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Existence results for a self-adjoint coupled system of nonlinear second-order ordinary differential inclusions with nonlocal integral boundary conditions

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Abstract

A coupled system of nonlinear self-adjoint second-order ordinary differential inclusions supplemented with nonlocal nonseparated coupled integral boundary conditions on an arbitrary domain is studied. The existence results for convex and nonconvex valued maps involved in the given problem are proved by applying the nonlinear alternative of Leray–Schauder for multivalued maps and Covitz–Nadler's fixed point theorem for contractive multivalued maps, respectively. Illustrative examples for the obtained results are presented. The paper concludes with some interesting observations.

MSC: 34A60; 34B10; 34B15

Keywords: Self-adjoint ordinary differential inclusions; Coupled; Nonlocal integral boundary conditions; Existence; Fixed point

1 Introduction

Inspired by the work of Bitsadze and Samarskii [1] on nonlocal elliptic boundary value problems, Il'in and Moiseev [2, 3] initiated the study of nonlocal boundary value problems for second-order ordinary differential equations. Nonlocal boundary value problems constitute an important area of research as such problems find their applications in chemical engineering, thermo-elasticity, underground water flow, and population dynamics; for details and examples, see [4, 5]. For a variety of interesting results on nonlocal boundary value problems, we refer the reader to the works [6–21] and the references cited therein. Self-adjoint differential equations are found to be of great interest in certain disciplines, for example, see [22–25]. In [26], a self-adjoint coupled system of nonlinear ordinary differential equations with nonlocal multi-point boundary conditions was studied. In a recent article [27], the authors established existence results for a self-adjoint coupled system of nonlinear second-order ordinary differential equations complemented with nonlocal nonseparated integral boundary conditions.

The aim of the present paper is to consider and investigate the existence of solutions for the multivalued case of the problem discussed in [27]. In precise terms, we consider a self-adjoint coupled system of second-order ordinary differential inclusions on an arbitrary domain, subject to the nonlocal nonseparated integral coupled boundary conditions given by

$$\begin{cases} (p(t)u'(t))' \in \mu_1 F(t,u(t),v(t)), & t \in [a,b], \\ (q(t)v'(t))' \in \mu_2 G(t,u(t),v(t)), & t \in [a,b], \\ \alpha_1 u(a) + \alpha_2 u(b) = \lambda_1 \int_a^\eta v(s)\,ds, & \alpha_3 u'(a) + \alpha_4 u'(b) = \lambda_2 \int_a^\eta v'(s)\,ds, \\ \beta_1 v(a) + \beta_2 v(b) = \lambda_3 \int_\xi^b u(s)\,ds, & \beta_3 v'(a) + \beta_4 v'(b) = \lambda_4 \int_\xi^b u'(s)\,ds, \end{cases} \tag{1.1}$$

where $a < \eta < \xi < b$, $p,q \in C([a,b],\mathbb{R}^+)$, α_i, β_i, $\lambda_i \in \mathbb{R}^+$, $i = 1,2,3,4$, $\mu_j \in \mathbb{R}^+$, $j = 1,2$, and $F,G : [a,b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ are given multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

We establish existence criteria for the solutions of problem (1.1) for convex and non-convex valued multivalued maps F and G by applying the nonlinear alternative of Leray–Schauder for multivalued maps in the convex case and Covitz and Nadler's fixed point theorem for contractive multivalued maps in the nonconvex case, respectively. The tools of the fixed point theory employed in our analysis are standard, however their application to problem (1.1) is new. We emphasize that the results derived in this paper are new and enrich the literature on self-adjoint multivalued nonlocal boundary value problems.

The rest of the paper is organized as follows. We present background material about multivalued analysis in Sect. 2, while the main results are presented in Sect. 3. Numerical examples illustrating the obtained results are constructed in Sect. 4.

2 Preliminaries

We begin this section by reviewing some basic definitions, lemmas, and theorems on multivalued maps from [28, 29] which are related to the study of problem (1.1).

Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. We denote the classes of all closed, bounded, compact, and compact and convex sets in $\mathcal{X}$ by $\mathcal{P}_{cl}$, $\mathcal{P}_b$, $\mathcal{P}_{cp}$, and $\mathcal{P}_{cp,c}$, respectively.

A multivalued map $F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ is (a) convex (closed) valued if $F(x)$ is convex (closed) for all $x \in \mathcal{X}$; (b) upper semicontinuous (u.s.c.) on $\mathcal{X}$ if for each $x_0 \in \mathcal{X}$, the set $F(x_0)$ is a nonempty closed subset of $\mathcal{X}$, and if for each open set $\mathcal{N}$ of $\mathcal{X}$ containing $F(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $F(\mathcal{N}_0) \subseteq \mathcal{N}$; (c) bounded on bounded sets if $F(\mathbb{B}) = \bigcup_{x\in\mathbb{B}} F(x)$ is bounded in $\mathcal{X}$ for all $\mathbb{B} \in \mathcal{P}_b(\mathcal{X})$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in F(x)\}\} < \infty$); (d) completely continuous if $F(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(\mathcal{X})$. F has a fixed point if there is $x \in \mathcal{X}$ such that $x \in F(x)$.

A multivalued map $F : W \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if, for every $b \in \mathbb{R}$, the function $t \longmapsto d(b,F(t)) = \inf\{|b-c| : c \in F(t)\}$ is measurable. We define the graph of F to be the set $Fr(F) = \{(x,y) \in X \times Y, y \in F(x)\}$. The fixed point set of the multivalued operator F will be denoted by $FixF$.

Remark 2.1 (The relationship between closed graphs and upper-semicontinuity) If $F : \mathcal{X} \to \mathcal{P}_{cl}(\mathcal{X})$ is u.s.c., then $Fr(F)$ is a closed subset of $X \times Y$ i.e. for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$, and $y_n \in F(x_n)$, then

$y_* \in F(x_*)$. Conversely, if F is completely continuous and has a closed graph, then it is upper semi-continuous.

Definition 2.2 A multivalued map $F : [a,b] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,u,v)$ is measurable for each $u, v \in \mathbb{R}$;

(ii) $(u,v) \longmapsto F(t,u,v)$ is upper semicontinuous for almost all $t \in [a,b]$;

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\Omega_\rho \in L^1([a,b],\mathbb{R}^+)$ such that

$$\|F(t,u,v)\| = \sup\{|x| : x \in F(t,u,v)\} \leq \Omega_\rho(t)$$

for all $\|u\|, \|v\| \leq \rho$ and for a. e. $t \in [a,b]$.

Definition 2.3 A function $(u,v) \in \mathcal{F} \times \mathcal{F}$, where $\mathcal{F} = C^2([a,b],\mathbb{R})$, is a solution of the self-adjoint coupled system (1.1) if it satisfies the coupled conditions of (1.1) and there exist functions $\hat{f}, \hat{g} \in L^1([a,b],\mathbb{R})$ such that $\hat{f}(t) \in F(t,u(t),v(t)), \hat{g}(t) \in G(t,u(t),v(t))$ a.e on $[a,b]$.

Let us now recall the following lemma from [27].

Lemma 2.4 *For $f_1, g_1 \in C([a,b],\mathbb{R})$ and $R \neq 0$, $E \neq 0$, the solution of the linear system*

$$\begin{cases} (p(t)u'(t))' = \mu_1 f_1(t), & t \in [a,b], \\ (q(t)v'(t))' = \mu_2 g_1(t), & t \in [a,b], \\ \alpha_1 u(a) + \alpha_2 u(b) = \lambda_1 \int_a^\eta v(s)\,ds, & \alpha_3 u'(a) + \alpha_4 u'(b) = \lambda_2 \int_a^\eta v'(s)\,ds, \\ \beta_1 v(a) + \beta_2 v(b) = \lambda_3 \int_\xi^b u(s)\,ds, & \beta_3 v'(a) + \beta_4 v'(b) = \lambda_4 \int_\xi^b u'(s)\,ds, \end{cases} \tag{2.1}$$

can be expressed by the formulas:

$$\begin{aligned} u(t) = {} & \int_a^t \left(\frac{\mu_1}{p(u)} \int_a^u f_1(z)\,dz\right) du + \frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2) \int_a^b \left(\frac{\mu_1}{p(u)} \int_a^u f_1(z)\,dz\right) du \\ & + \lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)} \int_a^u g_1(z)\,dz\right) du\,ds \\ & - \lambda_1\beta_2(\eta-a) \int_a^b \left(\frac{\mu_2}{q(u)} \int_a^u g_1(z)\,dz\right) du \\ & + \lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)} \int_a^u f_1(z)\,dz\right) du\,ds\Bigg] \\ & + \frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\ & + E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\ & - RE_4 \int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)} \int_a^b f_1(z)\,dz\right) + \Bigg(-E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}\,dz \\ & + E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}\,dz \end{aligned}$$

$$
\begin{aligned}
&+ E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\Bigg) \\
&\times \left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s g_1(z)\,dz\,ds\right) \\
&+ \Bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_2\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b g_1(z)\,dz\right) + \Bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t \frac{1}{p(z)}\,dz\Bigg) \\
&\times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s f_1(z)\,dz\,ds\right)\Bigg]
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
v(t) = {}& \int_a^t\left(\frac{\mu_2}{q(u)}\int_a^u g_1(z)\,dz\right)du + \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\left(\frac{\mu_1}{p(u)}\int_a^u f_1(z)\,dz\right)du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u g_1(z)\,dz\right)du\,ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u g_1(z)\,dz\right)du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u f_1(z)\,dz\right)du\,ds\Bigg] \\
&+ \frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b f_1(z)\,dz\right) + \Bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s g_1(z)\,dz\,ds\right) \\
&+ \Bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b g_1(z)\,dz\right) + \Bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz
\end{aligned}
$$

$$+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz$$
$$+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\Bigg)$$
$$\times\Bigg(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s f_1(z)\,dz\,ds\Bigg)\Bigg], \tag{2.3}$$

where

$$\begin{aligned} &R = (\alpha_1+\alpha_2)(\beta_1+\beta_2) - \lambda_1\lambda_3(\eta-a)(b-\xi),\\ &E = E_1E_4 - E_2E_3,\\ &E_1 = \frac{\alpha_3}{p(a)} + \frac{\alpha_4}{p(b)}, \qquad E_2 = \int_a^\eta \frac{\lambda_2}{q(s)}\,ds,\\ &E_3 = \int_\xi^b \frac{\lambda_4}{p(s)}\,ds, \qquad E_4 = \frac{\beta_3}{q(a)} + \frac{\beta_4}{q(b)}. \end{aligned} \tag{2.4}$$

Let us consider the set of selection functions F and G for each $(u,v)\in\mathcal{F}\times\mathcal{F}$ defined by

$$S_{F,(u,v)} := \big\{\hat{f}\in L^1\big([a,b],\mathbb{R}\big) : \hat{f}(t)\in F\big(t,u(t),v(t)\big) \text{ for a.e. } t\in[a,b]\big\}$$

and

$$S_{G,(u,v)} := \big\{\hat{g}\in L^1\big([a,b],\mathbb{R}\big) : \hat{g}(t)\in G\big(t,u(t),v(t)\big) \text{ for a.e. } t\in[a,b]\big\}.$$

Define the operators $\Theta_1,\Theta_2 : \mathcal{F}\times\mathcal{F}\to\mathcal{P}(\mathcal{F}\times\mathcal{F})$ by

$$\begin{aligned} \Theta_1(u,v) = \big\{&h_1\in\mathcal{F}\times\mathcal{F} : \text{there exists } \hat{f}\in S_{F,(u,v)}, \hat{g}\in S_{G,(u,v)} \text{ such that}\\ &h_1(u,v)(t) = \mathcal{Z}_1(u,v)(t), \forall t\in[a,b]\big\} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \Theta_2(u,v) = \big\{&h_2\in\mathcal{F}\times\mathcal{F} : \text{there exists } \hat{f}\in S_{F,(u,v)}, \hat{g}\in S_{G,(u,v)} \text{ such that}\\ &h_2(u,v)(t) = \mathcal{Z}_2(u,v)(t), \forall t\in[a,b]\big\}, \end{aligned} \tag{2.6}$$

where

$$\mathcal{Z}_1(u,v)(t)$$
$$= \int_a^t \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)\,dz\right)du + \frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)\,dz\right)du$$
$$+ \lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)\,dz\right)du\,ds$$
$$- \lambda_1\beta_2(\eta-a)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_1(z)\,dz\right)du$$
$$+ \lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_1(z)\,dz\right)du\,ds\Bigg]$$

$$
\begin{aligned}
&+\frac{1}{ER}\bigg[\bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}_1(z)\,dz\bigg)+\bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_4\int_a^t\frac{1}{p(z)}\,dz\bigg)\bigg(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}_1(z)\,dz\,ds\bigg)\\
&+\bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}_1(z)\,dz\bigg)+\bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_2\int_a^t\frac{1}{p(z)}\,dz\bigg)\\
&\times\bigg(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}_1(z)\,dz\,ds\bigg)\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{Z}_2(u,v)(t)\\
&=\int_a^t\bigg(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\bigg)\,du+\frac{1}{R}\bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\bigg(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)\,dz\bigg)\,du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\bigg(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\bigg)\,du\,ds\\
&-\beta_2(\alpha_1+\alpha_2)\int_a^b\bigg(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\bigg)\,du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\bigg(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)\,dz\bigg)\,du\,ds\bigg]\\
&+\frac{1}{ER}\bigg[\bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}\,dz-E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_3\int_a^t\frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}_1(z)\,dz\bigg)+\bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)\,dz\,ds\bigg)\\
&+\bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)\,dz\bigg) + \bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz\\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz\\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\bigg)\\
&\times\bigg(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)\,dz\,ds\bigg)\bigg].
\end{aligned}
$$

Next, we introduce an operator $\Theta : \mathcal{F}\times\mathcal{F} \to \mathcal{P}(\mathcal{F}\times\mathcal{F})$ as

$$
\Theta(u,v)(t) = \begin{pmatrix}\Theta_1(u,v)(t)\\ \Theta_2(u,v)(t)\end{pmatrix},
$$

where Θ_1 and Θ_2 are defined by (2.5) and (2.6) respectively.

