3.1). Then, using Definition 2.11, it yields

$$\begin{cases} 0 \in F(a,b) + M(a), \\ 0 \in G(a,b) + N(b), \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} \widehat{H}_1(a) - \lambda F(a,b) \in \widehat{H}_1(a) + \lambda M(a) = (\widehat{H}_1 + \lambda M)(a), \\ \widehat{H}_2(b) - \rho G(a,b) \in \widehat{H}_2(b) + \rho N(b) = (\widehat{H}_2 + \rho N)(b), \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} a = (\widehat{H}_1 + \lambda M)^{-1}(\widehat{H}_1(a) - \lambda F(a,b)), \\ b = (\widehat{H}_2 + \rho N)^{-1}(\widehat{H}_2(b) - \rho G(a,b)), \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} a = R^{\widehat{H}_1}_{M,\lambda}[\widehat{H}_1(a) - \lambda F(a,b)], \\ b = R^{\widehat{H}_2}_{N,\rho}[\widehat{H}_2(b) - \rho G(a,b)], \end{cases}$$

where $R^{\widehat{H}_1}_{M,\lambda} = (\widehat{H}_1 + \lambda M)^{-1}$ and $R^{\widehat{H}_2}_{N,\rho} = (\widehat{H}_2 + \rho N)^{-1}$.

The proof of "(ii) $\Rightarrow$ (iii)" is obvious.

"(iii) $\Rightarrow$ (i)" Suppose that for some $\lambda_0, \rho_0 > 0$, (a,b) satisfies

$$\begin{cases} a = R^{\widehat{H}_1}_{M,\lambda_0}[\widehat{H}_1(a) - \lambda_0 F(a,b)], \\ b = R^{\widehat{H}_2}_{N,\rho_0}[\widehat{H}_2(b) - \rho_0 G(a,b)]. \end{cases}$$

Then, in the light of Definition 2.11, we obtain

$$\begin{cases} a = (\widehat{H}_1 + \lambda_0 M)^{-1}(\widehat{H}_1(a) - \lambda_0 F(a,b)], \\ b = (\widehat{H}_2 + \rho_0 N)^{-1}(\widehat{H}_2(b) - \rho_0 G(a,b)], \end{cases}$$

which implies that

$$\begin{cases} \widehat{H}_1(a) - \lambda_0 F(a,b) \in \widehat{H}_1(a) + \lambda_0 M(a), \\ \widehat{H}_2(b) - \rho_0 G(a,b) \in \widehat{H}_2(b) + \rho_0 N(b), \end{cases}$$

and hence,

$$\begin{cases} 0 \in F(a,b) + M(a), \\ 0 \in G(a,b) + N(b), \end{cases}$$

i.e., $(a,b) \in X_1 \times X_2$ is a solution of the SVI (3.1). The proof is completed. □

Before proceeding to the main result of this section, we need to recall the following notion that will be used efficiently in its proof.

Definition 3.2 A mapping $F : X \times X \to X$ is said to be

(i) ς-Lipschitz continuous with respect to its first argument if there exists a constant $\varsigma > 0$ such that

$$\|F(x_1, y) - F(x_2, y)\| \leq \varsigma \|x_1 - x_2\|, \quad \forall x_1, x_2, y \in X;$$

(ii) ξ-Lipschitz continuous with respect to its second argument if there exists a constant $\xi > 0$ such that

$$\|F(x, y_1) - F(x, y_2)\| \leq \xi \|y_1 - y_2\|, \quad \forall x, y_1, y_2 \in X.$$

Theorem 3.3 *Let X_1 and X_2 be two real smooth Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, $\widehat{H}_1 : X_1 \to X_1$ be a ϱ_1-strongly accretive and r-Lipschitz continuous mapping, $\widehat{H}_2 : X_2 \to X_2$ be a ϱ_2-strongly accretive and k-Lipschitz continuous mapping, $M : X_1 \rightrightarrows X_1$ be an $\widehat{H}_1$-accretive set-valued mapping, and $N : X_2 \rightrightarrows X_2$ be an $\widehat{H}_2$-accretive set-valued mapping. Suppose further that the mapping $F : X_1 \times X_2 \to X_1$ is τ_1-Lipschitz continuous with respect to its first argument and τ_2-Lipschitz continuous with respect to its second argument, and the mapping $G : X_1 \times X_2 \to X_2$ is θ_1-Lipschitz continuous with respect to its first argument and θ_2-Lipschitz continuous with respect to its second argument. If $r < \varrho_1$ and $k < \varrho_2$ then, the* SVI (3.1) *admits a unique solution.*

Proof For any given $\lambda, \rho > 0$, define $T_\lambda : X_1 \times X_2 \to X_1$ and $S_\rho : X_1 \times X_2 \to X_2$ for all $(x, y) \in X_1 \times X_2$, by

$$T_\lambda(x, y) = R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(x) - \lambda F(x, y)\big] \tag{3.2}$$

and

$$S_\rho(x, y) = R^{\widehat{H}_2}_{N,\rho}\big[\widehat{H}_2(y) - \rho G(x, y)\big], \tag{3.3}$$

respectively. At the same time, for any given $\lambda, \rho > 0$, define $Q_{\lambda,\rho} : X_1 \times X_2 \to X_1 \times X_2$ by

$$Q_{\lambda,\rho}(x, y) = \big(T_\lambda(x, y), S_\rho(x, y)\big), \quad \forall (x, y) \in X_1 \times X_2. \tag{3.4}$$

Making use of (3.2) and Lemma 2.12, it follows that for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$,

$$\begin{aligned} &\big\|T_\lambda(x_1, y_1) - T_\lambda(x_2, y_2)\big\|_1 \\ &\quad = \big\|R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(x_1) - \lambda F(x_1, y_1)\big] - R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(x_2) - \lambda F(x_2, y_2)\big]\big\|_1 \\ &\quad \leq \frac{1}{\varrho_1}\big\|\widehat{H}_1(x_1) - \widehat{H}_1(x_2) - \lambda\big(F(x_1, y_1) - F(x_2, y_2)\big)\big\|_1 \\ &\quad \leq \frac{1}{\varrho_1}\big(\big\|\widehat{H}_1(x_1) - \widehat{H}_1(x_2)\big\|_1 + \lambda\big\|F(x_1, y_1) - F(x_2, y_2)\big\|_1\big). \end{aligned} \tag{3.5}$$

Taking into account that $\widehat{H}_1$ is r-Lipschitz continuous, and F is τ_1-Lipschitz continuous with respect to its first argument and τ_2-Lipschitz continuous with respect to its second argument, we obtain

$$\left\|\widehat{H}_1(x_1) - \widehat{H}_1(x_2)\right\|_1 \leq r\|x_1 - x_2\|_1 \tag{3.6}$$

and

$$\begin{aligned}\left\|F(x_1,y_1) - F(x_2,y_2)\right\|_1 &\leq \left\|F(x_1,y_1) - F(x_2,y_1)\right\|_1 \\ &\quad + \left\|F(x_2,y_1) - F(x_2,y_2)\right\|_1 \\ &\leq \tau_1\|x_1 - x_2\|_1 + \tau_2\|y_1 - y_2\|_2.\end{aligned} \tag{3.7}$$

Combining (3.5)–(3.7), we deduce that for all $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$,

$$\left\|T_\lambda(x_1,y_1) - T_\lambda(x_2,y_2)\right\|_1 \leq \frac{r+\lambda\tau_1}{\varrho_1}\|x_1 - x_2\|_1 + \frac{\lambda\tau_2}{\varrho_1}\|y_1 - y_2\|_2. \tag{3.8}$$

By arguments analogous to the previous inequalities (3.5)–(3.8), employing the assumptions, for all $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$, we obtain

$$\left\|S_\rho(x_1,y_1) - S_\rho(x_2,y_2)\right\|_2 \leq \frac{k+\rho\theta_2}{\varrho_2}\|y_1 - y_2\|_2 + \frac{\rho\theta_1}{\varrho_2}\|x_1 - x_2\|_1. \tag{3.9}$$

Define the function $\|\cdot\|_*$ on $X_1 \times X_2$ by

$$\left\|(x_1,x_2)\right\|_* = \|x_1\|_1 + \|x_2\|_2, \quad \forall (x_1,x_2) \in X_1 \times X_2. \tag{3.10}$$

It can be easily seen that $(X_1 \times X_2, \|\cdot\|_*)$ is a Banach space. Then, applying (3.4), (3.8), and (3.9), yields

$$\begin{aligned}&\left\|Q_{\lambda,\rho}(x_1,y_1) - Q_{\lambda,\rho}(x_2,y_2)\right\|_* \\ &\quad = \left\|\big(T_\lambda(x_1,y_1), S_\rho(x_1,y_1)\big) - \big(T_\lambda(x_2,y_2), S_\rho(x_2,y_2)\big)\right\|_* \\ &\quad = \left\|\big(T_\lambda(x_1,y_1) - T_\lambda(x_2,y_2), S_\rho(x_1,y_1) - S_\rho(x_2,y_2)\big)\right\|_* \\ &\quad = \left\|T_\lambda(x_1,y_1) - T_\lambda(x_2,y_2)\right\|_1 + \left\|S_\rho(x_1,y_1) - S_\rho(x_2,y_2)\right\|_2 \\ &\quad \leq \left(\frac{r+\lambda\tau_1}{\varrho_1} + \frac{\rho\theta_1}{\varrho_2}\right)\|x_1 - x_2\|_1 + \left(\frac{k+\rho\theta_2}{\varrho_2} + \frac{\lambda\tau_2}{\varrho_1}\right)\|y_1 - y_2\|_2 \\ &\quad = \vartheta_{\lambda,\rho}\big(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2\big) \\ &\quad = \vartheta_{\lambda,\rho}\left\|(x_1,y_1) - (x_2,y_2)\right\|_*,\end{aligned} \tag{3.11}$$

where

$$\vartheta_{\lambda,\rho} = \max\left\{\frac{r+\lambda\tau_1}{\varrho_1} + \frac{\rho\theta_1}{\varrho_2}, \frac{k+\rho\theta_2}{\varrho_2} + \frac{\lambda\tau_2}{\varrho_1}\right\}.$$

Since $r < \varrho_1$ and $k < \varrho_2$, we can choose $\lambda_0, \rho_0 > 0$ small enough such that

$$\frac{r+\lambda_0\tau_1}{\varrho_1} + \frac{\rho_0\theta_1}{\varrho_2} < 1 \quad \text{and} \quad \frac{k+\rho_0\theta_2}{\varrho_2} + \frac{\lambda_0\tau_2}{\varrho_1} < 1. \tag{3.12}$$

From (3.12) it follows that

$$\vartheta_{\lambda_0,\rho_0} = \max\left\{\frac{r+\lambda_0\tau_1}{\varrho_1}+\frac{\rho_0\theta_1}{\varrho_2}, \frac{k+\rho_0\theta_2}{\varrho_2}+\frac{\lambda_0\tau_2}{\varrho_1}\right\} \in (0,1) \tag{3.13}$$

and so Q_{λ_0,ρ_0} is a contraction mapping. Then, the Banach Fixed-Point Theorem ensures the existence of a unique $(a,b) \in X_1 \times X_2$ such that $Q_{\lambda,\rho}(a,b) = (a,b)$. Thereby, making use of (3.2)–(3.4) we conclude that for some $\lambda_0, \rho_0 > 0$,

$$\begin{cases} a = R^{\widehat{H}_1}_{M,\lambda_0}[\widehat{H}_1(a) - \lambda_0 F(a,b)], \\ b = R^{\widehat{H}_2}_{N,\rho_0}[\widehat{H}_2(b) - \rho_0 G(a,b)]. \end{cases}$$

Accordingly, Lemma 3.1 guarantees that $(a,b) \in X_1 \times X_2$ is the unique solution of the SVI (3.1). This completes the proof. □

Given a real normed space X with a norm $\|\cdot\|$, we recall that a nonlinear mapping $T : X \to X$ is called *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in X$. It is well known that the class of nonexpansive mappings has a deep and close relation with the classes of monotone and accretive operators that arise naturally in the theory of differential equations. On the other hand, the fixed-point theory is an attractive and interesting subject with a large number of applications in various fields of mathematics and other branches of science. At the same time, the study of nonexpansive mappings is a very interesting research area in fixed-point theory. These facts have motivated many researchers to extend the notion of nonexpansive mapping and several interesting generalized nonexpansive mappings in the framework of different spaces have appeared in the literature. For example, two classes of generalized nonexpansive mappings are recalled in the next definition.

Definition 3.4 A nonlinear mapping $T : X \to X$ is said to be

(i) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\|, \quad \forall x, y \in X;$$

(ii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\|, \quad \forall x, y \in X.$$

It is significant to emphasize that every uniformly L-Lipschitzian mapping is L-Lipschitzian but the converse need not be true. The following example illustrates that the class of L-Lipschitzian mappings contains properly the class of uniformly L-Lipschitzian mappings.

Example 3.5 Consider $X = \mathbb{R}$ with the Euclidean norm $\|\cdot\| = |\cdot|$ and let the self-mapping T of X be defined by $T(x) = kx$ for all $x \in X$, where $k > 1$ is an arbitrary real constant. Taking into account that for all $x, y \in X$, $|T(x) - T(y)| = k|x - y|$, it follows that T is a k-Lipschitzian mapping. However, thanks to the fact that $k > 1$, for any $x, y \in X$ and $n \in \mathbb{N}\backslash\{1\}$, we obtain $|T^n(x) - T^n(y)| = k^n|x - y| > k|x - y|$. This fact ensures that T is not a uniformly k-Lipschitzian mapping.

The introduction and study of the notion of asymptotically nonexpansive mapping as a generalization of the concept of nonexpansive mapping was first made by Goebel and Kirk [47].

Definition 3.6 ([47]) A nonlinear mapping $T : X \to X$ is said to be asymptotically nonexpansive if, there exists a sequence $\{a_n\} \subset (0, +\infty)$ with $\lim_{n\to\infty} a_n = 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq (1 + a_n)\|x - y\|, \quad \forall x, y \in X.$$

Equivalently, we say that the mapping T is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \quad \forall x, y \in X.$$

In recent decades, successful attempts in this direction have continued and several other interesting generalizations of nonexpansive mappings and asymptotically nonexpansive mappings are presented. For instance, in 2006, Alber et al. [49] succeeded in introducing a class of generalized nonexpansive mappings, the so-called total asymptotically nonexpansive mappings, which are more general than the classes of asymptotically nonexpansive mappings and nearly asymptotically nonexpansive mappings.

Definition 3.7 ([49]) A nonlinear mapping $T : X \to X$ is said to be total asymptotically nonexpansive (also referred to as $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive) if there exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in X$,

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + a_n\phi(\|x - y\|) + b_n, \quad \forall n \in \mathbb{N}.$$

Using a modified Mann iteration process, they also studied the iterative approximation of the fixed point of total asymptotically nonexpansive mappings under some appropriate conditions. Note, in particular, that every asymptotically nonexpansive mapping is total asymptotically nonexpansive with $b_n = 0$ (or equivalently $b_n = 0$ and $a_n = k_n - 1$) for all $n \in \mathbb{N}$ and $\phi(t) = t$ for all $t \geq 0$, but the converse need not be true. In other words, the class of total asymptotically nonexpansive mappings is more general than the class of asymptotically nonexpansive mappings. This fact is shown in the next example.

