Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Reilly-type inequality for the Φ-Laplace operator on semislant submanifolds of Sasakian space forms

Yanlin Li[1], Fatemah Mofarreh[2*], Ravi P. Agrawal[3] and Akram Ali[4]

*Correspondence:
fyalmofarrah@pnu.edu.sa
[2]Mathematical Science Department Faculty of Science, Princess Nourah bint Abdulrahman University, Riyadh 11546, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

This paper aims to establish new upper bounds for the first positive eigenvalue of the Φ-Laplacian operator on Riemannian manifolds in terms of mean curvature and constant sectional curvature. The first eigenvalue for the Φ-Laplacian operator on closed oriented m-dimensional semislant submanifolds in a Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ is estimated in various ways. Several Reilly-like inequalities are generalized from our findings for Laplacian to the Φ-Laplacian on semislant submanifolds in a sphere $\mathbb{S}^{2n+1}$ with $\epsilon = 1$ and $\Phi = 2$.

Keywords: Reilly-type inequality; Φ-Laplacian; Eigenvalues estimates; Semislant submanifolds; Sasakian space forms

1 Introduction and statement of main results

Let $\mathcal{N}^m$ be a complete noncompact Riemannian manifold and Σ be the compact domain in $\mathcal{N}^m$. Assume $\Lambda_1(\Sigma) > 0$ denotes the first eigenvalue of the Dirichlet boundary value problem

$$\Delta f + \Lambda f = 0, \quad \text{in } \Sigma \quad \text{and} \quad f = 0 \quad \text{on } \partial\Sigma, \tag{1.1}$$

where Δ denotes the Laplace operator on $\mathcal{N}^m$. Then, the first eigenvalue $\Lambda_1(\mathcal{N})$ is defined by $\Lambda_1(\mathcal{N}) = inf_\Sigma \Lambda_1(\Sigma)$. The Reilly formula relates exclusively to the intrinsic geometry of the manifold and certainly to the specific PDE under consideration. This can be simply understandable with the following example. Let $(\mathcal{N}^m, g)$ be a compact m-dimensional Riemannian manifold and let Λ_1 denote the first nonzero eigenvalue of the Neumann problem

$$\Delta f + \Lambda_1 f = 0, \quad \text{on } \mathcal{N} \quad \text{and} \quad \frac{\partial f}{\partial N} = 0, \quad \text{on } \partial\mathcal{N}, \tag{1.2}$$

where N is the outward normal on $\partial\mathcal{N}^m$. A result of Reilly [22], reads as the following.

Let $\mathcal{N}^m$ be a Riemannian manifold and $\mathbb{R}^k$ is the Euclidean space having dimensions m and k, respectively. The manifold $\mathcal{N}^m$ is connected, closed, and oriented. The $\mathcal{N}^m$ is isometrically immersed in $\mathbb{R}^k$ with condition $\partial\mathcal{N}^m = 0$. The mean curvature of this isometric

Springer

immersion is denoted by $\mathbb{H}$ and the first nonzero eigenvalue Λ_1^{∇} of the Laplacian on $\mathcal{N}^m$ can be written as in the sense of Reily [22]

$$\Lambda_1^{\nabla} \leq \frac{l}{Vol(\mathcal{N}^m)} \int_{\mathcal{N}^m} |\mathbb{H}|^2 \, dV, \tag{1.3}$$

where the volume element of $\mathcal{N}^m$ is denoted by dV. It can be seen in the literature that many authors were prompted to work in such inequalities for different ambient spaces after the breakthrough of inequality (1.3). In Minkowski spaces, the upper bound for a Finsler submanifold is proposed by both Zeng and He [29]. This upper bound relates to the 1st eigenvalue of the Φ-Laplacian. For a closed manifold, the first eigenvalue of the Φ-Laplace operator is presented by Seto and Wei [25] by using the condition of integral curvature. In the hyperbolic space, the bottom spectra of the Laplace manifold for a complete and a noncompact submanifold is calculated by Lin [19] and the mean curvature has the condition of integral pinching. In addition, Xiong [28] contributed his role on closed hyperspace to find the first Hodge Laplacian eigenvalue. Moreover, Xiong worked for a complete Riemannian manifold that included the Reilly-type sharp upper bounds for the eigenvalues in product manifolds. The generalized Reilly inequality (1.3) and first nonzero eigenvalue of the Φ-Laplace operator is calculated by Du et al. [16]. On a compact submanifold, they used the Wentzel–Laplace operator having a boundary in Euclidean space. Following the same pattern, for Dirichlet and Neumann boundary conditions, Blacker and Seto [6] evidenced a Lichnerowicz-type lower bound for the first nonzero eigenvalue of the Φ-Laplacian. They used the Hessian decomposition on Kaehler manifolds having positive Ricci curvature. A simply connected space form $\mathbb{M}^m(c)$ having constant curvature c is obtained by a well-known evaluation for the first nonzero eigenvalue of Laplacian by the immersion of a submanifold $\mathcal{N}^m$ in simply connected space having m-dimension. This space form included the Euclidean space $\mathbb{R}^m$, the unit sphere $\mathbb{S}^m(1)$, and the hyperbolic space $\mathbb{H}(-1)^m$ with $c = 0, 1$ and $c = -1$, respectively.

In [3, 4, 13, 15], the first nonzero eigenvalue of the Laplacian is evidenced that is considered as the generalization of the results in Reilly [22]. For various ambient spaces, the outcomes of different classes of Riemannian submanifolds indicate that the result of both 1st nonzero eigenvalues depict alike inequalities and ultimately have identical upper bounds [12, 13]. This result is valid for both Dirichlet and Neumann conditions. For an ambient manifold, it is obvious from the literature that Laplace and Φ-Laplace operators on Riemannian submanifolds helped to acquire different breakthroughs in Riemannian geometry (see [5, 8, 9, 11, 14, 17, 20, 21, 23, 26, 29]) through the work of [22]. To define the Φ-Laplacian that is a second-order quasilinear elliptic operator on $\mathcal{N}^m$ (compact Riemannian manifold $\mathcal{N}^m$ having m-dimension), we have

$$\Delta_{\Phi} f = div\big(|\nabla f|^{\Phi-2} \nabla f\big), \tag{1.4}$$

where $\Phi > 1$ to satisfy the above equation. We have the usual Laplacian for $\Phi = 2$. On the other hand, the eigenvalue of Δ_{Φ} has similarity with the Laplacian. For instance, if a nonzero function f satisfies the subsequent equation with the Dirichlet boundary condition (1.1) (or Neumann boundary condition (1.2)) then Λ (any real number) is a Dirichlet eigenvalue. Similarly, the above criteria also hold for Neumann boundary conditions (1.2)

$$\Delta_{\Phi} f = -\Lambda |f|^{\Phi-2} f. \tag{1.5}$$

Let us study a Riemannian manifold $\mathcal{N}^m$ with no boundary. The Rayleigh-type variational characterization is observed in the first nonzero eigenvalue of Δ_Φ that is given by $\Lambda_{1,\Phi}$. From (cf. [27]):

$$\Lambda_{1,\Phi} = \inf\left\{\frac{\int_{\mathcal{N}} |\nabla f|^q}{\int_{\mathcal{N}} \|f\|^q}\bigg| f \in W^{1,\Phi}(\mathcal{N}^l)\{0\}, \int_{\mathcal{N}} |f|^{\Phi-2}f = 0\right\}. \tag{1.6}$$

This naturally raises the question: Is it possible to generalize the Reilly-type inequalities for submanifolds in spheres through the class of almost contact manifolds that were proved in [1, 13, 15]? In Sasakian space form, our aim is to derive the 1st eigenvalue for the Φ-Laplacian on a slant submanifold. Following this opinion and motivated by the historical development in the analysis of the first nonnull eigenvalue of the Φ-Laplacian on a submanifold in various space forms, by using the Gauss equation and influenced by the studies of [12, 13, 16], our goal is to give a general view of the above Reilly conclusion for the Φ-Laplace operator and we going to provide a sharp estimate of the first eigenvalue for the Φ-Laplacian on a semislant submanifold of Sasakian space form $\mathbb{M}^{2k+1}(\epsilon)$. The main finding of this paper will be announced in the following theorem.

