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A novel study for hybrid pair of multivalued dominated mappings in b-multiplicative metric space with applications

Tahair Rasham[1*] and Manuel De La Sen[2]

*Correspondence:
tahir_resham@yahoo.com;
tahairrasham@upr.edu.pk
[1]Department of Mathematics, University of Poonch, Rawalakot, Azad Kashmir, Pakistan
Full list of author information is available at the end of the article

Abstract

The aim of this paper is to prove some new fixed point results for a pair of multivalued dominated locally contractive mappings in b-multiplicative metric space. Further, fixed point theorems for multigraph-dominated mappings are also established. Some new fixed point results on a closed ball are obtained for a pair of multigraph-dominated mappings endowed with graphic structure in b-multiplicative metric space. An illustrative example is given to show the validity of the hypothesis of our obtained result. Moreover, applications for a coupled system of nonlinear Volterra-type integral equations and functional equations in dynamic programming are presented to show the novelty of our results.

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1 Introduction and preliminaries

Fixed point theory is the mixture of modern mathematical analysis with broad applications in the study of various significant fields of mathematics. It has numerous applications in different areas of the mathematical sciences, such as modern optimization, theories of control, functional analysis, topology, geometry, and economic modeling in both pure and applied mathematics. In fixed point theorem, we deal with the self-mappings T from set X to itself has more than one or unique fixed points and the fundamental equation is $Tx = x$ has many solutions. Contractive conditions play a significant role in solving fixed point problems in the field of metric fixed point theory. In 1922 [6] Banach proved a fixed point result called the Banach contraction principle. Because of its significance, many authors demonstrated various generalizations of his result (see [1–31]).

Ozavsar and Cevikel [21] introduced the notion of multiplicative metric spaces and showed some new fixed point results satisfying contraction mappings in multiplicative metric space and some related topological properties. Mongkolkeha et al. [18] proved the best proximity fixed point results for multiplicative proximal contraction in such spaces. In 2017, Ali et al. [3] established the concept of b-multiplicative metric space and proved some new fixed point theorems endowed with graph and also applied their main result to solve a Fredholm type nonlinear multiplicative integral equations. As an application, they established an existence theorem for solving a system of Fredholm multiplicative integral equations. Shoaib et al. [29] investigated fixed point theorems for self-mappings fulfilling contractions on the closed ball in complete b-multiplicative metric space.

Wardowski [31] introduced the notion of F-contraction, which is the most important generalization of the Banach contraction. Lateral, a huge number of researchers proved different variants of F-contraction and introduced various important fixed point results that can be seen in [2, 11, 24, 25, 28]. Recently, Rasham et al. [23] established fixed point results for a pair of dominated fuzzy contractions on closed balls in b-metric like spaces. They also presented applications to find the unique solution of nonlinear integral equations and functional equations in dynamics programming. In this paper, we prove some new fixed point results for multivalued generalized rational type F-contractive dominated mappings on closed balls in b-multiplicative metric space. Furthermore, we demonstrate applications for a coupled system of nonlinear integral and functional equations. Let us state the following preliminary concepts.

Definition 1.1 ([3]) Let Λ be a nonempty set, and let $\ell \geq 1$ be a given real number. A mapping $\sigma : \Lambda \times \Lambda \to [1, \infty)$ is called a b-multiplicative metric with coefficient ℓ if the following conditions hold:

(i) $\sigma(\vartheta, \upsilon) > 1$ for all $\vartheta, \upsilon \in \Lambda$ with $\vartheta \neq \upsilon$ and $\sigma(\vartheta, \upsilon) = 1$ if and only if $\vartheta = \upsilon$;

(ii) $\sigma(\vartheta, \upsilon) = \sigma(\upsilon, \vartheta)$ for all $\vartheta, \upsilon \in \Lambda$;

(iii) $\sigma(\vartheta, \varrho) \leq [\sigma(\vartheta, \upsilon).\sigma(\upsilon, \varrho)]^{\ell}$ for all $\vartheta, \upsilon, \varrho \in \Lambda$.

The triplet (Λ, σ, ℓ) is a b-multiplicative metric space or shortly *BMMS*. For $e \in \Lambda$ and $r > 0$, $\overline{\beta_{\sigma_m}(e, r)} = \{\nu \in \Lambda : \sigma(e, \nu) \leq r\}$ is a closed ball in *BMMS*.

Example 1.2 ([3]) Let $\Lambda = [0, \infty)$. Define the mapping $\sigma : \Lambda \times \Lambda \to [1, \infty)$ by

$$\sigma(t, \upsilon) = \nu^{|t-\upsilon|^2},$$

where $\nu > 1$ is any fixed real number. Then, σ is a b-multiplicative metric on Λ with $\ell = 2$. Note that σ is not a multiplicative metric on Λ. Taking $\nu = 3$, $r = 3^4$, and $t = 1$, then $\overline{\beta_{\sigma_m}(\hbar_0, r)} = [0, 3]$ is a closed ball in Λ.

Example 1.3 ([3]) If $p \in (0, 1)$, then $l^p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ endowed with the functional

$$\sigma_e : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to [1, \infty), \qquad \sigma_e\bigl(\{x_n\}, \{y_n\}\bigr) = e^{(\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}},$$

for each $\{x_n\}, \{y_n\} \in l^p(\mathbb{R})$, is a b-multiplicative metric space with $s = 2^{\frac{1}{p}-1}$. Note that σ_e is neither a metric nor a b-metric on Λ.

Definition 1.4 ([29]) Let (Λ, σ) be a *BMMS*.

(i) A sequence $\{p_n\}$ is convergent if there exists a point $p \in \Lambda$ such that $\sigma(p_n, p) \to 1$ as $n \to +\infty$.

(ii) A sequence $\{p_n\}$ is said to be b-multiplicative Cauchy in Λ iff $\sigma(p_n, p_m) \to 1$ as $m, n \to +\infty$.

(iii) A b-multiplicative metric space (Λ,σ) is said to be complete if every multiplicative Cauchy sequence in Λ is convergent to some $p \in \Lambda$.

Definition 1.5 ([26]) Let Λ be a nonempty set and $\ell \geq 1$ be a real number. A mapping $\rho : \Lambda \times \Lambda \to R + \cup\{0\}$ is said to be b-metric with coefficient 'ℓ' if for all $\vartheta, \upsilon, \varrho \in \Lambda$, the following conditions hold:

(i) $\rho(\vartheta,\upsilon) = 0$ if and only if $\vartheta = \upsilon$;
(ii) $\rho(\vartheta,\upsilon) = \rho(\upsilon,\vartheta)$;
(iii) $\rho(\vartheta,\varrho) \leq \ell[\rho(\vartheta,\upsilon) + \rho(\upsilon,\varrho)]$.

The pair (Λ,ρ) is a b-metric space.

Remark 1.6 ([3]) Every b-metric space (Λ,ρ) generates a b-multiplicative metric space (Λ,σ) defined as $\sigma(x,\upsilon) = e^{\rho(x,\upsilon)}$.

Definition 1.7 Let Y is not an empty subset of a *BMMS*, and let $\ell \in \Lambda$. A point $c_0 \in Y$ is the best approximation of ℓ in Y if

$$\sigma(\ell,Y) = \sigma(\ell,c_0), \quad \text{where } \sigma(\ell,Y) = \inf_{c\in Y} \sigma(\ell,c).$$

We say that Y is a closed compact set if for any ℓ in Λ, there exists a point of the best approximation in Y.

Definition 1.8 Define the mapping $H_\sigma : \wp(\Lambda) \times \wp(\Lambda) \to R^+$ by

$$H_\sigma(Q,E) = \max\left\{\sup_{l\in Q} \sigma(l,E), \sup_{f\in E} \sigma(Q,f)\right\}.$$

H_σ is said to be a Hausdorff multiplicative metric on $\wp(\Lambda)$.