For the sake of computational convenience, we set the notation

$$
\mathcal{E}_1 = \mathcal{D}_1 + \mathcal{D}_3, \qquad \mathcal{E}_2 = \mathcal{D}_2 + \mathcal{D}_4, \tag{2.7}
$$

where

$$
\begin{aligned}
\mathcal{D}_1 &= \frac{\mu_1}{|R\bar{p}|}\bigg[\frac{(b-a)^2}{2}\big(|R| + \alpha_2(\beta_1+\beta_2)\big) + \frac{\lambda_1\lambda_2(\eta-a)[(b-a)^3-(\xi-a)^3]}{6}\bigg]\\
&+ \frac{1}{|RE|}\bigg[\bigg(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+ \frac{E_4\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\bigg)\bigg(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\bigg)\\
&+ \bigg(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+ \frac{E_2\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}} + \frac{RE_2(b-a)}{\bar{p}}\bigg)\\
&\times\bigg(\frac{\lambda_4\mu_1[(b-a)^2-(\xi-a)^2]}{2\bar{p}}\bigg)\bigg],\\
\mathcal{D}_2 &= \frac{\mu_2}{|2R\bar{q}|}\bigg[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3} + \lambda_1\beta_2(\eta-a)(b-a)^2\bigg]\\
&+ \frac{1}{|RE|}\bigg[\bigg(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}} + \frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}} + \frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+ \frac{E_4\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}} + \frac{RE_4(b-a)}{\bar{p}}\bigg)\bigg(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\right.\\
&\left.+\frac{E_2\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\right)\left(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\right)\Bigg],\\
\mathcal{D}_3={}&\frac{\mu_1}{|R\bar{p}|}\left[\frac{(b-a)^2}{2}\big(\alpha_2\lambda_3(b-\xi)\big)+\frac{\lambda_3(\alpha_1+\alpha_2)[(b-a)^3-(\xi-a)^3]}{6}\right]\\
&+\frac{1}{RE}\Bigg[\left(\frac{E_4\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_3\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_3\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}\right.\\
&\left.+\frac{E_4\lambda_3(\alpha_1+\alpha_2)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_3(b-a)}{\bar{p}}\right)\left(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\right)\\
&+\left(\frac{E_2\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_1\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_1\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}\right.\\
&\left.+\frac{E_2\lambda_3(\alpha_1+\alpha_2)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_1(b-a)}{\bar{p}}\right)\\
&\times\left(\frac{\lambda_4\mu_1[(b-a)^2-(\xi-a)^2]}{2\bar{p}}\right)\Bigg],\\
\mathcal{D}_4={}&\frac{\mu_2}{|R\bar{q}|}\left[\frac{(b-a)^2}{2}\big(|R|+\beta_2(\alpha_1+\alpha_2)\big)+\frac{\lambda_1\lambda_3(b-\xi)(\eta-a)^3}{6}\right]\\
&+\frac{1}{RE}\Bigg[\left(\frac{E_4\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_3\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_3\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}\right.\\
&\left.+\frac{E_4\lambda_3(\alpha_1+\alpha_2)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_3(b-a)}{\bar{p}}\right)\left(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\right)\\
&+\left(\frac{E_2\alpha_2\lambda_3(b-\xi)(b-a)}{\bar{p}}+\frac{E_1\lambda_1\lambda_3(b-\xi)(\eta-a)^2}{2\bar{q}}+\frac{E_1\beta_2(\alpha_1+\alpha_2)(b-a)}{\bar{q}}\right.\\
&\left.+\frac{E_2\lambda_3(\alpha_1+\alpha_2)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_1(b-a)}{\bar{p}}\right)\left(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\right)\Bigg],
\end{aligned}
\tag{2.8}
$$

$$
\bar{p}=\inf_{z\in[a,b]}\big|p(z)\big|,\qquad \bar{q}=\inf_{z\in[a,b]}\big|q(z)\big|. \tag{2.9}
$$

3 The Carathéodory case

To prove our first existence result for multivalued problem (1.1), we need the following known results.

Lemma 3.1 ([30]) *Let X be a Banach space. Let $F:[a,b]\times\mathbb{R}^2\to\mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1- Carathéodory multivalued map, and let φ be a linear continuous mapping from $L^1([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. Then the operator*

$$
\varphi\circ S_{F,u}:C\big([a,b],\mathbb{R}\big)\to P_{cp,c}\big(C\big([a,b],\mathbb{R}\big)\big),\quad u\mapsto(\varphi\circ S_{F,u})(u)=\varphi(S_{F,u})
$$

is a closed graph operator in $C([a,b],\mathbb{R})\times C([a,b],\mathbb{R})$.

Lemma 3.2 (Nonlinear alternative for Kakutani maps [31]) *Let $\mathcal{S}$ be a Banach space, $\mathcal{S}_1$ be a closed convex subset of $\mathcal{S}$, U be an open subset of $\mathcal{S}_1$, and $0\in U$. Suppose that $F:\overline{U}\to\mathcal{P}_{c,cv}(\mathcal{S}_1)$ is an upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(\mathcal{S}_1)$ denotes the family of nonempty, compact convex subsets of $\mathcal{S}_1$. Then either*

(i) *F has a fixed point in $\overline{U}$ or*
(ii) *there are $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Now we are in a position to present our first main result.

Theorem 3.3 *Assume that*

(H_1) *$F, G : [a,b] \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ are Carathéodory possessing compact and convex values;*

(H_2) *There exist continuous nondecreasing functions $\psi_1, \psi_2, \phi_1, \phi_2 : [0,\infty) \longrightarrow (0,\infty)$ such that*

$$\big\|F(t,u,v)\big\|_{\mathcal{P}} := \sup\big\{|\hat{f}| : \hat{f} \in F(t,u,v)\big\} \le p_1(t)\big[\psi_1\big(\|u\|\big) + \phi_1\big(\|v\|\big)\big]$$

and

$$\big\|G(t,u,v)\big\|_{\mathcal{P}} := \sup\big\{|\hat{g}| : \hat{g} \in G(t,u,v)\big\} \le p_2(t)\big[\psi_2\big(\|u\|\big) + \phi_2\big(\|v\|\big)\big]$$

for each $(t,u,v) \in [a,b] \times \mathbb{R}^2$, where $p_1, p_2 \in C([a,b], \mathbb{R}^+)$;

(H_3) *There exists a constant $N > 0$ such that*

$$\frac{N}{\mathcal{E}_1\|p_1\|[\psi_1(N) + \phi_1(N)] + \mathcal{E}_2\|p_2\|[\psi_2(N) + \phi_2(N)]} > 1,$$

where $\mathcal{E}_i$ $(i = 1,2)$ are given in (2.7).

Then problem (1.1) has at least one solution on $[a,b]$.

Proof Consider the operators $\Theta_1, \Theta_2 : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ defined by (2.5) and (2.6) respectively. It follows from assumption (H_1) that the sets $S_{F,(u,v)}$ and $S_{G,(u,v)}$ are nonempty for each $(u,v) \in \mathcal{F} \times \mathcal{F}$. Then, for $\hat{f} \in S_{F,(u,v)}$, $\hat{g} \in S_{G,(u,v)}$ and $\forall (u,v) \in \mathcal{F} \times \mathcal{F}$, we have

$$\begin{aligned}
h_1(u,v)(t) = {} & \int_a^t \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)\,dz\right) du + \frac{1}{R}\bigg[-\alpha_2(\beta_1 + \beta_2) \int_a^b \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)\,dz\right) du \\
& + \lambda_1(\beta_1 + \beta_2) \int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)\,dz\right) du\,ds \\
& - \lambda_1\beta_2(\eta - a) \int_a^b \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)\,dz\right) du \\
& + \lambda_1\lambda_3(\eta - a) \int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)\,dz\right) du\,ds\bigg] \\
& + \frac{1}{ER}\bigg[\bigg(E_4\alpha_2(\beta_1 + \beta_2) \int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1 + \beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
& + E_3\lambda_1\beta_2(\eta - a) \int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta - a) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
& - RE_4 \int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}(z)\,dz\right) + \bigg(-E_4\alpha_2(\beta_1 + \beta_2) \int_a^b \frac{1}{p(z)}\,dz \\
& + E_3\lambda_1(\beta_1 + \beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta - a) \int_a^b \frac{1}{q(z)}\,dz
\end{aligned}$$

$$
\begin{aligned}
&+ E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\Bigg) \\
&\times\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)\,dz\,ds\right) \\
&+ \Bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_2\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)\,dz\right) + \Bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t \frac{1}{p(z)}\,dz\Bigg) \\
&\times\left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)\,dz\,ds\right)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(u,v)(t) ={}& \int_a^t \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du \\
&+ \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du\,ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du\,ds\Bigg] \\
&+ \frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)\,dz\right) + \Bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg) \\
&\times\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)\,dz\,ds\right) \\
&+ \Bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)\,dz\biggr) + \biggl(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\biggr) \\
&\times \biggl(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)\,dz\,ds\biggr)\biggr],
\end{aligned}
$$

where $h_1 \in \Theta_1(u,v)$, $h_2 \in \Theta_2(u,v)$, and hence $(h_1,h_2) \in \Theta(u,v)$.