Theorem 1.1 *Let $\mathcal{N}^m$ be an $m(\geq 2)$-dimensional closed orientated semislant submanifold in a Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$. Then,*

(1) *The first nonnull eigenvalue $\Lambda_{1,\Phi}$ of the Φ-Laplacian satisfies*:

$$\begin{aligned}\Lambda_{1,\Phi} \leq{}& \left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(1-\frac{\Phi}{2})}m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{\Phi/2}}\right)\\ &\times\left\{\int_{\mathcal{N}^m}\left\{\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right\}dV\right\}^{\Phi/2}\\ &\textit{for } 1<\Phi\leq 2,\end{aligned}\tag{1.7}$$

and

$$\begin{aligned}\Lambda_{1,\Phi} \leq{}& \left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(\frac{\Phi}{2}-1)}m^{\frac{\Phi}{2}}}{Vol(\mathcal{N})}\right)\\ &\times\int_{\mathcal{N}^m}\left\{\left|\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right|\right\}^{\Phi/2}dV\\ &\textit{for } 2<\Phi\leq\frac{m}{2}+1.\end{aligned}\tag{1.8}$$

(2) *The equality carries in (1.7) and (1.8) if and only if $\Phi = 2$ and $\mathcal{N}^m$ is minimally immersed in a geodesic sphere of radius r_ϵ of $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ with the following equalities*

$$r_0 = \left(\frac{m}{\Lambda_1^\Delta}\right)^{1/2},$$

$$r_1 = \sin^{-1} r_0,$$

$$r_{-1} = \sinh^{-1} r_0.$$

Remark 1.1 For an immediate consequence of the above, we put $\Phi = 2$ in our estimate to find the corollary.

Corollary 1.1 *Let $\mathcal{N}^m$ be an m-dimensional closed orientated semislant submanifold in Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$. Then, Λ_1^{Δ} satisfies the following inequality for the Laplacian*

$$\Lambda_1^{\Delta} \leq \frac{m}{Vol(\mathcal{N})} \int_{\mathcal{N}} \left\{ |\mathbb{H}|^2 + \left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2 + d_3(6\cos^2\vartheta - 4)}{m(m-1)}\right)\right\} dV. \tag{1.9}$$

The equality's cases are the same as in Theorem 1.1 (2).

This is an immediate application of Theorem 1.1 by using $1 < \Phi \leq 2$, as the Sasakian space form.

Theorem 1.2 *Let $\mathcal{N}^m$ be an $m(\geq 2)$-dimensional closed orientated semislant submanifold in Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$. Then, $\Lambda_{1,\Phi}$ satisfies the following inequality for the Φ-Laplacian*

$$\begin{aligned}\Lambda_{1,p} \leq {} & \left(\frac{2^{1-\frac{\Phi}{2}}(m+1)^{(1-\frac{\Phi}{2})} m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{(\Phi-1)}}\right) \\ & \times \left\{\int_{\mathcal{N}^m} \left(\left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2 + d_3(6\cos^2\vartheta - 4)}{m(m-1)}\right) + |\mathbb{H}|^2\right)^{\frac{\Phi}{2(\Phi-1)}} dV\right\}^{(\Phi-1)}\end{aligned} \tag{1.10}$$

for $1 < \Phi \leq 2$.

Remark 1.2 Consider the inequality (1.10) and put $\Phi = 2$, then inequality (1.10) generalizes the Reilly-type inequality (1.9). This shows that the Reilly-type inequality calculates the first eigenvalue for the Laplace operator on a slant submanifold in Euclidean sphere $\mathbb{S}^{2k+1}$ (see Theorem 1.2 in [15] and Theorem 1.3 in [13]), are the same in the case of our Theorem 1.1 for $\epsilon = 1$ and $\Phi = 2$.

2 Preliminaries and notations

An almost contact manifold is an odd-dimensional C^∞-manifold $(\widetilde{\mathbb{M}}^{2k+1}, g)$ with almost contact structure (ψ, ξ, η) that satisfies the following properties, i.e.,

$$\begin{aligned}&\psi^2 = -I + \eta \otimes \xi,\\ &\eta(\xi) = 1,\\ &\psi(\xi) = 0,\\ &\eta \circ \psi = 0, && (2.1)\\ &g(\psi U_2, \psi V_2) = g(U_2, V_2) - \eta(U_2)\eta(V_2),\\ &\eta(U_2) = g(U_2, \xi) && (2.2)\end{aligned}$$

for any U_2, V_2 belong to $\widetilde{\mathbb{M}}^{2k+1}$. The three parameters of an almost contact structure can be individually elaborated as ψ is a $(1,1)$-type tensor field, whereas ξ is the structure vector field and η is dual 1-form. In the perspective of the Riemannian connection, an almost contact manifold can be a *Sasakian manifold* [2, 24] if

$$(\widetilde{\nabla}_{U_2}\psi)V_2 = g(U_2, V_2)\xi - \eta(V_2)U_2. \tag{2.3}$$

This indicates that

$$\widetilde{\nabla}_{U_2}\xi = -\psi U_2, \tag{2.4}$$

where ∇ indicates the Riemannian connection in regard to g and U_2, V_2 are any vector fields on $\widetilde{\mathbb{M}}^{2k+1}$. With this, we consider that $\widetilde{\mathbb{M}}^{2k+1}$ converts into a Sasakian space form if it has a ψ-sectional constant curvature ϵ and is represented by $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$. Thus, we can represent the curvature tensor $\widetilde{R}$ of $\widetilde{\mathbb{M}}^{2m+1}(\epsilon)$ as:

$$\begin{aligned}\widetilde{R}(X_2, Y_2, Z_2, W_2) = {} & \frac{\epsilon+3}{4}\big\{g(Y_2, Z_2)g(X_2, W_2) - g(X_2, Z_2)g(Y_2, W_2)\big\} \\ & + \frac{\epsilon-1}{4}\big\{\eta(X_2)\eta(Z_2)g(Y_2, W_2) + \eta(W_2)\eta(Y_2)g(X_2, Z_2) \\ & - \eta(Y_2)\eta(Z_2)g(X_2, W_2) - \eta(X_2)g(Y_2, Z_2)\eta(W_2) \\ & + g(\psi Y_2, Z_2)g(\psi X_2, W_2) - g(\psi X_2, Z_2)g(\psi Y_2, W_2) \\ & + 2g(X_2, \psi Y_2)g(\psi Z_2, W_2)\big\},\end{aligned} \tag{2.5}$$

for any arbitary X_2, Y_2, Z_2, W_2 belonging to $\widetilde{\mathbb{M}}^{2k+1}$. For more details, see [2, 10, 24].