Definition 1.9 ([27]) Let $Z, E : \Lambda \to \wp(\Lambda)$ are set-valued mappings and $\beta : \Lambda \times \Lambda \to [0,+\infty)$ be a function of positive real valued. Then both Z and R are called $\beta_\star$-admissible if for each $g, h \in \Lambda$,

$$\beta(g,h) \geq 1 \quad \Rightarrow \quad \beta_\star(Zg,Eh) \geq 1, \quad \text{and} \quad \beta_\star(Eh,Zg) \geq 1,$$

where $\beta_\star(Zg,Ee) = \inf\{\beta(a,s) : a \in Zg, s \in Ee\}$. When Z intersect with E, in this instance, we get the definition of α_*-admissible mapping shown by [5].

Definition 1.10 ([23]) Let (Λ,d_b) be a b-metric space. Let $B : \Lambda \to \wp(\Lambda)$ be a multi-valued mapping, and $\alpha : \Lambda \times \Lambda \to [0,+\infty)$ is a function, and $U \subseteq \Lambda$. Then, B is said to be α_*-dominated on U if for all $w \in U$, $\alpha_*(w,Bw) = \inf\{\alpha(w,y) : y \in Bw\} > 1$.

Definition 1.11 ([31]) Let (Λ,d) be a metric and $B : \Lambda \to \Lambda$ be a self-mapping. It is said to be an $\mathcal{F}$-contraction if for each $u, \ell \in \Lambda$, there is a $\tau > 0$ so that $d(Bu,B\ell) > 0$ implies

$$\tau + \mathcal{F}\big(d(Bu,B\ell)\big) \leq \mathcal{F}\big(d(u,\ell)\big),$$

where $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ is such that:

(F1) For each $c, d > 0$ so that $c < d$, $\mathcal{F}(c) < \mathcal{F}(d)$;
(F2) $\lim_{n\to+\infty} f_n = 0$ if and only if $\lim_{n\to+\infty} \mathcal{F}(f_n) = -\infty$, for every positive sequence $\{f_n\}_{n=1}^{\infty}$;
(F3) For some $r \in (0,1)$ then $\lim_{f\to 0^+} f^r \mathcal{F}(f) = 0$.

Let $\mathcal{F}$ denotes the set of mappings so that (F1)–(F3) hold.

Lemma 1.12 *Let (Λ,σ) be a BMMS. Let $(\wp(\Lambda), H_\sigma)$ be a Hausdorff b-multiplicative metric space. If for all $u \in M$ and for each $M, N \in \wp(\Lambda)$, there is $h_l \in N$ such that $\sigma(u,N) = \sigma(u,h_l)$, then $H_\sigma(M,N) \geq \sigma(u,h_l)$ holds.*

Example 1.13 Let $\Lambda = \mathbb{R}$. We define $\alpha : \Lambda \times \Lambda \to [0,+\infty)$ by

$$\alpha(y,h) = \begin{Bmatrix} 1 & \text{if } y > h \\ \frac{1}{2} & \text{otherwise} \end{Bmatrix}.$$

Define $L, M : \Lambda \to P(\Lambda)$ by

$$Lj = [j-2, j-1] \quad \text{and} \quad Mq = [q-5, q-4].$$

This means L and M are α_*-dominated but not α_*-admissible.

2 Main results

Let (Λ,σ) be a *BMMS*, $\hbar_0 \in \Lambda$, and let $\check{S}, \mathcal{H} : \Lambda \to \wp(\Lambda)$ be multivalued mappings on Λ. Let $\hbar_1 \in \check{S}\hbar_0$ be an element such that $\sigma(\hbar_0, \check{S}\hbar_0) = \sigma(\hbar_0,\hbar_1)$. Let $\hbar_2 \in \mathcal{H}\hbar_1$ be such that $\sigma(\hbar_1,\mathcal{H}\hbar_1) = \sigma(\hbar_1,\hbar_2)$. Let $\hbar_3 \in \check{S}\hbar_2$ be such that $\sigma(\hbar_2,\check{S}\hbar_2) = \sigma(\hbar_2,\hbar_3)$. In this way, we get a sequence $\{\mathcal{H}\check{S}(\hbar_n)\}$ in Λ, where $\hbar_{2n+1} \in \check{S}\hbar_{2n}$, $\hbar_{2n+2} \in \mathcal{H}\hbar_{2n+1}$, $n \in \mathbb{N}$. Also $\sigma(\hbar_{2n},\check{S}\hbar_{2n}) = \sigma(\hbar_{2n},\hbar_{2n+1})$, $\sigma(\hbar_{2n+1},\mathcal{H}\hbar_{2n+1}) = \sigma(\hbar_{2n+1},\hbar_{2n+2})$. Then, $\{\mathcal{H}\check{S}(\hbar_n)\}$ is said to be a sequence in Λ generated by $\hbar_0$. If $\check{S} = \mathcal{H}$, then we say $\{\Lambda\check{S}(\hbar_n)\}$ instead of $\{\mathcal{H}\check{S}(\hbar_n)\}$. For $e, \nu \in \Lambda$, $\kappa \in (0,\frac{1}{2})$, we define $\nabla(e,y)$ as

$$\nabla(e,y) = \left(\max\begin{Bmatrix} \sigma(e,y), \sigma(e,\check{S}e), \sigma(y,\mathcal{H}y), \\ \frac{\sigma^2(e,\check{S}e).\sigma(y,\mathcal{H}y)}{1+\sigma^2(e,y)} \end{Bmatrix}\right)^{\kappa}.$$

Theorem 2.1 *Let (Λ,σ) be a complete BMMS. Suppose there exists a function $\alpha : \Lambda\times\Lambda \to [0,\infty)$. Let $r > 0$, $\hbar_0 \in \overline{\beta_{\sigma_m}(\hbar_0,r)} \subseteq \Lambda$, and $\check{S},\mathcal{H} : \Lambda \to \wp(\Lambda)$ be α_*-dominated mappings on $\overline{\beta_{\sigma_m}(\hbar_0,r)}$. Assume that $\tau > 0$, and there exists $\kappa \in (0,\frac{1}{\ell})$ with $\eta = \frac{\kappa}{1-\kappa}$, and $\mathcal{F}$ is a strictly increasing function satisfying*:

$$\tau + \mathcal{F}\big(H_\sigma(\check{S}e,\mathcal{H}y)\big) \leq \mathcal{F}\big(\nabla(e,y)\big), \tag{2.1}$$

for all $e, y \in \overline{\beta_{\sigma_m}(\hbar_0,r)} \cap \{\mathcal{H}\check{S}(\hbar_n)\}$, $\alpha(e,y) \geq 1$, and $H_\sigma(\check{S}e,\mathcal{H}y) > 0$ such that,

$$\sigma(\hbar_0,\check{S}\hbar_0) \leq r^{\frac{1-\ell\eta}{\ell}}. \tag{2.2}$$

Then, $\{\mathcal{H}\check{S}(\hbar_n)\}$ is a sequence in $\overline{\beta_{\sigma_m}(\hbar_0,r)}$, $\alpha(\hbar_n,\hbar_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{\mathcal{H}\check{S}(\hbar_n)\} \to \mu \in \overline{\beta_{\sigma_m}(\hbar_0,r)}$. Also, if μ satisfies (2.1), $\alpha(\hbar_n,\mu) \geq 1$, and $\alpha(\mu,\hbar_n) \geq 1$ for all integers $n \geq 0$, then $\check{S}$ and $\mathcal{H}$ have a common fixed point μ in $\overline{\beta_{\sigma_m}(\hbar_0,r)}$.