Example 3.8 For $1 \leq p < \infty$, consider

$$l^p = \left\{ x = \{x_n\}_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty, x_n \in \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C} \right\},$$

the classical space consisting of all p-power summable sequences, with the p-norm $\|\cdot\|_p$ defined on it by

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}, \qquad x = \{x_n\}_{n\in\mathbb{N}} \in l^p.$$

Furthermore, let B denote the closed unit ball in the Banach space l^p and consider $X := \mathbb{R} \times B$ with the norm $\|\cdot\|_X = |\cdot|_{\mathbb{R}} + \|\cdot\|_p$ and define the self-mapping T of X by

$$T(u,x) = \begin{cases} \frac{1}{\beta}(1, x^\dagger), & \text{if } u \in [0,\beta], \\ (0, \frac{x^\dagger}{\beta}), & \text{if } u \in (-\infty, 0) \cup (\beta, +\infty), \end{cases}$$

where

$$x^\dagger = \big(\underbrace{0,0,\dots,0}_{\lambda \text{ times}}, \gamma \sin^{k_1}|x_1|, 0, \gamma |x_2|^{s_1}, 0, \gamma \sin^{k_2}|x_3|, 0, \gamma |x_4|^{s_2},$$
$$\dots, 0, \gamma \sin^{k_{\frac{m+1}{2}}}|x_m|, 0, \gamma |x_{m+1}|^{s_{\frac{m+1}{2}}}, 0, \gamma x_{m+2}, \dots\big),$$

$\gamma \in (0,1)$ and $\beta > 1$ are arbitrary real constants, $\lambda \in \mathbb{N}$ is an arbitrary constant, m is an arbitrary but fixed odd natural number, $\lambda \geq m+1$ is an arbitrary but fixed natural number and $k_i, s_i \in \mathbb{N}\backslash\{1\}$ ($i = 1, 2, \dots, \frac{m+1}{2}$) are arbitrary constants. Indeed, the element $x^\dagger$ of l^p can be written as $x^\dagger = \{x_n^\dagger\}_{n=1}^\infty$, where $x_i^\dagger = 0$ for all $1 \leq i \leq \lambda$, $x_{\lambda+2i}^\dagger = 0$ for all $i \in \mathbb{N}$,

$$x_{\lambda+2i-1}^\dagger = \begin{cases} \gamma \sin^{k_{\frac{i+1}{2}}}|x_i|, & \text{if } i \in \{2t-1 | t = 1, 2, \dots, \frac{m+1}{2}\}, \\ \gamma |x_i|^{s_{\frac{i}{2}}}, & \text{if } i \in \{2\sigma | \sigma = 1, 2, \dots, \frac{m+1}{2}\}, \end{cases}$$

and $x_{\lambda+2m+j}^\dagger = \gamma x_{m+\frac{j+1}{2}}$ for all $j \in \{2l+1 | l \in \mathbb{N}\}$. Taking into account that the mapping T is not continuous at the points (β, x) for all $x \in B$, we conclude that T is not Lipschitzian and so it is not an asymptotically nonexpansive mapping. For all $(u,x), (v,y) \in [0,\beta] \times B$ and $(u,x), (v,y) \in ((-\infty,0) \cup (\beta,+\infty)) \times B$, one can show that

$$\begin{aligned}
&\big\|T(u,x) - T(v,y)\big\|_X \\
&= \bigg\|(0, \frac{1}{\beta}\big((\underbrace{0,0,\dots,0}_{\lambda \text{ times}}, \gamma\big(\sin^{k_1}|x_1| - \sin^{k_1}|y_1|\big), 0, \\
&\quad \gamma\big(|x_2|^{s_1} - |y_2|^{s_1}\big), 0, \gamma\big(\sin^{k_2}|x_3| - \sin^{k_2}|y_3|\big), 0, \\
&\quad \gamma\big(|x_4|^{s_2} - |y_4|^{s_2}\big), \dots, 0, \gamma\big(\sin^{k_{\frac{m+1}{2}}}|x_m| - \sin^{k_{\frac{m+1}{2}}}|y_m|\big), 0, \\
&\quad \gamma\big(|x_{m+1}|^{s_{\frac{m+1}{2}}} - |y_{m+1}|^{s_{\frac{m+1}{2}}}\big), 0, \gamma(x_{m+2} - y_{m+2}), \dots\big)\bigg\|_X \\
&= \frac{1}{\beta}\Bigg(\gamma^p \sum_{i=1}^{\frac{m+1}{2}} \big|\sin^{k_i}|x_{2i-1}| - \sin^{k_i}|y_{2i-1}|\big|^p \\
&\quad + \gamma^p \sum_{i=1}^{\frac{m+1}{2}} \big||x_{2i}|^{s_i} - |y_{2i}|^{s_i}\big|^p + \gamma^p \sum_{i=m+2}^{\infty} |x_i - y_i|^p\Bigg)^{\frac{1}{p}} \\
&\leq \gamma\Bigg(\sum_{i=1}^{\frac{m+1}{2}}\Bigg(\sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}\Bigg)^p |x_{2i-1} - y_{2i-1}|^p \\
&\quad + \sum_{i=1}^{\frac{m+1}{2}}\Bigg(\sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}\Bigg)^p |x_{2i} - y_{2i}|^p + \sum_{i=m+2}^{\infty} |x_i - y_i|^p\Bigg)^{\frac{1}{p}}.
\end{aligned} \tag{3.14}$$

The fact that $x, y \in B$ implies that $0 \leq |x_{2i-1}|^{k_i-j}, |y_{2i-1}|^{j-1} \leq 1$ for each $j \in \{1, 2, \dots, k_i\}$ and $0 \leq |x_{2i}|^{s_i-r}, |y_{2i}|^{r-1} \leq 1$ for each $r \in \{1, 2, \dots, s_i\}$ and $i \in \{1, 2, \dots, \frac{m+1}{2}\}$. Relying on these facts, we conclude that $0 \leq \sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1} \leq k_i$ and $0 \leq \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1} \leq s_i$ for each $i \in \{1, 2, \dots, \frac{m+1}{2}\}$. Thereby, making use of (3.14) it follows that for all $(u, x), (v, y) \in [0, \beta] \times B$ and $(u, x), (v, y) \in ((-\infty, 0) \cup (\beta, +\infty)) \times B$,

$$\begin{aligned}
&\|T(u,x) - T(v,y)\|_X \\
&\leq \gamma \left(\max \left\{ \left(\sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1} \right)^p, \left(\sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1} \right)^p, 1 : \right.\right. \\
&\quad \left.\left. i = 1, 2, \dots, \frac{m+1}{2} \right\} \sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}} \\
&= \gamma \max \left\{ \sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : \right. \\
&\quad \left. i = 1, 2, \dots, \frac{m+1}{2} \right\} \|x - y\|_p.
\end{aligned} \tag{3.15}$$

If $u \in [0, \beta]$ and $v \in (-\infty, 0) \cup (\beta, +\infty)$, then in a similar fashion to the preceding analysis, one can prove that for all $x, y \in B$,

$$\begin{aligned}
&\|T(u,x) - T(v,y)\|_X \\
&= \left\| \frac{1}{\beta}(1, x^\dagger) - \left(0, \frac{1}{\beta} y^\dagger\right) \right\|_X = \frac{1}{\beta} \left\| (1, x^\dagger - y^\dagger) \right\|_X \\
&= \frac{1}{\beta} \left(1 + \gamma \max \left\{ \sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \right.\right. \\
&\quad \left.\left. \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \dots, \frac{m+1}{2} \right\} \|x - y\|_p \right) \\
&< |u - v| + \gamma \max \left\{ \sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \right. \\
&\quad \left. \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \dots, \frac{m+1}{2} \right\} \|x - y\|_p + \frac{1}{\beta}.
\end{aligned} \tag{3.16}$$

Now, applying (3.15) and (3.16), for all $(u, x), (v, y) \in X$, we obtain

$$\begin{aligned}
&\|T(u,x) - T(v,y)\|_X \\
&\leq |u - v| + \gamma \max \left\{ \sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \right. \\
&\quad \left. \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : i = 1, 2, \dots, \frac{m+1}{2} \right\} \|x - y\|_p + \frac{1}{\beta}
\end{aligned}$$

$$\leq |u-v| + \|x-y\|_p + \gamma \max\Bigg\{\sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : i = 1,2,\dots,\frac{m+1}{2}\Bigg\}\big(|u-v| + \|x-y\|_p\big) + \frac{1}{\beta}. \tag{3.17}$$

For all $n \geq 2$ and $(u,x) \in X$, we have

$$\begin{aligned} T^n(u,x) = \frac{1}{\beta}\Big(1, \big(&\underbrace{0,0,\dots,0}_{(2^n-1)\lambda \text{ times}}, \gamma^n \sin^{k_1}|x_1|, \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n |x_2|^{s_1}, \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \\ &\gamma^n \sin^{k_2}|x_3|, \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n |x_4|^{s_2}, \dots, \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n \sin^{k_{\frac{m+1}{2}}}|x_m|, \\ &\underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n |x_{m+1}|^{s_{\frac{m+1}{2}}}, \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n x_{m+2}, \dots\big)\Big). \end{aligned}$$

Then, by an argument analogous to those of (3.14) and (3.15), for all $(u,x),(v,y) \in X$ and $n \geq 2$, one can deduce that

$$\begin{aligned} \big\|T^n(u,x) - T^n(v,y)\big\|_X = \frac{1}{\beta}\Big\|\Big(0, \big(&\underbrace{0,0,\dots,0}_{(2^n-1)\lambda \text{ times}}, \gamma^n\big(\sin^{k_1}|x_1| - \sin^{k_1}|y_1|\big), \\ &\underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n\big(|x_2|^{s_1} - |y_2|^{s_1}\big), \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \\ &\gamma^n\big(\sin^{k_2}|x_3| - \sin^{k_2}|y_3|\big), \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n\big(|x_4|^{s_2} - |y_4|^{s_2}\big), \\ &\dots, \underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n\big(\sin^{k_{\frac{m+1}{2}}}|x_m| - \sin^{k_{\frac{m+1}{2}}}|y_m|\big), \\ &\underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n\big(|x_{m+1}|^{s_{\frac{m+1}{2}}} - |y_{m+1}|^{s_{\frac{m+1}{2}}}\big), \\ &\underbrace{0,0,\dots,0}_{(2^n-1) \text{ times}}, \gamma^n(x_{m+2} - y_{m+2}), \dots\big)\Big)\Big\|_X \\ &\leq \gamma^n \max\Bigg\{\sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \\ &\sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : i = 1,2,\dots,\frac{m+1}{2}\Bigg\}\|x-y\|_p \\ &\leq |u-v| + \|x-y\|_p + \gamma^n \max\Bigg\{\sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \\ &\sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : i = 1,2,\dots,\frac{m+1}{2}\Bigg\}\big(|u-v| \\ &+ \|x-y\|_p\big) + \frac{1}{\beta^n}. \end{aligned} \tag{3.18}$$

Employing (3.17) and (3.18) and by virtue of the fact that for each $i \in \{1,2,\dots,\frac{m+1}{2}\}$, $0 \leq \sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1} \leq k_i$ and $0 \leq \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1} \leq s_i$, we conclude that for all $(u,x),(v,y) \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|T^n(u,x)-T^n(v,y)\|_X &\leq |u-v| + \|x-y\|_p + \gamma^n \max\Biggl\{\sum_{j=1}^{k_i} |x_{2i-1}|^{k_i-j}|y_{2i-1}|^{j-1}, \\ &\qquad \sum_{r=1}^{s_i} |x_{2i}|^{s_i-r}|y_{2i}|^{r-1}, 1 : i = 1,2,\dots,\frac{m+1}{2}\Biggr\}\bigl(|u-v| \\ &\qquad + \|x-y\|_p\bigr) + \frac{1}{\beta^n} \\ &\leq \|(u,x)-(v,y)\|_X + \gamma^n \xi \|(u,x)-(v,y)\|_X + \frac{1}{\beta^n},\end{aligned}$$

where $\xi = \max\{k_i, s_i : i = 1,2,\dots,\frac{m+1}{2}\}$. Taking $a_n = \gamma^n$ and $b_n = \frac{1}{\beta^n}$ for all $n \in \mathbb{N}$, the fact that $0 < \gamma < 1 < \beta$ implies that $a_n, b_n \to 0$ as $n \to \infty$. Now, define the function $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = \xi t$ for all $t \in [0,+\infty)$. Then, for all $(u,x),(v,y) \in X$ and $n \in \mathbb{N}$, we obtain

$$\|T^n(u,x)-T^n(v,y)\|_X \leq \|(u,x)-(v,y)\|_X + a_n\phi\bigl(\|(u,x)-(v,y)\|_X\bigr) + b_n,$$

that is, T is a $(\{\gamma^n\},\{\frac{1}{\beta^n}\},\phi)$-total asymptotically nonexpansive mapping.

With the aim of presenting a unifying framework for generalized nonexpansive mappings available in the literature and verifying a general convergence theorem applicable to all these classes of nonlinear mappings, very recently, Kiziltunc and Purtas [50] introduced a new class of generalized nonexpansive mappings as follows.

Definition 3.9 ([50]) A nonlinear mapping $T : X \to X$ is said to be total uniformly L-Lipschitzian (or $(\{a_n\},\{b_n\},\phi)$-total uniformly L-Lipschitzian) if there exist a constant $L > 0$, nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x)-T^n(y)\| \leq L\bigl[\|x-y\| + a_n\phi\bigl(\|x-y\|\bigr) + b_n\bigr], \quad \forall x,y \in X.$$

It is essential to note that, for given nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, an $(\{a_n\},\{b_n\},\phi)$-total asymptotically nonexpansive mapping is $(\{a_n\},\{b_n\},\phi)$-total uniformly L-Lipschitzian with $L = 1$, but the converse may not be true. In the following example, the fact that the class of total uniformly L-Lipschitzian mappings contains properly the class of total asymptotically nonexpansive mappings is illustrated.