Assuming that $\mathcal{N}^m$ is an m-dimensional submanifold isometrically immersed in a Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, if ∇ and $\nabla^{\perp}$ are induced connections on the tangent bundle $T\mathcal{N}$ and the normal bundle $T^{\perp}\mathcal{N}$ of $\mathcal{N}$, respectively, then, the Gauss and Weingarten formulas are given by:

$$(i) \quad \widetilde{\nabla}_{U_2}V_2 = \nabla_{U_2}V_2 + h(U_2, V_2), \qquad (ii) \quad \widetilde{\nabla}_{U_2}\zeta = -A_{\zeta}U_2 + \nabla^{\perp}_{U_2}\zeta \tag{2.6}$$

for each $U_2, V_2 \in \Gamma(T\mathcal{N})$ and $\zeta \in \Gamma(T^{\perp}\mathcal{N})$, where h and A_{ζ} are the second fundamental form and shape operator (analogous to the normal vector field ζ), respectively, for the immersion of $\mathcal{N}^m$ into $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$. They are connected as: $g(h(U_2, V_2), \zeta) = g(A_{\zeta}U_2, V_2)$. Throughout the structure vector field ξ is assumed to be tangential to $\mathcal{N}$, otherwise $\mathcal{N}$ is simply antiinvariant. Now, for any $U \in \Gamma(T\mathcal{N})$ and $N \in \Gamma(T^{\perp}\mathcal{N})$, we have:

$$(i) \quad \psi U_2 = TU_2 + FU_2, \qquad (ii) \quad \psi\zeta = t\zeta + f\zeta, \tag{2.7}$$

where $TU_2(t\zeta)$ and $FU_2(f\zeta)$ are the tangential and normal components of $\psi U_2(\psi\zeta)$, respectively. From (2.7) it is not difficult to check that for each $U_2, V_2 \in \Gamma(T\mathcal{N})$

$$g(TU_2, V_2) = -g(U_2, TV_2).$$

A submanifold $\mathcal{N}^m$ is defined to be a slant submanifold if for any $x \in \mathcal{N}$ and for any vector field $U_2 \in \Gamma(T\mathcal{N}^m)$, linearly independent on ξ, the angle between ψU_2 and $T\mathcal{N}$ is a constant angle $\vartheta(U_2)$ that lies between zero and $\pi/2$.

This follows from the definition of slant immersions, where Cabrerizo [7] obtained the necessary and sufficient condition that a submanifold $\mathcal{N}^m$ is said to be a slant submanifold if and only if there exists a constant $C \in [0, \pi/2]$ and one tensor fled T is satisfied by the following:

$$T^2 = -C(I - \eta \otimes \xi), \tag{2.8}$$

such that $C = \cos^2\vartheta$. Also, we have a consequence of the above formula

$$g(TU_2, TV_2) = \cos^2\vartheta\{g(U_2, V_2) - \eta(U_2)\eta(V_2)\}. \tag{2.9}$$

With the help of the moving-frame method, we explore some of the interesting features of conformal geometry and slant submanifolds. The specific convection has been applied on indices range, though we exclude in a way that:

$$1 \leq i, j, s, \ldots \leq m; \qquad m+1 \leq \alpha, \beta, \gamma, \ldots \leq 2k+1, \qquad 1 \leq a, b, c, \ldots \leq 2k+1.$$

The mean curvature and squared norm of the mean curvature vector $H_{\mathcal{N}}$ of a Riemannian submanifold $\mathcal{N}^m$ are defined by:

$$\mathbb{H} = \frac{1}{m}\sum_{i=1}^{n} h(e_i, e_i) \quad \text{and,} \quad \|\mathbb{H}\|^2 = \frac{1}{m^2}\sum_{r=m+1}^{k}\left(\sum_{i=1}^{m} h_{ii}^r\right)^2. \tag{2.10}$$

Similarly, the length of the second fundamental form h is given by

$$h_{ij}^r = g\big(h(e_i, e_j), e_r\big), \quad \text{and} \quad S = \|h\|^2 = \sum_{r=m+1}^{k}\sum_{i,j=1}^{n}\big(h_{ij}^r\big)^2. \tag{2.11}$$

In addition, we denote the following:

$$\|T\|^2 = \sum_{i,j=1}^{m} g^2(Te_i, e_j). \tag{2.12}$$

Our main motivation comes from the following example:

Example 2.1 ([7]) Let $(\mathbb{R}^{2k+1}, \varphi, \xi, \eta, g)$ denote the Sasakian manifold with Sasakian structure

$$\eta = \frac{1}{2}\left(dz_1 - \sum_{i=1}^{k} y_1^i\, dx_1^i\right), \qquad \xi = 2\frac{\partial}{\partial z_1},$$

$$g = \eta \otimes \eta + \frac{1}{4}\left(\sum_{i=1}^{k}\big(dx_1^i \otimes dx_1^i \otimes dy_1^i \otimes dy_1^i\big)\right),$$

$$\varphi\left(\sum_{i=1}^{k}\left(X_i\frac{\partial}{\partial x_1^i} + Y_i\frac{\partial}{\partial y_1^i}\right) + Z\frac{\partial}{\partial z_1}\right) = \sum_{i=1}^{k}\left(Y_i\frac{\partial}{\partial y_1^i} - X_i\frac{\partial}{\partial x_1^i}\right) + \sum_{i=1}^{k} Y_i y_1^i\frac{\partial}{\partial z_1},$$

where (x_1^i, y_1^i, z_1), $i = 1\cdots k$ are the coordinates system. It is easy to explain that $(\mathbb{R}^{2k+1}, \varphi, \xi, \eta, g)$ is an almost contact metric manifold. Now, consider the 3-dimensional submanifold in $\mathbb{R}^5$ with Sasakian structure. For any $\vartheta \in [0, \frac{\pi}{2}]$ such that:

$$\psi(u_1, v_1, t) = 2(u_1\cos\vartheta, u_1\sin\vartheta, v_1, 0, t). \tag{2.13}$$

Under the above immersion $\mathcal{N}^3$ is a three-dimensional minimal slant submanifold containing slant angle ϑ and scalar curvature $\tau = -\frac{\cos^2\vartheta}{3}$.

Similarly, we give more examples for a nonminimal submanifold.

Example 2.2 ([7]) For any constant λ, we define an immersion:

$$\psi(u_1, v_1, t) = 2\big(e^{\lambda u_1}\cos u_1 \cos v_1, e^{\lambda u_1}\sin u_1 \cos v_1, e^{\lambda u_1}\cos u_1 \sin v_1, e^{\lambda u_1}\sin u_1 \sin v_1, t\big). \tag{2.14}$$

It is easy to see that the above immersion is a three-dimensional slant submanifold with slant angle $\vartheta = \cos^{-1}(\frac{|\lambda|}{\sqrt{1+\lambda^2}})$. Moreover, scalar curvature $\tau = -\frac{\lambda^2}{3(1+\lambda^2)}$ and mean curvature $|\mathbb{H}| = \frac{2e^{-\lambda u_1}}{3\sqrt{1+\lambda^2}}$.

A Riemannian submanifold $\mathcal{N}^m$ of an almost contact manifold $\widetilde{\mathbb{M}}$ is said to be a semislant submanifold if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\vartheta$ such that $T\mathcal{N} = \mathcal{D} \oplus \mathcal{D}^\vartheta \oplus \zeta$, the distribution $\mathcal{D}$ is invariant, i.e., $\varphi\mathcal{D} = \mathcal{D}$ and the distribution D^θ is slanted with slant angle $\vartheta \neq \frac{\pi}{2}$. If we denote the dimensions of $\mathcal{D}$ and $\mathcal{D}^\theta$ by d_2 and d_3, respectively, then it is clear that contact CR-submanifolds and slant submanifolds are semislant submanifolds with $\theta = \frac{\pi}{2}$ and $d_2 = 0$, respectively. If neither $d_2 = 0$ nor $\theta = \frac{\pi}{2}$, then $\mathcal{N}^m$ is a proper semislant submanifold.

Remark 2.1 It is clear that a semislant submanifold is generalized to a slant submanifold with $d_2 = 0$.

Remark 2.2 A totally real submanifold is a particular case of a semislant submanifold with slant angle $\vartheta = \frac{\pi}{2}$ and $d_2 = 0$.

