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The well-posedness problem of an anisotropic porous medium equation with a convection term

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Abstract

The initial boundary value problem of an anisotropic porous medium equation is considered in this paper. The existence of a weak solution is proved by the monotone convergent method. By showing that $\nabla u \in L^\infty(0,T;L^2_{\mathrm{loc}}(\Omega))$, according to different boundary value conditions, some stability theorems of weak solutions are obtained. The unusual thing is that the partial boundary value condition is based on a submanifold Σ of $\partial\Omega \times (0,T)$ and, in some special cases, $\Sigma = \{(x,t) \in \partial\Omega \times (0,T) : \prod a_i(x,t) > 0\}$.

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Keywords: Anisotropic porous medium equation; Stability theorem; Partial boundary condition; Submanifold

1 Introduction

The well-posedness and regularity of weak solutions to the porous medium equation

$$u_t = \Delta u^m \tag{1.1}$$

or

$$\beta(u)_t = \Delta u, \qquad \beta(u) = |u|^{\frac{1}{m}} \operatorname{sign} u \tag{1.2}$$

were addressed from the sixties to eighties in the twentieth century by many mathematicians, one can refer to [3, 6, 17, 18, 21, 22] and the references therein. Later, DiBenedetto [2] and Ziemer [28] studied the regularity to the more general equation

$$\beta(u)_t = \nabla \cdot \vec{a}(x,t,u,\nabla u) + b(x,t,u,\nabla u), \tag{1.3}$$

considering suitable assumptions on $\vec{a}$ and b. The proofs followed different approaches: DiBenedetto's proof was based on a parabolic version of De Giorgi's technique, while

Ziemer's approach was related to Moser's iteration technique. But, since many reaction-diffusion processes depend on different environments, one should consider a reaction-diffusion equation with anisotropic characteristic, then the anisotropic porous medium equation modeled by

$$u_t = \sum_{i=1}^{N} \left(u^{m_i}\right)_{x_i}, \quad (x,t) \in Q_T, \tag{1.4}$$

was introduced and studied since 1980s. Actually, Song [19, 20] studied the existence and uniqueness of the very weak solution of the anisotropic porous medium equation with singular advections and absorptions. Henriques [7] established an interior regularity result for the solutions of (1.4). Li [11] developed the finite element method to derive a special analytical solution for anisotropic porous medium equation for time-independent diffusion. Also, several applied models related to an anisotropic porous medium have been introduced recently. The first one is the flow diverter model. Since the explicit modeling of thin wires of simulation of flow diverter (FD) imposes extremely high demand of computational resources and time, such a fact limits its use in time-sensitive presurgical planning. One alternative approach is to model as a homogenous porous medium, which saves time but with compromise in accuracy. Then, Ou et al. [13] proposed a new method to model FD as a heterogeneous and anisotropic porous medium whose properties were determined from local porosity. The second one is a multiple-relaxation-time lattice Boltzmann model for the flow and heat transfer in a hydrodynamically and thermally anisotropic porous medium [8]. The third one arises from computational fluid dynamics (CFD). Doumbia et al. [5] gave a CFD modelling of an animal occupied zone using an anisotropic porous medium model with velocity-dependent resistance parameters. Another model comes from the physical characteristics of cracked rocks. By testing elastic velocities and Thomsen parameters—as a function of crack density for fixed values of aspect ratio—predicted by the model with data acquired from synthetic rock samples, Nascimento et al. [12] introduced a new ultrasonic physical model in an anisotropic porous cracked medium.

Moreover, in the theory of PDE, the anisotropic equation has provoked more people's attention in recent time. For example, the existence and multiplicity of nontrivial solutions to the anisotropic elliptic equation

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}\right) = f(x,u), \quad x \in \Omega, \tag{1.5}$$

has been an active topic in recent years (see [4, 15, 16], etc.), while the anisotropic parabolic equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\right) + f(x,t,u), \quad (x,t) \in Q_T, \tag{1.6}$$

was studied in [1, 14], etc.

In this paper, we consider the well-posedness of weak solutions to the following initial boundary value problem:

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\big(a_i(x,t)|u|^{\alpha_i} u_{x_i}\big) + \sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{1.7}$$

with

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.8}$$

and

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{1.9}$$

Compared with equation (1.1), we call equation (1.7) an anisotropic medium equation with a convection term. Apart from the anisotropic characteristic of equation (1.7), we are concerned with whether the homogeneous boundary value condition (1.9) is overdetermined or not. In our previous work [27], we made the usual exploration on the following porous medium equation:

$$u_t = \operatorname{div}\big(a(x)\nabla u^m\big) + \sum_{i=1}^{N} \frac{\partial b_i(u^m)}{\partial x_i}, \quad (x,t) \in Q_T. \tag{1.10}$$

We found that if one wants to prove the uniqueness (or the stability) of weak solutions to this equation, the homogeneous boundary value condition (1.9) can be replaced by that $a(x) = 0, x \in \partial\Omega$. Even much earlier, Yin and Wang [23, 24] studied the following equation:

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) - f_i(x)D_i u + c(x,t)u = g(x,t), \quad (x,t) \in Q_T, \tag{1.11}$$

divided the boundary value condition into three parts, and in particular they showed that if $a(x) = 0, f_i(x) = 0$ when $x \in \partial\Omega$, then the uniqueness of a weak solution to equation (1.11) can be proved independent of the boundary value condition (1.9). The optimal boundary value condition matching up with equation (1.11) was studied by the author recently in [26].

Instead of $a(x)|_{x\in\partial\Omega} = 0$ in [27], we only assume that $a_i(x,t) > 0$, $(x,t) \in \Omega \times (0,T)$ and do not emphasize that

$$a_i(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], i = 1, 2, \dots, N.$$

So, for any given $t \in [0,T]$, both

$$\big\{x \in \partial\Omega : a_i(x,t) = 0\big\}$$

and

$$\big\{x \in \partial\Omega : a_i(x,t) > 0\big\}$$

may have a positive $(N-1)$-dimensional Hausdorff measure in $\partial\Omega$. Naturally, based on past experience [23, 24, 27], we guess that a partial boundary value condition

$$u(x,t)=0, \quad (x,t)\in\Sigma\subseteq\partial\Omega\times(0,T), \tag{1.12}$$

is enough to ensure the well-posedness of weak solutions to equation (1.7). The further work is to specify the explicit expression of Σ in (1.12). Different from other related references [23, 24, 27] in which Σ is just a cylinder, we found that Σ appearing in (1.12) is a submanifold of $\partial\Omega\times(0,T)$ and, in some special cases, $\Sigma=\{(x,t)\in\partial\Omega\times(0,T):\prod a_i(x,t)>0\}$.

Actually, compared with [7, 19, 20], the degeneracy of diffusion coefficient $a_i(x,t)$ has brought more essential difficulties. For example, maybe it is not difficult to construct the fundamental solution of equation (1.4) by Barenblatt's method, but it is impossible to construct the corresponding fundamental solution of the simplest anisotropic porous medium equation

$$u_t=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\bigl(a_i(x,t)|u|^{\alpha_i}u_{x_i}\bigr) \tag{1.13}$$

by a similar method. However, the main aim of this paper is to study the well-posedness of weak solutions to equation (1.7), and we pay no attention to the fundamental solution for the time being. The local integrability $\nabla u\in L^\infty(0,T;L^2_{\mathrm{loc}}(\Omega))$, which was found for the first time in this paper, acts as an important role to overcome the above difficulties.