Example 3.10 Let $X = \mathbb{R}$ endowed with the Euclidean norm $\|\cdot\| = |\cdot|$ and let the self-mapping T of X be defined by

$$T(x) = \begin{cases} 0, & \text{if } x \in (-\infty,0), \\ \beta, & \text{if } x \in (0,\frac{1}{\beta}) \cup (\frac{1}{\beta},\alpha), \\ \frac{1}{\beta}, & \text{if } x \in [\alpha,+\infty) \cup \{0,\frac{1}{\beta}\}, \end{cases}$$

where $\alpha > 0$ and $\beta > \frac{\alpha+\sqrt{\alpha^2+4}}{2}$ are arbitrary real constants such that $\alpha\beta > 1$. Since the mapping T is discontinuous at the points $x = 0, \alpha, \frac{1}{\beta}$, it follows that T is not Lipschitzian and so it is not an asymptotically nonexpansive mapping. Take $a_n = \frac{\gamma}{n}$ and $b_n = \frac{\alpha}{k^n}$ for each $n \in \mathbb{N}$, where $\gamma > 0$ and $k > 1$ are arbitrary constants such that $k \neq \alpha\beta$. Let us now define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = \theta t^m$ for all $t \in \mathbb{R}^+$, where $m \in \mathbb{N}$ and $\theta \in (0, \frac{k^m(\beta^2-\alpha\beta-1)}{\beta\gamma(k-1)^m\alpha^m})$ are arbitrary constants. Selecting $x = \alpha$ and $y = \frac{\alpha}{k}$, we have $T(x) = \frac{1}{\beta}$ and $T(y) = \beta$. With the help of the fact that $0 < \theta < \frac{k^m(\beta^2-\alpha\beta-1)}{\beta\gamma(k-1)^m\alpha^m}$, it follows that

$$\begin{aligned}
\bigl|T(x) - T(y)\bigr| &= \beta - \frac{1}{\beta} \\
&> \alpha + \frac{\gamma\theta(k-1)^m\alpha^m}{k^m} \\
&= \frac{(k-1)\alpha}{k} + \frac{\gamma\theta(k-1)^m\alpha^m}{k^m} + \frac{\alpha}{k} \\
&= |x-y| + \gamma\theta|x-y|^m + \frac{\alpha}{k} \\
&= |x-y| + a_1\phi\bigl(|x-y|\bigr) + b_1,
\end{aligned}$$

which implies that T is not a $(\{\frac{\gamma}{n}\}, \{\frac{\alpha}{k^n}\}, \phi)$-total asymptotically nonexpansive mapping. However, for all $x, y \in X$, we obtain

$$\begin{aligned}
\bigl|T(x) - T(y)\bigr| &\leq \beta \\
&\leq \frac{k\beta}{\alpha}\left(|x-y| + \gamma\theta|x-y|^m + \frac{\alpha}{k}\right) \\
&= \frac{k\beta}{\alpha}\bigl(|x-y| + a_1\phi\bigl(|x-y|\bigr) + b_1\bigr)
\end{aligned} \tag{3.19}$$

and for all $n \geq 2$,

$$\begin{aligned}
\bigl|T^n(x) - T^n(y)\bigr| &< \frac{k\beta}{\alpha}\left(|x-y| + \frac{\gamma\theta}{n}|x-y|^m + \frac{\alpha}{k^n}\right) \\
&= \frac{k\beta}{\alpha}\bigl(|x-y| + a_n\phi\bigl(|x-y|\bigr) + b_n\bigr),
\end{aligned} \tag{3.20}$$

due to the fact that $T^n(z) = \frac{1}{\beta}$ for all $z \in X$ and $n \geq 2$. Making use of (3.19) and (3.20), we deduce that T is a $(\{\frac{\gamma}{n}\}, \{\frac{\alpha}{k^n}\}, \phi)$-total uniformly $\frac{k\beta}{\alpha}$-Lipschitzian mapping.

Lemma 3.11 *Let X_1 and X_2 be two real Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, and let $S_1 : X_1 \to X_1$ and $S_2 : X_2 \to X_2$ be $(\{a_i\}_{i=1}^{\infty}, \{b_i\}_{i=1}^{\infty}, \phi_1)$-total uniformly L_1-Lipschitzian and $(\{c_i\}_{i=1}^{\infty}, \{d_i\}_{i=1}^{\infty}, \phi_2)$-total uniformly L_2-Lipschitzian mappings, respectively. Moreover, let Q and ϕ be self-mappings of $X_1 \times X_2$ and $\mathbb{R}^+$, respectively, defined by*

$$Q(x_1, x_2) = (S_1x_1, S_2x_2), \quad \forall (x_1, x_2) \in X_1 \times X_2 \tag{3.21}$$

and

$$\phi(t) = \max\bigl\{\phi_j(t) : j = 1, 2\bigr\}, \quad \forall t \in \mathbb{R}^+. \tag{3.22}$$

Then, Q is an $(\{a_i + c_i\}_{i=1}^{\infty}, \{b_i + d_i\}_{i=1}^{\infty}, \phi)$-total uniformly $\max\{L_1, L_2\}$-Lipschitzian mapping.

Proof In view of the fact that for each $j \in \{1,2\}$, $\phi_j : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function, for all $(x_1,x_2),(y_1,y_2) \in X_1 \times X_2$ and $i \in \mathbb{N}$, yields

$$\begin{aligned}
\|Q^i(x_1,x_2) - Q^i(y_1,y_2)\|_* &= \|(S_1^i x_1, S_2^i x_2) - (S_1^i y_1, S_2^i y_2)\|_* \\
&= \|(S_1^i x_1 - S_1^i y_1, S_2^i x_2 - S_2^i y_2)\|_* \\
&= \|S_1^i x_1 - S_1^i y_1\|_1 + \|S_2^i x_2 - S_2^i y_2\|_2 \\
&\leq L_1\big(\|x_1 - y_1\|_1 + a_i\phi_1(\|x_1 - y_1\|_1) + b_i\big) \\
&\quad + L_2\big(\|x_2 - y_2\|_2 + c_i\phi_2(\|x_2 - y_2\|_2) + d_i\big) \\
&\leq \max\{L_1, L_2\}\big(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2 \\
&\quad + a_i\phi_1(\|x_1 - y_1\|_1) + c_i\phi_2(\|x_2 - y_2\|_2) + b_i + d_i\big) \\
&\leq \max\{L_1, L_2\}\big(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2 \\
&\quad + a_i\phi_1(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2) \\
&\quad + c_i\phi_2(\|x_1 - y_1\|_1 + \|x_2 - y_2\|_2) + b_i + d_i\big) \\
&\leq \max\{L_1, L_2\}\big(\|(x_1,x_2) - (y_1,y_2)\|_* \\
&\quad + (a_i + c_i)\phi(\|(x_1,x_2) - (y_1,y_2)\|_*) + b_i + d_i\big),
\end{aligned}$$

where $\|\cdot\|_*$ is a norm on $X_1 \times X_2$ defined by (3.10). This fact ensures that Q is an $(\{a_i + c_i\}_{i=1}^{\infty}, \{b_i + d_i\}_{i=1}^{\infty}, \phi)$-total uniformly $\max\{L_1, L_2\}$-Lipschitzian mapping. The proof is completed. □

Assume that X_1 and X_2 are two real smooth Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, $S_1 : X_1 \to X_1$ is an $(\{a_i\}_{i=1}^{\infty}, \{b_i\}_{i=1}^{\infty}, \phi_1)$-total uniformly L_1-Lipschitzian mapping and $S_2 : X_2 \to X_2$ is a $(\{c_i\}_{i=1}^{\infty}, \{d_i\}_{i=1}^{\infty}, \phi_2)$-total uniformly L_2-Lipschitzian mapping. Furthermore, let Q be a self-mapping of $X_1 \times X_2$ defined as (3.21). Denote by $\mathrm{Fix}(S_j)$ $(j = 1,2)$ and $\mathrm{Fix}(Q)$ the sets of all the fixed points of S_j $(j = 1,2)$ and Q, respectively. At the same time, denote by $\mathrm{SVI}(X_j, \widehat{H}_j, M, N, F, G : j = 1,2)$ the set of all the solutions of the SVI (3.1), where for $j = 1,2$, the nonlinear mappings $\widehat{H}_j : X_j \to X_j$ are strictly accretive, and the set-valued mappings $M : X \rightrightarrows X_1$ and $N : X_2 \rightrightarrows X_2$ are $\widehat{H}_1$-accretive and $\widehat{H}_2$-accretive, respectively. Using (3.21), we infer that for any $(x_1,x_2) \in X_1 \times X_2$, $(x_1,x_2) \in \mathrm{Fix}(Q)$ if and only if for $j = 1,2$, $x_j \in \mathrm{Fix}(S_j)$, that is, $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1,S_2) = \mathrm{Fix}(S_1) \times \mathrm{Fix}(S_2)$. If $(a,b) \in \mathrm{Fix}(Q) \cap \mathrm{SVI}(X_j, A_j, M, N, F, G : j = 1,2)$, then with the help of Lemma 3.1 it can be easily observed that for each $i \in \mathbb{N}$,

$$\begin{cases} a = S_1^i a = R_{M,\lambda}^{\widehat{H}_1}[\widehat{H}_1(a) - \lambda F(a,b)] = S_1^i R_{M,\lambda}^{\widehat{H}_1}[\widehat{H}_1(a) - \lambda F(a,b)], \\ b = S_2^i b = R_{N,\rho}^{\widehat{H}_2}[\widehat{H}_2(b) - \rho G(a,b)] = S_2^i R_{N,\rho}^{\widehat{H}_2}[\widehat{H}_2(b) - \rho G(a,b)]. \end{cases} \tag{3.23}$$

Using the fixed-point formulation (3.23) we now are able to construct the following iterative algorithm for finding a common element of the two sets of $\mathrm{SVI}(X_j, \widehat{H}_j, M, N, F, G : j = 1,2)$ and $\mathrm{Fix}(Q) = \mathrm{Fix}(S_1,S_2)$.

Algorithm 3.12 Assume that X_j ($j = 1,2$), F and G are the same as in the SVI (3.1). Let for $i \geq 0$ and $j = 1,2$, $\widehat{H}_{i,j} : X_j \to X_j$ be strictly accretive, $M_i : X_1 \rightrightarrows X_1$ be an $\widehat{H}_{i,1}$-accretive set-valued mapping and $N_i : X_2 \rightrightarrows X_2$ be an $\widehat{H}_{i,2}$-accretive set-valued mapping. Suppose further that for $j = 1,2$, $S_j : X_j \to X_j$ is a $(\{c_{i,j}\}_{i=0}^{\infty}, \{d_{i,j}\}_{i=0}^{\infty}, \phi_j)$-total uniformly L_j-Lipschitzian mapping. For an arbitrarily chosen initial point $(a_0, b_0) \in X_1 \times X_2$, compute the iterative sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ in $X_1 \times X_2$ by the iterative schemes

$$\begin{cases} a_{i+1} = \alpha_i a_i + (1-\alpha_i) S_1^i R_{M_i,\lambda_i}^{\widehat{H}_{i,1}} [\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i, b_i)], \\ b_{i+1} = \alpha_i b_i + (1-\alpha_i) S_2^i R_{N_i,\rho_i}^{\widehat{H}_{i,2}} [\widehat{H}_{i,2}(b_i) - \rho_i G(a_i, b_i)], \end{cases} \tag{3.24}$$

where $i \in \mathbb{N} \cup \{0\}$; $\lambda_i, \rho_i > 0$ are real constants; and $\{\alpha_i\}_{i=0}^{\infty}$ is a sequence in the interval $[0,1]$ such that $\limsup_i \alpha_i < 1$.

If $S_j \equiv I_j$ ($j = 1,2$), the identity mapping on X_j, then Algorithm 3.12 reduces to the following algorithm.

Algorithm 3.13 Let X_j, $\widehat{H}_{i,j}$, M_i, N_i, F, G ($j = 1,2$; $i \in \mathbb{N} \cup \{0\}$) be the same as in Algorithm 3.12. For any given $(a_0, b_0) \in X_1 \times X_2$, define the iterative sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ in $X_1 \times X_2$ by the iterative processes

$$\begin{cases} a_{i+1} = \alpha_i a_i + (1-\alpha_i) R_{M_i,\lambda_i}^{\widehat{H}_{i,1}} [\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i, b_i)], \\ b_{i+1} = \alpha_i b_i + (1-\alpha_i) R_{N_i,\rho_i}^{\widehat{H}_{i,2}} [\widehat{H}_{i,2}(b_i) - \rho_i G(a_i, b_i)], \end{cases}$$

where $i \in \mathbb{N} \cup \{0\}$; the constants $\lambda_i, \rho_i > 0$ and the sequence $\{\alpha_i\}_{i=0}^{\infty}$ are the same as in Algorithm 3.12.

If $\widehat{H}_{i,j} = \widehat{H}_j$, $\lambda_i = \lambda$ and $\rho_i = \rho$ for each $i \geq 0$ and $j \in \{1,2\}$, then Algorithm 3.13 collapses to the following algorithm.

Algorithm 3.14 Suppose that X_j ($j = 1,2$), F and G are the same as in Algorithm 3.12. Let for $j = 1,2$, $\widehat{H}_j : X_j \to X_j$ be strictly accretive mappings and let for each $i \geq 0$, $M_i : X_1 \rightrightarrows X_1$ be an $\widehat{H}_1$-accretive set-valued mapping and $N_i : X_2 \rightrightarrows X_2$ be an $\widehat{H}_2$-accretive set-valued mapping. For any given $(a_0, b_0) \in X_1 \times X_2$, compute the iterative sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ in $X_1 \times X_2$ by the iterative schemes

$$\begin{cases} a_{i+1} = \alpha_i a_i + (1-\alpha_i) R_{M_i,\lambda}^{\widehat{H}_1} [\widehat{H}_1(a_i) - \lambda F(a_i, b_i)], \\ b_{i+1} = \alpha_i b_i + (1-\alpha_i) R_{N_i,\rho}^{\widehat{H}_2} [\widehat{H}_2(b_i) - \rho G(a_i, b_i)], \end{cases}$$

where $i \in \mathbb{N} \cup \{0\}$; $\lambda, \rho > 0$ are two constants; and the sequence $\{\alpha_i\}_{i=0}^{\infty}$ is the same as in Algorithm 3.12.

4 Graph convergence and an application

Before turning to the main results of this paper, we need to recall the following definition.

Definition 4.1 ([6, 20]) Given set-valued mappings $M_i, M : X \rightrightarrows X$ ($i \geq 0$), the sequence $\{M_i\}_{i=0}^{\infty}$ is said to be graph-convergent to M, denoted by $M_i \xrightarrow{G} M$, if for every point $(x, u) \in$

Graph(M), there exists a sequence of points $(x_i, u_i) \in \mathrm{Graph}(M_i)$ such that $x_i \to x$ and $u_i \to u$ as $i \to \infty$.

We now establish a new equivalence relationship between the graph convergence of a sequence of $\widehat{H}$-accretive mappings and their associated resolvent operators, respectively, to a given $\widehat{H}$-accretive mapping and its associated resolvent operator under some appropriate conditions.