It is necessary to clarify the definition of the curvature tensor $\widetilde{R}$ for a slant submanifold in the Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ and is given by:

$$\begin{aligned}\widetilde{R}(e_i, e_j, e_i, e_j) &= \left(\frac{\epsilon+3}{4}\right)\big(m^2 - m\big) \\ &\quad + \left(\frac{\epsilon-1}{4}\right)\left\{3\sum_{ii,j=1}^{m} g^2(\varphi e_i, e_j) - 2(m-1)\right\}.\end{aligned} \tag{2.15}$$

On the other hand, let $\{e_1, \dots e_{d_2}, \dots e_m = \zeta\}$ be an orthonormal basis of $T_x\mathcal{N}$ such that

$$\begin{aligned}
&e_1, e_2 = \varphi e_1, \dots, e_{2d_2-1}, e_{2d_2} = \varphi e_{2d_2-1}, \\
&e_{2d_2+1}, e_{2d_2+2} = \sec\vartheta\, Te_{2d_2+1}, \dots, e_{2d_2+2d_3-1}, e_{2d_2+2d_3} = \sec\vartheta\, Te_{2d_2+2d_3-1}, \dots, \\
&\vdots \\
&e_{2d_2+2d_3}, e_{2d_2+2d_2+1} = \zeta.
\end{aligned}$$

Thus, we have

$$g(\varphi e_1, e_2) = g(\varphi e_1, \sec\vartheta\, Te_1) = \sec\vartheta\, g(\varphi e_1, Te_1) = \sec\vartheta\, g(Te_1, Te_1). \tag{2.16}$$

It is clear that the dimension of $\mathcal{N}^m$ can be decomposed as $m = 2d_2 + 2d_3 + 1$. Then, from (2.9), we derive that:

$$g(\varphi e_1, e_2) = \cos\vartheta. \tag{2.17}$$

In similar way, we repeat that then:

$$g^2(\varphi e_i, e_{i+1}) = \cos^2\vartheta \quad \Longrightarrow \quad \sum_{i,j=1}^{m} g^2(\varphi e_i, e_j) = 2\bigl(d_2 + d_3\cos^2\vartheta\bigr). \tag{2.18}$$

Merging (2.15) and (2.18) implies that

$$\begin{aligned}\widetilde{R}(e_i, e_j, e_i, e_j) = {} & \left(\frac{\epsilon+3}{4}\right)\bigl(m^2 - m\bigr) \\ & + \left(\frac{\epsilon-1}{4}\right)\bigl\{2d_2 + d_3\bigl(6\cos^2\vartheta - 4\bigr)\bigr\}.\end{aligned} \tag{2.19}$$

2.1 Structure equations for semislant submanifolds

Let x be a totally real embedding from $\mathcal{N}^m$ to an $2k+1$-dimensional Riemannian manifold $(\widetilde{\mathbb{M}}, \widetilde{g})$. Then, $\mathcal{N}^m$ has an induced metric $g_{\mathcal{N}} = x^*\widetilde{g}$. Let us consider $\widetilde{\mathbb{M}}^{2k+1} = \widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, then pulling back [[1] Eq. (12)] by x and using [[1] Eqs. (13), (14)], we obtain the Gauss equations for a slant submanifold in Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ and taking into account (2.15)

$$\begin{aligned}R_{ijtl} = {} & \left(\frac{\epsilon+3}{4}\right)(\delta_{it}\delta_{jl} - \delta_{il}\delta_{jt}) + \left(\frac{\epsilon-1}{4}\right)\bigl\{3(\varphi e_i, e_j) - 2(m-1)\bigr\} \\ & + \sum_{\alpha}\bigl(h_{it}^{\alpha}h_{jl}^{\alpha} - h_{il}^{\alpha}h_{jt}^{\alpha}\bigr).\end{aligned} \tag{2.20}$$

Taking the trace of the above equation and using (2.19), we obtain:

$$R = m^2|\mathbb{H}|^2 - S + \left(\frac{\epsilon+3}{4}\right)m(m-1) + \left(\frac{\epsilon-1}{4}\right)\bigl\{2d_2 + d_3\bigl(6\cos^2\vartheta - 4\bigr)\bigr\}, \tag{2.21}$$

where R is the scalar curvature of $\mathcal{N}^m$ and S is the length of the second fundamental form h.

2.2 Conformal relations

In this section, we will look at how the conformal transformation affects the curvature and the second fundamental form. Although these relationships are well known (cf. [1]), we use the moving-frame method to provide a quick proof for the readers' convenience.

Assume that $\widetilde{\mathbb{M}}^{2k+1}$ has a new metric $\tilde{g} = e^{2\rho}\tilde{g}$, that is conformal to $\tilde{g}$, and where $\rho \in C^{\infty}(\widetilde{\mathbb{M}})$. Then, $\tilde{\varpi}_a = e\rho\varpi_a$ is the dual coframe of $(\widetilde{\mathbb{M}}, \tilde{g})$, and $\tilde{e}_a = e^{\rho}e_a$ is the orthogonal frame of $(\widetilde{\mathbb{M}}, \tilde{g})$. The equality's equations of $(\widetilde{\mathbb{M}}, \tilde{g})$ are given in ([1], Eqs. (20), (21), (22) (23)) by:

$$\widetilde{\varpi}_{ab} = \varpi_{ab} + \rho_a\varpi_b - \rho_b\varpi_a, \tag{2.22}$$

where ρ_a is the covariant derivative of ρ with respect to e_a, that is, $d\rho = \sum_a \rho_a e_a$.

$$e^{2\rho}\widetilde{R}_{ijtl} = R_{ijtl} - (\rho_{it}\delta_{jl} + \rho_{jl}\delta_{it} - \rho_{il}\delta_{jt} - \rho_{jt}\delta_{il})$$

$$
\begin{aligned}
&+ (\rho_i\rho_t\delta_{jl} + \rho_j\rho_l\delta_{it} - \rho_j\rho_t\delta_{il} - \rho_i\rho_l\delta_{jt}) \\
&- |\nabla_\alpha|^2(\delta_{it}\delta_{jl} - \delta_{il}\delta_{jt}).
\end{aligned}
\tag{2.23}
$$

By pulling back (2.22) to $\mathcal{N}^m$ by x, we have:

$$
\begin{aligned}
\tilde{h}^\alpha_{ij} &= e^{-\rho}\big(h^\alpha_{ij} - \rho_\alpha\delta_{ji}\big), \\
\widetilde{\mathbb{H}}^\alpha &= e^{-\alpha}\big(\mathbb{H}^\alpha - \rho_\alpha\big),
\end{aligned}
\tag{2.24}
$$

from this, it is easy to obtain the useful relation:

$$
e^{2\rho}\big(\widetilde{S} - m|\widetilde{\mathbb{H}}|^2\big) + m|\mathbb{H}|^2 = S. \tag{2.25}
$$

3 Proof of main result

In this section we shall prove Theorem 1.1 announced in a previous section. First, some fundamental formulas will be presented and some useful lemmas from [20] will be recalled to our setting. For the proposes of this paper, we are going to provide an important lemma that was essentially motivated by the study in [1, 20].

Remark 3.1 A simply connected Sasakian space form $\mathbb{M}^{2k+1}$ is a $(2k+1)$-sphere $\mathbb{S}^{2k+1}$ and Euclidean space $\mathbb{R}^{2k+1}$ with constant φ-sectional curvature $\epsilon = 1$ and $\epsilon = -3$, respectively.

Based on the above arguments, we have a lemma.

Lemma 3.1 ([1]) *Let $\mathcal{N}^m$ be a slant submanifold of Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ that is closed and oriented with dimension m. If $x : \mathcal{N}^m \to \widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ is an embedding from $\mathcal{N}^m$ into $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, then there exists a regular conformal map $\Gamma : \widetilde{\mathbb{M}}^{2k+1}(\epsilon) \to \mathbb{S}^{2k+1}(1) \subset \mathbb{R}^{2k+2}$ such that the embedding $\varpi = \Gamma \circ x = (\varpi^1, \dots \varpi^{2k+2})$ satisfies that*:

$$
\int_{\mathcal{N}^m} \big|\varpi^a\big|^{\Phi-2}\varpi^a \, dV_{\mathcal{N}} = 0, \quad a = 1, \dots 2(k+1), \tag{3.1}
$$

for $\Phi > 1$.

