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# Optimal quadrature formulas for oscillatory integrals in the Sobolev space

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**Abstract**
This work studies the problem of construction of optimal quadrature formulas in the sense of Sard in the space $L_2^{(m)}(0,1)$ for numerical calculation of Fourier coefficients. Using Sobolev's method, we obtain new sine and cosine weighted optimal quadrature formulas of such type for $N+1 \geq m$, where $N+1$ is the number of nodes. Then, explicit formulas for the optimal coefficients of optimal quadrature formulas are obtained. The obtained optimal quadrature formulas in $L_2^{(m)}(0,1)$ space are exact for algebraic polynomials of degree $(m-1)$.

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## 1 Introduction

Methods based on the Fourier transform are virtually used in many areas of engineering and science. It is known that one of the most important and interesting discoveries in mathematics is that many math functions can be approximated by a series of sinusoids, called Fourier series. Furthermore, we know that the Fourier coefficients

$$F_s(\omega) = \int_0^1 f(t)\sin(2\pi\omega t)\,dt, \qquad F_c(\omega) = \int_0^1 f(t)\cos(2\pi\omega t)\,dt$$

are strongly oscillating integrals for sufficiently large values of $\omega$. Moreover, these weighted integrals can be applied to reconstruct X-ray Computed Tomography images [11, 13, 15]. It should be noted that standard methods are not suitable for numerical calculation of these integrals. Therefore, it is necessary to develop special methods for approximate calculation of such integrals. It should be noted that one of the first numerical integration formula for the integral

$$I[f,\omega] = \int_a^b e^{2\pi i\omega x} f(x)\,dx, \tag{1}$$

i.e., for the linear combination of $F_s(\omega)$ and $F_c(\omega)$, was obtained by Filon [5] in 1928 using a quadratic spline. Since then, for integrals of different types of highly oscillating functions many special effective methods have been developed, such as the Filon-type method, the Clenshaw–Curtis–Filon-type method, the Levin-type methods, the modified Clenshaw–Curtis method, the generalized quadrature rule, and the Gauss–Laguerre quadrature. Recently, in [1, 2], based on Sobolev's method, the problem of construction of optimal quadrature formulas in the sense of Sard for numerical calculation of integrals (1) with integer $\omega$ was studied in Hilbert spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$.

In [13], the authors deal with the construction of an optimal quadrature formula for approximation of Fourier integrals in the Sobolev space $L_2^{(1)}[a,b]$ of nonperiodic, complex-valued functions that are square integrable with first-order derivative. There, the quadrature sum consists of a linear combination of the given function values in a uniform grid. The difference between the integral and the quadrature sum is estimated by the norm of the error functional. The optimal quadrature formula is obtained by minimizing the norm of the error functional with respect to coefficients. Moreover, several numerical results are presented and the obtained optimal quadrature formula is applied to reconstruct the X-ray Computed Tomography image by approximating Fourier transforms. We note that the results of the paper were generalized for functions of the Sobolev space $L_2^{(m)}$ in [14].

In the work [15], the construction process of the optimal quadrature formulas for weighted integrals is presented in the Sobolev space $\widetilde{L}_2^{(m)}(0,1)$ of complex-valued periodic functions that are square integrable with $m$th-order derivative. In particular, optimal quadrature formulas are given for Fourier coefficients. There, using the optimal quadrature formulas the approximation formulas for Fourier integrals $\int_a^b e^{2\pi\omega x}f(x)\,dx$ with $\omega \in \mathbb{R}$ are obtained. In the cases $m = 1, 2$, and 3, the obtained approximation formulas are applied for reconstruction of Computed Tomography (CT) images coming from the filtered back-projection method. Compared with the optimal quadrature formulas in the nonperiodic case, the approximation formula for the periodic case is much simpler, therefore, it is easy to implement and involves less computation.

We note that quadrature and cubature formulas with extremal properties play an important role in applications. The works [7, 8] and [9] also deal with some extremality properties. In these works, the authors considered a sequence of positive linear operators that map $C(\Omega)$ into itself, where $\Omega$ is a compact convex subset of $\mathbb{R}^d$. In [8], they established Korovkin-type theorems. In the work [9], the authors studied cubature formulas on $\Omega$ that approximate the integral of every convex function from above. They are called negative-definite formulas. For aiming at "good" negative-definite formulas the authors introduced and studied three extremal properties named as minimal, best, and optimal.

The present work is devoted to numerical calculation of the integrals $F_s(\omega)$ and $F_c(\omega)$ with high accuracy.

For this, here in the space $L_2^{(m)}(0,1)$, we consider quadrature formulas of the forms

$$\int_0^1 \sin(2\pi\omega x)\varphi(x)\,dx \cong \sum_{\beta=0}^{N} C_s[\beta]\varphi[\beta] \tag{2}$$

and

$$\int_0^1 \cos(2\pi\omega x)\varphi(x)\,dx \cong \sum_{\beta=0}^{N} C_c[\beta]\varphi[\beta], \tag{3}$$

where $C_s[\beta]$ and $C_c[\beta]$ are coefficients, $[\beta] = h\beta$, $h = \frac{1}{N}$, $N$ is a natural number, $\omega \in \mathbb{R}$, and $\omega \neq 0$. $L_2^{(m)}(0,1)$ is the Sobolev space of function $\varphi$ that are square integrable with $m$th generalized derivative and equipped with the norm

$$\|\varphi\|_{L_2^{(m)}(0,1)} = \left(\int_0^1 \left(\varphi^{(m)}(x)\right)^2 dx\right)^{1/2}.$$

It should be noted that constructions of optimal quadrature formulas with sine and cosine weight functions of the forms (2) and (3) in the Sobolev space $L_2^{(m)}$ were considered in the works [3] and [12], respectively. In the present paper, for completeness, we give the results for optimal quadrature formulas of the form (2) obtained in [3] and we obtain a more simplified system for determining the coefficient of optimal quadrature formulas (3) that requires a smaller amount of calculation than the results of the work [12]. Along with these, we obtain a more simplified form of the results [14] by linear combination of optimal quadrature formulas of the form (2) and (3).

The rest of the paper is organized as follows. In Sect. 2 we state the problem of construction of weighted optimal quadrature formulas in the space $L_2^{(m)}(0,1)$. In Sect. 3 we give some definitions and preliminary results. In Sect. 4 we construct trigonometric weighted optimal quadrature formulas and find the optimal coefficients. Finally, in Sect. 5 we present some numerical results of the upper bounds for the errors of the optimal quadrature formulas in the forms (2) and (3).

## 2 Statement of the problem

In this section, we consider a weighted quadrature formula of the form

$$\int_0^1 p(x)\varphi(x)\,dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi[\beta], \tag{4}$$

where $p(x)$ is a weight function, $\int_0^1 p(x)\,dx < \infty$, $[\beta] = h\beta$, $h = 1/N$, $N$ is a natural number, $C[\beta]$ are coefficients of the formula (4), and $\varphi$ is a function of the space $L_2^{(m)}(0,1)$. In the following, for convenience we denote the space $L_2^{(m)}(0,1)$ as $L_2^{(m)}$.

The following difference is called the error of the quadrature formula (4)

$$(\ell,\varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)\,dx = \int_0^1 p(x)\varphi(x)\,dx - \sum_{\beta=0}^{N} C[\beta]\varphi[\beta]. \tag{5}$$

Here, $\ell$ is an error functional corresponding to the quadrature formula (4) and it belongs to the conjugate space $L_2^{(m)*}$. The functional $\ell$ has the form

$$\ell(x) = \varepsilon_{[0,1]}(x)p(x) - \sum_{\beta=0}^{N} C[\beta]\delta\bigl(x-[\beta]\bigr), \tag{6}$$

here, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, $\delta$ is the Dirac delta function.

The following conditions are imposed because the functional $\ell$ is defined on the space $L_2^{(m)}$

$$\bigl(\ell, x^{\alpha}\bigr) = 0, \quad \alpha = 0,1,2,\ldots,m-1. \tag{7}$$

The last equations mean exactness of the quadrature formula (4) for any polynomial of degree $(m-1)$.

It is known that by the Cauchy–Schwarz inequality

$$\left|(\ell,\varphi)\right| \leq \left|\|\varphi\|_{L_2^{(m)}} \cdot \|\ell\|\right|_{L_2^{(m)*}}$$

the error (5) can be estimated by the norm of the error functional (6)

$$\|\ell\|_{L_2^{(m)*}} = \sup_{\|\varphi\|_{L_2^{(m)}}=1} \left|(\ell,\varphi)\right|.$$

In this way, the error estimate of the quadrature formula (4) on the space $L_2^{(m)}$ is reduced to finding a norm of the error functional $\ell(x)$ in the conjugate space $L_2^{(m)*}$. Hence, we state the following.