Theorem 4.2 *Let X be a real smooth Banach space, and $\widehat{H}, \widehat{H}_i : X \to X$ $(i \geq 0)$ be ϱ-strongly accretive and ϱ_i-strongly accretive mappings, respectively, such that for each $i \geq 0$ the mapping $\widehat{H}_i$ is r_i-Lipschitz continuous. Suppose that M, $M_i : X \rightrightarrows X$ $(i \geq 0)$ are $\widehat{H}$-accretive and $\widehat{H}_i$-accretive mappings, respectively. Let the sequence $\{r_i\}_{i=0}^{\infty}$ be bounded and $\lim_{i\to\infty} \widehat{H}_i(x) = \widehat{H}(x)$ for any $x \in X$. Assume further that $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of real positive constants convergent to a positive real constant λ, and let the sequence $\{\frac{1}{\varrho_i}\}_{i=0}^{\infty}$ be bounded. Then, the following statements are equivalent*:

(i) $M_i \xrightarrow{G} M$;

(ii) *For each sequence $\{\lambda_i\}_{i=0}^{\infty}$ of real positive constants convergent to a positive real constant λ,*

$$R_{M_i,\lambda_i}^{\widehat{H}_i}(z) \to R_{M,\lambda}^{\widehat{H}}(z), \quad \forall z \in X,$$

where $R_{M_i,\lambda_i}^{\widehat{H}_i} = (\widehat{H}_i + \lambda_i M_i)^{-1}$ $(i \geq 0)$ and $R_{M,\lambda}^{\widehat{H}} = (\widehat{H} + \lambda M)^{-1}$;

(iii) *For some sequence $\{\lambda_{i,0}\}_{i=0}^{\infty}$ of real positive constants convergent to some positive real constant λ_0,*

$$R_{M_i,\lambda_{i,0}}^{\widehat{H}_i}(z) \to R_{M,\lambda_0}^{\widehat{H}}(z), \quad \forall z \in X.$$

Proof "(i) ⇒ (ii)" Suppose that $M_i \xrightarrow{G} M$ and let $\{\lambda_i\}_{i=0}^{\infty}$ be a sequence of real positive constants convergent to a constant $\lambda > 0$. Choose $z \in X$ arbitrarily but fixed. The fact that M is an $\widehat{H}$-accretive mapping implies that $(\widehat{H} + \lambda M)(X) = X$, which guarantees the existence of a point $(x, u) \in \mathrm{Graph}(M)$ such that $z = \widehat{H}(x) + \lambda u$. Then, thanks to Definition 4.1 there exists a sequence $\{(x_i, u_i)\}_{i=0}^{\infty} \subset \mathrm{Graph}(M_i)$ such that $x_i \to x$ and $u_i \to u$ as $i \to \infty$. Taking into account that $(x, u) \in \mathrm{Graph}(M)$ and $(x_i, u_i) \in \mathrm{Graph}(M_i)$, it follows that

$$x = R_{M,\lambda}^{\widehat{H}}\big[\widehat{H}(x) + \lambda u\big] \quad \text{and} \quad x_i = R_{M_i,\lambda_i}^{\widehat{H}_i}\big[\widehat{H}_i(x_i) + \lambda_i u_i\big]. \tag{4.1}$$

Picking $z_i = H_i(x_i) + \lambda_i u_i$ for each $i \geq 0$ and making use of Lemma 2.12, (4.1), and the assumptions, we derive that for all $i \geq 0$,

$$\begin{aligned}
&\big\|R_{M_i,\lambda_i}^{\widehat{H}_i}(z) - R_{M,\lambda}^{\widehat{H}}(z)\big\| \\
&\quad \leq \big\|R_{M_i,\lambda_i}^{\widehat{H}_i}(z) - R_{M_i,\lambda_i}^{\widehat{H}_i}(z_i)\big\| + \big\|R_{M_i,\lambda_i}^{\widehat{H}_i}(z_i) - R_{M,\lambda}^{\widehat{H}}(z)\big\| \\
&\quad \leq \frac{1}{\varrho_i}\|z_i - z\| + \big\|R_{M_i,\lambda_i}^{\widehat{H}_i}\big[\widehat{H}_i(x_i) + \lambda_i u_i\big] - R_{M,\lambda}^{\widehat{H}}\big[\widehat{H}(x) + \lambda u\big]\big\| \\
&\quad \leq \frac{1}{\varrho_i}\|z_i - z\| + \|x_i - x\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\varrho_i}\big\|\widehat{H}_i(x_i)+\lambda_i u_i-\widehat{H}(x)-\lambda u\big\|+\|x_i-x\| \\
&\leq \frac{1}{\varrho_i}\big(\big\|\widehat{H}_i(x_i)-\widehat{H}_i(x)\big\|+\big\|\widehat{H}_i(x)-\widehat{H}(x)\big\| \\
&\quad+\|\lambda_i u_i-\lambda_i u\|+\|\lambda_i u-\lambda u\|\big)+\|x_i-x\| \\
&\leq \left(1+\frac{r_i}{\varrho_i}\right)\|x_i-x\|+\frac{1}{\varrho_i}\big\|\widehat{H}_i(x)-\widehat{H}(x)\big\| \\
&\quad+\frac{\lambda_i}{\varrho_i}\|u_i-u\|+\frac{|\lambda_i-\lambda|}{\varrho_i}\|u\|.
\end{aligned}
\tag{4.2}
$$

Since $\lambda_i \to \lambda$ as $i \to \infty$ and the sequences $\{r_i\}_{i=0}^{\infty}$, $\{\frac{1}{\varrho_i}\}_{i=0}^{\infty}$ are bounded, it follows that the sequences $\{\frac{r_i}{\varrho_i}\}_{i=0}^{\infty}$ and $\{\frac{\lambda_i}{\varrho_i}\}_{i=0}^{\infty}$ are also bounded. By virtue of the facts that $x_i \to x$, $u_i \to u$ and $\lambda_i \to \lambda$ as $i \to \infty$, we conclude that the right-hand side of (4.2) tends to zero as $i \to \infty$, which implies that $R^{\widehat{H}_i}_{M_i,\lambda_i}(z) \to R^{\widehat{H}}_{M,\lambda}(z)$, as $i \to \infty$.

The proof of "(ii) $\Rightarrow$ (iii)" is obvious.

"(iii) $\Rightarrow$ (i)" Assume that for some sequence $\{\lambda_{i,0}\}_{i=0}^{\infty}$ of real positive constants convergent to some positive real constant λ_0, $R^{\widehat{H}_i}_{M_i,\lambda_{i,0}}(z) \to R^{\widehat{H}}_{M,\lambda_0}(z)$, as $i \to \infty$, for all $z \in X$. Then, for any $(x,u) \in \mathrm{Graph}(M)$, we have $x = R^{\widehat{H}}_{M,\lambda_0}[\widehat{H}(x)+\lambda_0 u]$ and so $R^{\widehat{H}_i}_{M_i,\lambda_{i,0}}[\widehat{H}(x)+\lambda_0 u] \to x$, as $i \to \infty$. Taking $x_i = R^{\widehat{H}_i}_{M_i,\lambda_{i,0}}[\widehat{H}(x)+\lambda_0 u]$ for each $i \geq 0$, we infer that for each $i \geq 0$, $\widehat{H}(x)+\lambda_0 u \in (\widehat{H}_i+\lambda_{i,0}M_i)(x_i)$. Thus, for each $i \geq 0$, we can choose $u_i \in M_i(x_i)$ such that $\widehat{H}(x)+\lambda_0 u = \widehat{H}_i(x)+\lambda_{i,0}u_i$. Since $x_i \to x$ as $i \to \infty$, it follows that $\lambda_{i,0}u_i \to \lambda_0 u$, as $i \to \infty$. Meanwhile, for all $i \geq 0$, it yields

$$
\begin{aligned}
\lambda_0\|u_i-u\| &= \|\lambda_0 u_i-\lambda_0 u\| \\
&\leq \|\lambda_{i,0}u_i-\lambda_0 u_i\|+\|\lambda_{i,0}u_i-\lambda_0 u\| \\
&= |\lambda_{i,0}-\lambda_0|\|u_i\|+\|\lambda_{i,0}u_i-\lambda_0 u\|.
\end{aligned}
\tag{4.3}
$$

Taking into account that $\lambda_{i,0} \to \lambda_0$ and $\lambda_{i,0}u_i \to \lambda_0 u$, as $i \to \infty$, we deduce that the right-hand side of (4.3) approaches zero, as $i \to \infty$, which ensures that $u_i \to u$ as $i \to \infty$. The proof is finished. □

We obtain the following corollary as a direct consequence of the above theorem immediately.

Corollary 4.3 *Suppose that X is a real smooth Banach space, and $\widehat{H}: X \to X$ is a ϱ-strongly accretive and γ-Lipschitz continuous mapping. Furthermore, let $M_i, M: X \rightrightarrows X$ be $\widehat{H}$-accretive mappings for $i = 1,2,\ldots$. Then, the following statements are equivalent*:

(i) $M_i \xrightarrow{G} M$;

(ii) *For each* $\lambda > 0$, $R^{\widehat{H}_i}_{M_i,\lambda}(z) \to R^{\widehat{H}}_{M,\lambda}(z)$, $\forall z \in X$;

(iii) *For some* $\lambda_0 > 0$, $R^{\widehat{H}_i}_{M_i,\lambda_0}(z) \to R^{\widehat{H}}_{M,\lambda_0}(z)$, $\forall z \in X$.

Lemma 4.4 ([59]) *Let $\{\delta_i\}_{i=0}^{\infty}$ be a sequence of real numbers and let there exist $\theta \in [0,1)$ and $\xi > 0$ such that*

$$
\delta_{i+1} \leq \theta\delta_i + \xi, \quad \forall i \geq 0.
$$

Then,

$$\delta_i \leq \frac{\xi}{1-\theta} + \left(\delta_0 - \frac{\xi}{1-\theta}\right)\theta^n.$$

The following lemma plays a prominent role in studying the convergence analysis of our iterative algorithms proposed in the previous section.

Lemma 4.5 *Suppose that* $\{\sigma_i\}_{i=0}^{\infty}$, $\{\gamma_i\}_{i=0}^{\infty}$ *and* $\{t_i\}_{i=0}^{\infty}$ *are three real sequences of nonnegative numbers that satisfy the following conditions*:

(i) $0 \leq \gamma_i < 1$ *for all* $i \geq 0$ *and* $\limsup_i \gamma_i < 1$;

(ii) $\sigma_{i+1} \leq \gamma_i\sigma_i + t_i$, *for all* $i \geq 0$;

(iii) $\lim_{i\to\infty} t_i = 0$.

Then, $\lim_{i\to\infty}\sigma_i = 0$.

Proof Let $\epsilon > 0$ be chosen arbitrarily but fixed. Taking into account that $\limsup_i \gamma_i < 1$ and $\lim_{i\to\infty} t_i = 0$, one can choose $i_0 \in \mathbb{N}$ such that we have $\limsup_i \gamma_i < 1 - \epsilon$ and $t_i < \epsilon^2$ for all $i \geq i_0$. In the light of (ii), we deduce that

$$\sigma_{i+1} \leq (1-\epsilon)\sigma_i + \epsilon^2, \quad \forall i \geq i_0.$$

Then, by taking $\theta = 1 - \epsilon$ and $\xi = \epsilon^2$, from Lemma 4.4, it follows that

$$\sigma_i \leq \epsilon + (\sigma_{i_0} - \epsilon)(1-\epsilon)^i, \quad \forall i \geq i_0,$$

which implies that $\limsup_i \sigma_i \leq \epsilon$. This completes the proof. □

Remark 4.6 (i) It should be pointed out that the condition $\limsup_i \sigma_i < 1$ imposed on the sequence $\{\sigma_i\}$ in Lemma 4.5 is essential and cannot be dropped. To illustrate this fact, let us take $\sigma_i = \beta$, $t_i = \frac{\beta}{i}$, and $\gamma_i = 1 - \frac{1}{i}$ for all $i \in \mathbb{N}$, where $\beta > 0$ is an arbitrary but fixed real number. Then, we have $\sigma_{i+1} \leq \gamma_i\sigma_i + t_i$ for all $i \in \mathbb{N}$, $\lim_{i\to\infty} t_i = 0$ and $\limsup_i \gamma_i = 1$, but $\lim_{i\to\infty}\sigma_i = \beta \neq 0$.

(ii) It is important to emphasize that Lemma 4.5 extends and unifies Lemma 5.1 in [13, 14] and Lemma 2.2 in [60].

We are now ready, as an application of the notion of graph convergence for $\widehat{H}$-accretive mappings, to present the most important result of this paper in which the strong convergence of the iterative sequence generated by Algorithm 3.12 to a common element of the two sets $\mathrm{SVI}(X_j, \widehat{H}_j, M, N, F, G : j = 1, 2)$ and $\mathrm{Fix}(Q)$, where Q is a self-mapping of $X_1 \times X_2$ defined by (3.23), is proved.

Theorem 4.7 *Suppose that* X_j, $\widehat{H}_j$, F, G, M, N $(j = 1, 2)$ *are the same as in Theorem* 3.3 *and let all the conditions of Theorem* 3.3 *hold. Assume that* $\widehat{H}_{i,j}$, M_i, N_i, S_j, λ_i, *and* ρ_i $(i \geq 0; j = 1, 2)$ *are the same as in Algorithm* 3.12 *such that for each* $i \geq 0$, $\widehat{H}_{i,1}$ *is a* $\varrho_{i,1}$*-strongly accretive and* r_i*-Lipschitz continuous and* $\widehat{H}_{i,2}$ *is a* $\varrho_{i,2}$*-strongly accretive and* k_i*-Lipschitz continuous mapping. Let* Q *be a self-mapping of* $X_1 \times X_2$ *defined by* (3.21) *such that* $\mathrm{Fix}(Q) \cap \mathrm{SVI}(X_j, \widehat{H}_j, M, N, F, G : j = 1, 2) \neq \emptyset$. *Suppose that* $\lim_{i\to\infty}\widehat{H}_{i,j}(x_j) = \widehat{H}_j(x_j)$,

$M_i \xrightarrow{G} M$, $N_i \xrightarrow{G} N$, $r_i \to r$, $k_i \to k$, $\varrho_{i,j} \to \varrho_i$, as $i \to \infty$, and $L_i(\vartheta_{\lambda_0,\rho_0} + 1) < 2$, where $\vartheta_{\lambda_0,\rho_0}$ is the same as in (3.13). Assume further that $r < \varrho_1$ and $k < \varrho_2$ and let there exist constants $\lambda, \rho > 0$ such that $\lambda_i \to \lambda$ and $\rho_i \to \rho$, as $i \to \infty$. Then, the iterative sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ generated by Algorithm 3.12 converges strongly to the only element $(a,b) \in \mathrm{Fix}(Q) \cap \mathrm{SVI}(X_j, \widehat{H}_j, M, N, F, G : j = 1,2)$.

Proof Since all the conditions of Theorem 3.3 hold, invoking Theorem 3.3, the SVI (3.1) admits the unique solution $(a,b) \in X_1 \times X_2$. Then, from Lemma 3.1(ii) we infer that

$$\begin{cases} a = R^{\widehat{H}_1}_{M,\lambda}[\widehat{H}_1(a) - \lambda F(a,b)], \\ b = R^{\widehat{H}_2}_{N,\rho}[\widehat{H}_2(b) - \rho G(a,b)], \end{cases}$$

which can be written, for each $i \geq 0$, as follows:

$$\begin{cases} a = \alpha_i a + (1-\alpha_i) S_1^i R^{\widehat{H}_1}_{M,\lambda}[\widehat{H}_1(a) - \lambda F(a,b)], \\ b = \alpha_i b + (1-\alpha_i) S_2^i R^{\widehat{H}_2}_{N,\rho}[\widehat{H}_2(b) - \rho G(a,b)], \end{cases} \tag{4.4}$$

where the sequence $\{\alpha_i\}_{i=0}^{\infty}$ is the same as in Algorithm 3.12. Applying (3.24), (4.4), Lemma 2.12, and considering the fact that S_1 is a $(\{c_{i,1}\}_{i=0}^{\infty}, \{d_{i,1}\}_{i=0}^{\infty}, \phi_1)$-total uniformly L_1-Lipschitzian mapping, we derive that for each $i \geq 0$,