In the above Lemma 3.1 by the constructed test function, we produce an upper bound for $\Lambda_{1,\Phi}$ in the form of the conformal function that is comparable with Lemma 2.7 in [20].

Proposition 3.1 *Let $\mathcal{N}^n$ be an $m \geq 2$-dimensional closed orientated slant submanifold into Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$. Then we have,*

$$
\Lambda_{1,\Phi} Vol\big(\mathcal{N}^m\big) \leq 2^{|1-\frac{\Phi}{2}|}(k+1)^{|1-\frac{\Phi}{2}|} m^{\frac{\Phi}{2}} \int_{\mathcal{N}^m} \big(e^{2\rho}\big)^{\frac{\Phi}{2}} \, dV, \tag{3.2}
$$

where Γ is the conformal map in Lemma 3.1 and for all $\Phi > 1$. Identified by Υ_ε is the standard metric on $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ and it is considered that $\Gamma^*\Upsilon_1 = e^{2\rho}\Upsilon_\varepsilon$,

Proof Considering ϖ^a as a test function along with Lemma 3.1, we derive

$$
\Lambda_{1,\Phi} \int_{\mathcal{N}^m} \big|\varpi^a\big|^\Phi \leq \big|\nabla\varpi^a\big|^\Phi \, dV, \quad 1 \leq a \leq 2(k+1). \tag{3.3}
$$

Observe that $\sum_{a=1}^{2k+2} |\varpi^a|^2 = 1$, then $|\varpi^a| \leq 1$. We accomplish:

$$\sum_{a=1}^{2k+2} \left|\nabla \varpi^a\right|^2 = \sum_{i=1}^{m} |\nabla_{e_i} \varpi|^2 = me^{2\rho}. \tag{3.4}$$

By using $1 < \Phi \leq 2$, then we derive:

$$\left|\varpi^a\right|^2 \leq \left|\varpi^a\right|^{\Phi}. \tag{3.5}$$

Using the Holder inequality along with (3.3), (3.4), and (3.5), we are able to obtain

$$\begin{aligned}
\Lambda_{1,\Phi} Vol(\mathcal{N}) &= \Lambda_{1,\Phi} \sum_{a=1}^{2k+2} \int_{\mathcal{N}^m} \left|\varpi^a\right|^2 dV \leq \Lambda_{1,\Phi} \sum_{a=1}^{2k+2} \int_{\mathcal{N}^m} \left|\varpi^a\right|^{\Phi} dV \\
&\leq \Lambda_{1,\Phi} \int_{\mathcal{N}^m} \sum_{a=1}^{2k+2} \left|\nabla \varpi^a\right|^{\Phi} dV \leq (2k+2)^{1-\Phi/2} \int_{\mathcal{N}^m} \left(\sum_{a=1}^{m} \left|\nabla \varpi^a\right|^2 \right)^{\frac{\Phi}{2}} dV \\
&= 2^{1-\frac{\Phi}{2}} (k+1)^{1-\frac{\Phi}{2}} \int_{\mathcal{N}^m} \left(me^{2\rho}\right)^{\frac{\Phi}{2}} dV.
\end{aligned}$$

This gives us the desired outcome (3.2). On the contrary, if we assume $\Phi \geq 2$, then by applying the Holder inequality we have

$$1 = \sum_{a=1}^{2k+2} \left|\varpi^a\right|^2 \leq (2k+2)^{1-\frac{2}{\Phi}} \left(\sum_{a=1}^{2k+2} \left|\varpi^a\right|^{\Phi} \right)^{\frac{2}{\Phi}}. \tag{3.6}$$

The outcome we obtain is

$$\Lambda_{1,\Phi} Vol\left(\mathcal{N}^m\right) \leq (2k+2)^{\frac{\Phi}{2}-1} \left(\sum_{a=1}^{2k+2} \Lambda_{1,\Phi} \int_{\mathcal{N}^m} \left|\varpi^a\right|^{\Phi} dV \right). \tag{3.7}$$

The Minkowski inequality gives

$$\sum_{a=1}^{2k+2} \left|\nabla \varpi^a\right|^{\Phi} \leq \left(\sum_{a=1}^{2k+2} \left|\nabla \varpi^a\right|^2 \right)^{\frac{\Phi}{2}} = \left(me^{2\rho}\right)^{\frac{\Phi}{2}}. \tag{3.8}$$

Hence, (3.2) follows from (3.3), (3.7), and (3.8). This completes the proof of the proposition. □

We are now in a position to prove Theorem 1.1.

3.1 Proof of Theorem 1.1

To begin with $1 < \Phi \leq 2$, then $\frac{\Phi}{2} \leq 1$. Taking help from Proposition 3.1 and implementing the Hölder inequality, we have:

$$\Lambda_{1,\Phi} Vol\left(\mathcal{N}^m\right) \leq 2^{1-\frac{\Phi}{2}} (k+1)^{1-\frac{\Phi}{2}} m^{\frac{\Phi}{2}} \int_{\mathcal{N}^m} \left(e^{2\rho}\right)^{\frac{\Phi}{2}} dV$$

$$\leq 2^{1-\frac{\Phi}{2}}(k+1)^{1-\frac{\Phi}{2}|}m^{\frac{\Phi}{2}}\left(Vol(\mathcal{N})\right)^{1-\frac{\Phi}{2}}\left(\int_{\mathcal{N}^m} e^{2\rho}\,dV\right)^{\frac{\Phi}{2}}.$$

By using both conformal relations and Gauss equations, it is possible to calculate $e^{2\rho}$. Let $\widetilde{\mathbb{M}}^{2k+1} = \widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, and $\tilde{g} = e^{-2\rho}\Upsilon_\epsilon, \tilde{g} = \Gamma^*\Upsilon_1$ in the above. From (2.21), the Gauss equations for the embedding x and the slant embedding $\varpi = \Gamma \circ x$ are, respectively:

$$\begin{aligned} R &= \left(\frac{\epsilon+3}{4}\right)m(m-1) + \left(\frac{\epsilon-1}{4}\right)\left\{2d_2 + d_3\left(6\cos^2\vartheta - 4\right)\right\} \\ &\quad + m(m-1)|\mathbb{H}|^2 + \left(m|\mathbb{H}|^2 - S\right), \end{aligned} \tag{3.9}$$

$$\tilde{R} = m(m-1) + m(m-1)|\tilde{\mathbb{H}}|^2 + \left(m|\tilde{\mathbb{H}}|^2 - \tilde{S}\right). \tag{3.10}$$

Tracing (2.23), it can be established that:

$$e^{2\rho}\tilde{R} = R - (m-2)(m-1)|\nabla_\rho|^2 - 2(m-1)\Delta_\rho, \tag{3.11}$$

which together with replacement of (3.9) and (3.10) into (3.11) gives:

$$\begin{aligned} & e^{2\rho}\left(m(m-1) + m(m-1)|\tilde{\mathbb{H}}|^2 + \left(m|\tilde{\mathbb{H}}|^2 - \tilde{S}\right)\right) \\ &\quad = \left(\frac{\epsilon+3}{4}\right)m(m-1) \\ &\qquad + \left(\frac{\epsilon-1}{4}\right)\left\{2d_2 + d_3\left(6\cos^2\vartheta - 4\right)\right\} \\ &\qquad + m(m-1)|\mathbb{H}|^2 + \left(m|\mathbb{H}|^2 - S\right) \\ &\qquad - (m-2)(m-1)|\nabla_\rho|^2 - 2(m-1)\Delta_\rho. \end{aligned}$$

This implies the following:

$$\begin{aligned} & e^{2\rho}\tilde{S} - S - (m-2)(m-1)|\nabla_\rho|^2 - 2(m-1)\Delta_\rho \\ &\quad = m(m-1)\left\{\left\{e^{2\rho} - \left(\frac{\epsilon+3}{4}\right) - \left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2 + d_3(6\cos^2\vartheta - 4)}{m(m-1)}\right)\right\}\right. \\ &\qquad \left. + \left(e^{2\rho}|\tilde{\mathbb{H}}|^2 - |\mathbb{H}|^2\right)\right\} + m\left(e^{2\rho}|\tilde{\mathbb{H}}|^2 - |\mathbb{H}|^2\right). \end{aligned}$$