The remainder of this paper is structured as follows. In Sect. 2, we present the definition of weak solution and the main results. In Sect. 3, the existence of a weak solution is proved. In Sect. 4, the stability of a weak solution to the usual initial boundary value problem is studied. In Sect. 5, the local integrability of ∇u is found and the uniqueness of a weak solution to the usual initial boundary value problem is obtained. In Sect. 6, when $\prod_{i=1}^N a_i(x,t)|_{x\in\partial\Omega}=0$, the stability of a weak solution based on a partial boundary value condition is proved. In Sect. 7, we prove the stability of weak solutions under the general condition $\prod_{i=1}^N a_i(x,t)\geq 0$.

2 The definition of the weak solution and the main results

The definition of weak solution and the main results of this paper are listed below.

Definition 2.1 A function $u(x,t)$ is said to be a weak solution of equation (1.7) if

$$\begin{aligned} &u\in L^\infty(Q_T), \qquad u_t\in L^1(Q_T),\\ &a_i(x,t)\bigl|u_{x_i}^{\frac{\alpha_i}{2}+1}\bigr|^2\in L^\infty\bigl(0,T;L^1(\Omega)\bigr), \quad i=1,2,\ldots,N, \end{aligned} \tag{2.1}$$

and for any function $\varphi\in C_0^1(Q_T)$, there holds

$$\iint_{Q_T}\left(-\frac{\partial\varphi}{\partial t}u+\sum_{i=1}^N a_i(x,t)u^{\alpha_i}u_{x_i}\varphi_{x_i}\right)dx\,dt+\sum_{i=1}^N\iint_{Q_T}b_i(u,x,t)\varphi_{x_i}(x,t)\,dx\,dt=0. \tag{2.2}$$

The initial value condition is satisfied in the sense of that

$$\lim_{t\to 0}\int_\Omega\bigl(u(x,t)-u_0(x)\bigr)\phi(x)\,dx=0, \tag{2.3}$$

where $\phi(x) \in C_0^\infty(\Omega)$. The boundary value condition (1.9) or the partial boundary value condition (1.12) is satisfied in the sense of trace.

Theorem 2.2 *If $\alpha_i > 0$, $b_i(s,x,t)$ is a C^1 function and $|\frac{\partial}{\partial x_i} b_i(s,x,t)| \leq c(M)$ when $|s| \leq M+1$, $u_0(x)$ satisfies*

$$u_0(x) \in L^\infty(\Omega), \qquad a_i(x,0)u_0^{\alpha_i}|u_{0x_i}| \in L^1(\Omega), \quad i = 1,2,\dots,N, \tag{2.4}$$

$a_i(x,t) \geq 0$ satisfies

$$\iint_{Q_T} \frac{\partial \sqrt{a_i}}{\partial x_i}\,dx\,dt \leq c, \quad i = 1,2,\dots,N, \tag{2.5}$$

then equation (1.7) with initial boundary values (1.8)–(1.9) has a nonnegative solution. Here and the after, M is a constant such that $\|u_0(x)\|_{L^\infty(\Omega)} \leq M$.

From Theorem 2.3 to Theorem 2.6, we all assume that $a_i(x,t) > 0$, $x \in \Omega$ and denote that

$$\alpha^+ = \max\{\alpha_1, \alpha_2, \dots, \alpha_N\}, \qquad \alpha^- = \max\{\alpha_1, \alpha_2, \dots, \alpha_N\}.$$

Theorem 2.3 *Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.7) with the initial value $u_0(x)$, $v_0(x)$ respectively, and with the boundary value condition (1.9). If $\alpha_i \geq 0$, and there is a constant $\alpha > \frac{1}{2}(\alpha^+ + 2)$ such that*

$$\bigl|b_i(u,x,t) - b_i(v,x,t)\bigr| \leq c a_i(x,t)^{\frac{1}{2}}|u-v|^\alpha, \quad i = 1,2,\dots,N, \tag{2.6}$$

then the solution of equation (1.7) is unique.

Theorem 2.4 *Let $u(x,t)$ and $v(x,t)$ be two nonnegative solutions of equation (1.7) with the initial value $u_0(x)$, $v_0(x)$ respectively, with the same boundary value condition (1.9). If $\alpha_i \geq 1$,*

$$\int_\Omega a_i(x,t)v^{\alpha_i-1}|v_{x_i}|^2 \leq c, \qquad \int_\Omega a_i(x,t)u^{\alpha_i-1}|u_{x_i}|^2 \leq c, \quad i = 1,2,\dots,N, \tag{2.7}$$

$$\bigl|b_i(u,x,t) - b_i(v,x,t)\bigr| \leq c a_i(x,t)^{\frac{1}{2}}|u-v|^2, \quad i = 1,2,\dots,N, \tag{2.8}$$

then

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr| \leq c\int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\,dx. \tag{2.9}$$

Theorem 2.4 implies that we only can show that the stability of weak solutions is true for a kind of solutions which satisfy (2.7). The following stability theorems are established on a partial boundary value condition.

Theorem 2.5 *Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.7) satisfying*

$$\frac{1}{\lambda}\int_{\Omega_{\lambda t}\setminus\Omega_{2\lambda t}} a_i(x,t)|u|^{\alpha_i}|u_{x_i}|^2\,dx < C(T), \tag{2.10}$$

$$\frac{1}{\lambda}\int_{\Omega_{\lambda t}\setminus\Omega_{2\lambda t}} a_i(x,t)|v|^{\alpha_i}|v_{x_i}|^2\,dx < C(T), \tag{2.11}$$

with the initial value $u_0(x)$, $v_0(x)$ respectively, and with a partial boundary value condition

$$v(x,t) = u(x,t) = 0, \quad (x,t) \in \Sigma. \tag{2.12}$$

If $\alpha^- \geq 1$, $b_i(\cdot,x,t)$ satisfies

$$\bigl|b_i(u,x,t) - b_i(v,x,t)\bigr| \leq c\sqrt{a_i(x,t)}|u-v|, \quad i = 1,2,\ldots,N, \tag{2.13}$$

then

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr| \leq c\int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\,dx. \tag{2.14}$$

Here,

$$\Sigma = \left\{(x,t) \in \partial\Omega \times (0,T) : \sum_{i=1}^N \sqrt{a_i(x,t)}\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i} \neq 0\right\} \tag{2.15}$$

and $\Omega_{\lambda t} = \{x \in \Omega : \prod_{i=1}^N a_i(x,t) > \lambda\}$.

Theorem 2.5 is based on the fact that we can show the first order partial derivative to the solution u is with the local integrability

$$u_{x_i} \in L^\infty\bigl(0,T;L^2_{\mathrm{loc}}(\Omega)\bigr), \quad i = 1,2,\ldots,N. \tag{2.16}$$

The weakness of Theorem 2.5 is that the expression of Σ, (2.15) seems too complicated. By choosing another test function, we can prove another stability theorem based on a simpler partial boundary value condition.