$$\begin{aligned}
\|a_{i+1} - a_i\|_1 &\leq \alpha_i \|a_i - a\|_1 + (1-\alpha_i) \Big\| S_1^i R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i,b_i)\big] \\
&\quad - S_1^i R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(a) - \lambda F(a,b)\big]\Big\|_1 \\
&\leq \alpha_i \|a_i - a\|_1 + (1-\alpha_i) L_1 \Big(\Big\| R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i,b_i)\big] \\
&\quad - R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(a) - \lambda F(a,b)\big]\Big\|_1 + c_{i,1}\phi_1\Big(\Big\| R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i,b_i)\big] \\
&\quad - R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(a) - \lambda F(a,b)\big]\Big\|_1\Big) + d_{i,1}\Big) \\
&\leq \alpha_i \|a_i - a\|_1 + (1-\alpha_i) L_1 \Big(\Big\| R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i,b_i)\big] \\
&\quad - R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_1(a) - \lambda F(a,b)\big]\Big\|_1 + \Big\| R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_1(a) - \lambda F(a,b)\big] \\
&\quad - R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(a) - \lambda F(a,b)\big]\Big\|_1 + c_{i,1}\phi_1\Big(\Big\| R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i,b_i)\big] \\
&\quad - R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_1(a) - \lambda F(a,b)\big]\Big\|_1 + \Big\| R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_1(a) - \lambda F(a,b)\big] \\
&\quad - R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(a) - \lambda F(a,b)\big]\Big\|_1\Big) + d_{i,1}\Big) \\
&\leq \alpha_i \|a_i - a\|_1 + (1-\alpha_i) L_1 \bigg(\frac{1}{\varrho_i} \Big\| \widehat{H}_{i,1}(a_i) - \lambda_i F(a_i,b_i) \\
&\quad - \big(\widehat{H}_1(a) - \lambda F(a,b)\big)\Big\|_1 + \|\varphi_{i,1}\|_1 + c_{i,1}\phi_1\bigg(\frac{1}{\varrho_{i,1}} \|\widehat{H}_{i,1}(a_i) - \lambda_i F(a_i,b_i) \\
&\quad - \big(\widehat{H}_1(a) - \lambda F(a,b)\big)\|_1 + \|\varphi_{i,1}\|_1\bigg) + d_{i,1}\bigg) \\
&\leq \alpha_i \|a_i - a\|_1 + (1-\alpha_i) L_1 \bigg(\frac{1}{\varrho_{i,1}} \Big\| \widehat{H}_{i,1}(a_i) - \widehat{H}_{i,1}(a)
\end{aligned}$$

$$
\begin{aligned}
&-\lambda_i\big(F(a_i,b_i)-F(a,b)\big)\big\|_1+\big\|\widehat{H}_{i,1}(a)-\widehat{H}_1(a)\big\|_1+|\lambda_i-\lambda|\big\|F(a,b)\big\|_1\Big)\\
&+\|\varphi_{i,1}\|_1+c_{i,1}\phi_1\bigg(\frac{1}{\varrho_{i,1}}\big\|\widehat{H}_{i,1}(a_i)-\widehat{H}_{i,1}(a)-\lambda_i\big(F(a_i,b_i)-F(a,b)\big)\big\|_1\\
&+\big\|\widehat{H}_{i,1}(a)-\widehat{H}_1(a)\big\|_1+|\lambda_i-\lambda|\big\|F(a,b)\big\|_1\bigg)+\|\varphi_{i,1}\|_1\big)+d_{i,1}\big)\\
\leq{}&\alpha_i\|a_i-a\|_1+(1-\alpha_i)L_1\bigg(\frac{1}{\varrho_{i,1}}\big(\big\|\widehat{H}_{i,1}(a_i)-\widehat{H}_{i,1}(a)\big\|_1\\
&+\lambda_i\big\|F(a_i,b_i)-F(a,b)\big\|_1+\big\|\widehat{H}_{i,1}(a)-\widehat{H}_1(a)\big\|_1\\
&+|\lambda_i-\lambda|\big\|F(a,b)\big\|_1\big)+\|\varphi_{i,1}\|_1+c_{i,1}\phi_1\bigg(\frac{1}{\varrho_{i,1}}\big(\big\|\widehat{H}_{i,1}(a_i)-\widehat{H}_{i,1}(a)\big\|_1\\
&+\lambda_i\big\|F(a_i,b_i)-F(a,b)\big\|_1+\big\|\widehat{H}_{i,1}(a)-\widehat{H}_1(a)\big\|_1\\
&+|\lambda_i-\lambda|\big\|F(a,b)\big\|_1\big)+\|\varphi_{i,1}\|_1\bigg)+d_{i,1}\bigg)\\
={}&\alpha_i\|a_i-a\|_1+(1-\alpha_i)L_1\bigg(\frac{1}{\varrho_{i,1}}\big(\big\|\widehat{H}_{i,1}(a_i)-\widehat{H}_{i,1}(a)\big\|_1\\
&+\lambda_i\big\|F(a_i,b_i)-F(a,b)\big\|_1\big)+\mu_{i,1}+c_{i,1}\phi_1\bigg(\frac{1}{\varrho_{i,1}}\big(\big\|\widehat{H}_{i,1}(a_i)-\widehat{H}_{i,1}(a)\big\|_1\\
&+\lambda_i\big\|F(a_i,b_i)-F(a,b)\big\|_1\big)+\mu_{i,1}\bigg)+d_{i,1}\bigg),
\end{aligned}
\tag{4.5}
$$

where for each $i \geq 0$,

$$
\mu_{i,1}=\frac{1}{\varrho_{i,1}}\big(\big\|\widehat{H}_{i,1}(a)-\widehat{H}_1(a)\big\|_1+|\lambda_i-\lambda|\big\|F(a,b)\big\|_1\big)+\|\varphi_{i,1}\|_1
$$

and

$$
\varphi_{i,1}=R^{\widehat{H}_{i,1}}_{M_i,\lambda_i}\big[\widehat{H}_1(a)-\lambda F(a,b)\big]-R^{\widehat{H}_1}_{M,\lambda}\big[\widehat{H}_1(a)-\lambda F(a,b)\big].
$$

Taking into account that for each $i \geq 0$, the mapping $\widehat{H}_{i,1}$ is r_i-Lipschitz continuous, and the mapping F is τ_1-Lipschitz continuous and τ_2-Lipschitz continuous with respect to its first and second arguments, respectively, it follows that

$$
\big\|\widehat{H}_{i,1}(a_i)-\widehat{H}_{i,1}(a)\big\|_1\leq r_i\|a_i-a\|_1 \tag{4.6}
$$

and

$$
\begin{aligned}
\big\|F(a_i,b_i)-F(a,b)\big\|_1&\leq\big\|F(a_i,b_i)-F(a,b_i)\big\|_1+\big\|F(a,b_i)-F(a,b)\big\|_1\\
&\leq\tau_1\|a_i-a\|_1+\tau_2\|b_i-b\|_2.
\end{aligned}
\tag{4.7}
$$

Substituting (4.6) and (4.7) into (4.5), for all $i \geq 0$, we obtain

$$
\begin{aligned}
\|a_{i+1}-a\|_1\leq{}&\alpha_i\|a_i-a\|_1+(1-\alpha_i)L_1\bigg(\frac{r_i+\lambda_i\tau_1}{\varrho_{i,1}}\|a_i-a\|_1\\
&+\frac{\lambda_i\tau_2}{\varrho_{i,1}}\|b_i-b\|_2+\mu_{i,1}+c_{i,1}\phi_1\bigg(\frac{r_i+\lambda_i\tau_1}{\varrho_{i,1}}\|a_i-a\|_1
\end{aligned}
\tag{4.8}
$$

$$+ \frac{\lambda_i \tau_2}{\varrho_{i,1}} \|b_i - b\|_2 + \mu_{i,1}\Bigg) + d_{i,1}\Bigg).$$

By following similar arguments as in the proofs of (4.5)–(4.8) with suitable changes, from (3.24), (4.5), Lemma 2.12, and the assumptions, one can deduce that

$$\begin{aligned} \|b_{i+1} - b\|_2 \leq{}& \alpha_i \|b_i - b\|_2 + (1-\alpha_i) L_2 \Bigg(\frac{k_i + \rho_i \theta_2}{\varrho_{i,2}} \|b_i - b\|_2 \\ &+ \frac{\rho_i \theta_1}{\varrho_{i,2}} \|a_i - a\|_1 + \mu_{i,2} + c_{i,2}\phi_2 \Bigg(\frac{k_i + \rho_i \theta_2}{\varrho_{i,2}} \|b_i - b\|_2 \\ &+ \frac{\rho_i \theta_1}{\varrho_{i,2}} \|a_i - a\|_1 + \mu_{i,2}\Bigg) + d_{i,2}\Bigg), \end{aligned} \tag{4.9}$$

where for each $i \geq 0$,

$$\mu_{i,2} = \frac{1}{\varrho_{i,2}} \big(\big\| \widehat{H}_{i,2}(b) - \widehat{H}_2(b) \big\|_2 + |\rho_i - \rho| \big\| G(a,b) \big\|_2 \big) + \|\varphi_{i,2}\|_2$$

and

$$\varphi_{i,2} = R^{\widehat{H}_{i,2}}_{N_i,\rho_i} \big[\widehat{H}_2(b) - \rho G(a,b) \big] - R^{\widehat{H}_2}_{N,\rho} \big[\widehat{H}_2(b) - \rho G(a,b) \big].$$

Taking $L = \max\{L_1, L_2\}$ and making use of (4.8) and (4.9), we conclude that for all $i \geq 0$,

$$\begin{aligned} &\big\| (a_{i+1}, b_{i+1}) - (a,b) \big\|_* \\ &\quad = \|a_{i+1} - a_i\|_1 + \|b_{i+1} - b\|_2 \\ &\quad \leq \alpha_i \big(\|a_i - a\|_1 + \|b_i - b\|_2 \big) \\ &\qquad + (1-\alpha_i) L \Bigg(\bigg(\frac{r_i + \lambda_i \tau_1}{\varrho_{i,1}} + \frac{\rho_i \theta_1}{\varrho_{i,2}} \bigg) \|a_i - a\|_1 \\ &\qquad + \bigg(\frac{k_i + \rho_i \theta_2}{\varrho_{i,2}} + \frac{\lambda_i \tau_2}{\varrho_{i,1}} \bigg) \|b_i - b_i\|_2 + \mu_{i,1} + \mu_{i,2} \\ &\qquad + c_{i,1}\phi_1 \bigg(\frac{r_i + \lambda_i \tau_1}{\varrho_{i,1}} \|a_i - a\|_1 + \frac{\lambda_i \tau_2}{\varrho_{i,1}} \|b_i - b\|_2 + \mu_{i,1} \bigg) \\ &\qquad + c_{i,2}\phi_2 \bigg(\frac{k_i + \rho_i \theta_2}{\varrho_{i,2}} \|b_i - b\|_2 + \frac{\rho_i \theta_1}{\varrho_{i,2}} \|a_i - a\|_1 + \mu_{i,2} \bigg) \\ &\qquad + d_{i,1} + d_{i,2} \Bigg) \\ &\quad \leq \alpha_i \big(\|a_i - a\|_1 + \|b_i - b\|_2 \big) \\ &\qquad + (1-\alpha_i) L \big(\vartheta_{\lambda_i,\rho_i}(i) \big(\|a_i - a\|_1 + \|b_i - b\|_2 \big) \\ &\qquad + \mu_{i,1} + \mu_{i,2} + c_{i,1}\phi_1 \big(\vartheta_{\lambda_i,\rho_i}(i) \big(\|a_i - a\|_1 + \|b_i - b\|_2 \big) \\ &\qquad + \mu_{i,1} \big) + c_{i,2}\phi_2 \big(\vartheta_{\lambda_i,\rho_i}(i) \big(\|a_i - a\|_1 + \|b_i - b\|_2 \big) \\ &\qquad + \mu_{i,2} \big) + d_{i,1} + d_{i,2} \big) \\ &\quad \leq \alpha_i \big\| (a_i, b_i) - (a,b) \big\|_* + (1-\alpha_i) L \big(\vartheta_{\lambda_i,\rho_i}(i) \big\| (a_i, b_i) - (a,b) \big\|_* \end{aligned} \tag{4.10}$$

$$\begin{aligned}&+\mu_{i,1}+\mu_{i,2}+c_{i,1}\phi\big(\vartheta_{\lambda_i,\rho_i}(i)\big\|(a_i,b_i)-(a,b)\big\|_*+\mu_{i,1}\big)\\&+c_{i,2}\phi\big(\vartheta_{\lambda_i,\rho_i}(i)\big\|(a_i,b_i)-(a,b)\big\|_*+\mu_{i,2}\big)+d_{i,1}+d_{i,2}\big),\end{aligned}$$

where ϕ is a self-mapping of $\mathbb{R}^+$ defined by (3.22), and for each $i \geq 0$,

$$\vartheta_{\lambda_i,\rho_i}(i)=\max\left\{\frac{r_i+\lambda_i\tau_1}{\varrho_{i,1}}+\frac{\rho_i\theta_1}{\varrho_{i,2}},\frac{k_i+\rho_i\theta_2}{\varrho_{i,2}}+\frac{\lambda_i\tau_2}{\varrho_{i,1}}\right\}.$$

Since $r_i \to r$, $k_i \to k$, $\lambda_i \to \lambda$, $\rho_i \to \rho$, $\varrho_{i,j} \to \varrho_j$ for $j = 1,2$, it follows that $\vartheta_{\lambda_i,\rho_i}(i) \to \vartheta_{\lambda,\rho}$, as $i \to \infty$, where $\vartheta_{\lambda,\rho}$ is the same as in (3.11). By virtue of the fact that $r < \varrho_1$ and $k < \varrho_2$, there are some $\lambda_0, \rho_0 > 0$ small enough such that $\vartheta_{\lambda_0,\rho_0} \in (0,1)$. Then, for $\widehat{\vartheta}_{\lambda_0,\rho_0} = \frac{\vartheta_{\lambda_0,\rho_0}+1}{2} \in (\vartheta_{\lambda_0,\rho_0}, 1)$ there exists $i_0 \geq 1$ such that $\vartheta_{\lambda_i,\rho_i}(i) < \widehat{\vartheta}_{\lambda_0,\rho_0}$ for all $i \geq i_0$. Thereby, from (4.10) we derive that for all $i \geq i_0$,

$$\begin{aligned}&\big\|(a_{i+1},b_{i+1})-(a,b)\big\|_*\\&\quad\le\alpha_i\big\|(a_i,b_i)-(a,b)\big\|_*+(1-\alpha_i)L\big(\widehat{\vartheta}_{\lambda_0,\rho_0}\big\|(a_i,b_i)-(a,b)\big\|_*\\&\qquad+\mu_{i,1}+\mu_{i,2}+c_{i,1}\phi\big(\widehat{\vartheta}_{\lambda_0,\rho_0}\big\|(a_i,b_i)-(a,b)\big\|_*+\mu_{i,1}\big)\\&\qquad+c_{i,2}\phi\big(\widehat{\vartheta}_{\lambda_0,\rho_0}\big\|(a_i,b_i)-(a,b)\big\|_*+\mu_{i,2}\big)+d_{i,1}+d_{i,2}\big)\\&\quad=\big(L\widehat{\vartheta}_{\lambda_0,\rho_0}+(1-L\widehat{\vartheta}_{\lambda_0,\rho_0})\alpha_i\big)\big\|(a_i,b_i)-(a,b)\big\|_*\\&\qquad+(1-\alpha_i)L\big(\mu_{i,1}+\mu_{i,2}+c_{i,1}\phi\big(\widehat{\vartheta}_{\lambda_0,\rho_0}\big\|(a_i,b_i)-(a,b)\big\|_*\\&\qquad+\mu_{i,1}\big)+c_{i,2}\phi\big(\widehat{\vartheta}_{\lambda_0,\rho_0}\big\|(a_i,b_i)-(a,b)\big\|_*+\mu_{i,2}\big)+d_{i,1}+d_{i,2}\big).\end{aligned}\tag{4.11}$$