Now, from (2.24) and (2.25), we derive:

$$\begin{aligned} & m(m-1)\left\{e^{2\rho} - \left(\frac{\epsilon+3}{4}\right) - \left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2 + d_3(6\cos^2\vartheta - 4)}{m(m-1)}\right)\right\} \\ &\qquad + m(m-1)\sum_\alpha\left(\mathbb{H}^\alpha - \rho_\alpha\right)^2 \\ &\quad = m(m-1)|\mathbb{H}|^2 - (m-2)(m-1)|\nabla_\rho|^2 - 2(m-1)\Delta_\rho. \end{aligned}$$

Dividing by $m(m-1)$ in the above equation, it implies that

$$e^{2\rho} = \left\{\left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2 + d_3(6\cos^2\vartheta - 4)}{m(m-1)}\right) + |\mathbb{H}|^2\right\}$$

$$-\frac{2}{m}\Delta_{\rho}-\frac{m-2}{m}|\Delta_{\rho}|^{2}-\left|(\tilde{\nabla}_{\rho})^{\perp}-\mathbb{H}\right|^{2}. \tag{3.12}$$

Taking integration along dV, it is not complicated to obtain the following

$$\begin{aligned}
&\Lambda_{1,\Phi} Vol(\mathcal{N}^m)\\
&\le 2^{1-\frac{\Phi}{2}}(k+1)^{1-\frac{\Phi}{2}|}m^{\frac{\Phi}{2}}\left(Vol(\mathcal{N}^m)\right)^{1-\frac{\Phi}{2}}\left(\int_{\mathcal{N}^m} e^{2\rho}\,dV\right)^{\frac{\Phi}{2}}\\
&\le \frac{(2k+2)^{1-\frac{\Phi}{2}|}m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{\frac{\Phi}{2}-1}}\\
&\quad\times\left\{\int_{\mathcal{N}^m}\left\{\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right\}dV\right\}^{\frac{\Phi}{2}}.
\end{aligned}$$

The above result is comparable to (1.7) as we desired to prove. In the case where $\Phi > 2$, it is not possible to apply the Holder inequality directly to govern $\int_{\mathcal{N}}(e^{2\nu})^{\frac{\Phi}{2}}$ by using $\int_{\mathcal{N}}(e^{2\rho})$. We did multiply both sides of (3.12) with the factor $e^{(\Phi-2)\rho}$ and then solve by using integration on $\mathcal{N}^m$ (cf. [11])

$$\begin{aligned}
\int_{\mathcal{N}^m} e^{\Phi\rho}\,dV &\le \int_{\mathcal{N}^m}\left\{\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right\}e^{(\Phi-2)\rho}\,dV\\
&\quad-\left(\frac{m-2-2\Phi+4}{m}\right)\int_{\mathcal{N}} e^{(\Phi-2)}|\Delta_{\rho}|^2\,dV\\
&\le \int_{\mathcal{N}^m}\left\{\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right\}e^{(\Phi-2)\rho}\,dV.
\end{aligned} \tag{3.13}$$

Next, it follows from the assumption that $m \ge 2\Phi - 2$, and we apply Young's inequality, then

$$\begin{aligned}
&\int_{\mathcal{N}^m}\left\{\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right\}e^{(\Phi-2)\rho}\,dV\\
&\le \frac{2}{\Phi}\int_{\mathcal{N}^m}\left\{\left|\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right|\right\}^{\frac{\Phi}{2}}dV\\
&\quad+\frac{(\Phi-2)}{\Phi}\int_{\mathcal{N}^m} e^{\frac{\Phi}{\rho}}\,dV.
\end{aligned} \tag{3.14}$$

From (3.13) and (3.14) we deduce the following inequality:

$$\int_{\mathcal{N}^m} e^{p\rho}\,dV \le \int_{\mathcal{N}^m}\left(\left|\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2+d_3(6\cos^2\vartheta-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right|\right)^{\frac{\Phi}{2}}dV. \tag{3.15}$$

Now, putting (3.15) into (3.2) we obtain (1.8). In the case of slant submanifolds, the equality case holds in (1.7), then considering the cases in (3.3) and (3.5), we obtain:

$$\left|\varpi^a\right|^2=\left|\varpi^a\right|^{\Phi},$$

$$\Delta_\Phi \varpi^a = -\Lambda_{1,\Phi} \left|\varpi^a\right|^{\Phi-2} \varpi^a,$$

for each $a = 1,\ldots,2k+2$. If $1 < \Phi < 2$ then $|\varpi^a| = 0$ or 1. Hence, there would be only one a for which $|\varpi^a| = 1$ and $\Lambda_{1,\Phi} = 0$, which seems to be a contradiction as the eigenvalue is nonzero. Hence, we consider $\Phi = 2$ and we are only restricted to the Laplacian case. Then, we are able to apply Theorem 1.5 from [15].

Let $\Phi > 2$ and the equality remains valid in (1.8), then it shows that (3.7) and (3.8) become the equalities that indicates

$$\left|\varpi^1\right|^\Phi = \cdots = \left|\varpi^{2k+2}\right|^\Phi$$

and condition $|\nabla \varpi^a| = 0$ holds for existing a. This shows that ϖ^a is a constant value and $\Lambda_{1,\Phi}$ is also equal to zero. This last result again represents a conflict in that $\Lambda_{1,\Phi}$ is a nonnull eigenvalue. This completes the proof of the theorem.

3.2 Proof of Theorem 1.2

Suppose that $1 < \Phi \le 2$, we have $\frac{\Phi}{2(\Phi-1)} \ge 1$. Then, by the *Hölder* inequality, we have:

$$\begin{aligned}
&\int_{\mathcal{N}^m} \left\{ \left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2 + d_3(6\cos^2\vartheta - 4)}{m(m-1)}\right) + |\mathbb{H}|^2 \right\} dV \\
&\le \left((Vol(\mathcal{N}))^{1-\frac{2(\Phi-1)}{\Phi}} \right) \\
&\quad \times \left\{ \int_{\mathcal{N}^m} \left\{ \left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2 + d_3(6\cos^2\vartheta - 4)}{m(m-1)}\right) + |\mathbb{H}|^2 \right\}^{\frac{\Phi}{2(\Phi-1)}} dV \right\}^{\frac{2(\Phi-1)}{\Phi}}.
\end{aligned} \tag{3.16}$$

Thus, combining the equations (1.7) with (3.16), we obtain the desired result (1.10). This completes the proof of the theorem.

Remark 3.2 As a result of the observations in Remark 3.1, the next result will be specified as a special version of Theorem 1.1. To be precise, we determine the following result by replacing $\epsilon = 1$ in (1.7) and (1.8), respectively.