Now, we will verify that the operator Θ satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. In the first step, we show that $\Theta(u,v)$ is convex valued for each $(u,v) \in \mathcal{F}\times\mathcal{F}$. Let $(h_i,\tilde{h}_i) \in (\Theta_1,\Theta_2)$, $i=1,2$. Then there exist $\hat{f}_i \in S_{F,(u,v)}$, $\hat{g}_i \in S_{G,(u,v)}$, $i=1,2$, such that, for each $t\in[a,b]$, we have

$$
\begin{aligned}
h_i(t) = {}& \int_a^t \biggl(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)\,dz\biggr)du + \frac{1}{R}\biggl[-\alpha_2(\beta_1+\beta_2)\int_a^b \biggl(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)\,dz\biggr)du \\
&+ \lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \biggl(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)\,dz\biggr)du\,ds \\
&- \lambda_1\beta_2(\eta-a)\int_a^b \biggl(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)\,dz\biggr)du \\
&+ \lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \biggl(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)\,dz\biggr)du\,ds\biggr] \\
&+ \frac{1}{ER}\biggl[\biggl(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_i(z)\,dz\biggr) + \biggl(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr) \\
&\times \biggl(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_i(z)\,dz\,ds\biggr) \\
&+ \biggl(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_2\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_i(z)\,dz\biggr) + \biggl(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+ E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\times\left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_i(z)\,dz\,ds\right)\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h}_i(t) = {}&\int_a^t\left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)\,dz\right)du \\
&+ \frac{1}{R}\bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)\,dz\right)du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)\,dz\right)du\,ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_i(z)\,dz\right)du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_i(z)\,dz\right)du\,ds\bigg] \\
&+ \frac{1}{ER}\bigg[\bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_i(z)\,dz\right) + \bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\times\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_i(z)\,dz\,ds\right) \\
&+ \bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_i(z)\,dz\right) + \bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\times\left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_i(z)\,dz\,ds\right)\bigg].
\end{aligned}
$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,1]$, we have

$$
\begin{aligned}
&\big[\omega h_1 + (1-\omega)h_2\big](t)\\
&\quad= \int_a^t \left(\frac{\mu_1}{p(u)}\int_a^u \big[\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)\big]\,dz\right)du\\
&\qquad+ \frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \left(\frac{\mu_1}{p(u)}\int_a^u \big[\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)\big]\,dz\right)du\\
&\qquad+ \lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \left(\frac{\mu_2}{q(u)}\int_a^u \big[\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)\big]\,dz\right)du\,ds\\
&\qquad- \lambda_1\beta_2(\eta-a)\int_a^b \left(\frac{\mu_2}{q(u)}\int_a^u \big[\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)\big]\,dz\right)du\\
&\qquad+ \lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \left(\frac{\mu_1}{p(u)}\int_a^u \big[\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)\big]\,dz\right)du\,ds\bigg]\\
&\qquad+ \frac{1}{ER}\bigg[\bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\qquad+ E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\qquad- RE_4\int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \big[\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)\big]\,dz\right)\\
&\qquad+ \bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz + E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\qquad- E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz + E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\qquad+ RE_4\int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \big[\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)\big]\,dz\,ds\right)\\
&\qquad+ \bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\qquad+ E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\qquad- RE_2\int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \big[\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)\big]\,dz\right)\\
&\qquad+ \bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz + E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\qquad- E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz + E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\qquad+ RE_2\int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \big[\omega\hat{f}_1(z) + (1-\omega)\hat{f}_2(z)\big]\,dz\,ds\right)\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\big[\omega\tilde{h}_1 + (1-\omega)\tilde{h}_2\big](t)\\
&\quad= \int_a^t \left(\frac{\mu_2}{q(u)}\int_a^u \big[\omega\hat{g}_1(z) + (1-\omega)\hat{g}_2(z)\big]\,dz\right)du
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{R}\biggl[-\alpha_2\lambda_3(b-\xi)\int_a^b\biggl(\frac{\mu_1}{p(u)}\int_a^u\bigl[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)\bigr]\,dz\biggr)\,du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\biggl(\frac{\mu_2}{q(u)}\int_a^u\bigl[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)\bigr]\,dz\biggr)\,du\,ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b\biggl(\frac{\mu_2}{q(u)}\int_a^u\bigl[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)\bigr]\,dz\biggr)\,du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\biggl(\frac{\mu_1}{p(u)}\int_a^u\bigl[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)\bigr]\,dz\biggr)\,du\,ds\biggr] \\
&+ \frac{1}{ER}\biggl[\biggl(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t\frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\bigl[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)\bigr]\,dz\biggr) \\
&+ \biggl(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz + E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds \\
&- E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}\,dz + E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t\frac{1}{p(z)}\,dz\biggr) \\
&\times\biggl(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\bigl[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)\bigr]\,dz\,ds\biggr) \\
&+ \biggl(E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t\frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\beta_4\mu_2}{q(b)}\int_a^b\bigl[\omega\hat{g}_1(z)+(1-\omega)\hat{g}_2(z)\bigr]\,dz\biggr) \\
&+ \biggl(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz + E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds \\
&- E_1\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}\,dz + E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds \\
&+ RE_1\int_a^t\frac{1}{p(z)}\,dz\biggr)\biggl(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\bigl[\omega\hat{f}_1(z)+(1-\omega)\hat{f}_2(z)\bigr]\,dz\,ds\biggr)\biggr].
\end{aligned}
$$

Since $S_{F,(u,v)}$, $S_{G,(u,v)}$ are convex valued as F and G are convex valued maps, therefore $\omega h_1 + (1-\omega)h_2 \in \Theta_1$, $\omega\tilde{h_1} + (1-\omega)\tilde{h_2} \in \Theta_2$, and hence $\omega(h_1,\tilde{h_1}) + (1-\omega)(h_2,\tilde{h_2}) \in \Theta$.

Now, we show that Θ maps bounded sets into bounded sets in $\mathcal{F}\times\mathcal{F}$. For a positive number ν^*, let $B_{\nu^*} = \{(u,v)\in\mathcal{F}\times\mathcal{F} : \|(u,v)\| \le \nu^*\}$ be a bounded set in $\mathcal{F}\times\mathcal{F}$. Then, for each $h_i \in \Theta_i$ $(i=1,2)$, $(u,v)\in B_{\nu^*}$, there exist $\hat{f}\in S_{F,(u,v)}$, $\hat{g}\in S_{G,(u,v)}$ such that

$$
\begin{aligned}
h_1(u,v)(t) = \int_a^t\biggl(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\biggr)\,du + \frac{1}{R}\biggl[-\alpha_2(\beta_1+\beta_2)\int_a^b\biggl(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\biggr)\,du \\
+ \lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\biggl(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\biggr)\,du\,ds
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda_1\beta_2(\eta-a)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du \\
&+ \lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\right)du\,ds\Bigg] \\
&+ \frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds \\
&+ E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds \\
&- RE_4\int_a^t\frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)\,dz\right) + \Bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz \\
&+ E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t\frac{1}{p(z)}\,dz\Bigg) \\
&\times\left(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)\,dz\,ds\right) \\
&+ \Bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds \\
&+ E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds \\
&- RE_2\int_a^t\frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)\,dz\right) + \Bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz \\
&+ E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t\frac{1}{p(z)}\,dz\Bigg) \\
&\times\left(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)\,dz\,ds\right)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(u,v)(t) = {}&\int_a^t\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du + \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\right)du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du\,ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\right)du\,ds\Bigg] \\
&+ \frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b\frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&- RE_3 \int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)\,dz\biggr) + \biggl(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\biggr) \\
&\times \biggl(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)\,dz\,ds\biggr) \\
&+ \biggl(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)\,dz\biggr) + \biggl(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\biggr) \\
&\times \biggl(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)\,dz\,ds\biggr)\biggr].
\end{aligned}
$$

Then, for $t \in [a,b]$, we have

$$
\begin{aligned}
&\bigl|h_1(u,v)(t)\bigr| \\
&\quad\le \int_a^t \biggl(\frac{|\mu_1|}{|p(u)|}\int_a^u \bigl|\hat{f}(z)\bigr|\,dz\biggr)du + \frac{1}{|R|}\biggl[\bigl|\alpha_2(\beta_1+\beta_2)\bigr|\int_a^b \biggl(\frac{|\mu_1|}{|p(u)|}\int_a^u \bigl|\hat{f}(z)\bigr|\,dz\biggr)du \\
&\qquad + \bigl|\lambda_1(\beta_1+\beta_2)\bigr|\int_a^\eta \int_a^s \biggl(\frac{|\mu_2|}{|q(u)|}\int_a^u \bigl|\hat{g}(z)\bigr|\,dz\biggr)du\,ds \\
&\qquad + \bigl|\lambda_1\beta_2(\eta-a)\bigr|\int_a^b \biggl(\frac{|\mu_2|}{|q(u)|}\int_a^u \bigl|\hat{g}(z)\bigr|\,dz\biggr)du \\
&\qquad + \bigl|\lambda_1\lambda_3(\eta-a)\bigr|\int_\xi^b \int_a^s \biggl(\frac{|\mu_1|}{|p(u)|}\int_a^u \bigl|\hat{f}(z)\bigr|\,dz\biggr)du\,ds\biggr] \\
&\qquad + \frac{1}{|ER|}\biggl[\biggl(\bigl|E_4\alpha_2(\beta_1+\beta_2)\bigr|\int_a^b \frac{1}{|p(z)|}\,dz + \bigl|E_3\lambda_1(\beta_1+\beta_2)\bigr|\int_a^\eta \int_a^s \frac{1}{|q(z)|}\,dz\,ds \\
&\qquad + \bigl|E_3\lambda_1\beta_2(\eta-a)\bigr|\int_a^b \frac{1}{|q(z)|}\,dz + \bigl|E_4\lambda_1\lambda_3(\eta-a)\bigr|\int_\xi^b \int_a^s \frac{1}{|p(z)|}\,dz\,ds \\
&\qquad + |RE_4|\int_a^t \frac{1}{|p(z)|}\,dz\biggr)\biggl(\frac{|\alpha_4\mu_1|}{|p(b)|}\int_a^b \bigl|\hat{f}(z)\bigr|\,dz\biggr) + \biggl(\bigl|E_4\alpha_2(\beta_1+\beta_2)\bigr|\int_a^b \frac{1}{|p(z)|}\,dz \\
&\qquad + \bigl|E_3\lambda_1(\beta_1+\beta_2)\bigr|\int_a^\eta \int_a^s \frac{1}{|q(z)|}\,dz\,ds + \bigl|E_3\lambda_1\beta_2(\eta-a)\bigr|\int_a^b \frac{1}{|q(z)|}\,dz \\
&\qquad + \bigl|E_4\lambda_1\lambda_3(\eta-a)\bigr|\int_\xi^b \int_a^s \frac{1}{|p(z)|}\,dz\,ds + |RE_4|\int_a^t \frac{1}{|p(z)|}\,dz\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_a^\eta \frac{|\lambda_2\mu_2|}{|q(s)|}\int_a^s\left|\hat{g}(z)\right|dz\,ds\right) \\
&+\left(\left|E_2\alpha_2(\beta_1+\beta_2)\right|\int_a^b \frac{1}{|p(z)|}\,dz+\left|E_1\lambda_1(\beta_1+\beta_2)\right|\int_a^\eta\int_a^s \frac{1}{|q(z)|}\,dz\,ds\right. \\
&+\left|E_1\lambda_1\beta_2(\eta-a)\right|\int_a^b \frac{1}{|q(z)|}\,dz+\left|E_2\lambda_1\lambda_3(\eta-a)\right|\int_\xi^b\int_a^s \frac{1}{|p(z)|}\,dz\,ds \\
&\left.+|RE_2|\int_a^t \frac{1}{|p(z)|}\,dz\right)\left(\frac{|\beta_4\mu_2|}{|q(b)|}\int_a^b\left|\hat{g}(z)\right|dz\right)+\left(\left|E_2\alpha_2(\beta_1+\beta_2)\right|\int_a^b \frac{1}{|p(z)|}\,dz\right. \\
&+\left|E_1\lambda_1(\beta_1+\beta_2)\right|\int_a^\eta\int_a^s \frac{1}{|q(z)|}\,dz\,ds+\left|E_1\lambda_1\beta_2(\eta-a)\right|\int_a^b \frac{1}{|q(z)|}\,dz \\
&\left.+\left|E_2\lambda_1\lambda_3(\eta-a)\right|\int_\xi^b\int_a^s \frac{1}{|p(z)|}\,dz\,ds+|RE_2|\int_a^t \frac{1}{|p(z)|}\,dz\right) \\
&\left.\times\left(\int_\xi^b \frac{|\lambda_4\mu_1|}{|p(s)|}\int_a^s\left|\hat{f}(z)\right|dz\,ds\right)\right] \\
\le{}&\left\{\frac{\mu_1}{|R\bar{p}|}\left[\frac{(b-a)^2}{2}\left(|R|+\alpha_2(\beta_1+\beta_2)\right)+\frac{\lambda_1\lambda_2(\eta-a)[(b-a)^3-(\xi-a)^3]}{6}\right]\right. \\
&+\frac{1}{|RE|}\left[\left(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\right.\right. \\
&\left.+\frac{E_4\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\right)\left(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\right) \\
&+\left(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\right. \\
&\left.+\frac{E_2\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\right) \\
&\left.\left.\times\left(\frac{\lambda_4\mu_1[(b-a)^2-(\xi-a)^2]}{2\bar{p}}\right)\right]\right\} \\
&\times\|p_1\|\left[\psi_1\left(\nu^*\right)+\phi_1\left(\nu^*\right)\right] \\
&+\left\{\frac{\mu_2}{|2R\bar{q}|}\left[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3}+\lambda_1\beta_2(\eta-a)(b-a)^2\right]\right. \\
&+\frac{1}{|RE|}\left[\left(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}\right.\right. \\
&+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}} \\
&\left.+\frac{E_4\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\right)\left(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\right) \\
&+\left(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\right. \\
&\left.\left.\left.+\frac{E_2\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\right)\left(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\right)\right]\right\} \\
&\times\|p_2\|\left[\psi_2\left(\nu^*\right)+\phi_2\left(\nu^*\right)\right] \\
={}&\mathcal{D}_1\|p_1\|\left[\psi_1\left(\nu^*\right)+\phi_1\left(\nu^*\right)\right]+\mathcal{D}_2\|p_2\|\left[\psi_2\left(\nu^*\right)+\phi_2\left(\nu^*\right)\right].
\end{aligned}
$$