Theorem 2.6 *Suppose $\alpha^- \geq 1$,*

$$\prod_{j=1}^N a_j(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{2.17}$$

Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.7) with the initial value $u_0(x)$, $v_0(x)$ respectively, but without the boundary value condition. If (2.13) is true and

$$\sum_{i=1}^N \int_{\Omega_{\lambda t}\setminus\Omega_{2\lambda t}} a_i(x,t)\left|\sum_{k=1}^N \frac{a_{kx_i}}{a_k}\right|^2 dx \leq c, \tag{2.18}$$

then

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr| \leq c\int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\,dx.$$

We find that the partial boundary value conditions with (2.12) are submanifold of $\partial\Omega \times (0,T)$, while in the previous works the corresponding partial boundary value conditions are the cylinder domains $\Sigma_1 \times (0,T)$, where $\Sigma_1 \subseteq \partial\Omega$ is a relatively open subset [9, 10, 23, 25, 27], etc.

Last but not least, once the well-posedness problem has been solved, we can consider the extinction, blow-up phenomena, the positivity, and the large time behavior of the weak solutions of anisotropic porous medium equation (1.7) in the future. However, different from the porous medium equation, because of the anisotropy, these problems are not so easy to be solved, the methods used in the usual porous medium equation (1.1) cannot be extended to the anisotropic porous medium equation (1.13) directly.

3 The existence of weak solution

Proof of Theorem 2.2 We consider the following normalized problem:

$$u_{nt} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\big(a_{in}(u_n,x,t)\nabla u_n\big) + \sum_{i=1}^{N} \frac{\partial b_i(u_n,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{3.1}$$
$$u_n(x,t) = \frac{1}{n}, \quad (x,t) \in \partial\Omega \times (0,T),$$
$$u_n(x,0) = u_{0n}(x) = u_0(x) + \frac{1}{n}, \quad x \in \Omega,$$

where $a_{in}(u,x,t) \geq c(n) > 0$, and

$$a_{in}(u_n,x,t) = \left(a_i(x,t) + \frac{1}{n}\right)u^{\alpha_i} \quad \text{if } u \in \left[\frac{1}{n}, M + \frac{1}{n}\right]. \tag{3.2}$$

Similar to the porous medium equation (1.1), we can show that problem (3.1) has a nonnegative solution u_n, which is called as a viscous solution generally and satisfies

$$u_n \in L^\infty(Q_T), \qquad u_{nt} \in L^2(Q_T), \qquad u_{nx_i} \in L^2(Q_T), \quad i = 1,2,\dots,N, \tag{3.3}$$

and by comparison theorem, we have

$$u_{n+1}(x,t) \leq u_n(x,t) \leq M + 1.$$

Thus

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \tag{3.4}$$

is well defined. Now, we will prove u is a weak solution of (1.7).

First, multiplying both sides of the first equation in (3.1) by $\phi = u_n - \frac{1}{n}$, denoting that $Q_t = \Omega \times (0,t)$ for $t \in (0,T)$, then

$$\begin{aligned} &\iint_{Q_t} u_{nt}\left(u_n - \frac{1}{n}\right)dx\,dt + \sum_{i=1}^{N} \iint_{Q_t} \left(a_i(x,t) + \frac{1}{n}\right)|u_n|^{\alpha_i}|u_{nx_i}|^2\,dx\,dt \\ &\quad = \iint_{Q_t} \iint_{Q_t} \frac{\partial b_i(u_n,x,t)}{\partial x_i}\left(u_n - \frac{1}{n}\right)dx\,dt. \end{aligned} \tag{3.5}$$

Since

$$\left|\frac{\partial}{\partial x_i}b_i(u_n,x,t)\right| \le c(M), \quad i=1,2,\dots,N,$$

we have

$$\begin{aligned}
&\left|\iint_{Q_t}\frac{\partial b_i(u_n,x,t)}{\partial x_i}\left(u_n-\frac{1}{n}\right)dx\,dt\right| \\
&\quad = \left|-\iint_{Q_t} b_i(u_n,x,t)\frac{\partial}{\partial x_i}\left(u_n-\frac{1}{n}\right)dx\,dt\right| \\
&\quad = \left|-\iint_{Q_t}\frac{\partial}{\partial x_i}\int_{\frac{1}{n}}^{u_n} b_i(s,x,t)\,ds\,dx\,dt + \iint_{Q_T}\int_{\frac{1}{n}}^{u_n}\frac{\partial}{\partial x_i}b_i(s,x,t)\,ds\,dx\,dt\right| \\
&\quad = \left|\iint_{Q_t}\int_{\frac{1}{n}}^{u_n}\frac{\partial}{\partial x_i}b_i(s,x,t)\,ds\,dx\,dt\right| \\
&\quad \le \iint_{Q_t}\left|\int_{\frac{1}{n}}^{u_n}\frac{\partial}{\partial x_i}b_i(s,x,t)\,ds\right|dx\,dt \\
&\quad \le c(M,T).
\end{aligned}$$

Thus, from (3.5), we can find that

$$\begin{aligned}
&\sum_{i=1}^{N}\iint_{Q_t}\left(a_i(x,t)+\frac{1}{n}\right)|u_n|^{\alpha_i}|u_{nx_i}|^2\,dx\,dt \\
&\quad = \int_{\Omega}\left(u_{0n}(x)-\frac{1}{n}\right)u_{0n}(x) - \int_{\Omega}u_n(x,T)\left(u_n(x,t)-\frac{1}{n}\right)dx \\
&\qquad + \iint_{Q_t}\frac{\partial b_i(u_n,x,t)}{\partial x_i}\left(u_n-\frac{1}{n}\right)dx\,dt \\
&\quad \le c,
\end{aligned}$$

which implies

$$\left(a_i(x,t)+\frac{1}{n}\right)|u_n|^{\alpha_i}|u_{nx_i}|^2\Big|_{L^1(Q_T)} \le c, \quad i=1,2,\dots,N. \tag{3.6}$$

By choosing a subsequence, we may assume that

$$\left(a_i(x,t)+\frac{1}{n}\right)^{\frac{1}{2}}\left(\frac{\alpha_i}{2}+1\right)^{-1}u_{nx_i}^{\frac{\alpha_i}{2}+1} \rightharpoonup \zeta_i, \tag{3.7}$$

weakly in $L^2(Q_T)$.

In the second step, we want to show that

$$\zeta_i = a_i(x,t)^{\frac{1}{2}}\left(\frac{\alpha_i}{2}+1\right)^{-1}u_{x_i}^{\frac{\alpha_i}{2}+1}. \tag{3.8}$$

For any $\forall\psi \in C_0^1(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_T} \left(a_i(x,t)+\frac{1}{n}\right)^{\frac{1}{2}} \left(\frac{\alpha_i}{2}+1\right)^{-1} u_{nx_i}^{\frac{\alpha_i}{2}+1} \cdot \psi \,dx\,dt \\
&= \left(\frac{\alpha_i}{2}+1\right)^{-1} \left[\iint_{Q_T} \frac{\partial}{\partial x_i}\left(\left(a(x,t)+\frac{1}{n}\right)^{\frac{1}{2}} u_n^{\frac{\alpha_i}{2}+1}\right)\psi \,dx\,dt \right.\\
&\quad \left. - \iint_{Q_T} \frac{\partial (a(x,t)+\frac{1}{n})^{\frac{1}{2}}}{\partial x_i} u_n^{\frac{\alpha_i}{2}+1} \psi \,dx\,dt\right] \\
&= \left(\frac{\alpha_i}{2}+1\right)^{-1} \left[-\iint_{Q_T} \left(a_i(x,t)+\frac{1}{n}\right)^{\frac{1}{2}} u_n^{\frac{\alpha_i}{2}+1} \psi_{x_i} \,dx\,dt \right.\\
&\quad \left. - \iint_{Q_T} \frac{\partial (a_i(x,t)+\frac{1}{n})^{\frac{1}{2}}}{\partial x_i} u_n^{\frac{\alpha_i}{2}+1} \psi \,dx\,dt\right].
\end{aligned} \tag{3.9}$$

