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Volterra integration operators from Hardy-type tent spaces to Hardy spaces

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Abstract

In this paper, we completely characterize the boundedness and compactness of the Volterra integration operators J_g acting from the Hardy-type tent spaces $\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)$ to the Hardy spaces $H^t(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$ for all $0 < p, q, t < \infty$ and $\alpha > -n - 1$. The duality and factorization techniques for tent spaces of sequences play an important role in the proof of the main results.

Keywords: Integration operator; Hardy-type tent space; Hardy space; Unit ball

1 Introduction

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$, and $\mathbb{S}_n$ the boundary of $\mathbb{B}_n$. Denote by $H(\mathbb{B}_n)$ the space of all holomorphic functions on $\mathbb{B}_n$. A function $g \in H(\mathbb{B}_n)$ induces an integration operator (or a Volterra operator) J_g given by the formula:

$$J_g f(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}, \quad z \in \mathbb{B}_n,$$

where f is holomorphic on $\mathbb{B}_n$ and Rg is the radial derivative of g, that is,

$$Rg(z) = \sum_{k=1}^{n} z_k \frac{\partial g}{\partial z_k}(z), \quad z = (z_1, \dots, z_n) \in \mathbb{B}_n.$$

In the one-dimensional case $n = 1$, the operator J_g was first studied in the setting of the Hardy spaces by Pommerenke [22] related to the functions of bounded mean oscillation. Some important papers include the pioneering works of Aleman, Cima and Siskakis [3, 5, 6], where they described the boundedness of the operators J_g acting on Hardy and Bergman spaces in the unit disk. Since then, much research on the Volterra operator J_g acting on many spaces of holomorphic functions has been carried out (see [2, 4, 10, 24] for example). The higher-dimensional variant of J_g was introduced by Hu [12]. A fundamental property of the operator J_g is the following basic formula involving the radial derivative R and the operator J_g:

$$R(J_g f)(z) = f(z) Rg(z), \quad z \in \mathbb{B}_n.$$

The boundedness and compactness of J_g have been extensively studied in many spaces of holomorphic functions in the unit ball (see [20] for the corresponding study between Hardy spaces, and [9, 19] from Bergman spaces to Hardy spaces, and others [16, 23, 25] for example).

For $0 < t < \infty$, the Hardy space $H^t(\mathbb{B}_n)$ consists of those holomorphic functions f on $\mathbb{B}_n$ with

$$\|f\|_{H^t(\mathbb{B}_n)}^t = \sup_{0<r<1} \int_{\mathbb{S}_n} \left|f(r\xi)\right|^t \, d\sigma(\xi) < \infty,$$

where $d\sigma$ is the surface measure on the unit sphere $\mathbb{S}_n := \partial\mathbb{B}_n$ normalized so that $\sigma(\mathbb{S}_n) = 1$.

For $0 < p, q < \infty$ and $\alpha > -n-1$, the weighted tent space $\mathcal{T}^p_{q,\alpha}(\mathbb{B}_n)$ consists of all measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|^p_{\mathcal{T}^p_{q,\alpha}(\mathbb{B}_n)} = \int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)} \left|f(z)\right|^q \left(1-|z|^2\right)^\alpha \, dv(z) \right)^{\frac{p}{q}} d\sigma(\xi) < \infty,$$

where dv is the volume measure on $\mathbb{B}_n$ normalized so that $v(\mathbb{B}_n) = 1$, and $\Gamma(\xi) = \{z \in \mathbb{B}_n : |1 - \langle z, \xi\rangle| < (1-|z|^2)\}$ is the admissible approach region. In particular, for $\alpha = 0$, we write $\mathcal{T}^p_q(\mathbb{B}_n)$ instead of $\mathcal{T}^p_{q,\alpha}(\mathbb{B}_n)$.

Analogously, $\mathcal{T}^p_\infty(\mathbb{B}_n)$ consists of all measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|^p_{\mathcal{T}^p_\infty(\mathbb{B}_n)} = \int_{\mathbb{S}_n} \left(\operatorname*{ess\,sup}_{z\in\Gamma(\xi)} \left|f(z)\right| \right)^p d\sigma(\xi) < \infty,$$

and $\mathcal{T}^\infty_{q,\alpha}(\mathbb{B}_n)$ consists of measurable functions f with

$$\|f\|_{\mathcal{T}^\infty_{q,\alpha}(\mathbb{B}_n)} = \operatorname*{ess\,sup}_{\xi\in\mathbb{S}_n} \left(\sup_{w\in\Gamma(\xi)} \frac{1}{(1-|w|^2)^n} \int_{Q(w)} \left|f(z)\right|^q \left(1-|z|^2\right)^{n+\alpha} dv(z) \right)^{\frac{1}{q}} < \infty,$$

where $Q(w) = \{z \in \mathbb{B}_n : |1 - \langle z, \frac{w}{|w|}\rangle| < 1-|w|^2\}$ for $w \in \mathbb{B}_n \backslash \{0\}$ and $Q(0) = \mathbb{B}_n$.