Similarly, we can obtain that

$$\left|h_2(u,v)(t)\right| \le \mathcal{D}_3\|p_1\|\left[\psi_1\left(v^*\right)+\phi_1\left(v^*\right)\right]+\mathcal{D}_4\|p_2\|\left[\psi_2\left(v^*\right)+\phi_2\left(v^*\right)\right].$$

Thus, we get

$$\begin{aligned}
\left\|h_1(u,v)\right\| &\le \mathcal{D}_1\|p_1\|\left[\psi_1\left(v^*\right)+\phi_1\left(v^*\right)\right]+\mathcal{D}_2\|p_2\|\left[\psi_2\left(v^*\right)+\phi_2\left(v^*\right)\right],\\
\left\|h_2(u,v)\right\| &\le \mathcal{D}_3\|p_1\|\left[\psi_1\left(v^*\right)+\phi_1\left(v^*\right)\right]+\mathcal{D}_4\|p_2\|\left[\psi_2\left(v^*\right)+\phi_2\left(v^*\right)\right],
\end{aligned}$$

where $\mathcal{D}_i$ $(i=1,\dots,4)$ are defined by (2.8). In consequence, we have

$$\begin{aligned}
\left\|(h_1,h_2)\right\| &= \left\|h_1(u,v)\right\| + \left\|h_2(u,v)\right\|\\
&\le (\mathcal{D}_1+\mathcal{D}_3)\|p_1\|\left[\psi_1\left(v^*\right)+\phi_1\left(v^*\right)\right]+(\mathcal{D}_2+\mathcal{D}_4)\|p_2\|\left[\psi_2\left(v^*\right)+\phi_2\left(v^*\right)\right]\\
&= \mathcal{E}_1\|p_1\|\left[\psi_1\left(v^*\right)+\phi_1\left(v^*\right)\right]+\mathcal{E}_2\|p_2\|\left[\psi_2\left(v^*\right)+\phi_2\left(v^*\right)\right]\\
&= \ell \quad \text{(constant)},
\end{aligned}$$

where $\mathcal{E}_i$, $i=1,2$, are defined in (2.7).

Next, we verify that $\Theta(u,v)$ is equicontinuous. Let $t_1,t_2\in[a,b]$ with $t_1<t_2$. Then, for $\hat{f}\in S_{F,(u,v)}$, $\hat{g}\in S_{G,(u,v)}$, we get

$$\begin{aligned}
&\left|h_1(u,v)(t_2)-h_1(u,v)(t_1)\right|\\
&= \left|\int_a^{t_2}\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(\tau)\,dz\right)du-\int_a^{t_1}\left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(\tau)\,dz\right)du\right.\\
&\quad+\left(\frac{E_4}{E}\left(\int_a^{t_2}\frac{1}{p(z)}\,dz-\int_a^{t_1}\frac{1}{p(z)}\,dz\right)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(\tau)\,dz\right)\right)\\
&\quad+\left(\frac{E_4}{E}\left(\int_a^{t_2}\frac{1}{p(z)}\,dz-\int_a^{t_1}\frac{1}{p(z)}\,dz\right)\left(\int_a^{\eta}\frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(\tau)\,dz\,ds\right)\right)\\
&\quad+\left(\frac{E_2}{E}\int_a^{t_2}\frac{1}{p(z)}\,dz-\int_a^{t_1}\frac{1}{p(z)}\,dz\right)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(\tau)\,dz\right))\\
&\quad+\left.\left(\frac{E_2}{E}\left(\int_a^{t_2}\frac{1}{p(z)}\,dz-\int_a^{t_1}\frac{1}{p(z)}\,dz\right)\left(\int_{\xi}^{b}\frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(\tau)\,dz\,ds\right)\right)\right|\\
&\le \left[\left(\frac{\mu_1}{|\bar{p}|}\right)\frac{(t_2-a)^2-(t_1-a)^2}{2}+\frac{E_4}{E|\bar{p}|}\left(\frac{\alpha_4\mu_1}{|p(b)|}\right)(t_2-t_1)(b-a)\right.\\
&\quad+\left.\frac{E_2}{E|\bar{p}|}\frac{(\lambda_4\mu_1)(t_2-t_1)[(b-a)^2-(\xi-a)^2]}{2}\right]\times\|p_1\|\left[\psi_1\left(v^*\right)+\phi_1\left(v^*\right)\right]\\
&\quad+\left[\frac{E_4}{E|\bar{p}|}\frac{(\lambda_2\mu_2)(t_2-t_1)(\eta-a)^2}{2\bar{q}}+\frac{E_2}{E|\bar{p}|}\left(\frac{\beta_4\mu_2}{|q(b)|}\right)(t_2-t_1)(b-a)\right]\\
&\quad\times\|p_2\|\left[\psi_2\left(v^*\right)+\phi_2\left(v^*\right)\right]\to 0 \quad \text{as } t_2\to t_1 \text{ independent of } (u,v).
\end{aligned}$$

Analogously, it can be shown that

$$\left|h_2(u,v)(t_2)-h_2(u,v)(t_1)\right|\to 0 \quad \text{as } t_2\to t_1 \text{ independent of } (u,v).$$

Obviously, the right-hand sides of the above inequalities tend to zero independently of $(u,v) \in B_{\nu^*}$ as $t_2 - t_1 \longrightarrow 0$. Therefore, the operator $\Theta(u,v)$ is equicontinuous, and hence we deduce that $\Theta(u,v) : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ is completely continuous by the Arzelá–Ascoli theorem.

In the next step, we show that $\Theta(u,v)$ is upper semicontinuous. Instead it will be established that $\Theta(u,v)$ has a closed graph in view of the fact that a completely continuous operator is upper semicontinuous if it has a closed graph. Let $(u_k,v_k) \longrightarrow (u_*,v_*)$ and $(h_k,\tilde{h_k}) \in \Theta(u_k,v_k)$ and $(h_k,\tilde{h_k}) \longrightarrow (h_*,\tilde{h_*})$. Then we have to show that $(h_*,\tilde{h_*}) \in \Theta(u_*,v_*)$. Associated with $(h_k,\tilde{h_k}) \in \Theta(u_k,v_k)$ and $\hat{f}_k \in S_{F,(u,v)}$, $\hat{g}_k \in S_{G,(u,v)}$, for each $t \in [a,b]$, we have

$$
\begin{aligned}
&h_k(u_k,v_k)(t) \\
&= \int_a^t \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z)\,dz\right) du + \frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2) \int_a^b \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z)\,dz\right) du \\
&\quad + \lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_k(z)\,dz\right) du\,ds \\
&\quad - \lambda_1\beta_2(\eta-a) \int_a^b \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_k(z)\,dz\right) du \\
&\quad + \lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_k(z)\,dz\right) du\,ds\bigg] \\
&\quad + \frac{1}{ER}\bigg[\bigg(E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_4 \int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}_k(z)\,dz\right) + \bigg(-E_4\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_3\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_4\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4 \int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)} \int_a^s \hat{g}_k(z)\,dz\,ds\right) \\
&\quad + \bigg(E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_2 \int_a^t \frac{1}{p(z)}\,dz\bigg)\left(\frac{\beta_4\mu_2}{q(b)} \int_a^b \hat{g}_k(z)\,dz\right) + \bigg(-E_2\alpha_2(\beta_1+\beta_2) \int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_1\lambda_1(\beta_1+\beta_2) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a) \int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_2\lambda_1\lambda_3(\eta-a) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2 \int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\quad \times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)} \int_a^s \hat{f}_k(z)\,dz\,ds\right)\bigg]
\end{aligned}
$$

and

$$\begin{aligned}
&\tilde{h}_k(u_k,v_k)(t)\\
&= \int_a^t \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\,dz\right)du + \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)\,dz\right)du\\
&\quad + \lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\,dz\right)du\,ds\\
&\quad - \beta_2(\alpha_1+\alpha_2)\int_a^b \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\,dz\right)du\\
&\quad + \lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)\,dz\right)du\,ds\Bigg]\\
&\quad + \frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\quad + E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\quad - RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)\,dz\right) + \Bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz\\
&\quad + E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz\\
&\quad + E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_k(z)\,dz\,ds\right)\\
&\quad + \Bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\quad + E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\quad - RE_1\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)\,dz\right) + \Bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz\\
&\quad + E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz\\
&\quad + E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\Bigg)\\
&\quad \times\left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)\,dz\,ds\right)\Bigg].
\end{aligned}$$

Consider the continuous linear operators $\Psi_1, \Psi_2 : L^1([a,b],\mathcal{F}\times\mathcal{F}) \longrightarrow C([a,b],\mathcal{F}\times\mathcal{F})$ given by

$$\begin{aligned}
\Psi_1(u,v)(t) &= \int_a^t \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du\\
&\quad + \frac{1}{R}\Bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du\,ds\\
&-\lambda_1\beta_2(\eta-a)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\right)du\,ds\Bigg]\\
&+\frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}(z)\,dz\right)+\Bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_4\int_a^t\frac{1}{p(z)}\,dz\Bigg)\\
&\times\left(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}(z)\,dz\,ds\right)\\
&+\Bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}(z)\,dz\right)+\Bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_2\int_a^t\frac{1}{p(z)}\,dz\Bigg)\\
&\times\left(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}(z)\,dz\,ds\right)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_2(u,v)(t)={}&\int_a^t\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du+\frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\right)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du\,ds\\
&-\beta_2(\alpha_1+\alpha_2)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}(z)\,dz\right)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}(z)\,dz\right)du\,ds\Bigg]\\
&+\frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)\,dz\bigg) + \bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\times \bigg(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)\,dz\,ds\bigg) \\
&+ \bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)\,dz\bigg) + \bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\times \bigg(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)\,dz\,ds\bigg)\bigg].
\end{aligned}
$$

From Lemma 3.1, we know that $(\Psi_1,\Psi_2)\circ(S_F,S_G)$ is a closed graph operator. Moreover, we have $(h_k,\tilde{h_k})\in(\Psi_1,\Psi_2)\circ(S_{F,(u_k,v_k)},S_{G,(u_k,v_k)})$ for all k. Since $(u_k,v_k)\longrightarrow(u_*,v_*)$, $(h_k,\tilde{h_k})\longrightarrow(h_*,\tilde{h_*})$, it follows that $\hat{f}_*\in S_{F,(u,v)}$, $\hat{g}_*\in S_{G,(u,v)}$ such that

$$
\begin{aligned}
&h_*(u_*,v_*)(t) \\
&= \int_a^t\bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)\,dz\bigg)du + \frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)\,dz\bigg)du \\
&\quad + \lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\bigg(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)\,dz\bigg)du\,ds \\
&\quad - \lambda_1\beta_2(\eta-a)\int_a^b\bigg(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)\,dz\bigg)du \\
&\quad + \lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)\,dz\bigg)du\,ds\bigg] \\
&\quad + \frac{1}{ER}\bigg[\bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_4\int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_*(z)\,dz\bigg) + \bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+ E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_*(z)\,dz\,ds\biggr) \\
&+ \biggl(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_2\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_*(z)\,dz\biggr) + \biggl(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t \frac{1}{p(z)}\,dz\biggr) \\
&\times \biggl(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_*(z)\,dz\,ds\biggr)\biggr]
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{h}_*(u_*,v_*)(t) \\
&= \int_a^t \biggl(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)\,dz\biggr)du + \frac{1}{R}\biggl[-\alpha_2\lambda_3(b-\xi)\int_a^b \biggl(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)\,dz\biggr)du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \biggl(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)\,dz\biggr)du\,ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b \biggl(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_*(z)\,dz\biggr)du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \biggl(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_*(z)\,dz\biggr)du\,ds\biggr] \\
&+ \frac{1}{ER}\biggl[\biggl(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_*(z)\,dz\biggr) + \biggl(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_*(z)\,dz\,ds\biggr) \\
&+ \biggl(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&- RE_1 \int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_*(z)\,dz\bigg) + \bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\times \bigg(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_*(z)\,dz\,ds\bigg)\bigg],
\end{aligned}
$$

which leads to the conclusion that $(h_k, \tilde{h}_k) \in \Theta(u_*, v_*)$.