Let $n \to \infty$. Then

$$\lim_{n\to\infty} \iint_{Q_T} \left(a_i(x,t)+\frac{1}{n}\right)^{\frac{1}{2}} \left(\frac{\alpha_i}{2}+1\right)^{-1} u_{nx_i}^{\frac{\alpha_i}{2}+1} \cdot \psi \,dx\,dt = \iint_{Q_T} \zeta\psi \,dx\,dt. \tag{3.10}$$

For the right-hand side of (3.9), by the assumption

$$\iint_{Q_T} \frac{\partial\sqrt{a}}{\partial x_i} \,dx\,dt \le c,$$

using the dominated convergent theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{\alpha_i}{2}+1\right)^{-1} \left[-\iint_{Q_T} \left(a_i(x,t)+\frac{1}{n}\right)^{\frac{1}{2}} u_n^{\frac{\alpha_i}{2}+1} \psi_{x_i} \,dx\,dt \right.\\
&\quad \left. - \iint_{Q_T} \frac{\partial (a_i(x,t)+\frac{1}{n})^{\frac{1}{2}}}{\partial x_i} u_n^{\frac{\alpha_i}{2}+1} \psi \,dx\,dt\right] \\
&= \left(\frac{\alpha_i}{2}+1\right)^{-1} \left[-\iint_{Q_T} a_i(x,t)^{\frac{1}{2}} u^{\frac{\alpha_i}{2}+1} \psi_{x_i} \,dx\,dt - \iint_{Q_T} \frac{\partial a_i(x,t)^{\frac{1}{2}}}{\partial x_i} u^{\frac{\alpha_i}{2}+1} \psi \,dx\,dt\right].
\end{aligned} \tag{3.11}$$

From (3.9)–(3.11), we obtain (3.8).

In the third step, since $b_i \in C^1$, by (3.4), we have

$$\lim_{n\to\infty} b_i(u_n,x,t) = b_i(u,x,t). \tag{3.12}$$

Moreover, by a BV estimate method [9, 10], we can show that

$$\iint_{Q_T} \left|\frac{\partial u_n}{\partial t}\right| \le c \tag{3.13}$$

and

$$\iint_{Q_T} \left|\frac{\partial u_n}{\partial x_i}\right| \le c, \quad i = 1,2,\dots,N. \tag{3.14}$$

Then $u_t \in L^1(Q_T)$ and (2.1) is true, and we can define the trace of u on the boundary $\partial\Omega$. The initial value condition true in the sense of (2.3) can be found in [22] etc.

Thus, u is a solution of equation (1.7) with the initial value (1.8) and the homogeneous boundary value condition (1.9). Theorem 2.2 is proved. □

Finally, we would like to point out that, though the viscous solution u_n satisfies (3.3), we cannot deduce that the solution u of equation (1.7) satisfies

$$u_t \in L^2(Q_T), \qquad |u_{x_i}| \in L^2(Q_T).$$

Actually, in the next section, we will show that

$$|u_{x_i}| \in L^2\big(0, T; L_{\mathrm{loc}}(\Omega)\big).$$

4 The uniqueness of weak solution of the usual initial boundary value problem

Proposition 4.1 *Let $u(x,t)$ be a solution of equation* (1.7). *Then*

$$\nabla u \in L^\infty\big(0, T; L^2_{\mathrm{loc}}(\Omega)\big). \tag{4.1}$$

Proof Let u_n be the viscous solution of the initial boundary value (3.1)–(3.3). If we choose $(u_n - u)\phi$ as the test function, where $\phi \in C_0^1(Q_T)$, then

$$\begin{aligned} &\int_0^T \int_\Omega (u_n - u)\phi \frac{\partial u_n}{\partial t}\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega a(x,t)\phi(x)|u_n|^{\alpha_i} u_{nx_i}(u_n - u)_{x_i}\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega a(x,t)|u_n|^{\alpha_i}(u_n - u)u_{nx_i}\phi_{x_i}\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega b_i(u_n,x,t)\phi(u_n - u)_{x_i}\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega b_i(u_n,x,t)\phi_{x_i}(u_n - u)\big)\,dx\,dt \\ &= 0. \end{aligned} \tag{4.2}$$

Let $n \to \infty$ in (4.2). We can deduce that

$$\lim_{n\to\infty} \iint_{Q_T} a(x,t)\phi|u_n|^{\alpha_i} u_{nx_i}(u_n - u)_{x_i}\,dx\,dt = 0, \quad i = 1, 2, \dots, N,$$

and this equality yields

$$\begin{aligned}&\lim_{n\to\infty}\iint_{Q_T} a(x,t)\phi|u_n|^{\alpha_i}u_{nx_i}u_{x_i}\,dx\,dt\\&\quad=\lim_{n\to\infty}\iint_{Q_T} a(x,t)\phi|u_n|^{\alpha_i}|u_{nx_i}|^2\,dx\,dt\\&\quad\le c.\end{aligned}$$

Due to the arbitrariness of ϕ and $|u_n|^{\alpha_i}u_{nx_i}\in L^1(0,T;L^2_{\mathrm{loc}}(\Omega))$, there holds

$$u_{x_i}\in L^\infty\bigl(0,T;L^2_{\mathrm{loc}}(\Omega)\bigr),\quad i=1,2,\dots,N.$$ □

Theorem 4.2 *If there is β, $1>\beta>0$, and there is a nonnegative function $g_i(x,t)$ such that*

$$\bigl|b_i(u,x,t)-b_i(v,x,t)\bigr|\le cg_i(x,t)|u-v|^{\frac{2-\beta+\alpha_i}{2}},\quad i=1,2,\dots,N, \tag{4.3}$$

$$\iint_{Q_T} g_i(x,t)^2a_i(x,t)^{-1}\,dx\,dt\le c,\quad i=1,2,\dots,N, \tag{4.4}$$

then the nonnegative solution of equation (1.7) *is unique*.

Proof For a small positive constant $\delta>0$, denoting $D_\delta=\{x\in\Omega:w=u-v>\delta\}$, we suppose that the measure $\mu(D_\delta)>0$. Let

$$F_\lambda(\xi)=\begin{cases}\frac{1}{1-\beta}\lambda^{\beta-1}-\frac{1}{1-\beta}\xi^{\beta-1}, & \text{if } \xi>\lambda,\\ 0, & \text{if } \xi\le\lambda,\end{cases} \tag{4.5}$$

where $\delta>2\lambda>0$, $1>\beta>0$.