Proof Consider a sequence $\{\mathcal{H}\check{S}(\hbar_n)\}$. From (2.2), we get

$$\sigma(\hbar_0,\hbar_1)=\sigma(\hbar_0,\check{S}\hbar_0)\leq r^{\frac{1-\ell\eta}{\ell}}<r.$$

It follows that,

$$\hbar_1\in\overline{\beta_{\sigma_m}(\hbar_0,r)}.$$

Let $\hbar_2,\ldots,\hbar_j\in\overline{\beta_{\sigma_m}(\hbar_0,r)}$ for some $j\in\mathbb{N}$. If j is odd, then $j=2\grave{\imath}+1$ for some $\grave{\imath}\in\mathbb{N}$. Since $\check{S},\mathcal{H}:\Lambda\to\wp(\Lambda)$ are α_*-dominated mappings on $\overline{\beta_{\sigma_m}(\hbar_0,r)}$, so $\alpha_*(\hbar_{2\grave{\imath}},\check{S}\hbar_{2\grave{\imath}})\geq 1$ and $\alpha_*(\hbar_{2\grave{\imath}+1},\mathcal{H}\hbar_{2\grave{\imath}+1})\geq 1$. As $\alpha_*(\hbar_{2\grave{\imath}},\check{S}\hbar_{2\grave{\imath}})\geq 1$, this implies $\inf\{\alpha(\hbar_{2\grave{\imath}},b):b\in\check{S}\hbar_{2\grave{\imath}}\}\geq 1$. Also $\hbar_{2\grave{\imath}+1}\in\check{S}\hbar_{2\grave{\imath}}$, so $\alpha(\hbar_{2\grave{\imath}},\hbar_{2\grave{\imath}+1})\geq 1$ and $\hbar_{2\grave{\imath}+2}\in\mathcal{H}\hbar_{2\grave{\imath}+1}$. Now using Lemma 1.12, we have

$$\begin{aligned}\tau+\mathcal{F}\big(\sigma(\hbar_{2\grave{\imath}+1},\hbar_{2\grave{\imath}+2})\big)&\leq\tau+\mathcal{F}\big(H_\sigma(\check{S}\hbar_{2\grave{\imath}},\mathcal{H}\hbar_{2\grave{\imath}+1})\big)\leq\mathcal{F}\big(\nabla(\hbar_{2\grave{\imath}},\hbar_{2\grave{\imath}+1})\big)\\&\leq\mathcal{F}\left(\max\left\{\begin{array}{c}\sigma(\hbar_{2\grave{\imath}},\hbar_{2\grave{\imath}+1}),\sigma(\hbar_{2\grave{\imath}},\hbar_{2\grave{\imath}+1}),\\\sigma(\hbar_{2\grave{\imath}+1},\hbar_{2\grave{\imath}+2}),\frac{\sigma(\hbar_{2\grave{\imath}},\hbar_{2\grave{\imath}+1}).\sigma(\hbar_{2\grave{\imath}+1},\hbar_{2\grave{\imath}+2})}{1+\sigma(\hbar_{2i},\hbar_{2i+1})}\end{array}\right\}^{\kappa}\right)\\&\leq\mathcal{F}\big(\max\big\{\sigma(\hbar_{2\grave{\imath}},\hbar_{2\grave{\imath}+1}),\sigma(\hbar_{2\grave{\imath}+1},\hbar_{2\grave{\imath}+2})\big\}^{\kappa}\big).\end{aligned}$$

Thus,

$$\tau+\mathcal{F}\big(\sigma(\hbar_{2\grave{\imath}+1},\hbar_{2\grave{\imath}+2})\big)\leq\mathcal{F}\big(\sigma(\hbar_{2\grave{\imath}},\hbar_{2i+1})\big)^{\eta},$$

for all $i\in N$, where $\eta=\frac{\kappa}{1-\kappa}$. As $\mathcal{F}:\mathbb{R}_+\to\mathbb{R}$ is a strictly increasing function then

$$\sigma(\hbar_{2\grave{\imath}+1},\hbar_{2\grave{\imath}+2})<\sigma(\hbar_{2\grave{\imath}},\hbar_{2i+1})^{\eta}.\tag{2.3}$$

Similarly, if j is even, we have

$$\sigma(\hbar_{2\grave{\imath}+2},\hbar_{2\grave{\imath}+3})<\sigma(\hbar_{2\grave{\imath}+1},\hbar_{2i+2})^{\eta}.\tag{2.4}$$

Now, we have

$$\sigma(\hbar_j,\hbar_{j+1})<\sigma(\hbar_{j-1},\hbar_j)^{\eta}\quad\text{for all } j\in\mathbb{N}.\tag{2.5}$$

Therefore,

$$\begin{aligned}\sigma(\hbar_j,\hbar_{j+1})&<\sigma(\hbar_{j-1},\hbar_j)^{\eta}<\sigma(\hbar_{j-2},\hbar_{j-1})^{\eta^2}<\sigma(\hbar_{j-3},\hbar_{j-2})^{\eta^3}\\&<\sigma(\hbar_{j-4},\hbar_{j-3})^{\eta^4}<\cdot\ldots\cdot<\sigma(\hbar_0,\hbar_1)^{j}.\end{aligned}\tag{2.6}$$

Now,

$$\begin{aligned}\sigma(\hbar_0,\hbar_{j+1})&\leq\sigma(\hbar_0,\hbar_1)^{\ell}\cdot\sigma(\hbar_1,\hbar_2)^{\ell^2}\cdot\sigma(\hbar_2,\hbar_3)^{\ell^3}\cdot\ldots\cdot\sigma(\hbar_j,\hbar_{j+1})^{\ell^{j+1}}\\&\leq\sigma(\hbar_0,\hbar_1)^{\ell}\cdot\sigma(\hbar_0,\hbar_1)^{\eta\ell^2}\cdot\sigma(\hbar_0,\hbar_1)^{\ell^3\eta^2}\cdot\sigma(\hbar_0,\hbar_1)^{\ell^4\eta^3}\\&\quad\cdot\sigma(\hbar_0,\hbar_1)^{\ell^5\eta^4}\cdot\ldots\cdot\sigma(\hbar_0,\hbar_1)^{\ell^{j+1}\eta^{j}}\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma(\hbar_0,\hbar_1)^{\ell(\eta^0+\ell\eta+\ell^2\eta^2+\ell^3\eta^3+\cdots+\ell^j\eta^j)}\\
&\leq \sigma(\hbar_0,\hbar_1)^{\ell(\frac{1}{1-\ell\eta})}.
\end{aligned}
$$

Then, we have

$$
\sigma(\hbar_0,\hbar_{j+1})\leq r^{\frac{1-l(\eta)\times l}{l\times 1-l(\eta)}}\leq r,
$$

which implies $\hbar_{j+1}\in\overline{\beta_{\sigma_m}(\hbar_0,r)}$. Hence, by induction $\hbar_n\in\overline{\beta_{\sigma_m}(\hbar_0,r)}$ for all $n\in\mathbb{N}$. Also, $\alpha(\hbar_n,\hbar_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$. Now,

$$
\sigma(\hbar_n,\hbar_{n+1})<\sigma(\hbar_0,\hbar_1)^{\eta^n}\quad\text{for all } n\in\mathbb{N}. \tag{2.7}
$$