Finally, we show that there exists an open set $U \subseteq \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ with $(u,v) \notin \epsilon\Theta(u,v)$ for any $\epsilon \in (0,1)$ and all $(u,v) \in \partial U$. Let $\epsilon \in (0,1)$ and $(u,v) \in \epsilon\Theta(u,v)$. Then there exist $\hat{f} \in S_{F,(u,v)}$ and $\hat{g} \in S_{G,(u,v)}$ such that, for $t \in [a,b]$, we have

$$
\begin{aligned}
u(t) = {}& \epsilon \int_a^t \bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\bigg)\,du + \frac{\epsilon}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b \bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\bigg)\,du \\
&+ \lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \bigg(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\bigg)\,du\,ds \\
&- \lambda_1\beta_2(\eta-a)\int_a^b \bigg(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\bigg)\,du \\
&+ \lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\bigg)\,du\,ds\bigg] \\
&+ \frac{\epsilon}{ER}\bigg[\bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_4\int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)\,dz\bigg) + \bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\bigg) \\
&\times \bigg(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)\,dz\,ds\bigg) \\
&+ \bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_2\int_a^t \frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)\,dz\bigg) + \bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+ E_2\lambda_1\lambda_3(\eta - a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t \frac{1}{p(z)}\,dz\Big) \\
&\times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)\,dz\,ds\right)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
v(t) = {}& \epsilon\int_a^t \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du + \frac{\epsilon}{R}\Big[-\alpha_2\lambda_3(b-\xi)\int_a^b \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du\,ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du\,ds\Big] \\
&+ \frac{\epsilon}{ER}\Big[\Big(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\,dz\Big)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)\,dz\right) + \Big(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\Big) \\
&\times \left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)\,dz\,ds\right) \\
&+ \Big(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\Big)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)\,dz\right) + \Big(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\Big) \\
&\times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)\,dz\,ds\right)\Big].
\end{aligned}
$$

Using the arguments employed in the second step, we find that

$$
\|u\| \le \mathcal{D}_1\|p_1\|\big[\psi_1\big(\|u\|\big) + \phi_1\big(\|v\|\big)\big] + \mathcal{D}_2\|p_2\|\big[\psi_2\big(\|u\|\big) + \phi_2\big(\|v\|\big)\big]
$$

and

$$\|v\| \leq \mathcal{D}_3\|p_1\|\big[\psi_1\big(\|u\|\big) + \phi_1\big(\|v\|\big)\big] + \mathcal{D}_4\|p_2\|\big[\psi_2\big(\|u\|\big) + \phi_2\big(\|v\|\big)\big].$$

Then we have

$$\begin{aligned}
\big\|(u,v)\big\| &= \|u\| + \|v\| \\
&\leq (\mathcal{D}_1 + \mathcal{D}_3)\|p_1\|\big[\psi_1\big(\|u\|\big) + \phi_1\big(\|v\|\big)\big] \\
&\quad + (\mathcal{D}_2 + \mathcal{D}_4)\|p_2\|\big[\psi_2\big(\|u\|\big) + \phi_2\big(\|v\|\big)\big] \\
&\leq \mathcal{E}_1\|p_1\|\big[\psi_1\big(\|u\|\big) + \phi_1\big(\|v\|\big)\big] + \mathcal{E}_2\|p_2\|\big[\psi_2\big(\|u\|\big) + \phi_2\big(\|v\|\big)\big],
\end{aligned}$$

where $\mathcal{E}_i$, $i = 1, 2$, are given by (2.7). Consequently, we have

$$\frac{\|(u,v)\|}{\mathcal{E}_1\|p_1\|[\psi_1(\|u\|) + \phi_1(\|v\|)] + \mathcal{E}_2\|p_2\|[\psi_2(\|u\|) + \phi_2(\|v\|)]} \leq 1.$$

According to (H_3), there exists N such that $\|(u,v)\| \neq N$. Let us set

$$U = \big\{(u,v) \in (\mathcal{F} \times \mathcal{F}) : \big\|(u,v)\big\| < N\big\}.$$

Observe that the operator $\Theta : \bar{U} \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{F}) \times \mathcal{P}_{cp,cv}(\mathcal{F})$ is completely continuous and upper semicontinuous. From the choice of U, there is no $(u,v) \in \partial U$ such that $(u,v) \in \epsilon\Theta(u,v)$ for some $\epsilon \in (0,1)$. Therefore, by the nonlinear alternative of Leray–Schauder type (Lemma 3.2), we deduce that Θ has a fixed point $(u,v) \in \bar{U}$ which is a solution of problem (1.1). □

4 The Lipschitz case

The forthcoming result is based on the fixed point theorem for contraction multivalued operators due to Covitz and Nadler [32], which is stated below.

Lemma 4.1 (Covitz and Nadler) *Let (X,d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then $FixG \neq \emptyset$.*

Remark 4.2 Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [33]).

Theorem 4.3 *Assume that the following conditions hold*:

(H_5) *$F, G : [a,b] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ are such that $F(\cdot,u,v), G(\cdot,u,v) : [a,b] \to \mathcal{P}_{cp}(\mathbb{R})$ are measurable for each $u, v \in \mathbb{R}$;*

(H_6) *For almost all $t \in [a,b]$ and $u,v,\bar{u},\bar{v} \in \mathbb{R}$ with $\mathcal{B}_1,\mathcal{B}_2 \in C([a,b],\mathbb{R}^+)$,*

$$H_d(F(t,u,v),F(t,\bar{u},\bar{v}) \le \mathcal{B}_1(t)\bigl(|u-\bar{u}|+|v-\bar{v}|\bigr),$$
$$H_d(G(t,u,v),G(t,\bar{u},\bar{v}) \le \mathcal{B}_2(t)\bigl(|u-\bar{u}|+|v-\bar{v}|\bigr),$$

and $d(0,F(t,0,0)) \le \mathcal{B}_1(t)$, $d(0,G(t,0,0)) \le \mathcal{B}_2(t)$.

Then the boundary value problem (1.1) *has at least one solution on $[a,b]$ if $\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| < 1$, where $\mathcal{E}_1$, $\mathcal{E}_2$ are given in* (2.7).

Proof Consider the multivalued map $\Theta : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F} \times \mathcal{F})$ defined at the beginning of the proof of Theorem 3.3. Observe that the fixed points of $\Theta(u,v)$ are solutions of problem (1.1).

Notice that the sets $S_{F,(u,v)}$ and $S_{G,(u,v)}$ are nonempty, and consequently $\Theta \neq \emptyset$ for each $(u,v) \in \mathcal{F} \times \mathcal{F}$. Then, by assumption (H_5), the multivalued maps $F(\cdot,(u,v))$ and $G(\cdot,(u,v))$ are measurable, and thus admit measurable selections.

Now we shall show that the operator $\Theta(u,v)$ satisfies the hypothesis of Lemma 4.1. Firstly, we verify that $\Theta(u,v) \in \mathcal{P}_{cl}(\mathcal{F}) \times \mathcal{P}_{cl}(\mathcal{F})$ for each $(u,v) \in \mathcal{F} \times \mathcal{F}$. Let $(h_k,\tilde{h}_k) \in \Theta(u_k,v_k)$ such that $(h_k,\tilde{h}_k)$ converges to $(h,\tilde{h})$ as $k \to \infty$ in $\mathcal{F} \times \mathcal{F}$. So $(h,\tilde{h}) \in \mathcal{F} \times \mathcal{F}$, and there exist $\hat{f}_k \in S_{F,(u_k,v_k)}$ and $\hat{g}_k \in S_{G,(u_k,v_k)}$ such that, for each $t \in [a,b]$, we have

$$\begin{aligned}
&h_k(u_k,v_k)(t)\\
&= \int_a^t \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)\,dz\right)du + \frac{1}{R}\biggl[-\alpha_2(\beta_1+\beta_2)\int_a^b \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)\,dz\right)du\\
&\quad + \lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\,dz\right)du\,ds\\
&\quad - \lambda_1\beta_2(\eta-a)\int_a^b \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\,dz\right)du\\
&\quad + \lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)\,dz\right)du\,ds\biggr]\\
&\quad + \frac{1}{ER}\biggl[\biggl(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\quad + E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\quad - RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)\,dz\right) + \biggl(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz\\
&\quad + E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz\\
&\quad + E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_k(z)\,dz\,ds\right)\\
&\quad + \biggl(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\quad + E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds
\end{aligned}$$

$$
\begin{aligned}
&- RE_2 \int_a^t \frac{1}{p(z)}\, dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)\, dz\bigg) + \bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\, dz \\
&+ E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\, dz\, ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\, dz \\
&+ E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\, dz\, ds + RE_2\int_a^t \frac{1}{p(z)}\, dz\bigg) \\
&\times \bigg(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)\, dz\, ds\bigg)\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{h}_k(u_k, v_k)(t) \\
&= \int_a^t \bigg(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\, dz\bigg)\, du + \frac{1}{R}\bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)\, dz\bigg)\, du \\
&+ \lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \bigg(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\, dz\bigg)\, du\, ds \\
&- \beta_2(\alpha_1+\alpha_2)\int_a^b \bigg(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_k(z)\, dz\bigg)\, du \\
&+ \lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \bigg(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_k(z)\, dz\bigg)\, du\, ds\bigg] \\
&+ \frac{1}{ER}\bigg[\bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\, dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\, dz\, ds \\
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\, dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\, dz\, ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\, dz\bigg)\bigg(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_k(z)\, dz\bigg) + \bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\, dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\, dz\, ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\, dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\, dz\, ds + RE_3\int_a^t \frac{1}{p(z)}\, dz\bigg)\bigg(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_k(z)\, dz\, ds\bigg) \\
&+ \bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\, dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\, dz\, ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\, dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\, dz\, ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\, dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_k(z)\, dz\bigg) + \bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\, dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\, dz\, ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\, dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\, dz\, ds + RE_1\int_a^t \frac{1}{p(z)}\, dz\bigg) \\
&\times \bigg(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_k(z)\, dz\, ds\bigg)\bigg].
\end{aligned}
$$

Since F and G have compact values, we pass onto subsequences (if necessary) to get that $\hat{f}_k$ and $\hat{g}_k$ converge to $\hat{f}$ and $\hat{g}$ in $L^1([a,b],\mathbb{R})$ respectively. Then $\hat{f} \in S_{F,(u,v)}$ and $\hat{g} \in S_{G,(u,v)}$, and for each $t \in [a,b]$, we have

$$\begin{aligned}
&h_k(u_k,v_k)(t)\\
&\quad\to h(u,v)(t)\\
&\quad= \int_a^t \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du + \frac{1}{R}\biggl[-\alpha_2(\beta_1+\beta_2)\int_a^b \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du\\
&\qquad + \lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du\,ds\\
&\qquad - \lambda_1\beta_2(\eta-a)\int_a^b \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}(z)\,dz\right)du\\
&\qquad + \lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}(z)\,dz\right)du\,ds\biggr]\\
&\qquad + \frac{1}{ER}\biggl[\biggl(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\qquad + E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\qquad - RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}(z)\,dz\right) + \biggl(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz\\
&\qquad + E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz\\
&\qquad + E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}(z)\,dz\,ds\right)\\
&\qquad + \biggl(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds\\
&\qquad + E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds\\
&\qquad - RE_2\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}(z)\,dz\right) + \biggl(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz\\
&\qquad + E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz\\
&\qquad + E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t \frac{1}{p(z)}\,dz\biggr)\\
&\qquad \times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}(z)\,dz\,ds\right)\biggr]
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{h}_k(u_k,v_k)(t)\\
&\quad\to \tilde{h}(u,v)(t)
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^t \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)\,dz \right) du + \frac{1}{R} \Bigg[-\alpha_2\lambda_3(b-\xi) \int_a^b \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)\,dz \right) du \\
&\quad + \lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)\,dz \right) du\,ds \\
&\quad - \beta_2(\alpha_1+\alpha_2) \int_a^b \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}(z)\,dz \right) du \\
&\quad + \lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}(z)\,dz \right) du\,ds \Bigg] \\
&\quad + \frac{1}{ER} \Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi) \int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_3\beta_2(\alpha_1+\alpha_2) \int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_3 \int_a^t \frac{1}{p(z)}\,dz \Bigg) \left(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}(z)\,dz \right) + \Bigg(-E_4\alpha_2\lambda_3(b-\xi) \int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_3\lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2) \int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_4\lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3 \int_a^t \frac{1}{p(z)}\,dz \Bigg) \left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)} \int_a^s \hat{g}(z)\,dz\,ds \right) \\
&\quad + \Bigg(E_2\alpha_2\lambda_3(b-\xi) \int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_1\beta_2(\alpha_1+\alpha_2) \int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_1 \int_a^t \frac{1}{p(z)}\,dz \Bigg) \left(\frac{\beta_4\mu_2}{q(b)} \int_a^b \hat{g}(z)\,dz \right) + \Bigg(-E_2\alpha_2\lambda_3(b-\xi) \int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_1\lambda_1\lambda_3(b-\xi) \int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2) \int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_2\lambda_3(\alpha_1+\alpha_2) \int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1 \int_a^t \frac{1}{p(z)}\,dz \Bigg) \\
&\quad \times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)} \int_a^s \hat{f}(z)\,dz\,ds \right) \Bigg].
\end{aligned}
$$

Therefore $(u,v) \in \Theta$, and hence $\Theta(u,v)$ is closed.