For $0 < p, q < \infty$ and $\alpha > -n-1$, the Hardy-type tent space $\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)$ consists of holomorphic functions on $\mathbb{B}_n$ that also belong to $\mathcal{T}^p_{q,\alpha}(\mathbb{B}_n)$, with the same quasinorm, and $\mathcal{HT}^p_\infty(\mathbb{B}_n)$ consists of holomorphic functions on $\mathbb{B}_n$ that also belong to $\mathcal{T}^p_\infty(\mathbb{B}_n)$. The space $\mathcal{CT}_{q,\alpha}(\mathbb{B}_n)$ consists of those holomorphic functions that belong to $\mathcal{T}^\infty_{q,\alpha}(\mathbb{B}_n)$ that is endowed with the same norm. We refer the reader to [21] for more details on Hardy-type tent spaces.

As useful tools, tent spaces play important roles in the study of harmonic analysis and partial differential equations. By the nontangential maximal function characterization of the Hardy space, $\mathcal{HT}^p_\infty(\mathbb{B}_n) = H^p(\mathbb{B}_n) \subseteq \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)$, see [26], and we can consider $H^p(\mathbb{B}_n)$ as the limit of $\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)$ when $q \to \infty$. Hence, we describe the boundedness and compactness of $J_g : \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \to H^t(\mathbb{B}_n)$ for all possible ranges $0 < p, q, t < \infty$ and $\alpha > -n-1$. Although only discrete characterizations are described in our theorems, continuous characterizations also can be obtained from subsequent proofs.

Our main results are as follows.

Theorem 1.1 *Let $0 < p, q, t < \infty$, $\alpha > -n-1$. Then, the integration operator $J_g : \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \to H^t(\mathbb{B}_n)$ is bounded if and only if for any $r \in (0,1)$ and an r-lattice $Z = \{a_k\}$*

in $\mathbb{B}_n$, the sequence

$$u = \{u_k\} = \left\{ \left| Rg(a_k) \right| \left(1 - |a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}} \right\}$$

satisfies one of the following conditions:

(a) *If $p > t$ and $q > 2$, then u belongs to $T^{\frac{pt}{p-t}}_{\frac{2q}{q-2}}(Z)$.*

(b) *If $p > t$ and $q \le 2$, then u belongs to $T^{\frac{pt}{p-t}}_{\infty}(Z)$.*

(c) *If $p = t$ and $q > 2$, then u belongs to $T^{\infty}_{\frac{2q}{q-2}}(Z)$.*

(d) *If $p = t$ and $q \le 2$ or $p < t$, then $\{u_k \cdot (1 - |a_k|^2)^{n(\frac{1}{t}-\frac{1}{p})}\}$ belongs to l^∞.*

Theorem 1.2 *Let $0 < p, q, t < \infty$, $\alpha > -n - 1$. Then, the integration operator $J_g : \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \to H^t(\mathbb{B}_n)$ is compact if and only if for any $r \in (0,1)$ and an r-lattice $Z = \{a_k\}$ in $\mathbb{B}_n$, the sequence*

$$u = \{u_k\} = \left\{ \left| Rg(a_k) \right| \left(1 - |a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}} \right\}$$

satisfies one of the following conditions:

(a) *If $p > t$ and $q > 2$, then*

$$\int_{\mathbb{S}_n} \left(\sup_{a_k \in \Gamma(\xi)} \left| Rg(a_k) \right|^{\frac{2q}{q-2}} \left(1 - |a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q} \cdot \frac{2q}{q-2}} \right)^{\frac{pt}{p-t} \cdot \frac{q-2}{2q}} d\sigma(\xi) < \infty.$$

(b) *If $p > t$ and $q \le 2$, then*

$$\lim_{\rho \to 1^-} \int_{\mathbb{S}_n} \left(\sup_{a_k \in \Gamma(\xi) \setminus \overline{D(0,\rho)}} \left| Rg(a_k) \right| \left(1 - |a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}} \right)^{\frac{pt}{p-t}} d\sigma(\xi) = 0.$$

(c) *If $p = t$ and $q > 2$, then*

$$\lim_{|w| \to 1^-} \frac{1}{(1 - |w|^2)^n} \sum_{a_k \in Q(w)} \left(\left| Rg(a_k) \right| \left(1 - |a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}} \right)^{\frac{2q}{q-2}} \left(1 - |a_k|^2\right)^n = 0.$$

(d) *If $p = t$ and $q \le 2$ or $p < t$, then*

$$\lim_{k \to \infty} \left| Rg(a_k) \right| \left(1 - |a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q} + n(\frac{1}{t} - \frac{1}{p})} = 0.$$

This paper is organized as follows: Sect. 2 contains some background materials and the tools used in the proofs. Theorems 1.1 and 1.2 are proved in Sect. 3 and Sect. 4, respectively.