Now, for any positive integers m, n $(n>m)$, we have

$$
\begin{aligned}
\sigma(\hbar_m,\hbar_n)&\leq\sigma(\hbar_m,\hbar_{m+1})^{\ell}\cdot\sigma(\hbar_{m+1},\hbar_{m+2})^{\ell^2}\cdot\sigma(\hbar_{m+2},\hbar_{m+3})^{\ell^3}\\
&\quad\cdot\ldots\cdot\sigma(\hbar_{n-1},\hbar_n)^{\ell^n}\\
&\leq\sigma(\hbar_0,\hbar_1)^{\ell\eta^m}\cdot\sigma(\hbar_0,\hbar_1)^{\ell^2\eta^{m+1}}\cdot\ldots\\
&\quad\cdot\sigma(\hbar_0,\hbar_1)^{\ell^n\eta^{n-1}}\quad(\text{by (2.7)})\\
&\leq\sigma(\hbar_0,\hbar_1)^{(\ell\eta^m+\ell^2\eta^{m+1}+\ell^3\eta^{m+2}+\cdots+\ell^n\eta^{n-1})}\\
&<\sigma(\hbar_0,\hbar_1)^{(\ell\eta^m+\ell^2\eta^{m+1}+\ell^3\eta^{m+2}+\cdots)},\\
\sigma(\hbar_m,\hbar_n)&<\sigma(\hbar_0,\hbar_1)^{(\frac{\ell\eta^m}{1-\ell\eta})}.
\end{aligned}
$$

Clearly, $\sigma(\hbar_m,\hbar_n)\to 1$ as $m,n\to\infty$. Hence, $\{\mathcal{H}\check{S}(\hbar_n)\}$ is a Cauchy sequence in a complete multiplicative metric space $(\overline{\beta_{\sigma_m}(\hbar_0,r)},\sigma)$. So, there is a $\mu\in\overline{\beta_{\sigma_m}(\hbar_0,r)}$ and $\{\mathcal{H}\check{S}(\hbar_n)\}\to\mu$ such that $n\to\infty$. Then

$$
\lim_{n\to\infty}\sigma(\hbar_n,\mu)=1. \tag{2.8}
$$

Now

$$
\sigma(\mu,\mathcal{H}\mu)\leq\sigma(\mu,\hbar_{2n+1})^{\ell}\cdot\sigma(\hbar_{2n+1},\mathcal{H}\mu)^{\ell}.
$$

So, there exists $\hbar_{2n+1}\in\check{S}\hbar_{2n}$ and $\sigma(\hbar_{2n},\check{S}\hbar_{2n})=\sigma(\hbar_{2n},\hbar_{2n+1})$. Using Lemma 1.12 and (2.1), we obtain

$$
\sigma(\mu,\mathcal{H}\mu)\leq\sigma(\mu,\hbar_{2n+1})^{\ell}\cdot H_{\sigma}(\check{S}\hbar_{2n},\mathcal{H}\mu)^{\ell} \tag{2.9}
$$

By assumption, $\alpha(\hbar_n,\mu)\geq 1$. Suppose that $\sigma(\mu,\mathcal{H}\mu)>0$, there exists a positive integer k such that $\sigma(\hbar_n,\mathcal{H}\mu)>0$ for all $n\geq k$. For $n\geq k$, we have

$$
\begin{aligned}
\sigma(\mu,\mathcal{H}\mu)&<\sigma(\mu,\hbar_{2n+1})^{\ell}.\left(\max\left\{\begin{array}{c}\sigma(\hbar_{2n},\mu),\sigma(\hbar_{2n},\mathcal{H}\mu),\\ \sigma(\hbar_{2n+1},\mathcal{H}\mu),\\ \frac{\sigma(\hbar_{2n},\hbar_{2n+1}).\sigma(\hbar_{2n+1},\mathcal{H}\mu)}{1+\sigma(\hbar_{2n},\hbar_{2n+1})}\end{array}\right\}^{\kappa}\right)^{\ell}\\
&<\sigma(\mu,\hbar_{2n+1})^{\ell}\cdot\left(\max\left\{\sigma(\hbar_{2n},\mu),\sigma(\hbar_{2n+1},\mathcal{H}\mu)\right\}\right)^{\ell\kappa}.
\end{aligned} \tag{2.10}
$$

Taking limit $n \to \infty$ and inequality (2.8) from both sides of (2.9), we get $\sigma(\mu, \mathcal{H}\mu) < \sigma(\mu, \mathcal{H}\mu)^{\ell\kappa}$ that is not true in general. Our supposition is wrong because $\ell\kappa < 1$. Hence, $\sigma(\mu, \mathcal{H}\mu) = 1$ or $\mu \in \mathcal{H}\mu$. Similarly, adopting the similar way and using Lemma 1.12 and inequality (2.8), we can get $\sigma(\mu, \check{S}\mu) = 1$ or $\mu \in \check{S}\mu$. So, $\check{S}$ and $\mathcal{H}$ have a common fixed point μ in $\overline{\beta_{\sigma_m}(\hbar_0, r)}$. Now,

$$\sigma(\mu,\mu) \le \left[\sigma(\mu,\check{S}\mu).\sigma(\check{S}\mu,\mu)\right]^{\ell}.$$

This implies that $\sigma(\mu,\mu) = 1$. □

Example 2.2 Let $\Lambda = R^+ \cup \{0\}$ and the function $\sigma : \Lambda \times \Lambda \to \Lambda$ defined by

$$\sigma(\imath, j) = e^{|i-j|^2} \quad \text{for all } i, j \in \Lambda.$$

Define the mappings $\check{S}, \mathcal{H} : \Lambda \times \Lambda \to \wp(\Lambda)$ by

$$\check{S}\hbar = \begin{cases} [\frac{\hbar}{5}, \frac{2}{5}\hbar] & \text{if } \hbar \in [0,15] \cap \Lambda, \\ [2\hbar, 3\hbar] & \text{if } \hbar \in (15,\infty) \cap \Lambda \end{cases}$$

and,

$$\mathcal{H}\hbar = \begin{cases} [\frac{\hbar}{7}, \frac{3}{7}\hbar] & \text{if } \hbar \in [0,15] \cap \Lambda, \\ [4\hbar, 5\hbar] & \text{if } \hbar \in (15,\infty) \cap \Lambda. \end{cases}$$

Suppose that, $\hbar_0 = 1$, $\ell = 2$, $r = 81$, $\overline{B_\sigma(\hbar_0, r)} = [0,15] \cap \Lambda$. Now, $\sigma(\hbar_0, \check{S}\hbar_0) = \sigma(1, \check{S}1) = \sigma(1, \frac{1}{5})$. So $\hbar_1 = \frac{1}{5}$. Now, $\sigma(\hbar_1, \mathcal{H}\hbar_1) = \sigma(\frac{1}{5}, \mathcal{H}\frac{1}{5}) = \sigma(\frac{1}{5}, \frac{1}{35})$. So $\hbar_2 = \frac{1}{35}$. Now, $\sigma(\hbar_2, \check{S}\hbar_2) = \sigma(\frac{1}{35}, \check{S}\frac{1}{35}) = \sigma(\frac{1}{35}, \frac{1}{175})$. So $\hbar_3 = \frac{1}{175}$. Continuing in this way, we have $\{\mathcal{H}\check{S}(\hbar_n)\} = \{1, \frac{1}{5}, \frac{1}{35}, \frac{1}{175}, \dots\}$. Moreover, taking $\kappa = \frac{7}{23} \in (0, \frac{1}{2})$ and $\eta = \frac{7}{17} \in (0,1)$. From (2.2), we also have

$$\sigma(\hbar_0, \check{S}\hbar_0) = e^{|1-\frac{1}{5}|^2} < 81^{\frac{9}{46}}.$$

Consider the mapping $\alpha : \Lambda \times \Lambda \to [0,\infty)$ by

$$\alpha(a,b) = \left\{ \begin{matrix} 1 & \text{if } a > b \\ \frac{1}{2} & \text{otherwise} \end{matrix} \right\}.$$