Now, by a process of limit, we can choose $F_\lambda(w)=F_\lambda(u-v)$ and integrate it over Q_t, $0\le t<T$, accordingly,

$$\begin{aligned}&\int_0^t\int_\Omega w_tF_\lambda(w)\,dx\,dt\\&\qquad+\sum_{i=1}^N\int_0^t\int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i}-v_{x_i})^2F_\lambda'(w)\,dx\,dt\\&\qquad+\sum_{i=1}^N\int_0^t\int_\Omega a_i(x,t)\bigl(|u|^{\alpha_i}-|v|^{\alpha_i}\bigr)v_{x_i}\bigr)(u_{x_i}-v_{x_i})F_\lambda'(w)\,dx\,dt\\&\qquad+\sum_{i=1}^N\int_0^t\int_\Omega\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr]\bigl(F_\lambda(w)\bigr)_{x_i}\,dx\,dt\\&\quad=0.\end{aligned} \tag{4.6}$$

In the first place,

$$\begin{aligned}&\int_0^t\int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i}-v_{x_i})^2F_\lambda'(w)\,dx\,dt\\&\quad\ge\int_0^t\int_\Omega a_i(x,t)|u|^{\alpha_i}(u-v)^{\beta-2}|w_{x_i}|^2\,dx\,dt.\end{aligned} \tag{4.7}$$

In the second place, by (4.3), (4.4), we have

$$
\begin{aligned}
&\int_0^t\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)v_{x_i}(u_{x_i}-v_{x_i})F_\lambda'(w)\,dx\,dt\\
&\quad=\int_0^t\int_{D_\lambda} a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)v_{x_i}(u_{x_i}-v_{x_i})F_\lambda'(w)\,dx\,dt\\
&\quad\le\int_0^t\int_{D_\lambda}\biggl[\frac{1}{4}a_i(x,t)|u|^{\alpha_i}(u-v)^{\beta-2}|w_{x_i}|^2\\
&\qquad+4a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)(u-v)^{\beta-2}|v_{x_i}|^2\biggr]\,dx\,dt\\
&\quad\le c+\frac{1}{4}\int_0^t\int_{D_\lambda} a_i(x,t)|u|^{\alpha_i}(u-v)^{\beta-2}|w_{x_i}|^2\,dx\,dt.
\end{aligned}
\tag{4.8}
$$

In the third place, by (4.3), since u and v both are nonnegative,

$$|u-v|\le u$$

we have

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega\big[b_i(u,x,t)-b_i(v,x,t)\big]F_\lambda'(u-v)(u-v)_{x_i}\,dx\,dt\right|\\
&\quad=\left|\int_0^t\int_{D_\lambda}\big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)^{\beta-2}(u-v)_{x_i}\,dx\,dt\right|\\
&\quad\le\int_0^t\int_\Omega w^{\beta-2}\biggl[4\Big(\big[b_i(u,x,t)-b_i(v,x,t)\big]u^{-\frac{\alpha_i}{2}}a_i(x,t)^{-\frac{1}{2}}\Big)^2\\
&\qquad+\frac{1}{4}a_i(x,t)u^{\alpha_i}|w_{x_i}|^2\biggr]\,dx\,dt\\
&\quad\le c+\frac{1}{2}\int_0^T\int_\Omega a_i(x,t)u^{\alpha_i}w^{\beta-2}|w_{x_i}|^2\,dx\,dt.
\end{aligned}
\tag{4.9}
$$

Last but not least, let $t_0=\inf\{\tau\in(0,t]:w>\lambda\}$. Then

$$
\begin{aligned}
\int_0^t\int_{D_\lambda} w_tF_\lambda(w)\,dx\,dt&=\int_{D_\lambda}\left(\int_0^{t_0}w_tF_\lambda(w)\,dt+\int_{t_0}^t w_tF_\lambda(w)\,dt\right)dx\\
&\ge\int_{D_\lambda}\int_\lambda^{w(x,t)}F_\lambda(s)\,ds\,dx\\
&\ge\int_{D_\lambda}(w-2\lambda)F_\lambda(2\lambda)\,dx\ge(\delta-2\lambda)F_\lambda(2\lambda)\mu(D_\lambda).
\end{aligned}
\tag{4.10}
$$

From (4.6)–(4.10), we have

$$(\delta-2\lambda)\frac{1-2^{\beta-1}}{1-\beta}\lambda^{\beta-1}\le c.$$

Letting $\lambda\to 0$, we get the contradiction. □

Proof of Theorem 2.3 If condition (2.6) is true, then conditions (4.3),(4.4) are true naturally. Thus, we have Theorem 2.3. □

5 The stability of weak solution of the usual initial boundary value problem

For any given positive integer n, let $g_n(s) = \int_0^s h_n(\tau)\,d\tau$, $h_n(s) = 2n(1-n|s|)_+$. Then $h_n(s) \in C(\mathbb{R})$, and

$$h_n(s) \geq 0, \qquad |sh_n(s)| \leq 1, \qquad |g_n(s)| \leq 1, \tag{5.1}$$

and

$$\lim_{n\to\infty} g_n(s) = \operatorname{sign} s, \qquad \lim_{n\to\infty} sg_n'(s) = 0. \tag{5.2}$$

As we have pointed out in the introduction section, for the classical porous medium equation

$$u_t = \Delta u^m,$$

if $u(x,t)$ and $v(x,t)$ are two nonnegative solutions of the initial boundary value problem, by choosing $g_n(u^m - v^m)$ as the test function, we easily show that

$$\int_\Omega |u(x,t) - v(x,t)|\,dx \leq \int_\Omega |u(x,0) - v(x,0)|\,dx, \quad t \in [0,T). \tag{5.3}$$

Now, for the anisotropic diffusion equation (1.7) considered in this paper, since α_i may be different from one to another, though for every i

$$u^{\alpha_i} u_{x_i} = \frac{1}{\alpha_i} u_{x_i}^{1+\alpha_i},$$

we cannot choose $g_n(u^{1+\alpha_i} - v^{1+\alpha_i})$ as a test function. If we insist on using a similar method to obtain the stability (5.3), then only for a kind of weak solution we can achieve the requirement.

Theorem 5.1 *Let $u(x,t)$ and $v(x,t)$ be two nonnegative solutions of the initial boundary value problem* (1.7)–(1.9) *satisfying* (2.7). *If $\alpha_i \geq 1$,*

$$\left|\frac{b_i(u,x,t) - b_i(v,x,t)}{(u-v)(u^{\frac{\alpha_i}{2}} - v^{\frac{\alpha_i}{2}})}\right| \leq cg_i(x,t), \quad i = 1,2,\ldots,N, \tag{5.4}$$

and

$$\int_\Omega a_i(x,t)^{-1} g_i(x,t)^2\,dx \leq c(T), \quad i = 1,2,\ldots,N, \tag{5.5}$$

then the stability (5.3) *is true.*

Proof By a process of limit, we can choose $g_n(u-v)$ as the test function, then

$$\begin{aligned}
&\int_\Omega g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx \\
&\quad+\sum_{i=1}^N\int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i}-v_{x_i})^2 g_n'(u-v)\,dx \\
&=-\sum_{i=1}^N\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)v_{x_i}(u_{x_i}-v_{x_i})g_n'(u-v)\,dx \\
&\quad-\sum_{i=1}^N\int_\Omega\big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)_{x_i}g_n'(u-v)\,dx\,dt.
\end{aligned}\tag{5.6}$$

Obviously,

$$\int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i}-v_{x_i})^2 g_n'(u-v)\,dx\ge 0.\tag{5.7}$$