Next, we show that Θ is a contraction on $\mathcal{P}_{cl}(\mathcal{F}) \times \mathcal{P}_{cl}(\mathcal{F})$, that is, there exists a positive number $\gamma < 1$ such that

$$
H_d\big(\Theta(u,v), \Theta(\bar{u},\bar{v})\big) \le \gamma \big(\|u-\bar{u}\| + \|v-\bar{v}\| \big) \quad \text{for each } u,v,\bar{u},\bar{v} \in \mathcal{F}.
$$

Let $(u,\bar{u}), (v,\bar{v}) \in \mathcal{F} \times \mathcal{F}$ and $(h_1, \tilde{h_1}) \in \Theta(u,v)$. Then there exist $\hat{f}_1(t) \in S_{F,(u,v)}$ and $\hat{g}_1(t) \in S_{G,(u,v)}$ such that, for each $t \in [a,b]$, we obtain

$$
\begin{aligned}
&h_1(u,v)(t) \\
&= \int_a^t \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_1(z)\,dz \right) du + \frac{1}{R} \Bigg[-\alpha_2(\beta_1+\beta_2) \int_a^b \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_1(z)\,dz \right) du
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\right)du\,ds\\
&-\lambda_1\beta_2(\eta-a)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\right)du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)\,dz\right)du\,ds\Bigg]\\
&+\frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\hat{f}_1(z)\,dz\right)+\Bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_4\int_a^t\frac{1}{p(z)}\,dz\Bigg)\left(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\hat{g}_1(z)\,dz\,ds\right)\\
&+\Bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b\hat{g}_1(z)\,dz\right)+\Bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_2\int_a^t\frac{1}{p(z)}\,dz\Bigg)\\
&\times\left(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\hat{f}_1(z)\,dz\,ds\right)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{h}_1(u,v)(t)\\
&=\int_a^t\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\right)du+\frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)\,dz\right)du\\
&+\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\right)du\,ds\\
&-\beta_2(\alpha_1+\alpha_2)\int_a^b\left(\frac{\mu_2}{q(u)}\int_a^u\hat{g}_1(z)\,dz\right)du\\
&+\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u\hat{f}_1(z)\,dz\right)du\,ds\Bigg]\\
&+\frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_3\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_1(z)\,dz\biggr) + \biggl(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_1(z)\,dz\,ds\biggr) \\
&+ \biggl(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds \\
&+ E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds \\
&- RE_1\int_a^t \frac{1}{p(z)}\,dz\biggr)\biggl(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_1(z)\,dz\biggr) + \biggl(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&+ E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta\int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&+ E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b\int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\biggr) \\
&\times \biggl(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_1(z)\,dz\,ds\biggr)\biggr].
\end{aligned}
$$

By (H_6), we have that

$$H_d\bigl(F(t,u,v),F(t,\bar{u},\bar{v})\bigr) \le \mathcal{B}_1(t)\bigl(\bigl|u(t)-\bar{u}(t)\bigr| + \bigl|v(t)-\bar{v}(t)\bigr|\bigr)$$

and

$$H_d\bigl(G(t,u,v),G(t,\bar{u},\bar{v})\bigr) \le \mathcal{B}_2(t)\bigl(\bigl|u(t)-\bar{u}(t)\bigr| + \bigl|v(t)-\bar{v}(t)\bigr|\bigr).$$

So there exist $\hat{\vartheta}_f \in F(t,u(t),v(t))$ and $\hat{\vartheta}_g \in G(t,u(t),v(t))$ such that

$$
\begin{aligned}
&\bigl|\hat{f}_1(t) - \hat{\vartheta}_f\bigr| \le \mathcal{B}_1(t)\bigl(\bigl|u(t)-\bar{u}(t)\bigr| + \bigl|v(t)-\bar{v}(t)\bigr|\bigr), \\
&\bigl|\hat{g}_1(t) - \hat{\vartheta}_g\bigr| \le \mathcal{B}_2(t)\bigl(\bigl|u(t)-\bar{u}(t)\bigr| + \bigl|v(t)-\bar{v}(t)\bigr|\bigr).
\end{aligned}
$$

Define $W_1, W_2 : [a,b] \to \mathcal{P}(\mathbb{R})$ by

$$W_1(t) = \bigl\{\hat{\vartheta}_f \in L^1\bigl([a,b),\mathbb{R}\bigr) : \bigl|\hat{f}_1(t) - \hat{\vartheta}_f\bigr| \le \mathcal{B}_1(t)\bigl(\bigl|u(t)-\bar{u}(t)\bigr| + \bigl|v(t)-\bar{v}(t)\bigr|\bigr)\bigr\}$$

and

$$W_2(t) = \bigl\{\hat{\vartheta}_g \in L^1\bigl([a,b),\mathbb{R}\bigr) : \bigl|\hat{g}_1(t) - \hat{\vartheta}_g\bigr| \le \mathcal{B}_2(t)\bigl(\bigl|u(t)-\bar{u}(t)\bigr| + \bigl|v(t)-\bar{v}(t)\bigr|\bigr)\bigr\}.$$

Since the multivalued operators $W_1(t) \cap F(t,u(t)v(t))$ and $W_2(t) \cap G(t,u(t),v(t))$ are measurable, there exist functions $\hat{f}_2(t)$, $\hat{g}_2(t)$ which are measurable selections for W_1 and W_2.

Thus $\hat{f}_2(t) \in F(t,u(t),v(t))$, $\hat{g}_2(t) \in G(t,u(t),v(t))$, and for each $t \in [a,b]$, we have

$$\left|\hat{f}_1(t) - \hat{f}_2(t)\right| \le \mathcal{B}_1(t)\left(\left|u(t) - \bar{u}(t)\right| + \left|v(t) - \bar{v}(t)\right|\right)$$

and

$$\left|\hat{g}_1(t) - \hat{g}_2(t)\right| \le \mathcal{B}_2(t)\left(\left|u(t) - \bar{u}(t)\right| + \left|v(t) - \bar{v}(t)\right|\right).$$

For each $t \in [a,b]$, let us define

$$\begin{aligned}
&h_2(u,v)(t) \\
&= \int_a^t \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)\,dz\right) du + \frac{1}{R}\biggl[-\alpha_2(\beta_1+\beta_2)\int_a^b \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)\,dz\right) du \\
&\quad + \lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)\,dz\right) du\,ds \\
&\quad - \lambda_1\beta_2(\eta-a)\int_a^b \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)\,dz\right) du \\
&\quad + \lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)} \int_a^u \hat{f}_2(z)\,dz\right) du\,ds\biggr] \\
&\quad + \frac{1}{ER}\biggl[\biggl(E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\frac{\alpha_4\mu_1}{p(b)} \int_a^b \hat{f}_2(z)\,dz\right) + \biggl(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_4\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)} \int_a^s \hat{g}_2(z)\,dz\,ds\right) \\
&\quad + \biggl(E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_2\int_a^t \frac{1}{p(z)}\,dz\biggr)\left(\frac{\beta_4\mu_2}{q(b)} \int_a^b \hat{g}_2(z)\,dz\right) + \biggl(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\lambda_1\beta_2(\eta-a)\int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_2\int_a^t \frac{1}{p(z)}\,dz\biggr) \\
&\quad \times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)} \int_a^s \hat{f}_2(z)\,dz\,ds\right)\biggr]
\end{aligned}$$