**Problem 1** Find the coefficients $\mathring{C}[\beta]$ that give a minimum value to the quantity $\|\ell\|_{L_2^{(m)*}}$ and find the following

$$\inf_{C[\beta]} \|\ell\|_{L_2^{(m)*}}. \tag{8}$$

The quadrature formula (4) with such coefficients $\mathring{C}[\beta]$ is called *the optimal quadrature formula in the sense of Sard* (see [17]), $\mathring{C}[\beta]$ are called *the optimal coefficients* and the corresponding error functional denoted by $\mathring{\ell}$ has the norm

$$\|\mathring{\ell}\|_{L_2^{(m)*}} = \inf_{C[\beta]} \|\ell\|_{L_2^{(m)*}}.$$

Thus, in order to solve Problem 1, first we should calculate $\|\ell\|_{L_2^{(m)*}}$ and then we have to find the optimal coefficients $\mathring{C}[\beta]$ that give the minimum to $\|\ell\|_{L_2^{(m)*}}$.

It is well known that for any functional $\ell$ in $L_2^{(m)*}$ the following equality holds (see [21, 23])

$$\|\ell\|^2_{L_2^{(m)*}} = (\ell,\psi_\ell) = \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)\,dx, \tag{9}$$

where

$$\psi_\ell(x) = \int_{-\infty}^{\infty} \ell(y)\frac{|x-y|^{2m-1}}{2(2m-1)!}\,dy + P_{m-1}(x) \tag{10}$$

and $\psi_\ell$ is *the extremal function* for the error functional $\ell$ defined on the space $L_2^{(m)}[0,1]$, $P_{m-1}(x)$ is any polynomial of degree $(m-1)$. We note that the extremal function was found by Sobolev [20].

Then from (9), taking (7) and (10) into account, one can obtain

$$\begin{aligned}\|\ell\|^2_{L_2^{(m)*}} = (-1)^m \Bigg( &\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C[\beta]C[\gamma]\frac{|[\beta]-[\gamma]|^{2m-1}}{2(2m-1)!} \\ &- 2\sum_{\beta=0}^{N} C[\beta]\int_0^1 p(x)\frac{|x-[\beta]|^{2m-1}}{2(2m-1)!}\,dx \\ &+ \int_0^1\int_0^1 p(x)p(y)\frac{|x-y|^{2m-1}}{2(2m-1)!}\,dx\,dy\Bigg). \end{aligned} \tag{11}$$

See, for example [19].

Thus, for construction of optimal quadrature formulas of the form (4) we should find the minimum value of the expression (11) under the conditions (7). For this, we need some definitions and preliminary results that are given in the next section.

## 3 Definitions and preliminary results

In this section we give some definitions and known results that are necessary in the proof of the main results.

Here, we use the concept of discrete argument functions and operations on them given in [20, 23].

Assume that $\phi(x)$ and $\psi(x)$ are real-valued functions of real variables and are defined in the real line $\mathbb{R}$. We recall that $[\beta] = h\beta$, $\beta \in \mathbb{Z}$, $h = \frac{1}{N}$, where $N$ is a natural number.

**Definition 1** Function $\phi[\beta]$ is *a function of a discrete argument*, if it is given on some set of integer values of $\beta$.

**Definition 2** *The inner product* of two discrete functions $\phi[\beta]$ and $\psi[\beta]$ is the number

$$[\phi,\psi] = \sum_{\beta=-\infty}^{\infty} \phi[\beta]\cdot\psi[\beta],$$

if the series on the right-hand side of the last equality converges absolutely.

**Definition 3** *The convolution* of two discrete functions $\phi[\beta]$ and $\psi[\beta]$ is the following inner product

$$\phi[\beta] * \psi[\beta] = \big[\phi[\gamma], \psi[\beta-\gamma]\big] = \sum_{\gamma=-\infty}^{+\infty} \phi[\gamma]\cdot\psi[\beta-\gamma].$$

In this work, the discrete analog $D_m[\beta]$ of the operator $d^{2m}/dx^{2m}$ plays an important role in the construction of optimal formulas in $L_2^{(m)}(0,1)$ space. This discrete operator satisfies the equality

$$hD_m[\beta] * \frac{|[\beta]|^{2m-1}}{2(2m-1)!} = \delta[\beta], \tag{12}$$

where $\delta[\beta] = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0, \end{cases}$ and $*$ is the convolution for the discrete argument functions.

It should be noted that the discrete analog $D_m[\beta]$ of the operator $d^{2m}/dx^{2m}$ was first introduced and studied by Sobolev [20]. In [18], the discrete analog $D_m[\beta]$ was constructed and the following theorem was proved.

**Theorem 1** *The discrete analog to the differential operator $\frac{d^{2m}}{dx^{2m}}$ has the form*

$$D_m[\beta] = p \cdot \begin{cases} \sum_{k=1}^{m-1} A_k q_k^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + \sum_{k=1}^{m-1} A_k, & |\beta| = 1, \\ C + \sum_{k=1}^{m-1} \frac{A_k}{q_k}, & \beta = 0, \end{cases} \tag{13}$$

*where*

$$p = \frac{(2m-1)!}{h^{2m}}, \qquad A_k = \frac{(1-q_k)^{2m+1}}{E_{2m-1}(q_k)}, \qquad C = -2^{2m-1},$$

*$E_{2m-1}(x)$ is the Euler–Frobenius polynomial of degree $(2m-1)$, $q_k$ are the roots of the Euler–Frobenius polynomials $E_{2m-2}(x)$* and satisfy the inequality $|q_k| < 1$, and *$h$ is a small positive parameter.*

Moreover, several properties of the discrete analog $D_m[\beta]$ were studied in [20, p. 732] and [18]. Here, we need the following of them.

The discrete argument function $D_m[\beta]$ and the monomials $[\beta]^k$ are related to each other as follows

$$D_m[\beta] * [\beta]^k = 0, \quad k = 0, 1, \dots, 2m-1. \tag{14}$$

The Euler–Frobenius polynomials $E_k(x)$, $k = 1, 2, \dots$ are defined by the following formula (see, e.g., [21, 22])

$$E_k(x) = \frac{(1-x)^{k+2}}{x} \cdot \left(x\frac{d}{dx}\right)^k \frac{x}{(1-x)^2}, \tag{15}$$

where $E_0(x) = 1$. The following identity holds for the polynomial $E_k(x)$:

$$E_k(x) = x^k E_k\left(\frac{1}{x}\right). \tag{16}$$

Moreover, the following takes place.

**Theorem 2** (Lemma 1.4.3 of [19]) *The Euler–Frobenius polynomial of degree $k$ is determined by the formula*

$$P_k(x) = (x-1)^{k+1} \sum_{i=0}^{k+1} \frac{\Delta^i 0^{k+1}}{(x-1)^i}, \tag{17}$$

*i.e., $P_k(x) = E_k(x)$, where $\Delta^i 0^k = \sum_{l=1}^{i} (-1)^{i-l} \binom{i}{l} l^k$.*

The coefficients of the Euler–Frobenius polynomial

$$E_k(x) = \sum_{m=0}^{k} a_{m,k} x^m \tag{18}$$

of degree $k$ satisfy the equality $a_{m,k} = a_{k-m,k}$, $m = 0, 1, \ldots, k$.

From [10] we use the following formula

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q}\right)^i \Delta^i \gamma^k|_{\gamma=n}, \tag{19}$$

where $\Delta^i \gamma^k$ is the finite difference of order $i$ of $\gamma^k$, and $q$ is a ratio of a geometric progression.

We also apply the following well-known formulas from [6]

$$\sum_{\gamma=0}^{\beta-1} \gamma^k = \sum_{j=1}^{k+1} \frac{k! B_{k+1-j}}{j!(k+1-j)!} \beta^j, \tag{20}$$

where $B_{k+1-j}$ are Bernoulli numbers, $k$ is a natural number, and

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu} \binom{\nu}{p} \Delta^\alpha 0^p x^{\nu-p}. \tag{21}$$

Further, we introduce the following notations

$$B_1 = \frac{e^{2\pi \mathrm{i}\omega h}}{(e^{2\pi \mathrm{i}\omega h} - 1)^{2m}} E_{2m-2}\left(e^{2\pi \mathrm{i}\omega h}\right), \tag{22}$$

$$B_2 = \frac{2e^{2\pi \mathrm{i}\omega h}}{2\mathrm{i}(e^{2\pi \mathrm{i}\omega h} - 1)^{2j+1}} E_{2j-1}\left(e^{2\pi \mathrm{i}\omega h}\right), \tag{23}$$

$$B_3 = \frac{e^{2\pi \mathrm{i}\omega h}(1 - e^{2\pi \mathrm{i}\omega}) - (-1)^\alpha e^{2\pi \mathrm{i}\omega h}(1 - e^{-2\pi \mathrm{i}\omega})}{(1 - e^{2\pi \mathrm{i}\omega h})^{\alpha+1}} E_{\alpha-1}\left(e^{2\pi \mathrm{i}\omega h}\right), \tag{24}$$

$$B_4 = \sum_{j=1}^{\alpha-1} \binom{\alpha}{j} h^{j+1} \frac{e^{2\pi \mathrm{i}\omega h + 2\pi \mathrm{i}\omega} - (-1)^j e^{2\pi \mathrm{i}\omega h - 2\pi \mathrm{i}\omega}}{(1 - e^{2\pi \mathrm{i}\omega h})^{j+1}} E_{j-1}\left(e^{2\pi \mathrm{i}\omega h}\right), \tag{25}$$

where $E_{2m-2}(x)$, $E_{\alpha-1}(x)$, $E_{2j-1}(x)$, and $E_{j-1}(x)$ are Euler–Frobenius polynomials, and $\mathrm{i}^2 = -1$.