Letting $\gamma_i = L\widehat{\vartheta}_{\lambda_0,\rho_0} + (1-L\widehat{\vartheta}_{\lambda_0,\rho_0})\alpha_i$ for each $i \geq 0$ and thanks to the facts that $L(\vartheta_{\lambda_0,\rho_0}+1) < 2$ and $\limsup_i \alpha_i < 1$, we deduce that

$$\begin{aligned}\limsup_i\gamma_i&=\limsup_i\big(L\widehat{\vartheta}_{\lambda_0,\rho_0}+(1-L\widehat{\vartheta}_{\lambda_0,\rho_0})\alpha_i\big)\\&=L\widehat{\vartheta}_{\lambda_0,\rho_0}+(1-L\widehat{\vartheta}_{\lambda_0,\rho_0})\limsup_i\alpha_i\\&<1.\end{aligned}$$

Owing to the facts that $M_i \xrightarrow{G} M$ and $N_i \xrightarrow{G} N$, from Theorem 4.2 it follows that for $j = 1,2$, $\|\varphi_{i,j}\|_j \to 0$ as $i \to \infty$. Meanwhile, since for $j = 1,2$, $\widehat{H}_{i,j}(x_j) \to \widehat{H}_j(x_j)$ for any $x_j \in X_j$, $\lambda_i \to \lambda$ and $\rho_i \to \rho$ as $i \to \infty$, we conclude that for $j = 1,2$, $\mu_{i,j} \to 0$ as $i \to \infty$. Relying on the fact that for $j = 1,2$, S_j is a $(\{c_{i,j}\}_{i=0}^{\infty}, \{d_{i,j}\}_{i=0}^{\infty}, \phi_j)$-total uniformly L_j-Lipschitzian mapping, invoking Definition 4.1, for $j = 1,2$ we have $c_{i,j}, d_{i,j} \to 0$ as $i \to \infty$. By assuming $\sigma_i = \|(a_i,b_i)-(a,b)\|_*$ and

$$\begin{aligned}t_i&=(1-\alpha_i)L\big(\mu_{i,1}+\mu_{i,2}+c_{i,1}\phi\big(\widehat{\vartheta}_{\lambda_0,\rho_0}\big\|(a_i,b_i)-(a,b)\big\|_*+\mu_{i,1}\big)\\&\quad+c_{i,2}\phi\big(\widehat{\vartheta}_{\lambda_0,\rho_0}\big\|(a_i,b_i)-(a,b)\big\|_*+\mu_{i,2}\big)+d_{i,1}+d_{i,2}\big),\end{aligned}$$

we infer that $\lim_{i\to\infty} t_i = 0$ and (4.11) can be written as $\sigma_{i+1} \leq \gamma_i\sigma_i + t_i$ for all $i \geq 0$. We now note that all the conditions of Lemma 4.5 are satisfied and thereby making use of (4.11) and Lemma 4.5 it follows that $\sigma_i \to 0$ as $i \to \infty$, i.e., $(a_i,b_i) \to (a,b)$, as $i \to \infty$. Accordingly, the sequence $\{(a_i,b_i)\}_{i=0}^{\infty}$ generated by Algorithm 3.12 converges strongly to the unique solution of the SVI (3.1), that is, the only element of $\mathrm{Fix}(Q)\cap \mathrm{SVI}(X_j,\widehat{H}_j,M,N,F,G:j=1,2)$. This completes the proof. □

Taking $S_j\equiv I_j$, the identity mapping on X_j, the following corollary follows from Theorem 4.7 immediately.

Corollary 4.8 *Assume that X_j, $\widehat{H}_j$, $\widehat{H}_{i,j}$, M_i, N_i, λ_i, ρ_i, F, G, M, N ($i\geq 0$ and $j=1,2$) are the same as in Theorem 3.3 and let all the conditions of Theorem 3.3 hold. Then, the iterative sequence $\{(a_i,b_i)\}_{i=0}^{\infty}$ generated by Algorithm 3.13 converges strongly to the unique solution of the* SVI (3.1).

Taking $S_j\equiv I_j$, $\widehat{H}_{i,j}=\widehat{H}_j$, $\lambda_i=\lambda$ and $\rho_i=\rho$ for each $i\geq 0$ and $j\in\{1,2\}$, we obtain the following corollary as a direct consequence of Theorem 4.7.

Corollary 4.9 *Let X_j, $\widehat{H}_j$, F, G, M, N ($j=1,2$) be the same as in Theorem 4.2 and let all the conditions of Theorem 4.2 hold. Suppose that for each $i\geq 0$, $M_i:X_1\rightrightarrows X_1$ is an $\widehat{H}_1$-accretive set-valued mapping and $N_i:X_2\rightrightarrows X_2$ is an $\widehat{H}_2$-accretive set-valued mapping such that $M_i\xrightarrow{G}M$ and $N_i\xrightarrow{G}N$. Assume further that $r<\varrho_1$ and $k<\varrho_2$. Then, the iterative sequence $\{(a_i,b_i)\}_{i=0}^{\infty}$ generated by Algorithm 3.14 converges strongly to the unique solution of the* SVI (3.1).

5 $H(\cdot,\cdot)$-Accretive operators and some comments

In this section, our attention is turned to investigate and analyze the notion of an $H(\cdot,\cdot)$-accretive operator and the related results available in [26]. Some remarks together with relevant commentaries are also pointed out.

Let us first remark that throughout [26], X is assumed to be a real Banach space such that J is single-valued. As we know, J is single-valued if and only if X is smooth. Hence, throughout the rest of paper, unless otherwise stated, we assume that X is a real smooth Banach space.

Definition 5.1 ([26]) For given single-valued mappings $A,B:X\to X$ and $H:X\times X\to X$,

(i) $H(A,.)$ is said to be α-generalized accretive with respect to A if there exists a constant $\alpha\in\mathbb{R}$ satisfying

$$\big\langle H(Ax,u)-H(Ay,u),J(x-y)\big\rangle\geq\alpha\|x-y\|^2,\quad \forall x,y,u\in X;$$

(ii) $H(.,B)$ is said to be β-generalized accretive with respect to B if there exists a constant $\beta\in\mathbb{R}$ such that

$$\big\langle H(u,Bx)-H(u,By),J(x-y)\big\rangle\geq\beta\|x-y\|^2,\quad \forall x,y,u\in X;$$

(iii) $H(\cdot,\cdot)$ is said to be ρ-Lipschitz continuous with respect to A if there exists a constant $\rho>0$ such that

$$\big\|H(Ax,u)-H(Ay,u)\big\|\leq\rho\|x-y\|,\quad \forall x,y,u\in X;$$

(iv) $H(\cdot,\cdot)$ is said to be ς-Lipschitz continuous with respect to B if there exists a constant $\varsigma > 0$ such that

$$\|H(u,Bx) - H(u,By)\| \leq \varsigma\|x-y\|, \quad \forall x,y,u \in X.$$

Here, it is to be noted that, as was pointed out in [26], in a similar way to cases (iv) and (v) of Definition 2.1 in [26], one can define the generalized accretivity of the mapping $H(\cdot,\cdot)$ with respect to B and the Lipschitz continuity of the mapping $H(\cdot,\cdot)$ with respect to B, as we have done, respectively, in parts (ii) and (iv) of Definition 5.1.

Proposition 5.2 *Let $A,B : X \to X$ and $H : X \times X \to X$ be the mappings and let $\widehat{H} : X \to X$ be a mapping defined by $\widehat{H}(x) := H(Ax,Bx)$ for all $x \in X$. Then, the following conclusions hold*:

(i) *If $H(\cdot,\cdot)$ is α, β-generalized accretive with respect to A, B, respectively, then $\widehat{H}$ is $(\alpha+\beta)$-strongly accretive and hence it is strictly accretive* (*resp., accretive and $-(\alpha+\beta)$-relaxed accretive*) *provided that* $\alpha+\beta > 0$ (*resp.*, $\alpha+\beta = 0$ *and* $\alpha+\beta < 0$);

(ii) *If $H(\cdot,\cdot)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B, then $\widehat{H}$ is (r_1+r_2)-Lipschitz continuous.*

Proof (i) Since $H(\cdot,\cdot)$ is α, β-generalized accretive with respect to A, B, respectively, it yields

$$\begin{aligned}
\langle \widehat{H}(x) - \widehat{H}(y), J(x-y)\rangle &= \langle H(Ax,Bx) - H(Ay,By), J(x-y)\rangle \\
&= \langle H(Ax,Bx) - H(Ay,Bx), J(x-y)\rangle \\
&\quad + \langle H(Ay,Bx) - H(Ay,By), J(x-y)\rangle \\
&\geq \alpha\|x-y\|^2 + \beta\|x-y\|^2 \\
&= (\alpha+\beta)\|x-y\|^2.
\end{aligned}$$

If $\alpha+\beta > 0$, the last inequality ensures that $\widehat{H}$ is $(\alpha+\beta)$-strongly accretive and so the fact that $\widehat{H}$ is strictly accretive is straightforward. For the case when $\alpha+\beta = 0$ (resp., $\alpha+\beta < 0$), thanks to the preceding inequality we infer that $\widehat{H}$ is accretive (resp., $-(\alpha+\beta)$-relaxed accretive).

(ii) Taking into account that the mapping $H(\cdot,\cdot)$ is r_1-Lipschitz continuous and r_2-Lipschitz continuous with respect to the mappings A and B, respectively, it follows that for all $x,y \in X$,

$$\begin{aligned}
\|\widehat{H}(x) - \widehat{H}(y)\| &= \|H(Ax,Bx) - H(Ay,By)\| \\
&\leq \|H(Ax,Bx) - H(Ay,Bx)\| \\
&\quad + \|H(Ay,Bx) - H(Ay,By)\| \\
&\leq (r_1+r_2)\|x-y\|;
\end{aligned}$$

i.e., $\widehat{H}$ is (r_1+r_2)-Lipschitz continuous. The proof is finished. □

It is significant to emphasize that every bifunction $H : X \times X \to X$ that is α, β-generalized accretive with respect to A, B, respectively, is actually a univariate $(\alpha + \beta)$-strongly accretive (resp., accretive and $-(\alpha + \beta)$-relaxed accretive) mapping provided that $\alpha + \beta > 0$ (resp., $\alpha + \beta = 0$ and $\alpha + \beta < 0$) and is not a new one. At the same time, thanks to Proposition 5.2(ii), the notion of Lipschitz continuity of the bifunction $H : X \times X \to X$ with respect to the mappings $A, B : X \to X$ presented in parts (iii) and (iv) of Definition 5.1 is exactly the same concept of Lipschitz continuity of a univariate mapping $\widehat{H} = H(A,B) : X \to X$ that appeared in Definition 2.1(v) and is not a new one.

Definition 5.3 ([26, 35]) For given single-valued mappings $A, B : X \to X$ and $H : X \times X \to X$, a set-valued mapping $M : X \rightrightarrows X$ is said to be $H(\cdot,\cdot)$-accretive with respect to mappings A and B (or simply $H(\cdot,\cdot)$-accretive in the following), if M is accretive and $(H(A,B) + \lambda M)(X) = X$ for every $\lambda > 0$.

Remark 5.4 It is worth mentioning that the concept of an $H(\cdot,\cdot)$-accretive operator was initially introduced by Zou and Huang [35] in 2008, and was studied for the case when $H(\cdot,\cdot)$ is α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$. Afterwards, several generalizations of this notion appeared in the literature. Recently, this notion has been considered by Tang and Wang [26] and has been studied in a more general case when $H(\cdot,\cdot)$ is α-generalized accretive with respect to A, β-generalized accretive with respect to B. It should be pointed out that by defining the mapping $\widehat{H} : X \to X$ as $\widehat{H}(x) := H(Ax, Bx)$ for all $x \in X$, Definition 5.3 coincides exactly with Definition 2.6. In other words, the concept of an $H(\cdot,\cdot)$-accretive operator is actually the same notion of the $\widehat{H}$-accretive operator introduced and studied by Fang and Huang [11] and is not a new one.

According to the following conclusion, the authors [26] deduced that every $H(\cdot,\cdot)$-accretive operator is maximal under some appropriate conditions.

Lemma 5.5 ([26, Theorem 2.1]) *Let $H(\cdot,\cdot)$ be α, β-generalized accretive with respect to A, B, respectively, such that $\alpha + \beta > 0$. Let $M : X \rightrightarrows X$ be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. If the inequality $\langle u - v, J(x - y)\rangle \geq 0$ holds for all $(y, v) \in \mathrm{Graph}(M)$, then $u \in M(x)$.*

Proof Defining the mapping $\widehat{H} : X \to X$ by $\widehat{H}(x) := H(Ax, Bx)$ for all $x \in X$, from the assumptions and using Proposition 5.2(i) it follows that $\widehat{H}$ is $(\alpha + \beta)$-strongly accretive and so it is a strictly accretive mapping. At the same time, invoking Remark 5.4, M is an $\widehat{H}$-accretive mapping. We now note that all the conditions of Lemma 2.9 are satisfied and so the conclusion follows from Lemma 2.9 immediately. □

It should be noted that the conclusion of Theorem 2.1 in [26] has been derived based on Theorem 3.1 in [35] without presenting any proof. In fact, in [35, Theorem 3.1], the authors proved that every $H(\cdot,\cdot)$-accretive operator with respect to mappings A and B satisfying the appropriate conditions, where $H(\cdot,\cdot)$ is α-strongly accretive with respect to A, β-relaxed accretive with respect to B and $\alpha > \beta$ is maximal. Tang and Wang [26] concluded the same assertion for $H(\cdot,\cdot)$-accretive mappings for the case when $H(\cdot,\cdot)$ is $\alpha\beta$-generalized accretive with respect to A, B, respectively, and $\alpha + \beta \neq 0$. However, by following a similar argument as in the proof of [35, Theorem 3.1] with suitable modifications, we found that the condition $\alpha + \beta \neq 0$ in the context of [26, Theorem 2.1] must be replaced by the condition $\alpha + \beta > 0$, as has been done in the context of Lemma 5.5.

Lemma 5.6 ([26, Theorem 2.2]) *Let $H(\cdot,\cdot)$ be α, β-generalized accretive with respect to A, B, respectively, such that $\alpha + \beta > 0$. Let $M : X \rightrightarrows X$ be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. Then, the operator $(H(A,B) + \lambda M)^{-1}$ is single-valued.*

Proof Let us define the mapping $\widehat{H} : X \to X$ as $\widehat{H}(x) := H(Ax, Bx)$ for all $x \in X$. Then, in the light of the assumptions, Proposition 5.2(i) implies that $\widehat{H}$ is $(\alpha + \beta)$-strongly accretive and so it is a strictly accretive mapping. Meanwhile, in view of Remark 5.4, M is an $\widehat{H}$-accretive operator. Thereby, all the conditions of Lemma 2.10 are satisfied. Hence, according to Lemma 2.10, the operator $(\widehat{H} + \lambda M)^{-1} = (H(\cdot,\cdot) + \lambda M)^{-1}$ is single-valued for every constant $\lambda > 0$. The proof is completed. □

Remark 5.7 It should be pointed out that the assertion of Theorem 2.2 in [26] is derived similarly to that of assertion of Theorem 3.3 in [35]. In fact, in Theorem 3.3 of [35], the authors proved that for a given $H(\cdot,\cdot)$-accretive operator $M : X \rightrightarrows X$ with respect to the mappings A and B, where $H(\cdot,\cdot)$ is α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$, the operator $(H(A,B) + \lambda M)^{-1}$ is single-valued for every constant $\lambda > 0$. Without giving any proof, Tang and Wang [26] claimed that the same assertion holds for the case when $H(\cdot,\cdot)$ is α, β-generalized accretive with respect to A, B, respectively, and $\alpha + \beta \neq 0$. However, by following a similar argument as in the proof of Theorem 3.3 presented in [35] with suitable changes, we inferred that the condition $\alpha + \beta \neq 0$ in the context of [26, Theorem 2.2] must be replaced by the condition $\alpha + \beta > 0$, as we have done in the context of Lemma 5.6.