Now, if $\hbar, \nu \in \overline{\beta_{\sigma_m}(\hbar_0, r)} \cap \{\mathcal{H}\check{S}(\hbar_n)\}$ with $\alpha(\hbar, \nu) \ge 1$, we have

$$\begin{aligned} H_\sigma(\check{S}\hbar, \mathcal{H}\nu) &= \max\left\{ \sup_{a \in \check{S}\hbar} \sigma(a, \mathcal{H}\nu), \sup_{b \in \mathcal{H}\nu} \sigma(\check{S}\hbar, b) \right\} \\ &= \max\left\{ \sup_{a \in S\hbar} \sigma\left(a, \left[\frac{\nu}{7}, \frac{3\nu}{7}\right]\right), \sup_{b \in T\nu} \sigma\left(\left[\frac{\hbar}{5}, \frac{2\hbar}{5}\right], b\right) \right\} \\ &= \max\left\{ \sigma\left(\frac{2\hbar}{5}, \left[\frac{\nu}{7}, \frac{3\nu}{7}\right]\right), \sigma\left(\left[\frac{\hbar}{5}, \frac{2\hbar}{5}\right], \frac{3\nu}{7}\right) \right\} \\ &= \max\left\{ \sigma\left(\frac{2\hbar}{5}, \frac{\nu}{7}\right), \sigma\left(\frac{\hbar}{5}, \frac{3\nu}{7}\right) \right\} \end{aligned}$$

$$= \max\left\{e^{|\frac{2\hbar}{5}-\frac{\nu}{7}|^2}, e^{|\frac{\hbar}{5}-\frac{3\nu}{7}|^2}\right\}$$
$$< \max\begin{pmatrix} e^{|\hbar-\nu|^2}, e^{|\hbar-\frac{\hbar}{5}|^2}, e^{|\nu-\frac{\nu}{7}|^2}, \\ \frac{e^{|\hbar-\frac{\hbar}{5}|^2}\cdot e^{|\nu-\frac{\nu}{7}|^2}}{1+e^{|\hbar-\frac{\hbar}{5}|^2}} \end{pmatrix}^{\kappa}$$
$$< \max\begin{pmatrix} \sigma(\hbar,\nu), \frac{\sigma(\hbar,[\frac{\hbar}{5},\frac{2}{5}\hbar]).\sigma(\nu,[\frac{\nu}{7},\frac{3}{7}\nu])}{1+\sigma(\hbar,\nu)}, \\ \sigma(\hbar,[\frac{\hbar}{5},\frac{2}{5}\hbar]), \sigma(\hbar,[\frac{\nu}{7},\frac{3}{7}\nu]) \end{pmatrix}^{\kappa}.$$

Thus,

$$H_\sigma(\check{S}\hbar,\mathcal{H}\nu)) < \big(\nabla(\hbar,\nu)\big),$$

this implies that if there is $\tau \in (0,\frac{11}{91}]$, and $\mathcal{F}$ is a strictly increasing function defined as $\mathcal{F}(\ell) = \ln\ell + \ell$, we have

$$H_\sigma(\check{S}\hbar,\mathcal{H}\nu)e^{H_\sigma(\check{S}\hbar,\mathcal{H}\nu)-\nabla(\hbar,\nu)+\tau} \leq \nabla(\hbar,\nu),$$
$$\ln\big(H_\sigma(\check{S}\hbar,\mathcal{H}\nu)\big) + H_\sigma(\check{S}\hbar,\mathcal{H}\nu) + \tau \leq \ln\big(\nabla(\hbar,\nu)\big) + \nabla(\hbar,\nu)),$$
$$\tau + \mathcal{F}\big(H_\sigma(\check{S}\hbar,\mathcal{H}\nu)\big) + H_\sigma(\check{S}\hbar,\mathcal{H}\nu) \leq \mathcal{F}\big(\nabla(\hbar,\nu)\big) + \nabla(\hbar,\nu).$$

Note that, taking $17, 18 \in \Lambda$, then $\alpha(17,18) \geq 1$. Now, we have

$$\tau + \mathcal{F}\big(H_\sigma(\check{S}18,\mathcal{H}17)\big) + H_\sigma(\check{S}18,\mathcal{H}17) > \mathcal{F}\big(\nabla(18,17)\big)) + \nabla(18,17).$$

So, the condition (2.1) is not satisfied on Λ. Hence, $\check{S}$ and $\mathcal{H}$ are satisfied all conditions of Theorem 2.1 for all $\hbar,\nu \in \overline{\beta_{\sigma_m}(\hbar_0,r)} \cap \{\mathcal{H}\check{S}(\hbar_n)\}$ with $\alpha(\hbar,\nu) \geq 1$. Hence, $\check{S}$ and $\mathcal{H}$ admit a common fixed point.

Corollary 2.3 *Let (Λ,σ) be a complete BMMS. Suppose there exists a function $\alpha : \Lambda \times \Lambda \to [0,\infty)$. Let $r > 0$, $\hbar_0 \in \overline{\beta_{\sigma_m}(\hbar_0,r)} \subseteq \Lambda$, and $\check{S},\mathcal{H} : \Lambda \to \wp(\Lambda)$ are α_*-dominated multifunctions on $\overline{\beta_{\sigma_m}(\hbar_0,r)}$. Assume that $\tau > 0$, and there exists $\kappa \in (0,\frac{1}{\ell})$ with $\eta = \frac{\kappa}{1-\kappa}$, and $\mathcal{F}$ be a strictly increasing function satisfying*:

$$\tau + \mathcal{F}\big(\sigma(\check{S}e,\mathcal{H}y)\big) \leq \mathcal{F}\big(\nabla(e,y)\big), \tag{2.11}$$

for all $e,y \in \overline{\beta_{\sigma_m}(\hbar_0,r)} \cap \{\hbar_n\}$, $\alpha(e,y) \geq 1$, and $\sigma(\check{S}e,\mathcal{H}y) > 0$ such that,

$$\sigma(\hbar_0,\check{S}\hbar_0) \leq r^{\frac{1-\ell\eta}{\ell}}.$$

Then, $\{\mathcal{H}\check{S}(\hbar_n)\}$ is a sequence in $\overline{\beta_{\sigma_m}(\hbar_0,r)}$, $\alpha(\hbar_n,\hbar_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{\mathcal{H}\check{S}(\hbar_n)\} \to \mu \in \overline{\beta_{\sigma_m}(\hbar_0,r)}$. Also, if μ satisfies (2.11) $\alpha(\hbar_n,\mu) \geq 1$ and $\alpha(\mu,\hbar_n) \geq 1$ for all naturals $n \geq 0$, then $\check{S}$ and $\mathcal{H}$ admit a common fixed point μ in $\overline{\beta_{\sigma_m}(\hbar_0,r)}$.

Corollary 2.4 *Let (Λ,σ) be a complete BMMS. Suppose there exists a function $\alpha : \Lambda \times \Lambda \to [0,\infty)$. Let $r > 0$, $\hbar_0 \in \overline{\beta_{\sigma_m}(\hbar_0,r)} \subseteq \Lambda$ and $\check{S} : \Lambda \to \wp(\Lambda)$ be a α_*-dominated multifunction on $\overline{\beta_{\sigma_m}(\hbar_0,r)}$. Assume that $\tau > 0$, and there exists $\kappa \in (0,\frac{1}{\ell})$ with $\eta = \frac{\kappa}{1-\kappa}$, and $\mathcal{F}$*

be a strictly increasing function satisfying:

$$\tau + \mathcal{F}\big(H_\sigma(\check{S}e, \check{S}y)\big) \le \mathcal{F}\big(\nabla(e,y)\big), \tag{2.12}$$

for all $e, y \in \overline{\beta_{\sigma_m}(\hbar_0, r)} \cap \{\Lambda\check{S}(\hbar_n)\}$, $\alpha(e,y) \ge 1$, *and* $H_\sigma(\check{S}e, \mathcal{H}y) > 0$, *such that*,

$$\sigma(\hbar_0, \check{S}\hbar_0) \le r^{\frac{1-\ell\eta}{\ell}}.$$

Then, $\{\Lambda\check{S}(\hbar_n)\}$ *is a sequence in* $\overline{\beta_{\sigma_m}(\hbar_0, r)}$, $\alpha(\hbar_n, \hbar_{n+1}) \ge 1$ *for all* $n \in \mathbb{N}$ *and* $\{\Lambda\check{S}(\hbar_n)\} \to \mu \in \overline{\beta_{\sigma_m}(\hbar_0, r)}$. *Also, if* μ *satisfies* (2.12), $\alpha(\hbar_n, \mu) \ge 1$ *and* $\alpha(\mu, \hbar_n) \ge 1$ *for all natural* $n \ge 0$, *then* $\check{S}$ *and* $\mathcal{H}$ *admit a fixed point* μ *in* $\overline{\beta_{\sigma_m}(\hbar_0, r)}$.