By (2.7) and $\alpha_i\ge 1$, using the Lebesgue dominated theorem, we have

$$\lim_{n\to\infty}\int_\Omega a_i(x,t)\big||u|^{\alpha_i}-|v|^{\alpha_i}\big||v_{x_i}|^2 g_n'(u-v)\,dx=0,\tag{5.8}$$

$$\lim_{n\to\infty}\int_\Omega a_i(x,t)\big||u|^{\alpha_i}-|v|^{\alpha_i}\big||u_{x_i}|^2 g_n'(u-v)\,dx=0.\tag{5.9}$$

From (5.8)–(5.9), we obtain

$$\lim_{n\to\infty}\left|\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)v_{x_i}(u_{x_i}-v_{x_i})g_n'(u-v)\,dx\right|=0.\tag{5.10}$$

We now prove that

$$\lim_{n\to\infty}\int_\Omega\big(b_i(u,x,t)-b_i(v,x,t)\big){g_n}'(u-v)(u-v)_{x_i}\,dx=0.\tag{5.11}$$

In detail, by (5.4), we have

$$\begin{aligned}
&\left|\int_\Omega\big(b_i(u,x,t)-b_i(v,x,t)\big)g_n'(u-v)(u-v)_{x_i}\,dx\right| \\
&=\left|\int_{D_{nt}}\big[b_i(u,x,t)-b_i(v,x,t)\big]g_n'(u-v)(u-v)_{x_i}\,dx\right| \\
&\le c\int_{D_{nt}}\left|\frac{b_i(u,x,t)-b_i(v,x,t)}{u-v}\right|\big|(u-v)_{x_i}\big|\,dx \\
&=c\int_{D_{nt}}\left|a_i(x,t)^{-\frac12}\frac{b_i(u,x,t)-b_i(v,x,t)}{(u-v)(u^{\frac{\alpha_i}{2}}-v^{\frac{\alpha_i}{2}})}\right|\big|a_i(x,t)^{\frac12}\big(u^{\frac{\alpha_i}{2}}-v^{\frac{\alpha_i}{2}}\big)(u-v)_{x_i}\big|\,dx \\
&\le c\left(\int_{D_{nt}}\left|a_i(x,t)^{-\frac12}\frac{b_i(u,x,t)-b_i(v,x,t)}{(u-v)(u^{\frac{\alpha_i}{2}}-v^{\frac{\alpha_i}{2}})}\right|^2dx\right)^{\frac12}
\end{aligned}\tag{5.12}$$

$$
\begin{aligned}
&\cdot c\biggl(\int_{D_{nt}} \bigl|a_i(x,t)^{\frac{1}{2}}\bigl(u^{\frac{\alpha_i}{2}}-v^{\frac{\alpha_i}{2}}\bigr)(u-v)_{x_i}\bigr|^2\,dx\biggr)^{\frac{1}{2}}\\
&\leq c\biggl(\int_{D_{nt}} a_i(x,t)^{-1}g_i(x,t)^2\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{D_{nt}} a_i(x,t)\bigl(\bigl|u_{x_i}^{\frac{\alpha_i}{2}+1}\bigr|^2+\bigl|u_{x_i}^{\frac{\alpha_i}{2}+1}\bigr|^2\bigr)\,dx\biggr)^{\frac{1}{2}}.
\end{aligned}
$$

Here, we have used the notation

$$
D_{nt}=\biggl\{x\in\Omega:|u-v|<\frac{1}{n}\biggr\}.
$$

Let $n\to\infty$ in (5.12). Since (5.5),

$$
\int_{\Omega} a_i(x,t)^{-1}g_i(x,t)^2\,dx\leq c,
$$

if $D_0=\{x\in\Omega:|u-v|=0\}$ is a set with 0 measure, by that

$$
\int_{D_{nt}} a_i(x,t)\bigl(\bigl|u_{x_i}^{\frac{\alpha_i}{2}+1}\bigr|^2+\bigl|u_{x_i}^{\frac{\alpha_i}{2}+1}\bigr|^2\bigr)\,dx\leq c,
$$

we have

$$
\lim_{n\to\infty}\int_{D_{nt}} a_i(x,t)^{-1}g_i(x,t)^2\,dx=\int_{D_0} a_i(x,t)^{-1}g_i(x,t)^2\,dx=0. \tag{5.13}
$$

While $D_0=\{x\in\Omega:|u-v|=0\}$ has a positive measure, by that

$$
\begin{aligned}
&\int_{D_{nt}}\biggl|a_i(x,t)^{-\frac{1}{2}}\frac{b_i(u)-b_i(v)}{(u-v)(u^{\frac{\alpha_i}{2}}-v^{\frac{\alpha_i}{2}})}\biggr|^2\,dx\\
&\quad\leq\int_{\Omega} a_i(x,t)^{-1}g_i(x,t)^2\,dx\\
&\quad\leq c,
\end{aligned}
$$

then

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{D_{nt}}\bigl|a(x,t)^{\frac{1}{2}}\bigl(u^{\frac{\alpha_i}{2}}-v^{\frac{\alpha_i}{2}}\bigr)(u-v)_{x_i}\bigr|^2\,dx\\
&\quad=\int_{D_0}\bigl|a(x,t)^{\frac{1}{2}}\bigl(u^{\frac{\alpha_i}{2}}-v^{\frac{\alpha_i}{2}}\bigr)(u-v)_{x_i}\bigr|^2\,dx=0.
\end{aligned}
$$

Thus, in both cases, the right-hand side of inequality (5.12) goes to 0 as $n\to\infty$.

Moreover,

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega} g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx\\
&\quad=\int_{\Omega}\operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}\,dx\\
&\quad=\int_{\Omega}\operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}\\
&\quad=\frac{d}{dt}\int_{\Omega}|u-v|\,dx.
\end{aligned} \tag{5.14}
$$

At last, let $n \to \infty$ in (5.6). Then

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr|\,dx \le \int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\,dx. \qquad \square$$

Proof of Theorem 2.4 Since we assume conditions (2.7)–(2.8), conditions (5.4)–(5.5) are true naturally, by Theorem 5.1, we clearly have Theorem 2.4. $\square$

6 The partial boundary value condition based on a submanifold

In this section, we consider equation (1.7) with the initial value condition (1.8) and with a partial boundary value condition (2.12). For a small positive constant $\lambda > 0$ and any $t \in [0,t)$, let

$$\Omega_{\lambda t} = \left\{x \in \Omega : \prod_{i=1}^{N} a_i(x,t) > \lambda\right\},$$

and set

$$\phi(x) = \begin{cases} 1, & \text{if } x \in \Omega_{2\lambda t}, \\ \frac{1}{\lambda}(\prod_{i=1}^{N} a_i(x,t) - \lambda), & \text{if } x \in \Omega_{\lambda t} \setminus \Omega_{2\lambda t}, \\ 0, & \text{if } x \in \Omega \setminus \Omega_{\lambda t}. \end{cases} \tag{6.1}$$