Based on Theorem 2.2 in [26], the authors defined the resolvent operator associated with an $H(\cdot,\cdot)$-accretive operator $M : X \rightrightarrows X$ as follows.

Definition 5.8 ([26, Definition 2.3]) Let $H(\cdot,\cdot)$ be α, β-generalized accretive with respect to A, B, respectively, such that $\alpha + \beta > 0$. Let $M : X \rightrightarrows X$ be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. For each $\lambda > 0$, the resolvent operator $R^{H(\cdot,\cdot)}_{M,\lambda} : X \to X$ is defined by

$$R^{H(\cdot,\cdot)}_{M,\lambda} = \bigl(H(A,B) + \lambda M\bigr)^{-1}(u), \quad \forall u \in X.$$

Note, in particular, that by defining the operator $\widehat{H} : X \to X$ as $\widehat{H}(x) := H(Ax, Bx)$ for all $x \in X$, thanks to the assumptions from Proposition 5.2(i) it follows that $\widehat{H}$ is a strictly accretive operator. Furthermore, by virtue of Remark 5.4, M is an $\widehat{H}$-accretive operator. Thus, based on Definition 2.11, for any constant $\lambda > 0$, the resolvent operator $R^{\widehat{H}}_{M,\lambda} = R^{H(\cdot,\cdot)}_{M,\lambda} : X \to X$ associated with an $\widehat{H} = H(\cdot,\cdot)$-accretive operator M is defined by

$$R^{H(\cdot,\cdot)}_{M,\lambda}(u) = R^{\widehat{H}}_{M,\lambda}(u) = (\widehat{H} + \lambda M)^{-1}(u) = \bigl(H(A,B) + \lambda M\bigr)^{-1}(u), \quad \forall u \in X.$$

Indeed, in view of the discussion mentioned above, the notion of the resolvent operator $R^{H(\cdot,\cdot)}_{M,\lambda}$ associated with an $H(\cdot,\cdot)$-accretive operator $M : X \rightrightarrows X$ and an arbitrary constant $\lambda > 0$, where $H(\cdot,\cdot)$ is α, β-generalized accretive with respect to A, B, respectively, and

$\alpha+\beta>0$ is actually the same notion of the resolvent operator $R^{\widehat{H}}_{M,\lambda}$ associated with the $\widehat{H}$-accretive operator M and the real constant $\lambda>0$ given in Definition 2.11 and is not a new one. At the same time, it should be remarked that in Definition 2.3 of [26], the notion of a resolvent operator associated with an $H(\cdot,\cdot)$-accretive operator $M: X \rightrightarrows X$ is defined based on Theorem 2.2 in [26]. However, as was pointed out in Remark 5.7, the condition $\alpha+\beta\neq 0$ in the context of Theorem 2.2 of [26] must be replaced by the condition $\alpha+\beta>0$. Hence, this correction must be done in the context of Definition 2.3 of [26], as has done in the context of Definition 5.8.

With the aim of proving the Lipschitz continuity of the resolvent operator $R^{H(\cdot,\cdot)}_{M,\lambda}$ and computing an estimate of its Lipschitz constant, Tang and Wang [26] presented one of the most important results of Sect. 2 of [26] without any proof as follows.

Lemma 5.9 ([26, Theorem 2.3]) *Let $H(\cdot,\cdot)$ be α, β-generalized accretive with respect to A, B, respectively, such that $\alpha+\beta>0$. Let $M: X \rightrightarrows X$ be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. Then, the resolvent operator $R^{H(\cdot,\cdot)}_{M,\lambda}: X\to X$ is $\frac{1}{\alpha+\beta}$-Lipschitz continuous, that is,*

$$\left\|R^{H(\cdot,\cdot)}_{M,\lambda}(u)-R^{H(\cdot,\cdot)}_{M,\lambda}(v)\right\| \leq \frac{1}{\alpha+\beta}\|u-v\|, \quad \forall u\in X.$$

Proof Defining the mapping $\widehat{H}: X\to X$ as $\widehat{H}(x) := H(Ax,Bx)$ for all $x\in X$, with the help of the assumptions, from Proposition 5.2(i) it follows that the operator $\widehat{H}$ is $(\alpha+\beta)$-strongly accretive. At the same time, by virtue of Remark 5.4, M is an $\widehat{H}$-accretive operator. Taking $r=\alpha+\beta$, Lemma 2.12 ensures that the resolvent operator $R^{\widehat{H}}_{M,\lambda}=R^{H(\cdot,\cdot)}_{M,\lambda}: X\to X$ is Lipschitz continuous with constant $\frac{1}{r}=\frac{1}{\alpha+\beta}$, i.e.,

$$\left\|R^{H(\cdot,\cdot)}_{M,\lambda}(u)-R^{H(\cdot,\cdot)}_{M,\lambda}(v)\right\| = \left\|R^{\widehat{H}}_{M,\lambda}(u)-R^{\widehat{H}}_{M,\lambda}(v)\right\| \leq \frac{1}{r}\|u-v\| = \frac{1}{\alpha+\beta}\|u-v\|,$$

for all $u,v\in X$. This gives the desired result. □

Example 5.10 ([26, Example 2.1]) Let $X=\mathbb{R}^2=(-\infty,+\infty)\times(-\infty,+\infty)$ and define $A,B: \mathbb{R}^2\to\mathbb{R}^2$, respectively, by

$$Ax=(-x_1,-x_2)=-x \quad \text{and} \quad Bx=(2x_1,2x_2)=2x, \quad \forall x=(x_1,x_2)\in\mathbb{R}^2.$$

Suppose that the bifunction $H(\cdot,\cdot):\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ is defined by $H(\cdot,\cdot)((x,y))=x+y$, for all $x,y\in X=\mathbb{R}^2$. Thanks to the facts that

$$\begin{aligned}
\left\langle H(Ax,y)-H(Ay,u),J(x-y)\right\rangle &= \langle Ax-Ay,x-y\rangle \\
&= \left\langle(-x_1+y_1,-x_2+y_2),(x_1-y_1,x_2-y_2)\right\rangle \\
&= \langle -x+y,x-y\rangle \\
&= -\|x-y\|^2
\end{aligned}$$

and

$$\begin{aligned}\langle H(u,Bx)-H(u,By),J(x-y)\rangle &= \langle Bx-By,x-y\rangle \\ &= \langle (2x_1-2y_1,2x_2-2y_2),(x_1-y_1,x_2-y_2)\rangle \\ &= \langle 2x-2y,x-y\rangle \\ &= 2\|x-y\|^2,\end{aligned}$$

Tang and Wang [26] deduced that $H(\cdot,\cdot)$ is -1, 2-generalized accretive with respective to A, B, respectively. By virtue of the facts that

$$\|H(Ax,u)-H(Ay,u)\| = \|Ax-Ay\| = \|-x+y\| \le \|x-y\|$$

and

$$\|H(u,Bx)-H(u,By)\| = \|Bx-By\| = \|2x-2y\| \le 2\|x-y\|,$$

it follows that $H(\cdot,\cdot)$ is 1-Lipschitz continuous and 2-Lipschitz continuous with respect to A and B, respectively. Taking into account that H is not strongly accretive with respect to A, they pointed out that the condition of Theorems 3.1, 3.3, 3.4, and Definition 3.2 of Zou and Huang [35] is not satisfied.

Let us define the mapping $\widehat{H}:X\to X$ by $\widehat{H}(x):=H(Ax,Bx)$ for all $x\in X$. Then, for all $x\in X$, we have $\widehat{H}(x)=Ax+Bx=-x+2x=x$. Now, taking $\alpha=-1$, $\beta=2$, $r_1=1$ and $r_2=2$, in light of the fact that $\alpha+\beta=-1+2>0$, from parts (i) and (ii) of Proposition 5.2, it is expected that the mapping $\widehat{H}$ is $(\alpha+\beta)=1$-strongly accretive and $(r_1+r_2)=3$-Lipschitz continuous. Since

$$\langle \widehat{H}(x)-\widehat{H}(y),J(x-y)\rangle = \langle x-y,x-y\rangle \ge \|x-y\|^2$$

and

$$\|\widehat{H}(x)-\widehat{H}(y)\| = \|x-y\| \le 3\|x-y\|,$$

for all $x,y\in X$, these facts confirm our expectations, i.e., our observations are compatible with our derived assertions in Proposition 5.2.

Definition 5.11 ([26, 60]) Let $A,B:X\to X$ and $H:X\times X\to X$ be three single-valued mappings. Let M_i, $M:X\rightrightarrows X$ be $H(\cdot,\cdot)$-accretive operators for $i=1,2,\ldots$. The sequence $\{M_i\}$ is said to be graph-convergent to M, denoted by $M_i\xrightarrow{G}M$, if for every $(x,u)\in$ Graph(M), there exists a sequence of points $(x_i,u_i)\in$ Graph(M_i) such that $x_i\to x$ and $u_i\to u$, as $i\to\infty$.

As was pointed out, by defining the mapping $\widehat{H}:X\to X$ by $\widehat{H}(x):=H(Ax,Bx)$ for all $x\in X$, the notion of $H(\cdot,\cdot)$-accretive mapping is exactly the same concept of $\widehat{H}$-accretive mapping and is not a new one. Thanks to this fact we found that if in Definition 5.11, the mappings M_i ($i \geq 0$) and M are assumed to be $\widehat{H}$-accretive, then Definition 5.11 becomes actually the same Definition 4.1. In fact, the notion of graph convergence for $H(\cdot,\cdot)$-accretive mappings introduced in [26, 60] is exactly the same concept of graph convergence for $\widehat{H}$-accretive mappings as a special case of Definition 4.1 and is not a new one.

Using the notion of graph convergence for $H(\cdot,\cdot)$-accretive operators, Tang and Wang [26] established an equivalence between the graph convergence of a sequence of $H(\cdot,\cdot)$-accretive operators and their associated resolvent operators, respectively, to a given $H(\cdot,\cdot)$-accretive mapping and its associated resolvent operator as follows.

Theorem 5.12 ([26, Theorem 2.4]) *Let $M_i, M : X \rightrightarrows X$ be $H(\cdot,\cdot)$-accretive operators for $i = 1, 2, \dots$. Assume that $H : X \times X \to X$ is a single-valued mapping such that*

(a) *$H(A,B)$ is α, β-generalized accretive with respect to A, B, respectively, with $\alpha + \beta > 0$;*

(b) *$H(A,B)$ is γ_1, γ_2-Lipschitz continuous with respect to A, B, respectively.*

Then, the following statements are equivalent:

(i) $M_i \xrightarrow{G} M$;

(ii) *For each* $\lambda > 0$, $R^{H(\cdot,\cdot)}_{M_i,\lambda}(u) \to R^{H(\cdot,\cdot)}_{M,\lambda}(u)$, $\forall u \in X$;

(iii) *For some* $\lambda_0 > 0$, $R^{H(\cdot,\cdot)}_{M_i,\lambda_0}(u) \to R^{H(\cdot,\cdot)}_{M,\lambda_0}(u)$, $\forall u \in X$.

Proof Let us define the mapping $\widehat{H} : X \to X$ by $\widehat{H} := H(Ax, Bx)$ for all $x \in X$. Thanks to the assumptions mentioned in parts (a) and (b), from parts (i) and (ii) of Proposition 5.2 it follows that $\widehat{H}$ is $(\alpha + \beta)$-strongly accretive and $(\gamma_1 + \gamma_2)$-Lipschitz continuous. Meanwhile, invoking Remark 5.4, M_i ($i \geq 0$) and M are $\widehat{H}$-accretive mappings and so the resolvent operators $R^{H(\cdot,\cdot)}_{M_i,\lambda}$ ($i \geq 0$) and $R^{H(\cdot,\cdot)}_{M,\lambda}$ become actually the same resolvent operators $R^{\widehat{H}}_{M_i,\lambda}$ ($i \geq 0$) and $R^{\widehat{H}}_{M,\lambda}$, respectively. Taking $\varrho = \alpha + \beta$ and $\gamma = \gamma_1 + \gamma_2$, we note that all the conditions of Corollary 4.9 are satisfied. Now, in the light of Corollary 4.9, it follows that the following statements are equivalent:

(i) $M_i \xrightarrow{G} M$;

(ii) For each $\lambda > 0$, $R^{H(\cdot,\cdot)}_{M_i,\lambda}(u) = R^{\widehat{H}}_{M_i,\lambda}(u) \to R^{\widehat{H}}_{M,\lambda}(u) = R^{H(\cdot,\cdot)}_{M,\lambda}(u)$, $\forall u \in X$;

(iii) For some $\lambda_0 > 0$, $R^{H(\cdot,\cdot)}_{M_i,\lambda_0}(u) = R^{\widehat{H}}_{M_i,\lambda_0}(u) \to R^{\widehat{H}}_{M,\lambda_0}(u) = R^{H(\cdot,\cdot)}_{M,\lambda_0}(u)$, $\forall u \in X$.

The proof is completed. □

Let for $i = 1, 2$, X_i be real Banach spaces and let $A_i, B_i : X_i \to X_i$, $H_i : X_i \times X_i \to X_i$, $F : X_1 \times X_2 \to X_1$ and $G : X_1 \times X_2 \to X_2$ be the nonlinear operators. Recently, Tang and Wang [26] considered and studied the SVI (3.1), where $M : X_1 \rightrightarrows X_1$ and $N : X_2 \rightrightarrows X_2$ are $H_1(A_1, B_1)$-accretive and $H_2(A_2, B_2)$-accretive set-valued operators, respectively. In order to present a characterization of the solution of the SVI (3.1) involving $H_i(\cdot,\cdot)$-accretive operators M and N ($i = 1, 2$), Tang and Wang [26] gave the following conclusion by using the notion of the resolvent operators $R^{H_1(\cdot,\cdot)}_{M,\lambda}$ and $R^{H_2(\cdot,\cdot)}_{N,\rho}$.