3 Results for multigraph dominated mappings

In this section, we will prove an application of Theorem 2.1 in graph theory. Jachymski [15] established a significant result related to the contraction mappings in a metric space endowed with a graph. Hussain et al. [13] showed fixed point results for graphic contractions and presented an application. For more results on graphical fixed point theory, see ([9, 30]).

Definition 3.1 Let $Y \neq \Phi$ and $\zeta = (V(\zeta), \Xi(\zeta))$ be a graph such that $V(\zeta) = X$, $D \subseteq R$. A function $F : R \to \wp(R)$ is said to be a multigraph dominated on D if $(\delta, \gamma) \in \Xi(\zeta)$, for each $\gamma \in F\delta$ and $\gamma \in D$.

Theorem 3.2 *Let* (Λ, σ) *be a complete BMMS endowed with a graph* ζ. *Let* $r > 0$, $\hbar_0 \in \overline{\beta_{\sigma_m}(\hbar_0, r)}$, *and* $\check{S}, \mathcal{H} : \Lambda \to \wp(\Lambda)$, *and* $\{\mathcal{H}\check{S}(\hbar_n)\}$ *be a sequence in* $\overline{\beta_{\sigma_m}(\hbar_0, r)}$ *generated by* $\hbar_0$. *Assume that the following satisfy*:

(i) $\check{S}$, $\mathcal{H}$ *are dominated on* $\overline{\beta_{\sigma_m}(\hbar_0, r)} \cap \{\mathcal{H}\check{S}(\hbar_n)\}$.

(ii) *There exists* $\tau > 0$ *and* $\mathcal{F}$ *is a strictly increasing function satisfying*

$$\tau + \mathcal{F}\big(H_\sigma(\check{S}x, \mathcal{H}q)\big) \le \mathcal{F}\big(\nabla(x,q)\big), \tag{3.1}$$

for all $x, q \in \overline{\beta_{\sigma_m}(\hbar_0, r)} \cap \{\mathcal{H}\check{S}(\hbar_n)\}$, $(x,q) \in \Xi(\zeta)$ *and* $H_\sigma(\check{S}x, \mathcal{H}q) > 0$.

(iii) $\sigma(\hbar_0, \check{S}\hbar_0) \le r^{\frac{1-\ell\eta}{\ell}}$.

Then, $\{\mathcal{H}\check{S}(\hbar_n)\}$ *is a sequence in* $\overline{\beta_{\sigma_m}(\hbar_0, r)}$, $(\hbar_n, \hbar_{n+1}) \in \Xi(\zeta)$ *and* $\{\mathcal{H}\check{S}(\hbar_n)\} \to \mu$. *Also, if* μ *satisfies* (3.1) *and* $(\hbar_n, \mu) \in \Xi(\zeta)$ *or* $(\mu, \hbar_n) \in \Xi(\zeta)$ *for all naturals* $n = 1, 2, 3, \ldots$, *then* $\check{S}$ *and* $\mathcal{H}$ *have a common fixed point* μ *in* $\overline{\beta_{\sigma_m}(\hbar_0, r)}$.

Proof Define the function $\alpha : \Lambda \times \Lambda \to [0, \infty)$ by

$$\alpha(x,q) = \begin{cases} 1, & \text{if } x \in \overline{\beta_{\sigma_m}(\hbar_0, r)}, (x,q) \in \Xi(\zeta), \\ 0, & \text{otherwise.} \end{cases}$$

Since $\check{S}$ and $\mathcal{H}$ are graph dominated on $\overline{\beta_{\sigma_m}(\hbar_0, r)}$, then for all $e \in \overline{\beta_{\sigma_m}(\hbar_0, r)}$, $(x,q) \in \Xi(\zeta)$ for each $q \in \check{S}x$ and $(x,q) \in \Xi(\zeta)$ for every $q \in \mathcal{H}x$. So, $\alpha(x,q) = 1$ for each $q \in \check{S}x$ and $\alpha(x,q) = 1$ for each $q \in \mathcal{H}x$. This implies that $\inf\{\alpha(x,q) : q \in \check{S}x\} = 1$ and $\inf\{\alpha(x,q) : q \in \mathcal{H}x\} = 1$.

Hence, $\alpha_*(x,\check{S}x) = 1$, $\alpha_*(x,\mathcal{H}x) = 1$ for each $x \in \overline{\beta_{\sigma_m}(\hbar_0,r)}$. So, $\check{S},\mathcal{H} : \Lambda \to \wp(\Lambda)$ are α_*-dominated multifunctions on $\overline{\beta_{\sigma_m}(\hbar_0,r)}$. Furthermore, inequality (3.1) can be rewritten as

$$\tau + \mathcal{F}\big(H_\sigma(\check{S}x,\mathcal{H}q)\big) \le \mathcal{F}\big(\nabla(x,q)\big)$$

for all $x,q \in \overline{\beta_{\sigma_m}(\hbar_0,r)} \cap \{\mathcal{H}\check{S}(\hbar_n)\}$, $\alpha(x,q) \ge 1$ and $H_\sigma(\check{S}x,\mathcal{H}q) > 0$. Also, (iii) holds. Then, from Theorem 2.1, we have $\{\mathcal{H}\check{S}(\hbar_n)\}$ is a sequence in $\overline{\beta_{\sigma_m}(\hbar_0,r)}$ and $\{\mathcal{H}\check{S}(\hbar_n)\} \to \mu \in \overline{\beta_{\sigma_m}(\hbar_0,r)}$. Now, $\hbar_n,\mu \in \overline{\beta_{\sigma_m}(\hbar_0,r)}$ and either $(\hbar_n,\mu) \in \Xi(\zeta)$ or $(\mu,\hbar_n) \in \Xi(\zeta)$ implies that either $\alpha(\hbar_n,\mu) \ge 1$ or $\alpha(\mu,\hbar_n) \ge 1$. Hence, all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1, $\check{S}$ and $\mathcal{H}$ have a common fixed point μ in $\overline{\beta_{\sigma_m}(\hbar_0,r)}$ and $\sigma(\mu,\mu) = 0$. □

4 Application to integral equations

Theorem 4.1 *Let* (Λ,σ) *be a complete BMMS. Let* $\hbar_0 \in \Lambda$ *and* $\check{S},\mathcal{H} : \Lambda \to \Lambda$ *be the self-mappings. Assume that there are* $\tau > 0$ *and a function* $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ *strictly increasing so that satisfying the following condition*:

$$\tau + \mathcal{F}\big(\sigma(\check{S}x,\mathcal{H}q)\big) \le \mathcal{F}\big(\nabla(x,q)\big), \tag{4.1}$$

whenever $x,q \in \{\hbar_n\}$ *and* $\sigma(\check{S}x,\mathcal{H}q) > 0$. *Then* $\{\hbar_n\} \to f \in \Lambda$. *Also, if* (4.1) *holds for* $x,q \in \{f\}$, *then* $\check{S}$ *and* $\mathcal{H}$ *have a unique common fixed point* f *in* Λ.