Corollary 3.1 *Assume $\mathcal{N}^m$ is an $m(\ge 2)$-dimensional closed orientated semislant submanifold in $(2k+1)$-sphere $\mathbb{S}^{2k+1}(1)$, then, $\Lambda_{1,\Phi}$ satisfies the following inequality for the Φ-Laplacian*

$$\begin{aligned}
\Lambda_{1,\Phi} &\le \frac{2^{1-\frac{\Phi}{2}}(k+1)^{(1-\frac{\Phi}{2})} m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{p/2}} \left\{ \int_{\mathcal{N}^m} \left(1 + |\mathbb{H}|^2\right) dV \right\}^{\frac{\Phi}{2}} \\
&\quad \text{for } 1 < \Phi \le 2,
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\Lambda_{1,\Phi} &\le \frac{2^{1-\frac{\Phi}{2}}(k+1)^{(1-\frac{\Phi}{2})} m^{\frac{\Phi}{2}}}{Vol(\mathcal{N})} \int_{\mathcal{N}^k} \left(\left|1 + |\mathbb{H}|^2\right|\right)^{\frac{\Phi}{2}} dV \\
&\quad \text{for } 2 < \Phi \le \frac{m}{2} + 1.
\end{aligned} \tag{3.18}$$

There is another corollary based on Corollary 1.2 and it is as follows:

Corollary 3.2 *Assuming that $\mathcal{N}^m$ is an $m(\geq 2)$-dimensional closed orientated semislant submanifold in $(2k+1)$-sphere $\mathbb{S}^{2k+1}(1)$, then, $\Lambda_{1,\Phi}$ satisfies the following inequality for the Φ-Laplacian*

$$\Lambda_{1,\Phi} \leq \frac{(2k+2)^{(1-\frac{\Phi}{2})} m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{(\Phi-1)}} \left\{ \int_{\mathcal{N}^m} \left(1 + |\mathbb{H}|^2\right)^{\frac{\Phi}{2(\Phi-1)}} dV \right\}^{(\Phi-1)} \tag{3.19}$$

for $1 < \Phi \leq 2$.

3.3 Application to slant submanifolds of Sasakian space forms

Using Remark 2.1 and Theorem 1.1, we have the following results:

Corollary 3.3 ([18]) *Let $\mathcal{N}^m$ be an $m(\geq 2)$-dimensional closed orientated slant submanifold in a Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, then, $\Lambda_{1,\Phi}$ satisfies the following inequality for the Φ-Laplacian*

$$\begin{aligned} \Lambda_{1,\Phi} \leq & \left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(1-\frac{\Phi}{2})} m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{\Phi/2}} \right) \\ & \times \left\{ \int_{\mathcal{N}^m} \left\{ \left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{3\cos^2\vartheta - 2}{m}\right) + |\mathbb{H}|^2 \right\} dV \right\}^{\Phi/2} \\ & \text{for } 1 < \Phi \leq 2, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \Lambda_{1,\Phi} \leq & \left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(\frac{\Phi}{2}-1)} m^{\frac{\Phi}{2}}}{Vol(\mathcal{N})} \right) \\ & \times \int_{\mathcal{N}^m} \left\{ \left| \left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{3\cos^2\vartheta - 2}{m}\right) + |\mathbb{H}|^2 \right| \right\}^{\Phi/2} dV \\ & \text{for } 2 < \Phi \leq \frac{m}{2} + 1. \end{aligned} \tag{3.21}$$

From Corollary 1.1 for $\Phi = 2$ and Remark 2.1, we have

Corollary 3.4 ([18]) *Assuming that $\mathcal{N}^m$ is an m-dimensional closed orientated slant submanifold in a Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, then, Λ_1^{Δ} satisfies the following inequality for the Laplacian*

$$\Lambda_1^{\Delta} \leq \frac{m}{Vol(\mathcal{N})} \int_{\mathcal{N}} \left\{ |\mathbb{H}|^2 + \left(\frac{\epsilon+3}{4}\right) + \left(\frac{\epsilon-1}{4}\right)\left(\frac{3\cos^2\vartheta - 2}{m}\right) \right\} dV. \tag{3.22}$$

Similarly, from Theorem 1.2, we find that:

Corollary 3.5 ([18]) *Assuming that $\mathcal{N}^m$ is an $m(\geq 2)$-dimensional closed orientated slant submanifold in a Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, then $\Lambda_{1,\Phi}$ satisfies the following inequality* *for the* Φ*-Laplacian*

$$\Lambda_{1,\Phi} \leq \left(\frac{2^{1-\frac{\Phi}{2}}(k+1)^{(1-\frac{\Phi}{2})}m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{(\Phi-1)}}\right) \times \left\{\int_{\mathcal{N}^m}\left(\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{3\cos^2\vartheta-2}{m}\right)+|\mathbb{H}|^2\right)^{\frac{\Phi}{2(\Phi-1)}} dV\right\}^{(\Phi-1)} \tag{3.23}$$

for $1 < \Phi \leq 2$.

3.4 Application to antiinvariant submanifolds of Sasakian space forms

Using Remark 2.2 and Theorem 1.1, we have the following results:

Corollary 3.6 *Let* $\mathcal{N}^m$ *be an* $m(\geq 2)$*-dimensional closed orientated antiinvariant submanifold in a Sasakian space form* $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$*, then* $\Lambda_{1,\Phi}$ *satisfies the following inequality for the* Φ*-Laplacian*

$$\Lambda_{1,\Phi} \leq \left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(1-\frac{\Phi}{2})}m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{\Phi/2}}\right)\left\{\int_{\mathcal{N}^m}\left\{\left(\frac{\epsilon+3}{4}\right)-\left(\frac{\epsilon-1}{2m}\right)+|\mathbb{H}|^2\right\} dV\right\}^{p/2} \quad \text{for } 1 < \Phi \leq 2, \tag{3.24}$$

and

$$\Lambda_{1,\Phi} \leq \left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(\frac{\Phi}{2}-1)}m^{\frac{\Phi}{2}}}{Vol(\mathcal{N})}\right)\int_{\mathcal{N}^m}\left\{\left|\left(\frac{\epsilon+3}{4}\right)-\left(\frac{\epsilon-1}{2m}\right)+|\mathbb{H}|^2\right|\right\}^{\Phi/2} dV \quad \text{for } 2 < \Phi \leq \frac{m}{2}+1. \tag{3.25}$$

From Corollary 1.1 for $\Phi = 2$ and Remark 2.2, we have:

Corollary 3.7 *Assuming that* $\mathcal{N}^m$ *is an m-dimensional closed orientated antiinvariant submanifold in a Sasakian space form* $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$*, then* Λ_1^{Δ} *satisfies the following inequality for the Laplacian*

$$\Lambda_1^{\Delta} \leq \frac{m}{Vol(\mathcal{N})}\int_{\mathcal{N}}\left\{|\mathbb{H}|^2+\left(\frac{\epsilon+3}{4}\right)-\left(\frac{\epsilon-1}{2m}\right)\right\} dV. \tag{3.26}$$

Similarly, from Theorem 1.2, we find that:

Corollary 3.8 *Assuming that* $\mathcal{N}^m$ *is an* $m(\geq 2)$*-dimensional closed orientated antiinvariant submanifold in a Sasakian space form* $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$*, then* $\Lambda_{1,\Phi}$ *satisfies the following inequality for the* Φ*-Laplacian*

$$\Lambda_{1,\Phi} \leq \left(\frac{2^{1-\frac{\Phi}{2}}(k+1)^{(1-\frac{\Phi}{2})}m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{(\Phi-1)}}\right)\left\{\int_{\mathcal{N}^m}\left(\left(\frac{\epsilon+3}{4}\right)-\left(\frac{\epsilon-1}{2m}\right)+|\mathbb{H}|^2\right)^{\frac{\Phi}{2(\Phi-1)}} dV\right\}^{(\Phi-1)} \tag{3.27}$$

for $1 < \Phi \leq 2$.