Throughout the paper, constants are used with no attempt to calculate their exact values, and the value of a constant C may change from one occurrence to the next. We also use the notion $A \lesssim B$ to indicate that there is a constant $C > 0$ with $A \le CB$. The converse relation $A \gtrsim B$ is defined in an analogous manner, and if $A \lesssim B$ and $A \gtrsim B$ both hold, we write $A \asymp B$. Given $p \in [1, \infty]$, we will denote by $p' = p/(p-1)$ its Hölder conjugate, and we agree that $1' = \infty$ and $\infty' = 1$ in this paper.

2 Preliminaries

In this section, we introduce some basic results that will be used for the proofs of our main theorems.

2.1 Area methods and equivalent norms

For $\xi \in \mathbb{S}_n$ and $\gamma > 1$, the admissible approach region $\Gamma_\gamma(\xi)$ is defined as

$$\Gamma_\gamma(\xi) = \left\{ z \in \mathbb{B}_n : \left|1 - \langle z, \xi\rangle\right| < \frac{\gamma}{2}\left(1 - |z|^2\right)\right\}.$$

In this paper we agree that $\Gamma(\xi) := \Gamma_2(\xi)$. It is known that for every $\delta > 1$ and $\gamma > 1$, there exists $\gamma' > 1$ so that

$$\bigcup_{z \in \Gamma_\gamma(\xi)} D(z, \delta) \subset \Gamma_{\gamma'}(\xi).$$

We will write $\widetilde{\Gamma}(\xi)$ to indicate this change of aperture. Given $z \in \mathbb{B}_n$, we can define the set $I(z) = \mathbb{S}_n$ for $z = 0$, and $I(z) = \{\xi \in \mathbb{S}_n : z \in \Gamma(\xi)\} \subset \mathbb{S}_n$ for $z \neq 0$. Obviously, $\sigma(I(z)) \asymp (1 - |z|^2)^n$, and it follows from Fubini's theorem that, for a positive measurable function φ, and a finite positive measure ν, one has

$$\int_{\mathbb{B}_n} \varphi(z)\, d\nu(z) \asymp \int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)} \varphi(z) \frac{d\nu(z)}{(1 - |z|^2)^n} \right) d\sigma(\xi).$$

We will need the following well-known Calderón's area theorem [8], which will be very important for our arguments, and the variant can be found in [1, 20].

Lemma A *Let* $0 < t < \infty$. *If* $f \in H(\mathbb{B}_n)$ *and* $f(0) = 0$, *then*

$$\|f\|_{H^t}^t \asymp \int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)} \left|Rf(z)\right|^2 \left(1 - |z|^2\right)^{1-n} d\nu(z) \right)^{t/2} d\sigma(\xi).$$

Note that Lemma A shows that $f \in H(\mathbb{B}_n)$ belongs to H^t if and only if $Rf \in \mathcal{HT}^t_{2,1-n}$. This explains the special role of number 2 in Theorem 1.1 and Theorem 1.2.

2.2 Embedding theorems

We need the following embedding theorems for Hardy-type tent spaces, which are the generalizations of Lemma 15 and Lemma 23 in [21]. We prove them by a similar method.

Lemma B *Let* $0 < t \le p < \infty$, $0 < q \le s < \infty$, $\alpha > -n - 1$, *and* $\beta = \alpha + (\frac{s}{q} - 1)(n + 1 + \alpha)$. *Then,*

$$\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \subset \mathcal{HT}^t_{s,\beta}(\mathbb{B}_n),$$

with bounded inclusion.

Proof Let $\xi \in \mathbb{S}_n$ and $r > 0$. For any $z \in \Gamma(\xi)$ and $f \in H(\mathbb{B}_n)$, by the subharmonicity, we have

$$\begin{aligned}|f(z)| &\lesssim \frac{1}{(1-|z|^2)^{\frac{n+1+\alpha}{q}}}\left(\int_{D(z,r)}|f(\omega)|^q(1-|\omega|^2)^\alpha\,dv(\omega)\right)^{\frac{1}{q}}\\ &\lesssim \frac{1}{(1-|z|^2)^{\frac{n+1+\alpha}{q}}}\left(\int_{\widetilde{\Gamma}(\xi)}|f(\omega)|^q(1-|\omega|^2)^\alpha\,dv(\omega)\right)^{\frac{1}{q}}.\end{aligned}$$