In the next section we present the main results, i.e., we find the analytic expressions for coefficients of the optimal quadrature formulas of the forms (2) and (3).

## 4 Main results

For the quadrature formulas of the forms (2) and (3) we have the error functionals

$$\ell_s(x) = \varepsilon_{[0,1]}(x) \sin(2\pi\omega x) - \sum_{\beta=0}^{N} C_s[\beta] \delta\left(x - [\beta]\right)$$

and

$$\ell_c(x) = \varepsilon_{[0,1]}(x)\cos(2\pi\omega x) - \sum_{\beta=0}^{N} C_c[\beta]\delta\big(x - [\beta]\big),$$

respectively. For the norms of the functionals $\ell_s$ and $\ell_c$ from (11) when $p(x) = \sin(2\pi\omega x)$ and $p(x) = \cos(2\pi\omega x)$, we obtain the following expressions, respectively:

$$\begin{aligned}
\|\ell_s\|^2_{L_2^{(m)*}} = (-1)^m \Bigg( & \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_s[\beta]C_s[\gamma]\frac{|[\beta]-[\gamma]|^{2m-1}}{2(2m-1)!} \\
& - 2\sum_{\beta=0}^{N} C_s[\beta]\int_0^1 \sin(2\pi\omega x)\frac{|x-[\beta]|^{2m-1}}{2(2m-1)!}\,dx \\
& + \int_0^1\int_0^1 \sin(2\pi\omega x)\sin(2\pi\omega y)\frac{|x-y|^{2m-1}}{2(2m-1)!}\,dx\,dy\Bigg)
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\|\ell_c\|^2_{L_2^{(m)*}} = (-1)^m \Bigg( & \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_c[\beta]C_c[\gamma]\frac{|[\beta]-[\gamma]|^{2m-1}}{2(2m-1)!} \\
& - 2\sum_{\beta=0}^{N} C_c[\beta]\int_0^1 \cos(2\pi\omega x)\frac{|x-[\beta]|^{2m-1}}{2(2m-1)!}\,dx \\
& + \int_0^1\int_0^1 \cos(2\pi\omega x)\cos(2\pi\omega y)\frac{|x-y|^{2m-1}}{2(2m-1)!}\,dx\,dy\Bigg).
\end{aligned} \tag{27}$$

In order to find the optimal coefficients $\mathring{C}_s[\gamma]$ and $\mathring{C}_c[\gamma]$ for $\gamma = 0,1,\dots,N$ that give the minimum to $\|\ell_s\|^2_{L_2^{(m)*}}$ and $\|\ell_c\|^2_{L_2^{(m)*}}$ under the conditions (7), respectively, we use the Lagrange method of undetermined multipliers. Then, we obtain the following systems of linear equations for the optimal coefficients $\mathring{C}_s[\gamma]$:

$$\sum_{\gamma=0}^{N} \mathring{C}_s[\gamma]\frac{|[\beta]-[\gamma]|^{2m-1}}{2(2m-1)!} + P_{s,m-1}[\beta] = f_{s,m}[\beta], \quad \beta = 0,1,\dots,N, \tag{28}$$

$$\sum_{\gamma=0}^{N} \mathring{C}_s[\gamma][\gamma]^\alpha = g_{s,\alpha}, \quad \alpha = 0,1,\dots,m-1, \tag{29}$$

where

$$\begin{aligned}
f_{s,m}[\beta] = {} & \frac{(-1)^m}{(2\pi\omega)^{2m}}\sin\big(2\pi\omega[\beta]\big) - \sum_{\alpha=0}^{2m-1}\frac{[\beta]^{2m-1-\alpha}\cdot(-1)^\alpha}{2\cdot\alpha!(2m-1-\alpha)!}\cdot g_{s,\alpha} \\
& + \sum_{\alpha=0}^{2m-1}\frac{(-1)^\alpha[\beta]^{2m-1-\alpha}}{(2m-1-\alpha)!}\cos\left(\frac{\alpha\pi}{2}\right),
\end{aligned} \tag{30}$$

$$g_{s,\alpha} = -\sum_{k=0}^{\alpha-1}\frac{\alpha!}{(\alpha-k)!}\frac{\cos(2\pi\omega+\frac{k\pi}{2})}{(2\pi\omega)^{k+1}} - \frac{2\alpha!\sin(\pi\omega)}{(2\pi\omega)^{\alpha+1}}\sin\left(\pi\omega+\frac{\alpha\pi}{2}\right) \tag{31}$$

and for $\mathring{C}_c[\gamma]$:

$$\sum_{\gamma=0}^{N} \mathring{C}_c[\gamma]\frac{|[\beta]-[\gamma]|^{2m-1}}{2(2m-1)!} + P_{c,m-1}[\beta] = f_{c,m}[\beta], \quad \beta = 0,1,\dots,N, \tag{32}$$

$$\sum_{\gamma=0}^{N} \mathring{C}_c[\gamma][\gamma]^{\alpha} = g_{c,\alpha}, \quad \alpha = 0,1,\dots,m-1, \tag{33}$$

where

$$\begin{aligned} f_{c,m}[\beta] &= \frac{(-1)^m}{(2\pi\omega)^{2m}}\cos\bigl(2\pi\omega[\beta]\bigr) - \sum_{\alpha=0}^{2m-1}\frac{[\beta]^{2m-1-\alpha}\cdot(-1)^{\alpha}}{2\cdot\alpha!(2m-1-\alpha)!}\cdot g_{c,\alpha} \\ &\quad + \sum_{\alpha=0}^{2m-1}\frac{(-1)^{\alpha}[\beta]^{2m-1-\alpha}}{(2m-1-\alpha)!(2\pi\omega)^{\alpha+1}}\sin\left(\frac{\alpha\pi}{2}\right), \end{aligned} \tag{34}$$

$$g_{c,\alpha} = \sum_{k=0}^{\alpha-1}\frac{\alpha!}{(\alpha-k)!}\frac{\sin(2\pi\omega+\frac{k\pi}{2})}{(2\pi\omega)^{k+1}} + \frac{2\alpha!\sin(\pi\omega)}{(2\pi\omega)^{\alpha+1}}\cos\left(\pi\omega+\frac{\alpha\pi}{2}\right). \tag{35}$$

In [20] it is proved that each of the systems (28), (29) and (32), (33) has a unique solution for any fixed $N$ satisfying the inequality $N+1 \geq m$.

We note that in the (28), (29) and (32), (33) the coefficients $\mathring{C}_s[\gamma]$ and $\mathring{C}_c[\gamma]$, polynomials $P_{s,m-1}[\beta]$ and $P_{c,m-1}[\beta]$ are unknowns.

Our aim is to obtain the exact solutions of the systems (28), (29) and (32), (33), respectively.

The following theorems hold.