and

$$
\begin{aligned}
&\tilde{h}_2(u,v)(t) \\
&= \int_a^t \left(\frac{\mu_2}{q(u)} \int_a^u \hat{g}_2(z)\,dz\right) du + \frac{1}{R}\Bigg[-\alpha_2\lambda_3(b-\xi)\int_a^b \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_2(z)\,dz\right) du \\
&\quad + \lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_2(z)\,dz\right) du\,ds \\
&\quad - \beta_2(\alpha_1+\alpha_2)\int_a^b \left(\frac{\mu_2}{q(u)}\int_a^u \hat{g}_2(z)\,dz\right) du \\
&\quad + \lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \left(\frac{\mu_1}{p(u)}\int_a^u \hat{f}_2(z)\,dz\right) du\,ds\Bigg] \\
&\quad + \frac{1}{ER}\Bigg[\Bigg(E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b \hat{f}_2(z)\,dz\right) + \Bigg(-E_4\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_3\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_3\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_4\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_3\int_a^t \frac{1}{p(z)}\,dz\Bigg) \\
&\quad \times \left(\int_a^\eta \frac{\lambda_2\mu_2}{q(s)}\int_a^s \hat{g}_2(z)\,dz\,ds\right) \\
&\quad + \Bigg(E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz - E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds \\
&\quad + E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz - E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds \\
&\quad - RE_1\int_a^t \frac{1}{p(z)}\,dz\Bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b \hat{g}_2(z)\,dz\right) + \Bigg(-E_2\alpha_2\lambda_3(b-\xi)\int_a^b \frac{1}{p(z)}\,dz \\
&\quad + E_1\lambda_1\lambda_3(b-\xi)\int_a^\eta \int_a^s \frac{1}{q(z)}\,dz\,ds - E_1\beta_2(\alpha_1+\alpha_2)\int_a^b \frac{1}{q(z)}\,dz \\
&\quad + E_2\lambda_3(\alpha_1+\alpha_2)\int_\xi^b \int_a^s \frac{1}{p(z)}\,dz\,ds + RE_1\int_a^t \frac{1}{p(z)}\,dz\Bigg) \\
&\quad \times \left(\int_\xi^b \frac{\lambda_4\mu_1}{p(s)}\int_a^s \hat{f}_2(z)\,dz\,ds\right)\Bigg].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|h_1(u,v)(t) - h_2(u,v)(t)\right| \\
&\leq \int_a^t \left(\frac{\mu_1}{p(u)}\int_a^u \left|\hat{f}_1(z) - \hat{f}_2(z)\right|dz\right) du
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\bigg(\frac{\mu_1}{p(u)}\int_a^u\big|\hat{f}_1(z)-\hat{f}_2(z)\big|\,dz\bigg)\,du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\bigg(\frac{\mu_2}{q(u)}\int_a^u\big|\hat{g}_1(z)-\hat{g}_2(z)\big|\,dz\bigg)\,du\,ds\\
&-\lambda_1\beta_2(\eta-a)\int_a^b\bigg(\frac{\mu_2}{q(u)}\int_a^u\big|\hat{g}_1(z)-\hat{g}_2(z)\big|\,dz\bigg)\,du\\
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\bigg(\frac{\mu_1}{p(u)}\int_a^u\big|\hat{f}_1(z)-\hat{f}_2(z)\big|\,dz\bigg)\,du\,ds\bigg]\\
&+\frac{1}{ER}\bigg[\bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\big|\hat{f}_1(z)-\hat{f}_2(z)\big|\,dz\bigg)\\
&+\bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_4\int_a^t\frac{1}{p(z)}\,dz\bigg)\\
&\times\bigg(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\big|\hat{g}_1(z)-\hat{g}_2(z)\big|\,dz\,ds\bigg)\\
&+\bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}\,dz\bigg)\bigg(\frac{\beta_4\mu_2}{q(b)}\int_a^b\big|\hat{g}_1(z)-\hat{g}_2(z)\big|\,dz\bigg)+\bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz\\
&+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz\\
&+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_2\int_a^t\frac{1}{p(z)}\,dz\bigg)\\
&\times\bigg(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\big|\hat{f}_1(z)-\hat{f}_2(z)\big|\,dz\,ds\bigg)\bigg]\\
\le{}&\int_a^t\bigg(\frac{\mu_1}{p(u)}\int_a^u\mathcal{B}_1(z)\big(\big|u(z)-\bar{u}(z)\big|+\big|v(z)-\bar{v}(z)\big|\big)\,dz\bigg)\,du\\
&+\frac{1}{R}\bigg[-\alpha_2(\beta_1+\beta_2)\int_a^b\bigg(\frac{\mu_1}{p(u)}\int_a^u\mathcal{B}_1(z)\big(\big|u(z)-\bar{u}(z)\big|+\big|v(z)-\bar{v}(z)\big|\big)\,dz\bigg)\,du\\
&+\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\bigg(\frac{\mu_2}{q(u)}\int_a^u\mathcal{B}_2(z)\big(\big|u(z)-\bar{u}(z)\big|+\big|v(z)-\bar{v}(z)\big|\big)\,dz\bigg)\,du\,ds\\
&-\lambda_1\beta_2(\eta-a)\int_a^b\bigg(\frac{\mu_2}{q(u)}\int_a^u\mathcal{B}_2(z)\big(\big|u(z)-\bar{u}(z)\big|+\big|v(z)-\bar{v}(z)\big|\big)\,dz\bigg)\,du
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\left(\frac{\mu_1}{p(u)}\int_a^u\mathcal{B}_1(z)\big(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|\big)\,dz\right)du\,ds\Bigg]\\
&+\frac{1}{ER}\Bigg[\bigg(E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_4\int_a^t\frac{1}{p(z)}\,dz\bigg)\left(\frac{\alpha_4\mu_1}{p(b)}\int_a^b\mathcal{B}_1(z)\big(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|\big)\,dz\right)\\
&+\bigg(-E_4\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz+E_3\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&-E_3\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz+E_4\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_4\int_a^t\frac{1}{p(z)}\,dz\bigg)\\
&\times\left(\int_a^\eta\frac{\lambda_2\mu_2}{q(s)}\int_a^s\mathcal{B}_2(z)\big(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|\big)\,dz\,ds\right)\\
&+\bigg(E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz-E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&+E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz-E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds\\
&-RE_2\int_a^t\frac{1}{p(z)}\,dz\bigg)\left(\frac{\beta_4\mu_2}{q(b)}\int_a^b\mathcal{B}_2(z)\big(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|\big)\,dz\right)\\
&+\bigg(-E_2\alpha_2(\beta_1+\beta_2)\int_a^b\frac{1}{p(z)}\,dz+E_1\lambda_1(\beta_1+\beta_2)\int_a^\eta\int_a^s\frac{1}{q(z)}\,dz\,ds\\
&-E_1\lambda_1\beta_2(\eta-a)\int_a^b\frac{1}{q(z)}\,dz+E_2\lambda_1\lambda_3(\eta-a)\int_\xi^b\int_a^s\frac{1}{p(z)}\,dz\,ds+RE_2\int_a^t\frac{1}{p(z)}\,dz\bigg)\\
&\times\left(\int_\xi^b\frac{\lambda_4\mu_1}{p(s)}\int_a^s\mathcal{B}_1(z)\big(|u(z)-\bar{u}(z)|+|v(z)-\bar{v}(z)|\big)|\,dz\,ds\right)\Bigg]\\
\le&\left\{\frac{\mu_1}{|R\bar{p}|}\left[\frac{(b-a)^2}{2}\big(|R|+\alpha_2(\beta_1+\beta_2)\big)+\frac{\lambda_1\lambda_2(\eta-a)[(b-a)^3-(\xi-a)^3]}{6}\right]\right.\\
&+\frac{1}{|RE|}\Bigg[\bigg(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\bigg)\left(\frac{\alpha_4\mu_1(b-a)}{|p(b)|}\right)\\
&+\bigg(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\bigg)\\
&\left.\times\left(\frac{\lambda_4\mu_1[(b-a)^2-(\xi-a)^2]}{2\bar{p}}\right)\Bigg]\right\}\\
&\times\|\mathcal{B}_1\|\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big)\\
&+\left\{\frac{\mu_2}{|2R\bar{q}|}\left[\frac{\lambda_1(\beta_1+\beta_2)(\eta-a)^3}{3}+\lambda_1\beta_2(\eta-a)(b-a)^2\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|RE|}\Bigg[\Bigg(\frac{E_4\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_3\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_3\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_4\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_4(b-a)}{\bar{p}}\Bigg)\Bigg(\frac{\lambda_2\mu_2(\eta-a)^2}{2\bar{q}}\Bigg)\\
&+\Bigg(\frac{E_2\alpha_2(\beta_1+\beta_2)(b-a)}{\bar{p}}+\frac{E_1\lambda_1(\beta_1+\beta_2)(\eta-a)^2}{2\bar{q}}+\frac{E_1\lambda_1\beta_2(\eta-a)(b-a)}{\bar{q}}\\
&+\frac{E_2\lambda_1\lambda_3(\eta-a)[(b-a)^2-(\xi-a)^2]}{2\bar{p}}+\frac{RE_2(b-a)}{\bar{p}}\Bigg)\Bigg(\frac{\beta_4\mu_2(b-a)}{|q(b)|}\Bigg)\Bigg]\Bigg\}\\
&\times\|\mathcal{B}_2\|\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big)\\
&\le\big(\mathcal{D}_1\|\mathcal{B}_1\|+\mathcal{D}_2\|\mathcal{B}_2\|\big)\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big),
\end{aligned}
$$

which implies that

$$
\big|h_1(u,v)(t)-h_2(u,v)(t)\big|\le\big(\mathcal{D}_1\|\mathcal{B}_1\|+\mathcal{D}_2\|\mathcal{B}_2\|\big)\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big).
$$

In a similar manner, one can establish that

$$
\big|\tilde{h}_1(u,v)(t)-\tilde{h}_2(u,v)(t)\big|\le\big(\mathcal{D}_3\|\mathcal{B}_1\|+\mathcal{D}_4\|\mathcal{B}_2\|\big)\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big).
$$

In consequence, we get

$$
\begin{aligned}
\big\|(h_1,h_2),(\tilde{h}_1,\tilde{h}_2)\big\|&\le\big[(\mathcal{D}_1+\mathcal{D}_3)\|\mathcal{B}_1\|+(\mathcal{D}_2+\mathcal{D}_4)\|\mathcal{B}_2\|\big)\big]\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big)\\
&\le\big[\big(\mathcal{E}_1\|\mathcal{B}_1\|+\mathcal{E}_2\|\mathcal{B}_2\|\big)\big]\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big).
\end{aligned}
$$

Similarly, by interchanging the roles of (u,v) and $(\bar{u},\bar{v})$, we can obtain that

$$
H_d\big(\Theta(u,v),\Theta(\bar{u},\bar{v})\big)\le\big[\big(\mathcal{E}_1\|\mathcal{B}_1\|+\mathcal{E}_2\|\mathcal{B}_2\|\big)\big]\big(\|u-\bar{u}\|+\|v-\bar{v}\|\big).
$$

Therefore, it follows by the assumption $\mathcal{E}_1\|\mathcal{B}_1\|+\mathcal{E}_2\|\mathcal{B}_2\|<1$ that Θ is a contraction, So, by Lemma 4.1, Θ has a fixed point (u,v), which is a solution of problem (1.1). The proof is finished. □

5 Examples

Example 5.1 Consider the following self-adjoint coupled system of second-order ordinary differential inclusions with boundary conditions:

$$
\begin{cases}
((\frac{1}{t+13})u'(t))'\in\mu_1F(t,u,v), & t\in[0,3],\\
(\frac{8}{4t^2+2t+12}v'(t))'\in\mu_2G(t,u,v), & t\in[0,3],\\
\frac{7}{3}u(0)+\frac{5}{3}u(3)=\frac{1}{7}\int_0^{\frac{1}{2}}v(s)\,ds, & \frac{4}{3}u'(0)+u'(3)=\frac{2}{7}\int_0^{\frac{1}{2}}v'(s)\,ds,\\
\frac{1}{9}v(0)+\frac{2}{9}v(3)=\frac{3}{7}\int_{\frac{5}{2}}^3u(s)\,ds, & \frac{3}{9}v'(0)+\frac{4}{9}v'(3)=\frac{4}{7}\int_{\frac{5}{2}}^3u'(s)\,ds.
\end{cases}
\tag{5.1}
$$

Here, $p(t)=1/(t+13)$, $q(t)=8/(4t^2+2t+12)$, $\mu_1=3/36$, $\mu_2=2/93$, $a=0$, $b=3$, $\eta=1/2$, $\xi=5/2$, $\lambda_1=1/7$, $\lambda_2=2/7$, $\lambda_3=3/7$, $\lambda_4=4/7$, $\alpha_1=7/3$, $\alpha_2=5/3$, $\alpha_3=4/3$, $\alpha_4=1$, $\beta_1=1/9$, $\beta_2=2/9$, $\beta_3=3/9$, $\beta_4=4/9$, and $F(t,u,v)$, $G(t,u,v)$ will be fixed later.

Using the given data, we find that $|R| \approx 1.323129 \neq 0$, $|E| \approx 115.6354 \neq 0$ (R and E are given in (2.4)), $\bar{p} \approx 0.0625$, $\bar{q} = 0.148148$, $\mathcal{D}_1 \approx 17.1389708$, $\mathcal{D}_2 \approx 0.06036034$, $\mathcal{D}_3 \approx 38.2023705$, $\mathcal{D}_4 \approx 4.565128967$, $\mathcal{E}_1 \approx 17.19933114$, and $\mathcal{E}_2 \approx 42.76749946$ ($\bar{p}$, $\bar{q}$ and $\mathcal{D}_i$ ($i = 1, \dots, 4$) are defined in (2.8), while $\mathcal{E}_1$, $\mathcal{E}_2$ are given in (2.7)).

For illustration of Theorem 3.3, we choose

$$F(t,u,v) = \left(\frac{t}{108t^2 + 32}\right)\left[\frac{\sqrt{|u(t)|}}{|u(t)| + 65}, \frac{|v(t)|^3}{|v(t)|^3 + 1}\right]$$

and

$$G(t,u,v) = \left(\frac{\cos^2(\pi t)}{t^3 + 120}\right)\left[\frac{|u(t)|}{(|u(t)| + 1)^2}, \frac{|v(t)|^5}{1 + |v(t)|^5}\right].$$

For $f \in F$, we have

$$\begin{aligned} |f| &\leq \max\left\{\left(\frac{t}{108t^2 + 32}\right)\left[\frac{\sqrt{|u(t)|}}{|u(t)| + 65}, \frac{|v(t)|^3}{|v(t)|^3 + 1}\right]\right\} \\ &\leq 2\left\{\frac{t}{108t^2 + 32}\right\}, \quad u, v \in \mathbb{R}, t \in [0,3], \end{aligned}$$

and for $g \in G$, we have

$$\begin{aligned} |g| &\leq \max\left\{\left(\frac{\cos^2(\pi t)}{t^3 + 120}\right)\left[\frac{|u(t)|}{(|u(t)| + 1)^2}, \frac{|v(t)|^5}{1 + |v(t)|^5}\right]\right\} \\ &\leq 2\left\{\frac{\cos^2(\pi t)}{t^3 + 120}\right\}, \quad u, v \in \mathbb{R}, t \in [0,3]. \end{aligned}$$

Thus

$$\left\|F(t,u,v)\right\|_{\mathcal{P}} := \sup\bigl\{|f| : f \in F(t,u,v)\bigr\} \leq 2\left[\frac{t}{108t^2 + 32}\right] = p_1(t)\bigl[\psi_1\bigl(\|u\|\bigr) + \phi_1\bigl(\|v\|\bigr)\bigr]$$

and

$$\left\|G(t,u,v)\right\|_{\mathcal{P}} := \sup\bigl\{|g| : g \in G(t,u,v)\bigr\} \leq 2\left[\frac{\cos^2(\pi t)}{t^3 + 120}\right] = p_2(t)\bigl[\psi_2\bigl(\|u\|\bigr) + \phi_2\bigl(\|v\|\bigr)\bigr],$$

with $p_1(t) = \frac{t}{108t^2+32}$, $p_2(t) = \frac{\cos^2(\pi t)}{t^3+120}$, $\psi_1(\|u\|) = \phi_1(\|v\|) = \psi_2(\|u\|) = \phi_2(\|v\|) = 1$. Furthermore, it is found that $N > N_1$, where $N_1 = 0.81272506$ (N is given in (H_3)). Clearly, all the hypotheses of Theorem 3.3 are satisfied. Thus, there exists at least one solution for problem (5.1) on $[0,3]$.