Proof of Theorem 2.5 If we choose $\phi g_n(u-v)$ as the test function, then

$$\begin{aligned} &\int_0^T \int_\Omega \phi(x) g_n(u-v) \frac{\partial(u-v)}{\partial t}\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i} - v_{x_i})^2 \phi g_n'(u-v)\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega a_i(x,t)\bigl(|u|^{\alpha_i} - |v|^{\alpha_i}\bigr) v_{x_i}(u-v)_{x_i} \phi g_n'(u-v)\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega a_i(x,t)|u|^{\alpha_i} u_{x_i} - v_{x_i}|v|^{\alpha_i})\phi_{x_i} g_n(u-v)\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega \bigl[b_i(u,x,t) - b_i(v,x,t)\bigr](u-v)_{x_i} g_n'(u-v)\phi\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega \int_\Omega \bigl[b_i(u,x,t) - b_i(v,x,t)\bigr] g_n(u-v)\phi_{x_i}\,dx\,dt \\ &= 0. \end{aligned} \tag{6.2}$$

Clearly, we have

$$\int_0^T \int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i} - v_{x_i})(u-v)_{x_i} g_n'(u-v)\phi\,dx \ge 0, \tag{6.3}$$

and from $\iint_{Q_T}|u_t|\,dx\,dt\le c$, we deduce

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\lambda\to 0}\iint_{Q_t}\phi(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&=\lim_{n\to\infty}\iint_{Q_t}g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&=\iint_{Q_t}\operatorname{sign}(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&=\int_0^t\int_\Omega\frac{d}{dt}|u-v|\,dx\,dt.
\end{aligned}\tag{6.4}$$

Since

$$u_{x_i},v_{x_i}\in L^\infty\bigl(0,T;L^2_{\mathrm{loc}}(\Omega)\bigr),\quad i=1,2,\dots,N,\tag{6.5}$$

by that $\alpha_i\ge 1$, using the Lebesgue dominated theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_0^T\int_\Omega a_i(x,t)\bigl(|u|^{\alpha_i}-|v|^{\alpha_i}\bigr)v_{x_i}(u-v)_{x_i}\phi g_n'(u-v)\,dx\,dt\right|\\
&\le c\lim_{n\to\infty}\int_0^T\int_{\Omega_{\lambda t}}a_i(x,t)|u-v|\bigl(|v_{x_i}|^2+|u_{x_i}|^2\bigr)\phi g_n'(u-v)\,dx\,dt\\
&=0.
\end{aligned}\tag{6.6}$$

At the same time, if we denote that

$$\Omega_\phi=\bigl\{x\in\Omega:1>\phi(x)>0\bigr\}=\Omega_{\lambda t}\setminus\Omega_{2\lambda t},$$

then by (2.10) we have

$$\begin{aligned}
&\lim_{\lambda\to0}\lim_{n\to\infty}\left|\int_0^T\int_\Omega a_i(x,t)\bigl(|u|^{\alpha_i}u_{x_i}-|v|^{\alpha_i}v_{x_i}\bigr)\cdot\phi_{x_i}g_n(u-v)\,dx\,dt\right|\\
&=\lim_{\lambda\to0}\lim_{n\to\infty}\left|\int_0^T\int_{\Omega_\phi}a_i(x,t)\bigl(|u|^{\alpha_i}u_{x_i}-|v|^{\alpha_i}v_{x_i}\bigr)\cdot\frac{\bigl(\prod_{j=1}^N a_j(x,t)\bigr)_{x_i}}{\lambda}g_n(u-v)\,dx\,dt\right|\\
&\le c\lim_{\lambda\to0}\left(\int_0^T\frac{1}{\lambda}\int_{\Omega_\phi}a_i(x,t)\bigl(\bigl|u^{\alpha_i}\bigr||u_{x_i}|^2+\bigl|v^{\alpha_i}\bigr||u_{x_i}|^2\bigr)\,dx\,dt\right)^{\frac12}\\
&\quad\cdot\left(\int_0^T\frac{1}{\lambda}\int_{\Omega_\phi}a_i(x,t)\left|\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\operatorname{sign}(u-v)\right|^2\,dx\,dt\right)^{\frac12}\\
&\le c\left(\int_0^T\frac{1}{\lambda}\int_{\Omega_\phi}\left|\sqrt{a_i(x,t)}\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\operatorname{sign}(u-v)\right|^2\,dx\,dt\right)^{\frac12}\\
&=c\left(\int_0^T\int_{\partial\Omega}\left|\sqrt{a_i(x,t)}\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\operatorname{sign}(u-v)\right|^2\,d\Sigma\,dt\right)^{\frac12}\\
&=0.
\end{aligned}\tag{6.7}$$

For the convection term, by (6.5) and (2.13), we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_0^T\int_\Omega \big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)_{x_i}g_n'(u-v)\phi\,dx\,dt\right| \\
&\quad\le c\lim_{n\to\infty}\int_0^T\int_\Omega (u-v)_{x_i}|u-v|g_n'(u-v)\phi\,dx\,dt \\
&\quad\le c\lim_{n\to\infty}\left(\int_0^T\int_{\Omega_\phi}\big(|u_{x_i}|^2+|v_{x_i}|^2\big)\,dx\,dt\right)^{\frac12}\left(\int_0^T\int_\Omega\big[|u-v|g_n'(u-v)\big]^2\,dx\,dt\right)^{\frac12} \\
&\quad= 0.
\end{aligned}\tag{6.8}$$

By (2.13), using the homogeneous boundary value condition (2.12), we have

$$\begin{aligned}
&\lim_{\lambda\to0}\lim_{n\to\infty}\left|\int_0^T\int_\Omega\big[b_i(u,x,t)-b_i(v,x,t)\big]\phi_{x_i}g_n(u-v)\,dx\,dt\right| \\
&\quad\le\lim_{\lambda\to0}\int_0^T\frac{1}{\lambda}\int_{\Omega_{\lambda t}\setminus\Omega_{2\lambda t}}\big|b_i(u,x,t)-b_i(v,x,t)\big|\left|\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\right|\,dx\,dt \\
&\quad\le c\int_0^T\int_{\partial\Omega}|u-v|\left|\sqrt{a_i(x,t)}\left(\prod_{j=1}^N a_j(x,t)\right)_{x_i}\right|\,d\Sigma\,dt \\
&\quad= 0.
\end{aligned}\tag{6.9}$$

Now, after letting $n\to\infty$, let $\lambda\to0$ in (6.2). Then

$$\int_\Omega\big|u(x,t)-v(x,t)\big|\,dx\le\int_\Omega\big|u_0(x)-v_0(x)\big|\,dx.$$

Theorem 2.5 is proved. □

7 The proof of Theorem 2.6

In this section, we prove Theorem 2.6. For a small positive constant $\lambda>0$ and any $t\in[0,t)$, set $\Omega_{\lambda t}$ and $\phi(x)$ as (6.1).