**Theorem 3** *The optimal quadrature formulas in the sense of Sard of the form* (2) *in the space $L_2^{(m)}(0,1)$ when $\omega \in \mathbb{R}$ and $\omega h \notin \mathbb{Z}$ have the coefficients with the following analytic expressions*

$$\begin{aligned} \mathring{C}_s[0] &= h\left[\frac{1}{2\pi\omega h} - K_{m,\omega}\frac{\cos(\pi\omega h)}{2\sin(\pi\omega h)} + \sum_{k=1}^{m-1}\frac{m_{s,k}q_k - n_{s,k}q_k^N}{q_k-1}\right], \\ \mathring{C}_s[\beta] &= h\left[K_{m,\omega}\sin(2\pi\omega h\beta) + \sum_{k=1}^{m-1}\bigl(m_{s,k}q_k^{\beta} + n_{s,k}q_k^{N-\beta}\bigr)\right], \quad \beta = \overline{1,N-1}, \\ \mathring{C}_s[N] &= h\left[-\frac{\cos(2\pi\omega)}{2\pi\omega h} + K_{m,\omega}\frac{\cos(2\pi\omega - \pi\omega h)}{2\sin(\pi\omega h)} + \sum_{k=1}^{m-1}\frac{-m_{s,k}q_k^N + n_{s,k}q_k}{q_k-1}\right], \end{aligned} \tag{36}$$

*where*

$$K_{m,\omega} = \left(\frac{\sin(\pi\omega h)}{\pi\omega h}\right)^{2m}\frac{(2m-1)!}{2\sum_{k=0}^{m-2}a_{k,2m-2}\cos(2\pi\omega h(m-1-k)) + a_{m-1,2m-2}},$$

*$q_k$ are the roots of the Euler–Frobenius polynomial $E_{2m-2}(x)$ with $|q_k| < 1$, $a_{k,\alpha}$ are the kth coefficients of the Euler–Frobenius polynomial of degree $\alpha$, and $m_{s,k}$ and $n_{s,k}$ satisfy the*

*system of linear equations*

$$\sum_{k=1}^{m-1}\frac{m_{s,k}q_k-(-1)^j n_{s,k}q_k^{N+1}}{(q_k-1)^{j+1}}E_{j-1}(q_k)$$
$$=-\frac{(-1)^j j!\cos(\frac{\pi j}{2})}{(2\pi\omega h)^{j+1}}-\frac{(1+(-1)^j)K_{m,\omega}e^{2\pi \mathrm{i}\omega h}}{2\mathrm{i}(e^{2\pi \mathrm{i}\omega h}-1)^{j+1}}E_{j-1}\big(e^{2\pi \mathrm{i}\omega h}\big),\quad j=\overline{1,m-1},$$
$$\sum_{k=1}^{m-1}\frac{m_{s,k}q_k^{N+1}-(-1)^j n_{s,k}q_k}{(1-q_k)^{j+1}}E_{j-1}(q_k)$$
$$=\frac{j!\cos(2\pi\omega+\frac{\pi j}{2})}{(2\pi\omega h)^{j+1}}-\frac{(e^{2\pi \mathrm{i}\omega}+(-1)^j e^{-2\pi \mathrm{i}\omega})K_{m,\omega}e^{2\pi \mathrm{i}\omega h}}{2\mathrm{i}(1-e^{2\pi \mathrm{i}\omega h})^{j+1}}E_{j-1}\big(e^{2\pi \mathrm{i}\omega h}\big),\quad j=\overline{1,m-1}.$$

**Theorem 4** *The optimal quadrature formulas in the sense of Sard of the form* (2) *in the space* $L_2^{(m)}(0,1)$ *when* $\omega h\in\mathbb{Z}$ *and* $\omega\neq 0$ *have the coefficients with the following analytic expressions*

$$\mathring{C}_s[0]=h\left[\frac{1}{2\pi\omega h}+\sum_{k=1}^{m-1}\frac{m_{s,k}q_k-n_{s,k}q_k^N}{q_k-1}\right],$$
$$\mathring{C}_s[\beta]=h\left[\sum_{k=1}^{m-1}\big(m_{s,k}q_k^{\beta}+n_{s,k}q_k^{N-\beta}\big)\right],\quad \beta=\overline{1,N-1},$$
$$\mathring{C}_s[N]=h\left[-\frac{1}{2\pi\omega h}+\sum_{k=1}^{m-1}\frac{-m_{s,k}q_k^N+n_{s,k}q_k}{q_k-1}\right],$$

*where* $q_k$ *are the roots of the Euler–Frobenius polynomial* $E_{2m-2}(x)$ *with* $|q_k|<1$, *and* $m_{s,k}$ *and* $n_{s,k}$ *satisfy the system of linear equations*

$$\sum_{k=1}^{m-1}\frac{m_{s,k}q_k-(-1)^j n_{s,k}q_k^{N+1}}{(q_k-1)^{j+1}}E_{j-1}(q_k)=-\frac{(-1)^j j!\cos(\frac{\pi j}{2})}{(2\pi\omega h)^{j+1}},\quad j=\overline{1,m-1},$$
$$\sum_{k=1}^{m-1}\frac{m_{s,k}q_k^{N+1}-(-1)^j n_{s,k}q_k}{(1-q_k)^{j+1}}E_{j-1}(q_k)=\frac{j!\cos(2\pi\omega+\frac{\pi j}{2}))}{(2\pi\omega h)^{j+1}},\quad j=\overline{1,m-1}.$$

**Theorem 5** *The optimal quadrature formulas in the sense of Sard of the form* (3) *in the space* $L_2^{(m)}(0,1)$ *when* $\omega\in\mathbb{R}$ *and* $\omega h\notin\mathbb{Z}$ *have the coefficients with the following analytical expressions*

$$\mathring{C}_c[0]=h\left(\frac{K_{m,\omega}}{2}+\sum_{k=1}^{m-1}\frac{m_{c,k}q_k-n_{c,k}q_k^N}{q_k-1}\right),$$
$$\mathring{C}_c[\beta]=h\left(K_{m,\omega}\cos(2\pi\omega h\beta)+\sum_{k=1}^{m-1}\big(m_{c,k}q_k^{\beta}+n_{c,k}q_k^{N-\beta}\big)\right),\quad \beta=\overline{1,N-1},\tag{37}$$
$$\mathring{C}_c[N]=h\left(-\frac{\sin(2\pi\omega)}{2\pi\omega h}-K_{m,\omega}\frac{\sin(2\pi\omega-\pi\omega h)}{2\sin(\pi\omega h)}+\sum_{k=1}^{m-1}\frac{-m_{c,k}q_k^N+n_{c,k}q_k}{q_k-1}\right),$$

*where* $m_{c,k}$ and $n_{c,k}$ satisfy the following system of linear equations

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{m_{c,k}q_k - (-1)^j n_{c,k}q_k^{N+1}}{(q_k-1)^{j+1}} E_{j-1}(q_k) \\
&\quad = \frac{(-1)^j j! \sin(\frac{\pi j}{2})}{(2\pi\omega h)^{j+1}} - \frac{(1-(-1)^j)K_{m,\omega}e^{2\pi i\omega h}}{2(e^{2\pi i\omega h}-1)^{j+1}} E_{j-1}\big(e^{2\pi i\omega h}\big), \quad j = \overline{1, m-1}, \\
&\sum_{k=1}^{m-1} \frac{m_{c,k}q_k^{N+1} - (-1)^j n_{c,k}q_k}{(1-q_k)^{j+1}} E_{j-1}(q_k) \\
&\quad = -\frac{j! \sin(2\pi\omega + \frac{\pi j}{2})}{(2\pi\omega h)^{j+1}} - \frac{(e^{2\pi i\omega} - (-1)^j e^{-2\pi i\omega})K_{m,\omega}e^{2\pi i\omega h}}{2(1-e^{2\pi i\omega h})^{j+1}} E_{j-1}\big(e^{2\pi i\omega h}\big), \quad j = \overline{1, m-1},
\end{aligned}$$

*$q_k$ are the roots of the Euler–Frobenius polynomial $E_{2m-2}(x)$ with $|q_k| < 1$, $a_{k,\alpha}$ is the kth coefficient of the Euler–Frobenius polynomial of degree $\alpha$, and $K_{m,\omega}$ is defined in Theorem 3.*

**Theorem 6** *The optimal quadrature formulas in the sense of Sard of the form (3) in the space $L_2^{(m)}(0,1)$ when $\omega h \in \mathbb{Z}$ and $\omega \neq 0$ have the coefficients with the following analytical expressions*

$$\begin{aligned}
\mathring{C}_c[0] &= h\left(\sum_{k=1}^{m-1} \frac{m_{c,k}q_k - n_{c,k}q_k^N}{q_k - 1}\right), \\
\mathring{C}_c[\beta] &= h\left(\sum_{k=1}^{m-1} \left(m_{c,k}q_k^{\beta} + n_{c,k}q_k^{N-\beta}\right)\right), \quad \beta = \overline{1, N-1}, \\
\mathring{C}_[N] &= h\left(\sum_{k=1}^{m-1} \frac{-m_{c,k}q_k^N + n_{c,k}q_k}{q_k - 1}\right),
\end{aligned}$$

*where $q_k$ are the roots of the Euler–Frobenius polynomial $E_{2m-2}(x)$ with $|q_k| < 1$, $m_{c,k}$ and $n_{c,k}$ satisfy the system of linear equations*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{m_{c,k}q_k - (-1)^j n_{c,k}q_k^{N+1}}{(q_k-1)^{j+1}} E_{j-1}(q_k) = \frac{(-1)^j j! \sin(\frac{j\pi}{2})}{(2\pi\omega h)^{j+1}}, \quad j = \overline{1, m-1}, \\
&\sum_{k=1}^{m-1} \frac{m_{c,k}q_k^{N+1} - (-1)^j n_{c,k}q_k}{(1-q_k)^{j+1}} E_{j-1}(q_k) = -\frac{j! \sin(2\pi\omega + \frac{j\pi}{2})}{(2\pi\omega h)^{j+1}}, \quad j = \overline{1, m-1}.
\end{aligned}$$

Now, we prove Theorem 3. Theorems 4, 5, and 6 are proved similarly (see, for example [12]).