Writing $|f|^s = |f|^q|f|^{s-q}$ and applying this estimate to the second factor gives

$$\begin{aligned}&\int_{\Gamma(\xi)}|f(z)|^s(1-|z|^2)^\beta\,dv(z)\\ &\quad\lesssim \int_{\Gamma(\xi)}|f(z)|^q(1-|z|^2)^\alpha\left(\int_{\widetilde{\Gamma}(\xi)}|f(\omega)|^q(1-|\omega|^2)^\alpha\,dv(\omega)\right)^{s/q-1}dv(z)\\ &\quad\lesssim \left(\int_{\widetilde{\Gamma}(\xi)}|f(z)|^q(1-|z|^2)^\alpha\,dv(z)\right)^{\frac{s}{q}}.\end{aligned}$$

Then, for $t \le p$, we obtain $\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \subset \mathcal{HT}^p_{s,\beta}(\mathbb{B}_n) \subset \mathcal{HT}^t_{s,\beta}(\mathbb{B}_n)$. □

Lemma C *If* $0 < p < t < \infty$, $0 < q < \infty$ *and* $\alpha > -n-1$, *then*

$$\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \subset A^t_\eta(\mathbb{B}_n)$$

with bounded inclusion, where $A^t_\eta(\mathbb{B}_n)$ *is the weighted Bergman space and* $\eta = (\frac{t}{p}-1)n - 1 + \frac{t(n+1+\alpha)}{q}$.

Proof First, recall that if $p < t$, then $H^p(\mathbb{B}_n) \subset A^t_{(\frac{t}{p}-1)n-1}(\mathbb{B}_n)$ with bounded inclusion. Applying this to a fractional differential operator $\mathcal{R}^{s,\frac{n+1+\alpha}{2}}$ and according to [21, Theorem G], we have

$$\mathcal{HT}^p_{2,\alpha}(\mathbb{B}_n) \subset A^t_{(\frac{t}{p}-1)n-1+\frac{t(n+1+\alpha)}{2}}(\mathbb{B}_n).$$

For any natural number k, we have $f \in \mathcal{HT}^p_{2k,\alpha}(\mathbb{B}_n)$ if and only if $f^k \in \mathcal{HT}^{\frac{p}{k}}_{2,\alpha}(\mathbb{B}_n)$, and then

$$\mathcal{HT}^p_{2k,\alpha}(\mathbb{B}_n) \subset A^t_{(\frac{t}{p}-1)n-1+\frac{t(n+1+\alpha)}{2k}}(\mathbb{B}_n).$$

Let k be large enough such that $2k > q$. Then, by Lemma B, we have

$$\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \subset \mathcal{HT}^p_{2k,\alpha+(\frac{2k}{q}-1)(n+1+\alpha)}(\mathbb{B}_n) \subset A^t_\eta(\mathbb{B}_n).$$ □

We will also need the following Dirichlet-type embedding theorem, which can be found in [7].

Lemma D *Assume that $f \in H(\mathbb{B}_n)$ with $f(0) = 0$. If $0 < p < q < \infty$, then*

$$\|f\|_{H^q(\mathbb{B}_n)} \lesssim \|Rf\|_{A^p_{p-n-1+np/q}(\mathbb{B}_n)},$$

where $\|f\|^p_{A^p_\alpha(\mathbb{B}_n)} = \int_{\mathbb{B}_n} |f(z)|^p (1-|z|^2)^\alpha \, dv(z)$.

2.3 Khinchine and Kahane inequalities

Let $r_k(u)$ be a sequence of Rademacher functions. We recall first the classical Khinchine's inequality (see [11, Appendix A] for example).

Khinchine's inequality: Let $0 < p < \infty$. Then, for any sequence $\{c_k\} \in l^2$, we have

$$\left(\sum_k |c_k|^2\right)^{p/2} \asymp \int_0^1 \left|\sum_k c_k r_k(u)\right|^p dt.$$

The next result is known as Kahane's inequality, see for instance Lemma 5 of Luecking [18].

Kahane's inequality: Let X be a Banach space, and $0 < p, q < \infty$. For any sequence $\{x_k\} \subset X$, one has

$$\left(\int_0^1 \left\|\sum_k r_k(u) x_k\right\|_X^q dt\right)^{1/q} \asymp \left(\int_0^1 \left\|\sum_k r_k(u) x_k\right\|_X^p dt\right)^{1/p}.$$

2.4 Separated sequences and lattices

A sequence of points $\{z_j\} \subset \mathbb{B}_n$ is said to be separated if there exists $\delta > 0$ such that $\beta(z_i, z_j) \geq \delta$ for all i and j with $i \neq j$, where $\beta(z, w)$ denotes the Bergman metric on $\mathbb{B}_n$. This implies that there is $\delta > 0$ such that the Bergman metric balls $D_j = \{z \in \mathbb{B}_n : \beta(z, z_j) < \delta\}$ are pairwise disjoint.