Lemma 5.13 ([26, Lemma 3.1]) *Let X_1 and X_2 be two real smooth Banach spaces. Let $A_1, B_1 : X_1 \to X_1$, $A_2, B_2 : X_2 \to X_2$ be four single-valued operators, $H_1 : X_1 \times X_1 \to X_1$ be a single-valued mapping such that $H_1(A_1, B_1)$ is α_1, β_1-generalized accretive with respect to A_1, B_1, respectively, with $\alpha_1 + \beta_1 > 0$, and $H_2 : X_2 \times X_2 \to X_2$ be a single-valued mapping such that $H_2(A_2, B_2)$ is α_2, β_2-generalized accretive with respect to A_2, B_2, respectively, with $\alpha_2 + \beta_2 > 0$. Let $M : X_1 \rightrightarrows X_1$ be an $H_1(\cdot,\cdot)$-accretive set-valued mapping and $N : X_2 \rightrightarrows X_2$ be an $H_2(\cdot,\cdot)$-accretive set-valued mapping. Then, the following statements are equivalent*:

(i) $(a,b) \in X_1 \times X_2$ *is a solution of the problem* (3.1) (*involving an* $H_1(\cdot,\cdot)$-*accretive operator* M *and an* $H_2(\cdot,\cdot)$-*accretive operator* N, *that is*, [26, *problem* (3.1)]);

(ii) *For any* $\lambda, \rho > 0$, (a,b) *satisfies*

$$\begin{cases} a = R_{M,\lambda}^{H_1(\cdot,\cdot)}[H_1(A_1(a),B_1(a)) - \lambda F(a,b)], \\ b = R_{N,\rho}^{H_2(\cdot,\cdot)}[H_2(A_2(b),B_2(b)) - \rho G(a,b)]; \end{cases}$$

(iii) *For some* $\lambda_0 > 0$ *and* $\rho_0 > 0$, (a,b) *satisfies*

$$\begin{cases} a = R_{M,\lambda_0}^{H_1(\cdot,\cdot)}[H_1(A_1(a),B_1(a)) - \lambda_0 F(a,b)], \\ b = R_{N,\rho_0}^{H_2(\cdot,\cdot)}[H_2(A_2(b),B_2(b)) - \rho_0 G(a,b)]. \end{cases}$$

Proof Defining the mappings $\widehat{H}_i : X_i \to X_i$ for $i = 1,2$ as $\widehat{H}_i(x_i) := H_i(A_i x_i, B_i x_i)$ for all $x_i \in X_i$, in the light of the assumptions it follows from Proposition 5.2(i) that the operators $\widehat{H}_i$ ($i = 1,2$) are strictly accretive. At the same time, invoking Remark 5.4, we infer that M and N are $\widehat{H}_1$-accretive and $\widehat{H}_2$-accretive operators, respectively, and so the resolvent operators $R_{M,\lambda}^{H_1(\cdot,\cdot)}$ and $R_{N,\rho}^{H_2(\cdot,\cdot)}$ become actually the same resolvent operators $R_{M,\lambda}^{\widehat{H}_1}$ and $R_{N,\rho}^{\widehat{H}_2}$, respectively. Now, we note that all the conditions of Lemma 3.1 are satisfied. Hence, Lemma 3.1 ensures that the following statements are equivalent:

(i) $(a,b) \in X_1 \times X_2$ is a solution of the SVI (3.1);

(ii) For any $\lambda, \rho > 0$, (a,b) satisfies

$$\begin{cases} a = R_{M,\lambda}^{\widehat{H}_1}[\widehat{H}_1(a) - \lambda F(a,b)] = R_{M,\lambda}^{H_1(\cdot,\cdot)}[H_1(A_1(a),B_1(a)) - \lambda F(a,b)], \\ b = R_{N,\rho}^{\widehat{H}_2} R_{N,\rho}^{\widehat{H}_2}[\widehat{H}_2(b) - \rho G(a,b)] = R_{N,\rho}^{H_2(\cdot,\cdot)}[H_2(A_2(b),B_2(b)) - \rho G(a,b)]; \end{cases}$$

(iii) For some $\lambda_0 > 0$ and $\rho_0 > 0$, (a,b) satisfies

$$\begin{cases} a = R_{M,\lambda_0}^{\widehat{H}_1}[\widehat{H}_1(a) - \lambda_0 F(a,b)] = R_{M,\lambda_0}^{H_1(\cdot,\cdot)}[H_1(A_1(a),B_1(a)) - \lambda_0 F(a,b)], \\ b = R_{N,\rho_0}^{\widehat{H}_2} R_{N,\rho_0}^{\widehat{H}_2}[\widehat{H}_2(b) - \rho_0 G(a,b)] = R_{N,\rho_0}^{H_2(\cdot,\cdot)}[H_2(A_2(b),B_2(b)) - \rho_0 G(a,b)]. \end{cases}$$

This completes the proof. □

Taking into account the above-mentioned argument, it is significant to emphasize that contrary to the claim of the authors in [26], Lemma 5.13 (that is, [26, Lemma 3.1]) gives actually a characterization of the solution of the SVI (3.1) involving an $\widehat{H}_1$-accretive mapping M and an $\widehat{H}_2$-accretive mapping N not the SVI (3.1) involving $H_1(\cdot,\cdot)$-accretive and $H_2(\cdot,\cdot)$-accretive mappings M and N (that is, [26, the problem (3.1)]). Meanwhile, it should be remarked that throughout Sect. 3 of [26], the spaces X_i ($i = 1,2$) are assumed to be real Banach spaces such that for each $i \in \{1,2\}$, the normalized duality mapping $J_i : X_i \rightrightarrows X_i^*$ is single-valued. It is known that, in general, J_i ($i = 1,2$) is single-valued if and only if X_i is smooth. Hence, in the following, we may assume that X_i ($i = 1,2$) are real smooth Banach spaces, as we have assumed in the context of Lemma 5.13.

Under some suitable conditions, Tang and Wang [26] proved the existence of a unique solution for [26, problem (3.1)] (that is, the SVI (3.1) involving an $H_1(\cdot,\cdot)$-accretive mapping M and an $H_2(\cdot,\cdot)$-accretive mapping N) as follows.

Theorem 5.14 ([26, Theorem 3.1]) *Let $X_1, X_2, A_1, B_1, A_2, B_2, H_1, H_2, M, N$ be the same as in Lemma 5.13. Furthermore, assume that $H_1(A_1, B_1)$ is r_1, r_2-Lipschitz continuous with respect to A_1, B_1, respectively, $H_2(A_2, B_2)$ is k_1, k_2-Lipschitz continuous with respect to A_2, B_2, respectively, $F : X_1 \times X_2 \to X_1$ is τ_1-Lipschitz continuous with respect to its first argument and τ_2-Lipschitz continuous with respect to its second argument, and $G : X_1 \times X_2 \to X_2$ is θ_1-Lipschitz continuous with respect to its first argument and θ_2-Lipschitz continuous with respect to its second argument. If the following inequalities hold*:

$$\frac{r_1 + r_2}{\alpha_1 + \beta_1} < 1 \quad \textit{and} \quad \frac{k_1 + k_2}{\alpha_2 + \beta_2} < 1,$$

then the SVI (3.1) (*with an $H_1(\cdot,\cdot)$-accretive mapping M and an $H_2(\cdot,\cdot)$-accretive mapping N, that is,* [26, *the problem* (3.1)]) *admits a unique solution.*

Proof Let us define for $i = 1, 2$, the mapping $\widehat{H}_i : X_i \to X_i$ by $\widehat{H}_i(x_i) = H_i(A_i x_i, B_i x_i)$ for all $x_i \in X_i$. Since $H_1(A_1, B_1)$ is α_1, β_1-generalized accretive with respect to A_1, B_1, respectively, with $\alpha_1 + \beta_1 > 0$, and r_1, r_2-Lipschitz continuous, from parts (i) and (ii) of Proposition 5.2 we conclude that $\widehat{H}_1$ is $(\alpha_1 + \beta_1)$-strongly accretive and $(r_1 + r_2)$-Lipschitz continuous. By an argument analogous to the previous one, from the assumptions and Proposition 5.2 it follows that the operator $\widehat{H}_2$ is $(\alpha_2 + \beta_2)$-strongly accretive and $(k_1 + k_2)$-Lipschitz continuous. Furthermore, thanks to Remark 5.4 we deduce that M and N are $\widehat{H}_1$-accretive and $\widehat{H}_2$-accretive mappings, respectively. Then, the problem (3.1) in [26] involving an $H_1(\cdot,\cdot)$-accretive mapping M and an $H_2(\cdot,\cdot)$-accretive mapping N coincides exactly with the SVI (3.1) involving an $\widehat{H}_1$-accretive mapping M and an $\widehat{H}_2$-accretive mapping N. Taking $\varrho_i = \alpha_i + \beta_i$ $(i = 1, 2)$, $r = r_1 + r_2$ and $k = k_1 + k_2$, we have $\frac{r}{\varrho_1} = \frac{r_1 + r_2}{\alpha_1 + \beta_1} < 1$ and $\frac{k}{\varrho_2} = \frac{k_1 + k_2}{\alpha_2 + \beta_2} < 1$. We now note that all the conditions of Theorem 3.3 are satisfied and so in accordance with Theorem 3.3, the SVI (3.1) ([26, the problem (3.1)]) admits a unique solution. This completes the proof. □

Based on Lemma 5.13 and by assuming that for all $i \geq 0$, M_i is an $H_1(\cdot,\cdot)$-accretive mapping with respect to A_1 and B_1, and N_i is an $H_2(\cdot,\cdot)$-accretive mapping with respect to A_2 and B_2, Tang and Wang [26] constructed the following iterative algorithm for finding an approximate solution of the SVI (3.1) involving an $H_1(\cdot,\cdot)$-accretive mapping M and an $H_2(\cdot,\cdot)$-accretive mapping N (that is, [26, the problem (3.1)]).

Algorithm 5.15 ([26, Algorithm 3.1]) Step 0: Choose some $\lambda_0 > 0$ and $\rho_0 > 0$ to satisfy the two inequalities presented in (3.16) (of [26]). Select an initial point $(a_0, b_0) \in X_1 \times X_2$. Set $i := 0$.

Step i: Given $(a_i, b_i) \in X_1 \times X_2$, compute $(a_{i+1}, b_{i+1}) \in X_1 \times X_2$ by

$$\begin{aligned} a_{i+1} &= \alpha_i a_i + (1 - \alpha_i) R^{H_1(\cdot,\cdot)}_{M_i,\lambda_0}\bigl[H_1\bigl(A_1(a_i), B_1(a_i)\bigr) - \lambda_0 F(a_i, b_i)\bigr], \\ b_{i+1} &= \alpha_i b_i + (1 - \alpha_i) R^{H_2(\cdot,\cdot)}_{N_i,\rho_0}\bigl[H_2\bigl(A_2(b_i), B_2(b_i)\bigr) - \rho_0 G(a_i, b_i)\bigr], \end{aligned}$$

for $i = 0, 1, 2, \dots$, where $0 \leq \alpha_i < 1$ with $\limsup_i \alpha_i < 1$.

It is also remarkable that by defining the mapping $\widehat{H}_i : X_i \to X_i$ for $i = 1, 2$ by $\widehat{H}_i(x_i) := H_i(A_i x_i, B_i x_i)$ for all $x_i \in X_i$, with the help of the assumptions and utilizing Proposition 5.2(i) we infer that the operators $\widehat{H}_i$ $(i = 1, 2)$ are strictly accretive. In the light of

Remark 5.4 we also conclude that M and N are $\widehat{H}_1$-accretive and $\widehat{H}_2$-accretive operators, respectively. Meanwhile, the resolvent operators $R^{H_1(\cdot,\cdot)}_{M_i,\lambda_0}$ and $R^{H_2(\cdot,\cdot)}_{N_i,\rho_0}$ ($i \geq 0$) become actually the same resolvent operators $R^{\widehat{H}_1}_{M_i,\lambda_0}$ and $R^{\widehat{H}_2}_{N_i,\rho_0}$, respectively. Then, for each $i \geq 0$, it yields

$$\begin{cases} a_{i+1} = \alpha_i a_i + (1-\alpha_i) R^{H_1(\cdot,\cdot)}_{M_i,\lambda_0}[H_1(A_1(a_i), B_1(a_i)) - \lambda_0 F(a_i, b_i)] \\ \qquad = \alpha_i a_i + (1-\alpha_i) R^{\widehat{H}_1}_{M_i,\lambda_0}[\widehat{H}_1(a_i) - \lambda_0 F(a_i, b_i)], \\ b_{i+1} = \alpha_i b_i + (1-\alpha_i) R^{H_2(\cdot,\cdot)}_{N_i,\rho_0}[H_2(A_2(b_i), B_2(b_i)) - \rho_0 G(a_i, b_i)] \\ \qquad = \alpha_i b_i + (1-\alpha_i) R^{\widehat{H}_2}_{N_i,\rho_0}[\widehat{H}_2(b_i) - \rho_0 G(a_i, b_i)]. \end{cases}$$

Thereby, we find that Algorithm 5.15 actually becomes the same Algorithm 3.14 and is not a new one.

Finally, Tang and Wang [26] closed their paper with the most important result that appeared in it related to the strong convergence of the iterative sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ generated by Algorithm 5.15 to the unique solution of the SVI (3.1) involving an $H_1(\cdot,\cdot)$-accretive mapping M and an $H_2(\cdot,\cdot)$-accretive mapping N (that is, [26, the problem (3.1)]).

Theorem 5.16 ([26, Theorem 3.2]) *Let $X_1, X_2, A_1, B_1, A_2, B_2, H_1, H_2, M, N, F, G$ be the same as in Theorem 5.14 (that is, [26, Theorem 3.1]). Assume that the following inequalities hold*:

$$\frac{r_1 + r_2}{\alpha_1 + \beta_1} < 1 \quad \textit{and} \quad \frac{k_1 + k_2}{\alpha_2 + \beta_2} < 1.$$

Furthermore, let $M_i : X_1 \rightrightarrows X_1$ ($i = 0, 1, 2, \ldots$) be $H_1(\cdot,\cdot)$-accretive set-valued mappings such that $M_i \xrightarrow{G} M$ and $N_i : X_2 \rightrightarrows X_2$ be $H_2(\cdot,\cdot)$-accretive set-valued mappings such that $N_i \xrightarrow{G} N$. Then, the sequence generated by Algorithm 5.15 (that is, [26, Algorithm 3.1]) converges strongly to the unique solution of the SVI (3.1) (involving an $H_1(\cdot,\cdot)$-accretive mapping M and an $H_2(\cdot,\cdot)$-accretive mapping N, that is, [26, the problem (3.1)]).

Proof Define for $i = 1, 2$, the operators $\widehat{H}_i : X_i \to X_i$ by $\widehat{H}_i(x_i) := H_i(A_i x_i, B_i x_i)$ for all $x_i \in X_i$. In light of the assumptions and by the same arguments used in Theorem 5.14, we conclude that $\widehat{H}_1$ is an $(\alpha_1 + \beta_1)$-strongly accretive and $(r_1 + r_2)$-Lipschitz continuous mapping, $\widehat{H}_2$ is an $(\alpha_2 + \beta_2)$-strongly accretive and $(k_1 + k_2)$-Lipschitz continuous mapping, for all $i \geq 0$, the mappings M_i and N_i are $\widehat{H}_1$-accretive and $\widehat{H}_2$-accretive, respectively, and the problem (3.1) in [26] involving an $H_1(\cdot,\cdot)$-accretive mapping M and an $H_2(\cdot,\cdot)$-accretive mapping N coincides exactly with the SVI (3.1) involving an $\widehat{H}_1$-accretive mapping M and an $\widehat{H}_2$-accretive mapping N. At the same time, Algorithm 5.15 becomes actually the same Algorithm 3.14. Taking $\varrho_i = \alpha_i + \beta_i$ ($i = 1, 2$), $r = r_1 + r_2$ and $k = k_1 + k_2$, we obtain $\frac{r}{\varrho_1} = \frac{r_1 + r_2}{\alpha_1 + \beta_1} < 1$ and $\frac{k}{\varrho_2} = \frac{k_1 + k_2}{\alpha_2 + \beta_2} < 1$. In view of the fact that all the conditions of Corollary 4.9 are satisfied, Corollary 4.9 ensures that the iterative sequence $\{(a_i, b_i)\}_{i=0}^{\infty}$ generated by Algorithm 5.15 converges strongly to the unique solution of the problem (3.1) in [26]. This completes the proof. □

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Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
JB was a major contributor in writing the manuscript. JCY performed the validation and formal analysis. All authors read and approved the final manuscript.

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