*Proof of Theorem 3* First, we denote the left-hand-side of (28) by

$$u_s[\beta] = \sum_{\gamma=0}^{N} C_s[\gamma] \frac{|[\beta]-[\gamma]|^{2m-1}}{2 \cdot (2m-1)!} + P_{s,m-1}[\beta].$$

We assume that $C_s[\gamma]=0$ for $\gamma<0$ and $\gamma>N$. Then, using Definition 3, for the function $u_s[\beta]$ we have the following representation

$$u_s[\beta]=C_s[\beta]*\frac{|[\beta]|^{2m-1}}{2\cdot(2m-1)!}+P_{s,m-1}[\beta]. \tag{38}$$

Now, we should express the coefficients $C_s[\beta]$ through the function $u_s[\beta]$.

Then, taking into account (12), (38), and (14), we obtain

$$C_s[\beta]=hD_m[\beta]*u_s[\beta] \quad \text{for } \beta\in\mathbb{Z}. \tag{39}$$

Using (38), for the right-hand side of (39) we have

$$\begin{aligned}hD_m[\beta]*u_s[\beta]&=hD_m[\beta]*\left(C_s[\beta]*\frac{|[\beta]|^{2m-1}}{2\cdot(2m-1)!}+P_{s,m-1}[\beta]\right)\\&=hD_m[\beta]*\left(C_s[\beta]*\frac{|[\beta]|^{2m-1}}{2\cdot(2m-1)!}\right)+hD_m[\beta]*P_{s,m-1}[\beta],\end{aligned}$$

where $P_{s,m-1}[\beta]$ is a polynomial of degree $(m-1)$ with respect to $[\beta]$. Hence, taking into account equalities (12) and (14), and keeping in mind the finiteness of the discrete argument function $C_s[\beta]$, we obtain

$$\begin{aligned}hD_m[\beta]*u_s[\beta]&=C_s[\beta]*\left(hD_m[\beta]*\frac{|[\beta]|^{2m-1}}{2\cdot(2m-1)!}\right)\\&=C_s[\beta]*\delta[\beta]\\&=C_s[\beta].\end{aligned}$$

From the other side, for calculating the convolution in the right-hand side of (39), i.e., to obtain optimal coefficients $C_s[\beta]$, the function $u_s[\beta]$ should be determined for all integer values of $\beta$. It is clear from (28) and (38) that

$$u_s[\beta]=f_{s,m}[\beta] \quad \text{for } \beta=0,1,\dots,N. \tag{40}$$

Now, we have to find the representation of $u_s[\beta]$ for $\beta=-1,-2,\dots$ and $\beta=N+1,N+2,\dots$. First, we consider the cases $\beta=-1,-2,\dots$. Then, from (38), using the binomial formula, we have

$$\begin{aligned}u_s[\beta]&=-\sum_{\gamma=0}^{N}C_s[\gamma]\frac{([\beta]-[\gamma])^{2m-1}}{2\cdot(2m-1)!}+P_{s,m-1}[\beta]\\&=-\sum_{\gamma=0}^{N}C_s[\gamma]\sum_{\alpha=0}^{2m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^\alpha[\gamma]^\alpha}{2\cdot\alpha!(2m-1-\alpha)!}+P_{s,m-1}[\beta]\\&=-\sum_{\alpha=0}^{m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^\alpha}{2\alpha!(2m-1-\alpha)!}\sum_{\gamma=0}^{N}C_s[\gamma][\gamma]^\alpha\\&\quad-\sum_{\alpha=m}^{2m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^\alpha}{2\alpha!(2m-1-\alpha)!}\sum_{\gamma=0}^{N}C_s[\gamma][\gamma]^\alpha+P_{s,m-1}[\beta].\end{aligned}$$

Hence, using (29), taking into account (31) and denoting

$$Q_{s,m-1}[\beta]=\sum_{\alpha=m}^{2m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^{\alpha}}{2\alpha!(2m-1-\alpha)!}\sum_{\gamma=0}^{N}C_s[\gamma][\gamma]^{\alpha}, \tag{41}$$

we obtain

$$u_s[\beta]=-\sum_{\alpha=0}^{m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^{\alpha}}{2\alpha!(2m-1-\alpha)!}g_{s,\alpha}-Q_{s,m-1}[\beta]+P_{s,m-1}[\beta]\quad \text{for } \beta=-1,-2,\dots. \tag{42}$$

Next, we turn our attention to the cases $\beta=N+1,N+2,\dots$. Then, from (38), similarly using the binomial formula, we obtain

$$\begin{aligned}u_s[\beta]&=\sum_{\gamma=0}^{N}C_s[\gamma]\frac{([\beta]-[\gamma])^{2m-1}}{2\cdot(2m-1)!}+P_{s,m-1}[\beta]\\&=\sum_{\alpha=0}^{m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^{\alpha}}{2\alpha!(2m-1-\alpha)!}\sum_{\gamma=0}^{N}C_s[\gamma][\gamma]^{\alpha}\\&\quad+\sum_{\alpha=m}^{2m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^{\alpha}}{2\alpha!(2m-1-\alpha)!}\sum_{\gamma=0}^{N}C_s[\gamma][\gamma]^{\alpha}+P_{s,m-1}[\beta].\end{aligned}$$

From here, using (29), keeping in mind (41), we obtain

$$u_s[\beta]=\sum_{\alpha=0}^{m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^{\alpha}}{2\alpha!(2m-1-\alpha)!}g_{s,\alpha}+Q_{s,m-1}[\beta]+P_{s,m-1}[\beta]\quad \text{for } \beta=N+1,N+2,\dots. \tag{43}$$

By combining equations (40), (42), and (43) for $u_s[\beta]$ we obtain

$$u_s[\beta]=\begin{cases}-\sum_{\alpha=0}^{m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^{\alpha}}{2\cdot\alpha!(2m-1-\alpha)!}g_{s,\alpha}-Q_{s,m-1}[\beta]+P_{s,m-1}[\beta], & \beta<0,\\ f_{s,m}[\beta], & \beta=\overline{0,N},\\ \sum_{\alpha=0}^{m-1}\frac{[\beta]^{2m-1-\alpha}(-1)^{\alpha}}{2\cdot\alpha!(2m-1-\alpha)!}g_{s,\alpha}+Q_{s,m-1}[\beta]+P_{s,m-1}[\beta], & \beta>N,\end{cases} \tag{44}$$

where $Q_{s,m-1}[\beta]$ and $P_{s,m-1}[\beta]$ are unknown polynomials of degree $(m-1)$ with respect to $[\beta]$.

Now, using Theorem 1 and equality (44), from (39), after some calculations for optimal coefficients we obtain

$$\mathring{C}_s[\beta]=h\left[K_{m,\omega}\sin(2\pi\omega h\beta)+\sum_{k=1}^{m-1}\left(m_{s,k}q_k^{\beta}+n_{s,k}q_k^{N-\beta}\right)\right],\quad \beta=1,2,\dots,N-1, \tag{45}$$

where

$$m_{s,k}=\frac{A_kp}{q_k}\sum_{\gamma=1}^{\infty}q_k^{\gamma}\left(-\sum_{\alpha=0}^{m-1}\frac{[-\gamma]^{2m-1-\alpha}(-1)^{\alpha}}{2\cdot\alpha!(2m-1-\alpha)!}g_{s,\alpha}+P_{s,m-1}[-\gamma]-Q_{s,m-1}[-\gamma]-f_{s,m}[-\gamma]\right),$$

$$n_{s,k} = \frac{A_k p}{q_k} \sum_{\gamma=1}^{\infty} q_k^{\gamma} \left( \sum_{\alpha=0}^{m-1} \frac{[N+\gamma]^{2m-1-\alpha}(-1)^{\alpha}}{2\cdot\alpha!(2m-1-\alpha)!} g_{s,\alpha} + P_{s,m-1}[N+\gamma] + Q_{s,m-1}[N+\gamma] \right.$$
$$\left. - f_{s,m}[N+\gamma] \right),$$

and $q_k$, $p$, $C$, and $A_k$ are given in Theorem 1, $K_{m,\omega}$ is unknown and it will be found below.