We need a well-known result on decomposition of the unit ball $\mathbb{B}_n$. By Theorem 2.23 in [26], there exists a positive integer N such that for any $0 < r < 1$ we can find a sequence $\{a_k\}$ in $\mathbb{B}_n$ with the following properties:

(i) $\mathbb{B}_n = \bigcup_k D(a_k, r)$.
(ii) The sets $D(a_k, r/4)$ are mutually disjoint.
(iii) Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 4r)$.

Any sequence $\{a_k\}$ satisfying the above conditions is called an r-lattice (in the Bergman metric). Obviously any r-lattice is a separated sequence.

2.5 Tent spaces of sequences

Let $Z = \{a_k\}$ be an r-lattice. We consider the complex-valued sequences enumerated by this lattice: $\lambda_k = f(a_k)$. For $0 < p, q < \infty$, the tent space $T^p_q(Z)$ consists of those sequences $\lambda = \{\lambda_k\}$ satisfying

$$\|\lambda\|_{T^p_q(Z)} = \left(\int_{\mathbb{S}_n} \left(\sum_{a_k \in \Gamma(\xi)} |\lambda_k|^q\right)^{\frac{p}{q}} d\sigma(\xi)\right)^{\frac{1}{p}} < \infty.$$

Analogously, the tent space $T^p_\infty(Z)$ consists of λ with

$$\|\lambda\|_{T^p_\infty(Z)} = \left(\int_{\mathbb{S}_n} \sup_{a_k \in \Gamma(\xi)} |\lambda_k|^p \, d\sigma(\xi)\right)^{\frac{1}{p}} < \infty.$$

Another tent space $T_q^\infty(Z)$ consists of λ such that

$$\|\lambda\|_{T_q^\infty(Z)} = \operatorname*{ess\,sup}_{\xi\in\mathbb{S}_n}\left(\sup_{w\in\Gamma(\xi)}\frac{1}{(1-|w|^2)^n}\sum_{a_k\in Q(w)}|\lambda_k|^q\big(1-|a_k|^2\big)^n\right)^{\frac{1}{q}} < \infty.$$

We will need the following duality results for the tent spaces of sequences. The proof can be found in [13, 14, 17].

Lemma E *Let $1 \le p < \infty$ and $Z = \{a_k\}$ be an r-lattice. If $1 < q < \infty$, then the dual of $T_q^p(Z)$ is isomorphic to $T_{q'}^{p'}(Z)$ under the pairing*

$$\langle c, d\rangle_{T_2^2(Z)} = \sum_k c_k\overline{d_k}\big(1-|a_k|^2\big)^n, \quad c = \{c_k\} \in T_q^p(Z), d = \{d_k\} \in T_{q'}^{p'}(Z).$$

If $0 < q \le 1$, then the dual of $T_q^p(Z)$ is isomorphic to $T_\infty^{p'}(Z)$ under the pairing above.

The following result originates from [20], which will be used to construct our test functions.

Lemma F *Let $0 < p, q < \infty$ and $Z = \{a_k\}$ be an r-lattice. If $\theta > n\max(1, \frac{q}{p}, \frac{1}{p}, \frac{1}{q})$, then the operator*

$$S_Z\{\lambda_k\}(z) = \sum_{k=1}^\infty \lambda_k\frac{(1-|a_k|^2)^\theta}{(1-\langle z, a_k\rangle)^{\theta+\frac{n+1+\alpha}{q}}}$$

is bounded from $T_q^p(Z)$ to $\mathcal{HT}_{q,\alpha}^p(\mathbb{B}_n)$.

We will also need the following result concerning factorization of sequence tent spaces, which can be found in [19].

Theorem G *Let $0 < p, q < \infty$ and $Z = \{a_k\}$ be a δ-lattice. If $p < p_1, p_2 < \infty$, $q < q_1, q_2 < \infty$ and satisfying*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q},$$

then

$$T_q^p(Z) = T_{q_1}^{p_1}(Z) \cdot T_{q_2}^{p_2}(Z).$$

2.6 Discretization

We will use Khinchine's and Kahane's inequalities throughout the proof of our main results. These tools provide discrete version of the conditions we really need, hence, we need to obtain the continuous characterizations from the discrete ones. The following two results can be found in [19].

Lemma H *Let $0 < p,q < \infty$ and $\alpha > -n-1$. There exist $r_0 \in (0,1)$ so that if $0 < r < r_0$ and $Z = \{a_k\}$ is an r-lattice, then*

$$\begin{aligned}&\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^q\left(1-|z|^2\right)^\alpha\,dv(z)\right)^{p/q}d\sigma(\xi)\\&\quad\lesssim\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}\left|f(a_k)\right|^q\left(1-|a_k|^2\right)^{n+1+\alpha}\right)^{p/q}d\sigma(\xi),\end{aligned}$$

whenever f is holomorphic on $\mathbb{B}_n$ and in $\mathrm{T}^p_{q,\alpha}$.