Proof The proof of this theorem is similar to that of Theorem 2.1. □

In this section, we discuss the application of fixed point Theorem 4.1 in form of unique solution of two operator of Volterra-type integral equations given below:

$$f(k) = \int_0^k H(k,h,f)\,dh, \tag{4.2}$$

$$\varkappa(k) = \int_0^k G(k,h,\varkappa))\,dh \tag{4.3}$$

for all $k \in [0,1]$ and H, G are mappings from $[0,1] \times [0,1] \times £([0,1],\mathbb{R}_+)$ to $\mathbb{R}$. We find the solution of (4.2) and (4.3). Let $\Lambda = £([0,1],\mathbb{R}_+)$ be the set of all continuous functions on $[0,1]$. For $f \in £([0,1],\mathbb{R}_+)$, define a norm as: $\|f\|_\tau^2 = \sup_{k\in[0,1]}\{e^{|f(k)|^2}e^{-\tau k}\}$, where $\tau > 0$. Then define

$$\sigma_\tau(f,\varkappa) = \Big[\sup_{k\in[0,1]}\big\{e^{|f(k)-\varkappa(k)|}e^{-\tau k}\big\}\Big]^2 = e^{\|f-\varkappa\|_\tau^2}$$

for all $f,\varkappa \in £([0,1],\mathbb{R}_+)$, with these settings, $(£([0,1],\mathbb{R}_+),\sigma_\tau)$ becomes a complete *BMMS*.

Now, we are proving an existence theorem to investigate the solution of a coupled system of nonlinear integral equations.

Theorem 4.2 *Assume that the following are satisfied*:

(i) $H,G : [0,1] \times [0,1] \times £([0,1],\mathbb{R}_+) \to \mathbb{R}$;

(ii) *Define*

$$(\check{S}f)(k) = \int_0^k H(k,h,f)\,dh,$$
$$(\mathcal{H}\varkappa)(k) = \int_0^k G(k,h,\varkappa)\,dh.$$

Suppose that there exist $\tau > 0$ *such that*

$$e^{|H(k,h,f)-G(k,h,c)|^2} \le \frac{\tau\nabla(f,\varkappa)(e^{\nabla(f,\varkappa)-e^{\|\check{S}f-\mathcal{H}x\|_\tau^2}})}{e^{\tau(1-h)}}$$

for all $k,h \in [0,1]$ *and* $f,\varkappa \in £([0,1],\mathbb{R})$, *where*

$$\nabla(f,\varkappa) = \max\left\{\begin{array}{c} e^{\|f-\varkappa\|_\tau^2}, e^{\|f-\check{S}f)\|_\tau^2}, e^{\|\varkappa-\mathcal{H}\varkappa)\|_\tau^2}, \\ \frac{e^{\|f-\check{S}f)\|_\tau^4}.e^{\|\varkappa-\mathcal{H}\varkappa)\|_\tau^2}}{1+e^{\|f(h)-\varkappa(h)\|_\tau^4}} \end{array}\right\}.$$

Then, integral equations (4.2) *and* (4.3) *have a common solution in* $£([0,1],\mathbb{R}_+)$.

Proof By assumption (ii)

$$\begin{aligned} e^{|\check{S}f-\mathcal{H}\varkappa|^2} &= \int_0^k e^{|H(k,h,f)-G(k,h,\varkappa)|^2}\,dh \\ &\le \int_0^k \frac{\tau\nabla(f,\varkappa)(e^{\nabla(f,\varkappa)-e^{\|\check{S}f-\mathcal{H}x\|_\tau^2}})}{e^{\tau(1-h)}}\,dh \\ &\le \frac{\tau\nabla(f,\varkappa)(e^{\nabla(f,\varkappa)-e^{\|\check{S}f-\mathcal{H}x\|_\tau^2}})}{e^{\tau}}\int_0^k e^{\tau h}\,dh \\ &\le \frac{\nabla(f,\varkappa)(e^{\nabla(f,\varkappa)-e^{\|\check{S}f-\mathcal{H}x\|_\tau^2}})}{e^{\tau}}e^{\tau k}. \end{aligned}$$

This implies

$$\begin{aligned} &e^{|\check{S}f-\mathcal{H}\varkappa|^2}e^{-\tau k} \le \frac{e^{\nabla(f,\varkappa)-e^{\|\check{S}f-\mathcal{H}x\|_\tau^2}}}{e^{\tau}}, \\ &\frac{e^{\|\check{S}f-\mathcal{H}\varkappa\|_\tau^2}}{\nabla(f,\varkappa)} \le \frac{e^{\nabla(f,\varkappa)-e^{\|\check{S}f-\mathcal{H}x\|_\tau^2}}}{e^{\tau}}, \\ &\frac{e^{\|\check{S}f-\mathcal{H}\varkappa\|_\tau^2}}{e^{\nabla(f,\varkappa)-e^{\|\check{S}f-\mathcal{H}x\|_\tau^2}}} \le \frac{\nabla(f,\varkappa)}{e^{\tau}}. \end{aligned}$$

Taking ln both sides, we have

$$\ln\left(e^{\|\check{S}f-\mathcal{H}\varkappa\|_\tau^2}\right) - \nabla(f,\varkappa) + e^{\|\check{S}f-\mathcal{H}\varkappa\|_\tau^2} \le \ln\left(\nabla(f,\varkappa)\right) + \ln e^{-\tau},$$

which further implies

$$\tau + \ln\left(e^{\|\check{S}f-\mathcal{H}\varkappa\|_\tau^2}\right) + e^{\|\check{S}f-\mathcal{H}\varkappa\|_\tau^2} \le \ln\left(\nabla(f,\varkappa)\right) + \nabla(f,\varkappa).$$

So, all the requirements of Theorem 4.1 are satisfied for $\mathcal{F}(\varkappa) = \ln\varkappa + \varkappa$; $\varkappa > 0$ and $d_\tau(f,\varkappa) = e^{\|f-\varkappa\|_\tau^2}$. Hence, integral equations in (4.2) and (4.3) have a common solution. □

5 Application to functional equations

Here, we present an application for the solution of a functional equation in dynamic programming. Let $\hat{A}$ and Γ be two Banach spaces, $\pounds \subseteq \hat{A}$, $\Omega \subseteq \Gamma$ and

$$\sigma : \pounds \times \Omega \to \pounds,$$
$$\hbar, \varpi : \pounds \times \Omega \to \mathbb{R},$$
$$L, M : \pounds \times \Omega \times \mathbb{R} \to \mathbb{R}.$$

Further useful results relevant to dynamic programming are shown in [7, 8, 22]. We can assume that £ and Ω appear for the decisions spaces. The problem related to dynamic programming is to find the solution of the given equations:

$$p(\vartheta) = \sup_{\vartheta\in\Omega}\bigl\{\hbar(\vartheta,\Theta) + L\bigl(\vartheta,\Theta,p\bigl(\sigma(\vartheta,\Theta)\bigr)\bigr)\bigr\}, \tag{5.1}$$

$$q(\vartheta) = \sup_{\Theta\in\Omega}\bigl\{\varpi(\vartheta,\Theta) + M\bigl(\vartheta,\Theta,q\bigl(\sigma(\vartheta,\Theta)\bigr)\bigr)\bigr\}, \tag{5.2}$$

for $\vartheta \in \pounds$. We want to show that equations (5.1) and (5.2) unique solution. Suppose $R(\pounds)$ represents the class of all positive valued functions on £. Consider,

$$\sigma(\upsilon,w) = \bigl\|e^{\upsilon-w}\bigr\|_\infty^2 = \sup_{\vartheta\in N} e^{|\upsilon(\vartheta)-w(\vartheta)|^2} \tag{5.3}$$

for all $\upsilon, w \in R(\pounds)$, $(R(\pounds),\sigma)$ becomes a complete *BMMS*. Assume that:

($\hat{C}1$): L, M, $\hbar$, and ϖ are bounded.