Now, putting the representation (45) of the coefficients $\mathring{C}_s[\beta]$, $\beta = 1, 2, \ldots, N-1$, into the left-hand side of equality (28), using identities (19), (20), (21), and equality (30), after some simplifications we obtain the following identity with respect to $[\beta]$:

$$\mathring{C}_s[0]\frac{[\beta]^{2m-1}}{(2m-1)!} + \sin(2\pi\omega h\beta)\frac{K_{m,\omega}h^{2m}e^{2\pi i\omega h}}{(2m-1)!(e^{2\pi\omega h}-1)^{2m}}E_{2m-2}\bigl(e^{2\pi i\omega h}\bigr)$$
$$- \frac{[\beta]^{2m-1}h}{(2m-1)!}\left(-\frac{K_{m,\omega}\cos(\pi\omega h)}{2\sin(\pi\omega h)} + \sum_{k=1}^{m-1}\frac{m_{s,k}q_k - n_{s,k}q_k^N}{q_k-1}\right)$$
$$- \sum_{j=1}^{2m-1}\binom{2m-1}{j}[\beta]^{2m-1-j}\frac{h^{j+1}}{(2m-1)!}\left(\frac{K_{m,\omega}(1+(-1)^j)e^{2\pi i\omega h}}{2i(e^{2\pi i\omega h}-1)^{j+1}}E_{j-1}\bigl(e^{2\pi i\omega h}\bigr)\right.$$
$$\left. + \sum_{k=1}^{m-1}\frac{m_{s,k}q_k - (-1)^j n_{s,k}q_k^{N+1}}{(q_k-1)^{j+1}}E_{j-1}(q_k)\right)$$
$$- \frac{1}{2}\sum_{j=m}^{2m-1}\frac{[\beta]^{2m-1-j}(-1)^j}{j!(2m-1-j)!}\sum_{\gamma=0}^{N}C_s[\gamma][\gamma]^j + P_{s,m-1}[\beta]$$
$$= -\frac{1}{2}\sum_{j=m}^{2m-1}\frac{[\beta]^{2m-1-j}(-1)^j}{j!(2m-1-j)!}g_{s,j} - \sin(2\pi\omega h\beta)\frac{(-1)^{m+1}}{(2\pi\omega h)^{2m}}$$
$$+ \sum_{j=0}^{2m-1}\frac{[\beta]^{2m-1-j}(-1)^j}{(2m-1-j)!(2\pi\omega)^{j+1}}\cos\left(\frac{j\pi}{2}\right).$$

From here, equating the terms consisting of $\sin(2\pi\omega h\beta)$ we obtain

$$K_{m,\omega} = \left(\frac{\sin\pi\omega h}{\pi\omega h}\right)^{2m}\frac{(2m-1)!}{2\sum_{k=0}^{m-2}a_{k,2m-2}\cos(2\pi\omega h(m-1-k)) + a_{m-1,2m-2}},$$

which is given in Theorem 1 and equating the similar terms of $[\beta]^j$ for $j = \overline{0, m-1}$, $j = \overline{m, 2m-2}$ and $j = 2m-1$, separately, we obtain $P_{s,m-1}[\beta]$, the system of $(m-1)$ linear equations for $m_{s,k}$, $n_{s,k}$ and analytic expression for $\mathring{C}_s[0]$, respectively:

$$P_{s,m-1}[\beta] = \sum_{j=m}^{2m-1}\frac{[\beta]^{2m-1-j}}{j!(2m-1-j)!}\left[\frac{(1+(-1)^j)h^{j+1}K_{m,\omega}e^{2\pi i\omega h}}{2i(e^{2\pi i\omega h}-1)^{j+1}}E_{j-1}\bigl(e^{2\pi i\omega h}\bigr)\right.$$
$$+ h^{j+1}\sum_{k=1}^{m-1}\frac{m_{s,k}q_k - (-1)^j n_{s,k}q_k^{N+1}}{(q_k-1)^{j+1}}E_{j-1}(q_k) + \frac{(-1)^j}{2}\sum_{\gamma=0}^{N}C_s[\gamma][\gamma]^j$$
$$\left. + \frac{(-1)^j j!\cos(\frac{\pi j}{2})}{(2\pi\omega)^{j+1}} - \frac{(-1)^j}{2}g_j\right],$$

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{m_{s,k}q_k - (-1)^j n_{s,k} q_k^{N+1}}{(q_k-1)^{j+1}} E_{j-1}(q_k) \\
&\quad = -\frac{(-1)^j j! \cos(\frac{\pi j}{2})}{(2\pi\omega h)^{j+1}} \\
&\qquad - \frac{(1+(-1)^j) K_{m,\omega} e^{2\pi \mathrm{i}\omega h}}{2\mathrm{i}(e^{2\pi \mathrm{i}\omega h}-1)^{j+1}} E_{j-1}\bigl(e^{2\pi \mathrm{i}\omega h}\bigr), \quad j=\overline{1,m-1},
\end{aligned} \tag{46}$$

and

$$\mathring{C}_s[0] = h\Biggl[\frac{1}{2\pi\omega h} - K_{m,\omega}\frac{\sin 2\pi\omega h}{2-2\cos 2\pi\omega h} + \sum_{k=1}^{m-1}\biggl(m_{s,k}\frac{q_k}{q_k-1} + n_{s,k}\frac{q_k^N}{1-q_k}\biggr)\Biggr]. \tag{47}$$

Next, from (29) when $\alpha = 0$, using equalities (45) and (47) for $\mathring{C}_s[N]$ we have

$$\mathring{C}_s[N] = h\Biggl[\frac{\cos 2\pi\omega}{2\pi\omega h} - K_{m,\omega}\frac{\cos(2\pi\omega - \pi\omega h)}{2\sin\pi\omega h} + \sum_{k=1}^{m-1}\biggl(m_{s,k}\frac{q_k^N}{q_k-1} + n_{s,k}\frac{q_k}{1-q_k}\biggr)\Biggr]. \tag{48}$$

Finally, from (29) when $\alpha = 1, 2, \dots, m-1$, taking into account (45), (47), and (48), using (15)–(21) and (22)–(25), after several calculations, we obtain the following system of $(m-1)$ linear equations for $m_{s,k}$ and $n_{s,k}$

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{m_{s,k}q_k^{N+1} - (-1)^j n_{s,k} q_k}{(1-q_k)^{j+1}} E_{j-1}(q_k) \\
&\quad = \frac{j! \cos(2\pi\omega + \frac{\pi j}{2})}{(2\pi\omega h)^{j+1}} \\
&\qquad - \frac{(e^{2\pi \mathrm{i}\omega} + (-1)^j e^{-2\pi \mathrm{i}\omega}) K_{m,\omega} e^{2\pi \mathrm{i}\omega h}}{2\mathrm{i}(1-e^{2\pi \mathrm{i}\omega h})^{j+1}} E_{j-1}\bigl(e^{2\pi \mathrm{i}\omega h}\bigr), \quad j=\overline{1,m-1}.
\end{aligned} \tag{49}$$

Hence, combining systems (46) and (49) we come to the system of $(2m-2)$ linear equations that is given in the statement of Theorem 3.

Theorem 3 is proved. □

Now, we consider the cases $m = 1$ and $m = 2$. We have the following results for the same $\omega \in \mathbb{R}$.

**Corollary 1** *Coefficients of the optimal quadrature formulas of the form* (2) *in the sense of Sard in the space* $L_2^{(1)}(0,1)$ *when* $\omega \in \mathbb{R}$ *and* $\omega h \notin \mathbb{Z}$ *have the form*

$$\begin{aligned}
C_s[0] &= h\biggl[\frac{1}{2\pi\omega h} - \biggl(\frac{\sin(\pi\omega h)}{\pi\omega h}\biggr)^2 \frac{\cos(\pi\omega h)}{2\sin(\pi\omega h)}\biggr], \\
C_s[\beta] &= h\biggl(\frac{\sin(\pi\omega h)}{\pi\omega h}\biggr)^2 \sin\bigl(2\pi\omega[\beta]\bigr), \quad \beta = \overline{1,N-1}, \\
C_s[N] &= h\biggl[-\frac{\cos(2\pi\omega)}{2\pi\omega h} + \biggl(\frac{\sin(\pi\omega h)}{\pi\omega h}\biggr)^2 \frac{\cos(2\pi\omega - \pi\omega h)}{2\sin(\pi\omega h)}\biggr],
\end{aligned}$$

*where* $[\beta] = h\beta$ and $h = \frac{1}{N}$.

It should be noted that Corollary 1 is Corollary 2 of the work [13].