Proof of Theorem 2.6 Let $u(x,t)$, $v(x,t)$ be two solutions of equation (1.7) with the initial boundary values $u_0(x)$, $v_0(x)$ respectively, but without the partial boundary value condition (2.12). By assumption (2.17), we can choose $g_n(\phi(u-v))$ as the test function and get

$$\begin{aligned}
&\int_\Omega g_n\big(\phi(u-v)\big)\frac{\partial(u-v)}{\partial t}\,dx \\
&\qquad+\sum_{i=1}^N\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}u_{x_i}-|v|^{\alpha_i}v_{x_i}\big)(u_{x_i}-v_{x_i})g_n'\big(\phi(u-v)\big)\phi\,dx \\
&\qquad+\sum_{i=1}^N\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}u_{x_i}-|v|^{\alpha_i}v_{x_i}\big)\phi_{x_i}(u-v)g_n'\big(\phi(u-v)\big)\,dx \\
&\qquad+\sum_{i=1}^N\int_\Omega\big[b_i(u,x,t)-b_i(v,x,t)\big]\big[\phi_{x_i}(u-v)+\phi(u-v)_{x_i}\big]g_n'\big(\phi(u-v)\big)\,dx \\
&\quad= 0.
\end{aligned}\tag{7.1}$$

In the first place, we have

$$\begin{aligned}
&\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}u_{x_i}-|v|^{\alpha_i}v_{x_i}\big)(u_{x_i}-v_{x_i})g_n'\big(\phi(u-v)\big)\phi\,dx \\
&\quad=\int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i}-v_{x_i})^2g_n'\big(\phi(u-v)\big)\phi\,dx \\
&\qquad+\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)v_{x_i}(u_{x_i}-v_{x_i})g_n'\big(\phi(u-v)\big)\phi\,dx.
\end{aligned} \tag{7.2}$$

Clearly,

$$\int_\Omega a_i(x,t)|u|^{\alpha_i}(u_{x_i}-v_{x_i})^2g_n'\big(\phi(u-v)\big)\phi\,dx\ge 0,$$

and

$$\begin{aligned}
&\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)v_{x_i}(u_{x_i}-v_{x_i})g_n'\big(\phi(u-v)\big)\phi\,dx \\
&\quad\le\frac14\int_\Omega a_i(x,t)\big||u|^{\alpha_i}-|v|^{\alpha_i}\big|(u_{x_i}-v_{x_i})^2g_n'\big(\phi(u-v)\big)\phi\,dx \\
&\qquad+4\int_\Omega a_i(x,t)\big||u|^{\alpha_i}-|v|^{\alpha_i}\big||v_{x_i}|^2g_n'\big(\phi(u-v)\big)\phi\,dx.
\end{aligned} \tag{7.3}$$

Since $a_i(x,t)$ satisfies (2.17) and

$$\phi(x,t)=0,\quad x\in\Omega\setminus\Omega_{\lambda t}$$

by that $\alpha^-\ge 1$ and

$$|\nabla u|\in L^\infty\big(0,T;L^2_{\mathrm{loc}}(\Omega)\big),\qquad |\nabla v|\in L^\infty\big(0,T;L^2_{\mathrm{loc}}(\Omega)\big),$$

using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\Omega a_i(x,t)\big(u^{\alpha_i}-v^{\alpha_i}\big)(u_{x_i}-v_{x_i})^2g_n'\big(\phi(u-v)\big)\phi\,dx\right| \\
&\quad=\lim_{n\to\infty}\left|\int_{\Omega_{\lambda t}} a_i(x,t)\big(u^{\alpha_i}-v^{\alpha_i}\big)(u_{x_i}-v_{x_i})^2g_n'\big(\phi(u-v)\big)\phi\,dx\right| \\
&\quad=0,
\end{aligned} \tag{7.4}$$

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\Omega a_i(x,t)\big(u^{\alpha_i}-v^{\alpha_i}\big)|v_{x_i}|^2g_n'\big(\phi(u-v)\big)\phi\,dx\right| \\
&\quad=\lim_{n\to\infty}\left|\int_{\Omega_{\lambda t}} a_i(x,t)\big(u^{\alpha_i}-v^{\alpha_i}\big)|v_{x_i}|^2g_n'\big(\phi(u-v)\big)\phi\,dx\right| \\
&\quad=0.
\end{aligned} \tag{7.5}$$

In the second place, we have

$$\begin{aligned}
&\int_\Omega a_i(x,t)\big(|u|^{\alpha_i}u_{x_i}-|v|^{\alpha_i}v_{x_i}\big)\phi_{x_i}(u-v)g_n'\big(\phi(u-v)\big)\,dx\\
&\quad=\int_{\Omega_{\lambda t}} a_i(x,t)|u|^{\alpha_i}(u_{x_i}-v_{x_i})\frac{\phi_{x_i}}{\phi}\phi(u-v)g_n'\big(\phi(u-v)\big)\,dx\\
&\qquad+\int_{\Omega_{\lambda t}} a_i(x,t)\big(|u|^{\alpha_i}-|v|^{\alpha_i}\big)v_{x_i}\frac{\phi_{x_i}}{\phi}\phi(u-v)g_n'\big(\phi(u-v)\big)\,dx.
\end{aligned}\tag{7.6}$$

While by (2.18)

$$\int_\Omega a_i(x,t)\left|\frac{\phi_{x_i}}{\phi}\right|^2dx=\int_{\Omega_{\lambda t}\setminus\Omega_{2\lambda t}} a_i(x,t)\left|\sum_{k=1}^N\frac{a_{kx_i}}{a_k}\right|^2dx\le c,$$

we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega a_i(x,t)u^{\alpha_i}(u_{x_i}-v_{x_i})\frac{\phi_{x_i}}{\phi}\phi(u-v)g_n'\big(\phi(u-v)\big)\,dx\\
&\quad\le c\int_\Omega a_i(x,t)u^{\alpha_i}(u_{x_i}-v_{x_i})^2\big|\phi(u-v)g_n'\big(\phi(u-v)\big)\big|\,dx\\
&\qquad+c\lim_{n\to\infty}\int_\Omega a_i(x,t)\left|\frac{\phi_{x_i}}{\phi}\right|^2\big|\phi(u-v)g_n'\big(\phi(u-v)\big)\big|\,dx\\
&\quad=0.
\end{aligned}\tag{7.7}$$

Similarly, we have

$$\lim_{n\to\infty}\left|\int_\Omega a_i(x,t)\big(u^{\alpha_i}-v^{\alpha_i}\big)v_{x_i}\frac{\phi_{x_i}}{\phi}\phi(u-v)g_n'\big(\phi(u-v)\big)\right|=0.\tag{7.8}$$

In the third place, since (1.13) $|b_i(u,x,t)-b_i(v,x,t)|\le c\sqrt{a_i(x,t)}$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\Omega\big[b_i(u,x,t)-b_i(v,x,t)\big]g_n'\big(\phi(u-v)\big)(u-v)\phi_{x_i}\,dx\right|\\
&\quad=\lim_{n\to\infty}\int_\Omega\frac{|\phi_{x_i}|}{\phi}\big|b_i(u,x,t)-b_i(v,x,t)\big|\big|g_n'\big(\phi(u-v)\big)\phi(u-v)\big|\,dx\\
&\quad\le c\lim_{n\to\infty}\int_\Omega\left(\left|\frac{|\phi_{x_i}|}{\phi}\right|^2a_i(x,t)+1\right)\big|g_n'\big(\phi(u-v)\big)\phi(u-v)\big|\,dx\\
&\quad=0.
\end{aligned}\tag{7.9}$$

Moreover, since (4.1), we clearly have

$$\lim_{n\to\infty}\int_\Omega\big(b_i(u,x,t)-b_i(v,x,t)\big)g_n{}'\big(\phi(u-v)\big)(u-v)_{x_i}\phi(x)\,dx=0.$$

Now, let $n\to\infty$ in (7.1). Then

$$\int_\Omega\big|u(x,t)-v(x,t)\big|\,dx\le\int_\Omega\big|u_0(x)-v_0(x)\big|\,dx.$$

Theorem 2.6 is proved. □

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Author contribution
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