($\hat{C}2$): For $\vartheta \in \pounds$, $\upsilon \in R(\pounds)$, let $P, \hat{A} : R(\pounds) \to R(\pounds)$ be multivalued mappings, so that

$$P\upsilon(\vartheta) = \sup_{\Theta\in\Omega}\bigl\{\hbar(\vartheta,\Theta) + L\bigl(\vartheta,\Theta,\upsilon\bigl(\sigma(\vartheta,\Theta)\bigr)\bigr)\bigr\}, \tag{5.4}$$

$$\hat{A}\upsilon(\vartheta) = \sup_{\Theta\in\Omega}\bigl\{\varpi(\vartheta,\Theta) + M\bigl(\vartheta,\Theta,\upsilon\bigl(\sigma(\vartheta,\Theta)\bigr)\bigr)\bigr\}. \tag{5.5}$$

Furthermore, for each $(\vartheta,\Theta) \in \pounds \times \Omega$, $\upsilon, w \in R(\pounds)$, $t \in \pounds$ and for a $\tau > 0$,

$$e^{|L(\vartheta,\Theta,\upsilon(t))-M(\vartheta,\Theta,w(t)|^2} \leq \nabla(\upsilon,w)\bigl(e^{\nabla(\upsilon,w)-e^{|(\Upsilon\upsilon_1)(\vartheta)-(\hat{A}\upsilon_2)(\vartheta)|^2-\tau}}\bigr) \tag{5.6}$$

where

$$\nabla(\upsilon,w) = \sup\left\{\begin{pmatrix} e^{|\upsilon(t)-w(t)|^2}, \\ e^{|\upsilon(t)-\Upsilon\upsilon(t)|^2}, e^{|\upsilon(t)-\hat{A}w(t)|^2} \\ \frac{e^{|\upsilon(t)-\Upsilon\upsilon(t)|^4}.e^{|\upsilon(t)-\hat{A}w(t)|^2}}{1+e^{|\upsilon(t)-w(t)|^4}}, \end{pmatrix}^{\kappa}\right\}.$$

Theorem 5.1 *Assume that* ($\hat{C}1$), ($\hat{C}2$), *and* (5.6) *hold. Then, the equations* (5.1) *and* (5.2) *have a unique common and bounded solution in* $R(\pounds)$.

Proof Take any $c > 0$. From (5.4) and (5.5), there are $\upsilon_1, \upsilon_2 \in R(£)$, and $\Theta_1, \Theta_2 \in \Omega$ such that

$$(P\upsilon_1) < \hbar(\vartheta, \Theta_1) + L\big(\vartheta, \Theta_1, \upsilon_1\big(\sigma(\vartheta, \Theta_1)\big)\big) + c, \tag{5.7}$$

$$(\hat{A}\upsilon_2) < \hbar(\vartheta, \Theta_2) + M\big(\vartheta, \Theta_2, \upsilon_2\big(\sigma(\vartheta, \Theta_2)\big)\big) + c. \tag{5.8}$$

Using the definition of supremum, we get

$$(P\upsilon_1) \geq \hbar(\vartheta, \Theta_2) + L\big(\vartheta, \Theta_2, \upsilon_1\big(\sigma(\vartheta, \Theta_2)\big)\big), \tag{5.9}$$

$$(\hat{A}\upsilon_2) \geq \hbar(\vartheta, \Theta_1) + M\big(\vartheta, \Theta_1, \upsilon_2\big(\sigma(\vartheta, \Theta_1)\big)\big). \tag{5.10}$$

Then, from (5.6), (5.7), and (5.10), we have

$$\begin{aligned}
&e^{|(P\upsilon_1)(\vartheta)-(\hat{A}\upsilon_2)(\vartheta)|^2} \\
&\quad \leq e^{|L(\vartheta,\Theta_1,\upsilon_1(\sigma(\vartheta,\Theta_1)))-M(\vartheta,\Theta_1,\upsilon_2(\sigma(\vartheta,\Theta_1)))|} + c \\
&\quad \leq e^{|L(\vartheta,\Theta_1,\upsilon_1(\sigma(\vartheta,\Theta_1)))-M(\vartheta,\Theta_1,\upsilon_2(\sigma(\vartheta,\Theta_1)))|^2} + c \\
&\quad \leq \nabla(\upsilon, w)\big(e^{\nabla(\upsilon,w)-e^{|(P\upsilon_1)(\vartheta)-(\hat{A}\upsilon_2)(\vartheta)|^2-\tau}}\big) \\
&\quad \leq \nabla(\upsilon, w)e^{-\tau}\big(e^{\nabla(\upsilon,w)-e^{|(P\upsilon_1)(\vartheta)-(\hat{A}\upsilon_2)(\vartheta)|^2}}\big) + c.
\end{aligned}$$

Since, $c > 0$ is arbitrary, we obtain

$$\begin{aligned}
&e^{|P\upsilon_1(\vartheta)-\hat{A}\upsilon_2(\vartheta)|^2} \leq \nabla(\upsilon, w)e^{-\tau}\big(e^{\nabla(\upsilon,w)-e^{|(P\upsilon_1)(\vartheta)-(\hat{A}\upsilon_2)(\vartheta)|^2}}\big), \\
&e^{\tau}e^{|P\upsilon_1(\vartheta)-\hat{A}\upsilon_2(\vartheta)|^2} \leq \nabla(\upsilon, w)e^{\nabla(\upsilon,w)-e^{|(P\upsilon_1)(\vartheta)-(\hat{A}\upsilon_2)(\vartheta)|^2}}.
\end{aligned}$$

It implies that,

$$\tau + \ln\big(e^{|P\upsilon_1(\vartheta)-\hat{A}\upsilon_2(\vartheta)|^2}\big) + e^{|P\upsilon_1(\vartheta)-\hat{A}\upsilon_2(\vartheta)|^2} \leq \ln\big(\nabla(\upsilon, w)\big) + \nabla(\upsilon, w)).$$

Hence, all the conditions of Theorem 4.1 are satisfied for $\mathcal{F}(\varpi) = \ln\varpi + \varpi$; $\varpi > 0$ and $\sigma_\tau(\upsilon, w) = e^{\|\upsilon-w\|_\tau^2}$. Thus, P and $\hat{A}$ both have a unique common bounded solution of the equations (5.1) and (5.2). □

6 Conclusion

In this research, we have achieved some new fixed point results for a pair of multifunctions satisfying a generalized contractive conditions only on a closed ball with an intersection of an iterative sequence in complete b-multiplicative metric space. We have used a strictly increasing mapping $\mathcal{F}$ instead of the class of mappings used by Wardowski [31]. The notion of multigraph-dominated mappings is introduced. Furthermore, some new fixed point results are obtained for graphic contraction in a b-multiplicative metric space. Applications are given to approximate the unique common bounded solution for a coupled system of nonlinear integral equations and functional equations in dynamical programming. Our results extended and generalized many results appearing in the literature, such as Rasham et al. [23–25], Wordowski's result [31], Acar et al. [2], and many more classical results [11, 28, 29].

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Author contribution
TR and MDLS together discussed the problem and prepared this research article. MDLS analyzed all multivalued hybrid pair results for dominated mappings and made necessary improvements in applications section. TR is the major contributor in writing the paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of Poonch, Rawalakot, Azad Kashmir, Pakistan. [2]Institute of Research and Development of Processes IIDP, University of the Basque Country, Campus of Leioa, PO Box 48940, Leioa, Bizkaia 48940, Spain.

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