Lemma I *Let $0 < p < \infty$ and $\alpha \geq 0$. There exist $r_0 \in (0,1)$ so that if $0 < r < r_0$ and $Z = \{a_k\}$ is an r-lattice, then*

$$\int_{\mathbb{S}_n}\sup_{z\in\Gamma(\xi)}\left|f(z)\right|^p\left(1-|z|^2\right)^\alpha\,d\sigma(\xi)\lesssim\int_{\mathbb{S}_n}\sup_{a_k\in\Gamma(\xi)}\left|f(a_k)\right|^p\left(1-|a_k|^2\right)^\alpha\,d\sigma(\xi),$$

whenever f is holomorphic on $\mathbb{B}_n$ such that the left-hand side is finite.

We also need the following similar result.

Lemma J *Let $0 < p < \infty$, and $\alpha > -n-1$, $\beta > 0$. There exist $r_0 \in (0,1)$ so that if $0 < r < r_0$ and $Z = \{a_k\}$ is an r-lattice, then for any $a \in \mathbb{B}_n$, we have*

$$\begin{aligned}&\int_{\mathbb{B}_n}\frac{(1-|a|^2)^\beta}{|1-\langle a,z\rangle|^{n+\beta}}\left|f(z)\right|^p\left(1-|z|^2\right)^\alpha\,dv(z)\\&\quad\lesssim\sum_k\frac{(1-|a|^2)^\beta}{|1-\langle a,a_k\rangle|^{n+\beta}}\left|f(a_k)\right|^p\left(1-|a_k|^2\right)^{n+1+\alpha},\end{aligned}$$

whenever f is holomorphic on $\mathbb{B}_n$ such that the left-hand side is finite.

Proof For any $a \in \mathbb{B}_n$ and $\beta > 0$, note that

$$\begin{aligned}&\int_{\mathbb{B}_n}\frac{(1-|a|^2)^\beta}{|1-\langle a,z\rangle|^{n+\beta}}\left|f(z)\right|^p\left(1-|z|^2\right)^\alpha\,dv(z)\\&\quad\asymp\sum_k\int_{D(a_k,r)}\frac{(1-|a|^2)^\beta}{|1-\langle a,z\rangle|^{n+\beta}}\left|f(z)\right|^p\left(1-|z|^2\right)^\alpha\,dv(z)\\&\quad\lesssim\sum_k\int_{D(a_k,r)}\frac{(1-|a|^2)^\beta}{|1-\langle a,z\rangle|^{n+\beta}}\left|f(z)-f(a_k)\right|^p\left(1-|z|^2\right)^\alpha\,dv(z)\\&\qquad+\sum_k\int_{D(a_k,r)}\frac{(1-|a|^2)^\beta}{|1-\langle a,z\rangle|^{n+\beta}}\left|f(a_k)\right|^p\left(1-|z|^2\right)^\alpha\,dv(z).\end{aligned}$$

By [15, Lemma 2.2], there exist $r_0 \in (r, 4r)$, such that for any $z \in D(a_k, r)$,

$$\left|f(z)-f(a_k)\right|^p\lesssim\frac{r^p}{(1-|a_k|^2)^{n+1}}\int_{D(a_k,r_0)}\left|f(\omega)\right|^p\,dv(\omega).$$

Thus, we deduce that

$$\begin{aligned}
&\int_{\mathbb{B}_n} \frac{(1-|a|^2)^\beta}{|1-\langle a,z\rangle|^{n+\beta}} \big|f(z)\big|^p \big(1-|z|^2\big)^\alpha \, dv(z) \\
&\lesssim r^p \sum_k \int_{D(a_k,r)} \frac{(1-|a|^2)^\beta}{|1-\langle a,z\rangle|^{n+\beta}} \frac{1}{(1-|a_k|^2)^{n+1}} \int_{D(a_k,r_0)} \big|f(\omega)\big|^p \, dv(\omega) \big(1-|z|^2\big)^\alpha \, dv(z) \\
&\quad + \sum_k \frac{(1-|a|^2)^\beta}{|1-\langle a,a_k\rangle|^{n+\beta}} \big|f(a_k)\big|^p \big(1-|a_k|^2\big)^{n+1+\alpha} \\
&\lesssim r^p \sum_k \int_{D(a_k,4r)} \frac{(1-|a|^2)^\beta}{|1-\langle a,\omega\rangle|^{n+\beta}} \big|f(\omega)\big|^p \big(1-|\omega|^2\big)^\alpha \, dv(\omega) \\
&\quad + \sum_k \frac{(1-|a|^2)^\beta}{|1-\langle a,a_k\rangle|^{n+\beta}} \big|f(a_k)\big|^p \big(1-|a_k|^2\big)^{n+1+\alpha} \\
&\lesssim r^p \int_{\mathbb{B}_n} \frac{(1-|a|^2)^\beta}{|1-\langle a,\omega\rangle|^{n+\beta}} \big|f(\omega)\big|^p \big(1-|\omega|^2\big)^\alpha \, dv(\omega) \\
&\quad + \sum_k \frac{(1-|a|^2)^\beta}{|1-\langle a,a_k\rangle|^{n+\beta}} \big|f(a_k)\big|^p \big(1-|a_k|^2\big)^{n+1+\alpha}.
\end{aligned}$$