3.5 Application to contact CR-submanifolds of Sasakian space forms

Corollary 3.9 *Let $\mathcal{N}^m$ be an $m(\geq 2)$-dimensional closed orientated contact CR-submanifold in a Sasakian space form $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$, then $\Lambda_{1,\Phi}$ satisfies the following inequalities for the Φ-Laplacian*

(1) *The first nonnull eigenvalue $\Lambda_{1,\Phi}$ of the Φ-Laplacian satisfies*:

$$\begin{aligned}\Lambda_{1,\Phi} \leq &\left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(1-\frac{\Phi}{2})}m^{\frac{\Phi}{2}}}{(Vol(\mathcal{N}))^{\Phi/2}}\right)\\ &\times\left\{\int_{\mathcal{N}^m}\left\{\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right\}dV\right\}^{\Phi/2}\\ &for\ 1<\Phi\leq 2,\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\Lambda_{1,\Phi} \leq &\left(\frac{2^{(1-\frac{\Phi}{2})}(k+1)^{(\frac{\Phi}{2}-1)}m^{\frac{\Phi}{2}}}{Vol(\mathcal{N})}\right)\\ &\times\int_{\mathcal{N}^m}\left\{\left|\left(\frac{\epsilon+3}{4}\right)+\left(\frac{\epsilon-1}{4}\right)\left(\frac{2d_2-4)}{m(m-1)}\right)+|\mathbb{H}|^2\right|\right\}^{\Phi/2}dV\\ &for\ 2<\Phi\leq\frac{m}{2}+1.\end{aligned} \tag{3.29}$$

(2) *The equality carries in (1.7) and (1.8) if and only if $\Phi=2$ and $\mathcal{N}^m$ is minimally immersed in a geodesic sphere of radius r_ϵ of $\widetilde{\mathbb{M}}^{2k+1}(\epsilon)$ with the following equalities*

$$\begin{aligned}&r_0=\left(\frac{m}{\Lambda_1^{\Delta}}\right)^{1/2},\\ &r_1=\sin^{-1}r_0,\\ &r_{-1}=\sinh^{-1}r_0.\end{aligned}$$

Acknowledgements
The authors would like to express their gratitude to the Deanship of Scientific Research at King Khalid University, Saudi Arabia for providing a funding research group under the research grant R. G. P. 2/130/43. The authors also express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This research was funded by National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Funding
The authors would like to express their gratitude to the Deanship of Scientific Research at King Khalid University, Saudi Arabia for providing a funding research group under the research grant R. G. P. 2/130/43. The authors also express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This research was funded by National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Availability of data and materials
There is no data use for this manuscript.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution

Conceptualization of the paper was carried out by YL, FM, and AA. Methodology, formal analysis and investigation were carried out by RPA. Writing the original draft preparation was carried out by YL, FM, and AA. Writing, reviewing, and editing were carried out by AA and FM. Project administration was carried out by AA, FM, and YL. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Hangzhou Normal University, 311121 Hangzhou, P.R. China. [2]Mathematical Science Department Faculty of Science, Princess Nourah bint Abdulrahman University, Riyadh 11546, Saudi Arabia. [3]Department of Mathematics, Texas A and M University-Kingsville, Kingsville, TX 78363-8202, USA. [4]Department of Mathematics, College of Sciences, King Khalid University, 9004 Abha, Saudi Arabia.

Publisher's Note

Received: 17 August 2021 Accepted: 15 July 2022 Published online: 04 August 2022

References

1. Ali, A., Alkhaldi, A.H., Laurian-Ioan, P., Ali, R.: Eigenvalue inequalities for the p-Laplacian operator on C-totally real submanifolds in Sasakian space forms. Appl. Anal. **101**(2), 702–713 (2022)
2. Ali, A., Laurian-Ioan, P.: Geometric classification of warped products isometrically immersed in Sasakian space forms. Math. Nachr. **292**, 234–251 (2018)
3. Ali, A., Lee, J.W., Alkhaldi, A.H.: The first eigenvalue for the *p*-Laplacian on Lagrangian submanifolds in complex space forms. Int. J. Math. **33**(2), 2250016 (2022)
4. Alluhaibi, N., Ali, A.: The eigenvalue estimates of *p*-Laplacian of totally real submanifolds in generalized complex space forms. Ric. Mat. (2021). https://doi.org/10.1007/s11587-021-00670-5
5. Andrews, B.: Moduli of continuity, isoperimetric profiles, and multi-point estimates in geometric heat equations. Surveys in differential geometric 2014. Regularity and evolution of nonlinear equation, 1–47 (2015)
6. Blacker, C., Seto, S.: First eigenvalue of the *p*-Laplacian on Kaehler manifolds. Proc. Am. Math. Soc. **147**, 2197–2206 (2019)
7. Cabrerizo, J.L., Carriazo, A., Fernandez, L.M., Fernandez, M.: Slant submanifolds in Sasakian manifolds. Glasg. Math. J. **42**(1), 125–138 (2000)
8. Cavalletti, F., Mondino, A.: Sharp geometry and functional inequalities in metric measure spaces with lower Ricci curvature bounds. Geom. Topol. **21**, 603–645 (2017)
9. Chang, D.C., Frieder, O., Hung, C.F., Yao, H.R.: The analysis from nonlinear distance metric to kernel-based prescription prediction system. J. Nonlinear Var. Anal. **5**, 179–199 (2021)
10. Chen, B.Y., Ogiue, K.: On totally real submanifolds. Trans. Am. Math. Soc. **193**, 257–266 (1974)
11. Chen, D., Li, H.: Second eigenvalue of Paneitz operators and mean curvature. Commun. Math. Phys. **305**, 555–562 (2011)
12. Chen, H., Wang, X.: Sharp Reilly-type inequalities for a class of elliptic operators on submanifolds. Differ. Geom. Appl. **63**, 1–29 (2019)
13. Chen, H., Wei, G.: Reilly-type inequalities for *p*-Laplacian on submanifolds in space forms. Nonlinear Anal. **184**, 210–217 (2019)
14. Cheng, S.Y.: Eigenvalue comparison theorems and its geometric applications. Math. Z. **143**(3), 289–297 (1975)
15. Du, F., Mao, J.: Reilly-type inequalities for *p*-Laplacian on compact Riemannian manifolds. Front. Math. China **10**(3), 583–594 (2015)
16. Du, F., Wang, Q., Xia, C.: Estimates for eigenvalue of the Wentzel-Laplacian operator. J. Geom. Phys. **129**, 25–33 (2018)
17. He, Y.: Reilly type inequality for the first eigenvalue of the $L_{r;F}$ operator. Differ. Geom. Appl. **31**, 321–330 (2013)
18. Li, Y., Mofarreh, F., Ali, A., Abolrinwa, A.: Some eigenvalues estimate for the ϕ-Laplace operator on slant submanifolds of Sasakian space forms. J. Funct. Spaces **2021**, Art. ID 6195939 (2021)
19. Lin, H.: Eigenvalue estimate and gap theorems for submanifolds in the hyperbolic space. Nonlinear Anal. **148**, 126–137 (2017)
20. Matei, A.M.: Conformal bounds for the first eigenvalue of the *p*-Laplacian. Nonlinear Anal. **80**, 88–95 (2013)
21. Naber, A., Valtorta, D.: Sharp estimates on the first eigenvalue of the *p*-Laplacian with negative Ricci lower bound. Math. Z. **277**(3–4), 867–891 (2014)
22. Reilly, R.C.: On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space. Comment. Math. Helv. **52**(4), 525–533 (1977)
23. Sahu, D.R., Babu, F., Sharma, S.: The S-iterative techniques on Hadamard manifolds and applications. J. Appl. Numer. Optim. **2**, 353–371 (2020)
24. Sasahara, T.: A class of biminimal Legendrian submanifolds in Sasakian space forms. Math. Nachr. **287**(1), 79–90 (2014)
25. Seto, S., Wei, G.: First eigenvalue of the *p*-Laplacian under integral curvature condition. Nonlinear Anal. **163**, 60–70 (2017)
26. Valtorta, D.: Sharp estimates on the first eigenvalue of the *p*-Laplacian. Nonlinear Anal. **75**(13), 4974–4994 (2012)
27. Veron, L.: Some existence and uniqueness results for solution of some quasilinear elliptic equations on compact Riemannian manifolds. In: Differential Equation and Its Applications, Budapest, 1991, pp. 317–352 (1991)
28. Xiong, C.: Eigenvalue estimates of Reilly type in product manifolds and eigenvalue comparison of strip domains. Differ. Geom. Appl. **60**, 104–115 (2018)
29. Zeng, F., He, Q.: Reilly-type inequalities for the first eigenvalue of *p*-Laplacian of submanifolds in Minkowski spaces. Mediterr. J. Math. **14**, 218 (2017)