**Corollary 2** *Coefficients of the optimal quadrature formulas of the form* (2) *in the sense of Sard in the space* $L_2^{(2)}(0,1)$ *when* $\omega \in \mathbb{R}$ *and* $\omega h \notin \mathbb{Z}$ *have the form*

$$C_s[0] = h\left(\frac{1}{2\pi\omega h} - \frac{K_{2,\omega}\cos(\pi\omega h)}{2\sin(\pi\omega h)} + \frac{m_{s,1}q_1 - n_{s,1}q_1^N}{q_1 - 1}\right),$$
$$C_s[\beta] = h\bigl(K_{2,\omega}\sin\bigl(2\pi\omega[\beta]\bigr) + m_{s,1}q_1^\beta + n_{s,1}q_1^{N-\beta}\bigr), \quad \beta = \overline{1, N-1},$$
$$C_s[N] = h\left(-\frac{\cos(2\pi\omega)}{2\pi\omega h} + \frac{K_{2,\omega}\cos(2\pi\omega - \pi\omega h)}{2\sin(\pi\omega h)} + \frac{-m_{s,1}q_1^N + n_{s,1}q_1}{q_1 - 1}\right),$$

*where*

$$K_{2,\omega} = \left(\frac{\sin(\pi\omega h)}{\pi\omega h}\right)^4 \frac{3}{2 + \cos(2\pi\omega h)},$$
$$m_{s,1} = \frac{q_1^N(1-q_1)^2\sin(2\pi\omega)}{q_1(q_1^{2N}-1)}\left[-\frac{1}{(2\pi\omega h)^2} + \frac{K_{2,\omega}}{2(1-\cos(2\pi\omega h))}\right],$$
$$n_{s,1} = \frac{(1-q_1)^2\sin(2\pi\omega)}{q_1(1-q_1^{2N})}\left[-\frac{1}{(2\pi\omega h)^2} + \frac{K_{2,\omega}}{2(1-\cos(2\pi\omega h))}\right].$$

$[\beta] = h\beta$, $h = \frac{1}{N}$ and $q_1 = \sqrt{3} - 2$.

**Corollary 3** *Coefficients of the optimal quadrature formulas of the form* (3) *in the sense of Sard in the space* $L_2^{(1)}(0,1)$ *when* $\omega \in \mathbb{R}$ *and* $\omega h \notin \mathbb{Z}$ *have the form*

$$C_c[0] = \frac{h}{2}\left(\frac{\sin(\pi\omega h)}{\pi\omega h}\right)^2,$$
$$C_c[\beta] = h\left(\frac{\sin(\pi\omega h)}{\pi\omega h}\right)^2 \cos\bigl(2\pi\omega[\beta]\bigr), \quad \beta = \overline{1, N-1},$$
$$C_c[N] = h\left(\frac{\sin(2\pi\omega)}{2\pi\omega h} - \left(\frac{\sin(\pi\omega h)}{\pi\omega h}\right)^2 \frac{\sin(2\pi\omega - \pi\omega h)}{2\sin(\pi\omega h)}\right),$$

*where* $[\beta] = h\beta$ and $h = \frac{1}{N}$.

We note that Corollary 3 is Corollary 1 of the work [13].

**Corollary 4** *Coefficients of the optimal quadrature formulas of the form* (3) *in the sense of Sard in the space* $L_2^{(2)}(0,1)$ *when* $\omega \in \mathbb{R}$ *and* $\omega h \notin \mathbb{Z}$ *have the form*

$$C_c[0] = h\left(\frac{K_{2,\omega}}{2} + \frac{m_{c,1}q_1 - n_{c,1}q_1^N}{q_1 - 1}\right),$$
$$C_c[\beta] = h\bigl(K_{2,\omega}\cos\bigl(2\pi\omega[\beta]\bigr) + m_{c,1}q_1^\beta + n_{c,1}q_1^{N-\beta}\bigr), \quad \beta = \overline{1, N-1},$$
$$C_c[N] = h\left(\frac{\sin(2\pi\omega)}{2\pi\omega h} - \frac{K_{2,\omega}\sin(2\pi\omega - \pi\omega h)}{2\sin(\pi\omega h)} + \frac{-m_{c,1}q_1^N + n_{c,1}q_1}{q_1 - 1}\right),$$

*where*

$$K_{2,\omega} = \left(\frac{\sin(\pi\omega h)}{\pi\omega h}\right)^4 \frac{3}{2+\cos(2\pi\omega h)},$$
$$n_{c,1} = -\frac{(\cos(2\pi\omega) - q_1^N)(1-q_1)^2}{q_1}\left[\frac{1}{(2\pi\omega h)^2} + \frac{K_{2,\omega}}{2(\cos(2\pi\omega h)-1)}\right],$$
$$m_{c,1} = \frac{(1-q_1)^2(\cos(2\pi\omega) - q_1^N - 1 - q_1^{2N})}{q_1}\left[\frac{1}{(2\pi\omega h)^2} + \frac{K_{2,\omega}}{2(\cos(2\pi\omega h)-1)}\right],$$

$[\beta] = h\beta$, $h = \frac{1}{N}$ *and* $q_1 = \sqrt{3} - 2$.

*Remark* 1 Multiplying both sides of the approximate equality (2) by i (where $\mathrm{i}^2 = -1$) and adding to the left- and right-hand sides of the approximate equality (3), respectively, we obtain the quadrature formula of the following form

$$\int_0^1 e^{2\pi \mathrm{i}\omega x}\varphi(x)\,dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi[\beta]. \tag{50}$$

It should be noted that the construction of optimal quadrature formulas of the form (50) in the space $L_2^{(m)}$ was solved in [14]. The coefficients of the optimal quadrature formulas in the form (50) can be also defined as follows

$$\mathring{C}[\beta] = \mathring{C}_c[\beta] + \mathrm{i}\mathring{C}_s[\beta], \quad \beta = 0, 1, \dots, N,$$

where the optimal coefficients $\mathring{C}_s[\beta]$ and $\mathring{C}_c[\beta]$ are given in Theorems 3–6.

Thus, from the results of the present work one can obtain the results on optimal quadrature formulas of the form (50) of the work [14] with a more simplified system of linear equations for determining the optimal coefficients.

## 5 Numerical results

In this section we present numerical results of comparison for absolute errors of the optimal quadrature formula of the form (2) with sine weight in the case $m = 2$ and a composite trapezoidal formula. We note that both of these formulas are exact for linear functions. We obtain the numerical results of this section using Maple.

It should be noted that the composite trapezoidal quadrature formula is the Newton–Cotes rule of order 1.

As an example, we consider calculation of the following integral

$$I = \int_0^1 x^2 \sin(2\pi\omega x)\,dx. \tag{51}$$

For convenience, we denote the integrand as $f(x)$, i.e., here $f(x) = x^2\sin(2\pi\omega x)$.

We approximately calculate the integral $I$ by the composite trapezoidal rule. Then, the approximate value for the integral (51) is calculated as follows using the composite trapezoidal rule

$$A_{\mathrm{tr}} = \sum_{i=0}^{N-1} \frac{x_i^2\sin(2\pi\omega x_i) + x_{i+1}^2\sin(2\pi\omega x_{i+1})}{2}\cdot(x_{i+1} - x_i). \tag{52}$$

**Table 1** The absolute values of the error (53) of the composite trapezoidal rule for $N = 1, 10, 100, 1000$ and $\omega = 1.1, 10.1, 100.1, 1000.1$

| | $\omega = 1.1$ | $\omega = 10.1$ | $\omega = 100.1$ | $\omega = 1000.1$ |
|---|---|---|---|---|
| | $\lvert I - A_{\mathrm{tr}}\rvert$ | $\lvert I - A_{\mathrm{tr}}\rvert$ | $\lvert I - A_{\mathrm{tr}}\rvert$ | $\lvert I - A_{\mathrm{tr}}\rvert$ |
| $N = 1$ | 3.874936(−1) | 3.063506(−1) | 2.951759(−1) | 2.940213(−1) |
| $N = 10$ | 5.701761(−3) | 1.641521(−1) | 1.529774(−1) | 1.518228(−1) |
| $N = 100$ | 5.639861(−5) | 4.407336(−4) | 1.529774(−1) | 1.504333(−1) |
| $N = 1000$ | 5.639251(−7) | 4.376634(−6) | 4.278453(−5) | 1.504194(−1) |