Example 5.2 Consider the following boundary value problem of self-adjoint coupled second-order ordinary differential inclusions:

$$\begin{cases} (\frac{1}{t^2+2}u'(t))' \in \mu_1 F(t,u,v), \quad t \in [0,2], \\ (\frac{2}{t+6}v'(t))' \in \mu_2 G(t,u,v), \quad t \in [0,2], \\ \frac{1}{2}u(0) + u(2) = \frac{2}{3}\int_0^{\frac{1}{4}} v(s)\,ds, \qquad \frac{5}{8}u'(0) + \frac{4}{7}u'(2) = \int_0^{\frac{1}{4}} v'(s)\,ds, \\ 2v(0) + \frac{1}{6}v(2) = \frac{3}{4}\int_1^2 u(s)\,ds, \qquad \frac{1}{5}v'(0) + \frac{3}{5}v'(2) = \frac{5}{3}\int_1^2 u'(s)\,ds, \end{cases} \tag{5.2}$$

where $p(t) = 1/(t^2 + 2)$, $q(t) = 2/(t + 6)$, $\mu_1 = 1/16$, $\mu_2 = 3/43$, $a = 0$, $b = 2$, $\eta = 1/4$, $\xi = 1$, $\lambda_1 = 2/3$, $\lambda_2 = 1$, $\lambda_3 = 4/3$, $\lambda_4 = 5/3$, $\alpha_1 = 1/2$, $\alpha_2 = 1$, $\alpha_3 = 5/8$, $\alpha_4 = 4/7$, $\beta_1 = 2$, $\beta_2 = 1/6$, $\beta_3 = 1/5$, $\beta_4 = 3/5$, and $F(t,u,v)$, $G(t,u,v)$ will be fixed later.

Using the given values, it is found that $|R| \approx 3.083 \neq 0$, $|E| \approx 8.506200 \neq 0$ (R and E are given in (2.4)), $\bar{p} \approx 0.16$, $\bar{q} = 0.25$, $\mathcal{D}_1 \approx 6.31401038$, $\mathcal{D}_2 \approx 0.72123977$, $\mathcal{D}_3 \approx 12.94560512$, $\mathcal{D}_4 \approx 3.23687872$, $\mathcal{E}_1 \approx 7.035250153$, and $\mathcal{E}_2 \approx 16.18248385$ ($\bar{p}$, $\bar{q}$ and $\mathcal{D}_i$ $(i = 1,\dots,4)$ are defined in (2.8), while $\mathcal{E}_1$, $\mathcal{E}_2$ are given in (2.7)).

For illustrating Theorem 4.3, we take the following multivalued maps $F, G : [0,2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$:

$$
\begin{aligned}
F(t,u,v) &= \left[\left(\frac{1}{3t + 160}\right)\left(\frac{|u(t)|}{|u(t)| + 1}, \frac{|v(t)|}{3\sqrt{t + |v(t)|}}\right) + \frac{1}{190}\right], \\
G(t,u,v) &= \left[\left(\frac{1}{t^2 + 188}\right)\left(\tan^{-1} u(t), \frac{|v(t)|}{1 + |v(t)|^4}\right) + \frac{1}{200}\right].
\end{aligned}
\tag{5.3}
$$

Letting $\mathcal{B}_1(t) = \frac{1}{3t+160}$ and $\mathcal{B}_2(t) = \frac{1}{t^2+188}$, we find that $H_d(F(t,u,v), F(t,\bar{u},\bar{v})) \leq \mathcal{B}_1(t)(|u - \bar{u}| + |v - \bar{v}|)$ and $H_d(G(t,u,v), G(t,\bar{u},\bar{v})) \leq \mathcal{B}_2(t)(|u - \bar{u}| + |v - \bar{v}|)$. Clearly, $d(0, F(t,0,0)) = \frac{1}{190} \leq \mathcal{B}_1(t)$ and $d(0, G(t,0,0)) = \frac{1}{200} \leq \mathcal{B}_2(t)$ for almost all $t \in [0,2]$. Moreover, $\|\mathcal{B}_1\| = 1/160$ and $\|\mathcal{B}_2\| = 1/188$ and $\mathcal{E}_1\|\mathcal{B}_1\| + \mathcal{E}_2\|\mathcal{B}_2\| \approx 0.1300473552 < 1$. Thus all the assumptions of Theorem 4.3 hold true. Therefore, by conclusion of Theorem 4.3, problem (5.2) with F, G given by (5.3) has at least one solution on $[0,2]$.

6 Conclusions

We have developed the existence theory for a self-adjoint coupled system of nonlinear second-order ordinary differential inclusions supplemented with nonlocal integral multi-strip coupled boundary conditions on an arbitrary domain. Our study includes the cases of convex as well as nonconvex multivalued maps. Nonlinear alternative of Leray–Schauder type for multivalued maps and Covitz and Nadler's fixed point theorem for contractive multivalued maps are applied to prove the main results. Numerical examples are constructed for the illustration of the obtained results. Our results are new in the given configuration and enrich the related literature. Moreover, several new results can be recorded as special cases of the present work by fixing the parameters appearing in the system. For example, we obtain the existence results for an antiperiodic multivalued boundary value problem of self-adjoint coupled second-order ordinary differential inclusions by fixing $\alpha_i = 1$, $\beta_i = 1$, $\lambda_i = 0$, $i = 1,2,3,4$, in the results of this paper, which are indeed new.

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Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
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References
1. Bitsadze, A.V., Samarskii, A.A.: Some elementary generalizations of linear elliptic boundary value problems. Dokl. Akad. Nauk SSSR **185**, 739–740 (1969)
2. Il'in, V.A., Moiseev, E.I.: Nonlocal boundary value problems of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects. Differ. Equ. **23**, 803–810 (1987)
3. Il'in, V.A., Moiseev, E.I.: Nonlocal boundary value problems of the second kind for a Sturm- Liouville operator in its differential and finite difference aspects. Differ. Equ. **23**, 979–987 (1987)
4. Asif, N.A., Eloe, P.W., Khan, R.A.: Positive solutions for a system of singular second order nonlocal boundary value problems. J. Korean Math. Soc. **47**, 985–1000 (2010)
5. Zhang, X., Feng, M., Ge, W.: Existence result of second-order differential equations with integral boundary conditions at resonance. J. Math. Anal. Appl. **353**, 311–319 (2009)
6. Greguš, M., Neumann, F., Arscott, F.M.: Three-point boundary value problems in differential equations. Proc. Lond. Math. Soc. **3**, 459–470 (1964)
7. Gupta, C.P.: Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations. J. Math. Anal. Appl. **168**, 540–551 (1998)
8. Eloe, P.W., Ahmad, B.: Positive solutions of a nonlinear *n*th order boundary value problem with nonlocal conditions. Appl. Math. Lett. **18**, 521–527 (2005)
9. Ntouyas, S.K.: Nonlocal initial and boundary value problems: a survey. In: Canada, A., Drabek, P., Fonda, A. (eds.) Handbook on Differential Equations: Ordinary Differential Equations, pp. 459–555. Elsevier, Amsterdam (2005)
10. Clark, S., Henderson, J.: Uniqueness implies existence and uniqueness criterion for non local boundary value problems for third-order differential equations. Proc. Am. Math. Soc. **134**, 3363–3372 (2006)
11. Webb, J.R.L., Infante, G.: Positive solutions of nonlocal boundary value problems: a unified approach. J. Lond. Math. Soc. **74**, 673–693 (2006)
12. Graef, J.R., Webb, J.R.L.: Third order boundary value problems with nonlocal boundary conditions. Nonlinear Anal. **71**, 1542–1551 (2009)
13. Wang, L., Pei, M., Ge, W.: Existence and approximation of solutions for nonlinear second-order four-point boundary value problems. Math. Comput. Model. **50**, 1348–1359 (2009)
14. Sun, Y., Liu, L., Zhang, J., Agarwal, R.P.: Positive solutions of singular three-point boundary value problems for second-order differential equations. J. Comput. Appl. Math. **230**, 738–750 (2009)
15. Feng, M., Zhang, X., Ge, W.: Existence theorems for a second order nonlinear differential equation with nonlocal boundary conditions and their applications. J. Appl. Math. Comput. **33**, 137–153 (2010)
16. Asif, N.A., Talib, I., Tunc, C.: Existence of solution for first-order coupled system with nonlinear coupled boundary conditions. Bound. Value Probl. **2015**, 134 (2015)
17. Ahmad, B., Alsaedi, A., Al-Malki, N.: On higher-order nonlinear boundary value problems with nonlocal multipoint integral boundary conditions. Lith. Math. J. **56**, 143–163 (2016)
18. Ahmad, B., Alsaedi, A., Alsulami, M.: Existence theory for coupled nonlinear third-order ordinary differential equations with nonlocal multi-point anti-periodic type boundary conditions on an arbitrary domain. AIMS Math. **4**, 1634–1663 (2019)
19. Khan, H., Tunc, C., Khan, A.: Stability results and existence theorems for nonlinear delay-fractional differential equations with φ_p^*-operator. J. Appl. Anal. Comput. **10**, 584–597 (2020)
20. Alsaedi, A., Hamdan, S., Ahmad, B., Ntouyas, S.K.: Existence results for coupled nonlinear fractional differential equations of different orders with nonlocal coupled boundary conditions. J. Inequal. Appl. **2021**, 95 (2021)
21. Chauhan, H.V.S., Singh, B., Tunc, C., Tunc, O.: On the existence of solutions of non-linear 2D Volterra integral equations in a Banach space. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **116**, 101 (2022)
22. Brüning, J., Geyler, V., Pankrashkin, K.: Spectra of self-adjoint extensions and applications to solvable Schrödinger operators. Rev. Math. Phys. **20**, 1–70 (2008)
23. Dolgii, Y.F.: Application of self-adjoint boundary value problems to investigation of stability of periodic delay systems. Proc. Steklov Inst. Math. **255**, S16–S25 (2006)
24. Sugie, J.: Interval criteria for oscillation of second-order self-adjoint impulsive differential equations. Proc. Am. Math. Soc. **148**, 1095–1108 (2020)

25. Vladimirov, A.A.: Variational principles for self-adjoint Hamiltonian systems (Russian). Mat. Zametki **107**, 633–636 (2020)
26. Srivastava, H.M., Ntouyas, S.K., Alsulami, M., Alsaedi, A., Ahmad, B.: A self-adjoint coupled system of nonlinear ordinary differential equations with nonlocal multi-point boundary conditions on an arbitrary domain. Appl. Sci. **11**, 4798 (2021)
27. Alsaedi, A., Almalki, A., Ntouyas, S.K., Ahmad, B., Agarwal, R.P.: Existence results for a self-adjoint coupled system of nonlinear ordinary differential equations with nonlocal non-separated integral boundary conditions. Dyn. Syst. Appl. **30**, 1479–1501 (2021)
28. Deimling, K.: Multivalued Differential Equations. de Gruyter, Berlin (1992)
29. Hu, S., Papageorgiou, N.: Handbook of Multivalued Analysis, Theory I. Kluwer Academic, Dordrecht (1997)
30. Lasota, A., Opial, Z.: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781–786 (1965)
31. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2005)
32. Covitz, H., Nadler, S.B. Jr.: Multivalued contraction mappings in generalized metric spaces. Isr. J. Math. **8**, 5–11 (1970)
33. Kisielewicz, M.: Differential Inclusions and Optimal Control. Kluwer Academic, Dordrecht (1991)