Since the constants in "$\lesssim$" do not depend on r, we can find the desired r_0, which completes the proof. □

3 Proof of Theorem 1.1

3.1 Necessity

Suppose that the integration operator $J_g : \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \to H^t(\mathbb{B}_n)$ is bounded. We consider first the case $p = t$, $q \le 2$ or $p < t$. In this case, for any $a \in \mathbb{B}_n$ and $\theta > 0$, consider the test functions

$$F_a(z) = \frac{(1-|a|^2)^\theta}{(1-\langle z,a\rangle)^{\theta+\frac{n+1+\alpha}{q}+\frac{n}{p}}}, \quad z \in \mathbb{B}_n. \tag{1}$$

By the standard estimate for $H^t(\mathbb{B}_n)$ functions, we have

$$\big|Rg(z)\big|\big|F_a(z)\big| \lesssim \frac{\|J_g(F_a)\|_{H^t(\mathbb{B}_n)}}{(1-|z|^2)^{\frac{n+t}{t}}} \lesssim \|J_g\| \cdot \|F_a\|_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)} \big(1-|z|^2\big)^{-\frac{n}{t}-1}.$$

Replacing z by a in the inequality above, we obtain

$$\big|Rg(a)\big|\big(1-|a|^2\big)^{\frac{q-(n+1+\alpha)}{q}+n(\frac{1}{t}-\frac{1}{p})} \lesssim \|J_g\| < \infty.$$

In particular, we deduce that $\sup_k |Rg(a_k)|(1-|a_k|^2)^{\frac{q-(n+1+\alpha)}{q}+n(\frac{1}{t}-\frac{1}{p})} < \infty$ as desired.

Finally, it remains to deal with the other cases. Let $Z = \{a_k\}$ be an r-lattice and r be small enough. Consider the test functions

$$F_Z(z) = \sum_{k=1}^\infty \lambda_k \frac{(1-|a_k|^2)^\theta r_k(x)}{(1-\langle z,a_k\rangle)^{\theta+\frac{n+1+\alpha}{q}}},$$

where $\lambda = \{\lambda_k\} \in T_q^p(Z)$, $r_k(x)$ are the Rademacher functions, and θ is large enough such that Lemma F holds. Then, by Lemma A and Lemma F, we have

$$\begin{aligned}\|J_g(F_Z)\|_{H^t}^t &\asymp \int_{\mathbb{S}_n} \Big(\int_{\Gamma(\xi)} |R\big(J_g(F_Z)\big)|^2 \big(1-|z|^2\big)^{1-n}\, dv(z)\Big)^{t/2}\, d\sigma(\xi) \\ &\lesssim \|J_g\|^t \|F_Z\|^t_{\mathcal{H}\mathcal{T}^p_{q,\alpha}(\mathbb{B}_n)} \lesssim \|J_g\|^t \|\lambda\|^t_{T^p_q(Z)},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|Rg(z)\sum_{k=1}^{\infty}\frac{\lambda_k r_k(x)(1-|a_k|^2)^{\theta}}{(1-\langle z,a_k\rangle)^{\theta+\frac{(n+1+\alpha)}{q}}}\right|^2\big(1-|z|^2\big)^{1-n}\,dv(z)\right)^{t/2} d\sigma(\xi)\\ &\quad\lesssim \|J_g\|^t\|\lambda\|^t_{T^p_q(Z)}.\end{aligned}$$

Integrating with respect to x from 0 to 1, and using Fubini's theorem, Khinchine's inequality, and Kahane's inequality as in the proof of Theorem 7 in [19], we obtain

$$\begin{aligned}&\int_{\mathbb{S}_n}\left(\sum_{k=1}^{\infty}|\lambda_k|^2\int_{\Gamma(\xi)}\frac{(1-|a_k|^2)^{2\theta}}{|1-\langle z,a_k\rangle|^{2\theta+\frac{2(n+1+\alpha)}{q}}}\big|Rg(z)\big|^2\big(1-|z|^2\big)^{1-n}\,dv(z)\right)^{t/2} d\sigma(\xi)\\ &\quad\lesssim \|J_g\|^t\|\lambda\|^t_{T^p_q(Z)}.\end{aligned}$$