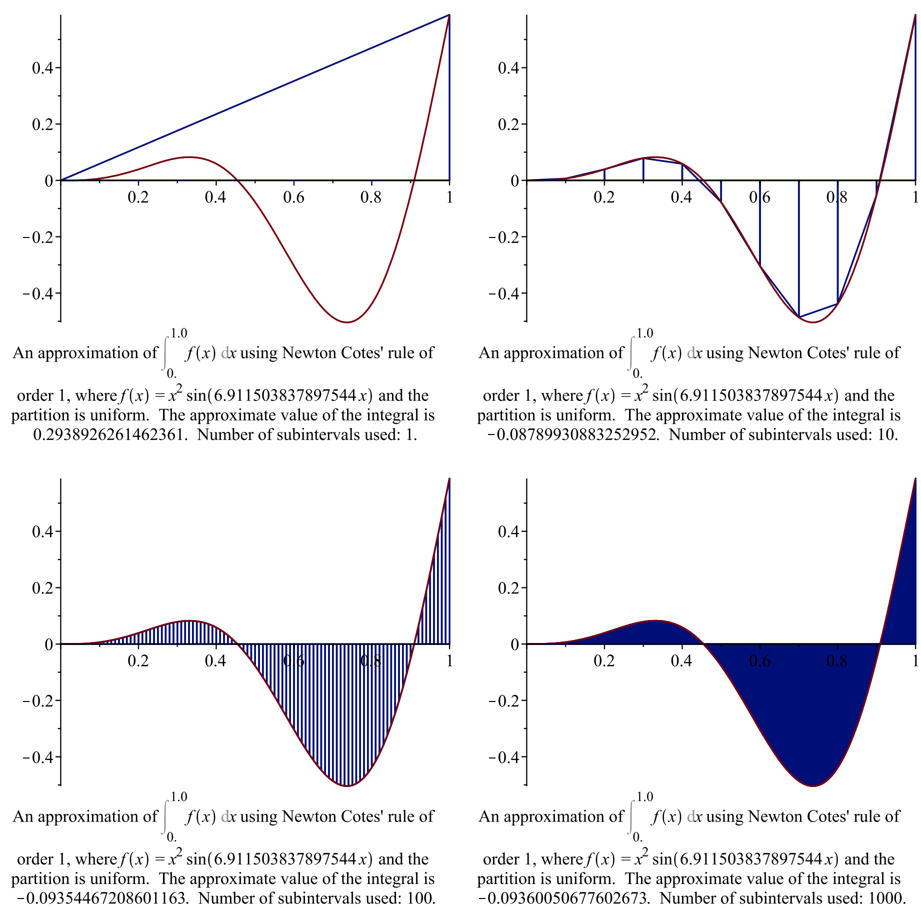


**Figure 1** The process of convergence for the composite trapezoidal rule for the case $\omega = 1.1$ and $N = 1, 10, 100, 1000$

Hence, for the function $f(x) = x^2 \sin(2\pi\omega x)$ the error of the composite trapezoidal rule (52) is

$$I - A_{\mathrm{tr}} = \int_0^1 x^2 \sin(2\pi\omega x)\,dx - \sum_{i=0}^{N-1} \frac{x_i^2 \sin(2\pi\omega x_i) + x_{i+1}^2 \sin(2\pi\omega x_{i+1})}{2} \cdot (x_{i+1} - x_i). \quad (53)$$

In Table 1 we give the absolute values of the error (53) of the composite trapezoidal rule for $N = 1, 10, 100, 1000$ and $\omega = 1.1, 10.1, 100.1, 1000.1$.

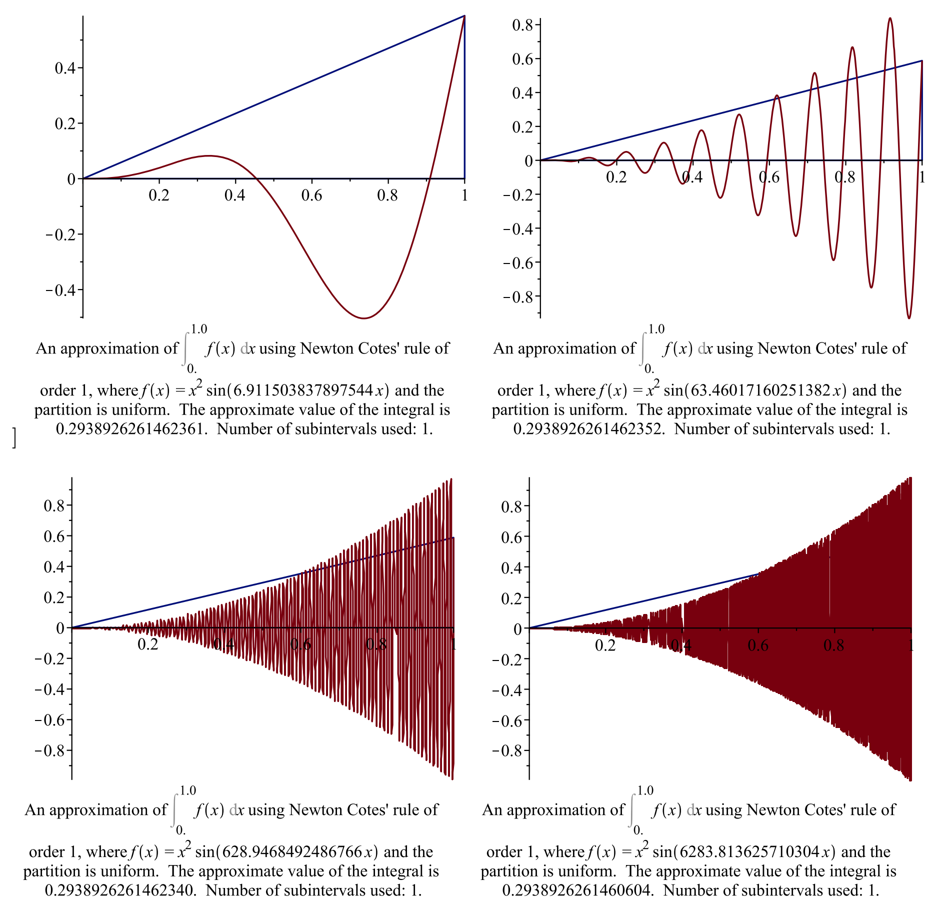


**Figure 2** The composite trapezoidal rule does not converge for $\omega > N$

It can be seen from the results given in Table 1 that the composite trapezoidal rule converges for $N \geq \omega$. In Fig. 1 the process of this convergence is graphically shown for the case $\omega = 1.1$ and $N = 1,\ 10, 100, 1000$.

In Fig. 2 are given the graphs of numerical calculation of the integral (51) by the composite trapezoidal rule for the case $\omega = 1.1, 10.1, 100.1, 1000.1$, and $N = 1$. Here, we can see that the composite quadrature process does not converge for $\omega > N$.

Now, we approximate the above integral (51) using the optimal quadrature formula of the form (2) with sine weight function in the case $m = 2$. Then, we have the following approximate equality

$$\int_0^1 \sin(2\pi\omega x)\varphi(x)\,dx \cong \sum_{\beta=0}^{N} C_s[\beta]\varphi[\beta], \tag{54}$$

for $\varphi(x) = x^2$ with optimal coefficients given in Corollary 2. The approximate value for the integral (51) is calculated as follows using the optimal quadrature formula

$$A_{\text{opt}} = \sum_{\beta=0}^{N} C_s[\beta][\beta]^2.$$

**Table 2** The absolute values for the error (55) of the optimal quadrature formula (54) for $N = 1, 10, 100, 1000$ and $\omega = 1.1, 10.1, 100.1, 1000.1$

| | $\omega = 1.1$ | $\omega = 10.1$ | $\omega = 100.1$ | $\omega = 1000.1$ |
|---|---|---|---|---|
| | $\|I - A_{\text{opt}}\|$ | $\|I - A_{\text{opt}}\|$ | $\|I - A_{\text{opt}}\|$ | $\|I - A_{\text{opt}}\|$ |
| $N = 1$ | 1.114784(−2) | 1.444594(−4) | 1.484368(−6) | 1.488425(−8) |
| $N = 10$ | 3.025851(−5) | 6.946015(−6) | 8.425457(−8) | 8.578893(−10) |
| $N = 100$ | 2.843105(−8) | 3.005545(−8) | 7.045133(−9) | 8.440393(−11) |
| $N = 1000$ | 2.845974(−11) | 2.842208(−11) | 3.003591(−11) | 7.055061(−12) |

Hence, for the function $\varphi(x) = x^2$ the error of the optimal quadrature formula (54) is

$$I - A_{\text{opt}} = \int_0^1 x^2 \sin(2\pi\omega x)\,dx - \int_0^1 x^2 \sin(2\pi\omega x)\,dx - \sum_{\beta=0}^{N} C_s[\beta][\beta]^2. \tag{55}$$

Thus, the numerical results of Table 2 show convergence of the optimal quadrature formula (54) for $N \geq \omega$ and $N < \omega$.

## 6 Conclusion

In the present paper we constructed the optimal quadrature formulas for numerical calculation of Fourier sine and cosine integrals, when $\omega \in \mathbb{R}$, $\omega \neq 0$. We obtained analytic forms of coefficients for the constructed optimal quadrature formulas in the Sobolev space. In order to obtain the analytic forms of the optimal coefficients we used the Sobolev method that is based on the discrete analog of the differential operator $d^{2m}/dx^{2m}$. The obtained optimal quadrature formulas in the space $L_2^{(m)}$ are exact for any algebraic polynomial of degree $m-1$. We presented numerical results of comparison for absolute errors of the optimal quadrature formula of the form (2) with sine weight in the case $m = 2$ and composite trapezoidal formula that show the advantage of the optimal quadrature formula.

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## Declarations

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The problem of the manuscript was stated by KS and AH. Proofs of the main theoretical results were obtained by AH and BB. The numerical results were obtained by BB. All authors read and approved the manuscript.

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