Write $u = \{u_k\}$ and $u_k = |Rg(a_k)|(1-|a_k|^2)^{\frac{q-(n+1+\alpha)}{q}}$. Using subharmonicity and bearing in mind $\bigcup_{z\in\Gamma(\xi)} D(z,4r) \subset \widetilde{\Gamma}(\xi)$, we obtain

$$\begin{aligned}&\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|\lambda_k|^2\big|Rg(a_k)\big|^2\big(1-|a_k|^2\big)^{\frac{2q-2(n+1+\alpha)}{q}}\right)^{t/2} d\sigma(\xi)\\ &\quad\lesssim \int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|\lambda_k|^2\int_{D(a_k,4r)}\big|Rg(z)\big|^2\frac{(1-|z|^2)^{1-n}(1-|a_k|^2)^{2\theta}}{|1-\langle z,a_k\rangle|^{2\theta+\frac{2(n+1+\alpha)}{q}}}\,dv(z)\right)^{t/2} d\sigma(\xi)\\ &\quad\lesssim \int_{\mathbb{S}_n}\left[\int_{\widetilde{\Gamma}(\xi)}\sum_{k=1}^{\infty}|\lambda_k|^2\frac{(1-|a_k|^2)^{2\theta}}{|1-\langle z,a_k\rangle|^{2\theta+\frac{2(n+1+\alpha)}{q}}}\big|Rg(z)\big|^2\big(1-|z|^2\big)^{1-n}\,dv(z)\right]^{t/2} d\sigma\xi)\\ &\quad\lesssim \|J_g\|^t\|\lambda\|^t_{T^p_q(Z)}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|\lambda_k|^2|u_k|^2\right)^{t/2} d\sigma(\xi) \lesssim \|J_g\|^t\|\lambda\|^t_{T^p_q(Z)}. \tag{2}$$

(a) If $p > t$ and $q > 2$, for some s large enough such that $2s > 1$ and $ts > 1$, we want to prove $u^{1/s} \in T^{\frac{pts}{p-t}}_{\frac{2qs}{q-2}}(Z)$, which is equivalent to $u \in T^{\frac{pt}{p-t}}_{\frac{2q}{q-2}}(Z)$. By the factorization result in Lemma G, we have

$$T^{\frac{pts}{p-t}}_{\frac{2qs}{q-2}}(Z) = \big(T^{\frac{pts}{pts-p+t}}_{\frac{2qs}{2qs-q+2}}(Z)\big)^* = \big(T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)\cdot T^{ps}_{qs}(Z)\big)^*.$$

Take any $\nu = \{\nu_k\} \in T^{\frac{pts}{pts-p+t}}_{\frac{2qs}{2qs-q+2}}(Z)$ and factor it as $\nu_k = \rho_k \cdot \lambda_k^{1/s}$, where $\rho = \{\rho_k\} \in T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)$, $\lambda = \{\lambda_k\} \in T^p_q(Z)$. Then, by (2) and Hölder's inequalities, we obtain

$$\begin{aligned}
&\sum_k \bigl|\nu_k u_k^{1/s}\bigr|\bigl(1-|a_k|^2\bigr)^n \\
&\asymp \int_{\mathbb{S}_n}\Biggl(\sum_{a_k\in\Gamma(\xi)} |\rho_k|\cdot|\lambda_k|^{1/s}\cdot|u_k|^{1/s}\Biggr)\,d\sigma(\xi) \\
&\lesssim \int_{\mathbb{S}_n}\Biggl(\sum_{a_k\in\Gamma(\xi)} |\rho_k|^{\frac{2s}{2s-1}}\Biggr)^{\frac{2s-1}{2s}}\Biggl(\sum_{a_k\in\Gamma(\xi)} |\lambda_k|^2|u_k|^2\Biggr)^{\frac{1}{2s}}\,d\sigma(\xi) \\
&\lesssim \Biggl(\int_{\mathbb{S}_n}\Biggl(\sum_{a_k\in\Gamma(\xi)} |\rho_k|^{\frac{2s}{2s-1}}\Biggr)^{\frac{2s-1}{2s}\cdot\frac{ts}{ts-1}}\,d\sigma(\xi)\Biggr)^{\frac{ts-1}{ts}}\Biggl(\int_{\mathbb{S}_n}\Biggl(\sum_{a_k\in\Gamma(\xi)} |\lambda_k|^2|u_k|^2\Biggr)^{\frac{t}{2}}\,d\sigma(\xi)\Biggr)^{\frac{1}{ts}} \\
&\lesssim \|\rho\|_{T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)}\|J_g\|^{1/s}\|\lambda\|^{1/s}_{T^p_q(Z)} \\
&\asymp \|J_g\|^{1/s}\|\nu\|_{T^{\frac{pts}{pts-p+t}}_{\frac{2qs}{2qs-q+2}}(Z)}.
\end{aligned}$$

By the duality of tent spaces of sequences given in Lemma E, we have that u belongs to $T^{\frac{pt}{p-t}}_{\frac{2q}{q-2}}(Z)$.

(b) If $p > t$ and $q \le 2$, it is sufficient to show that $u^{1/s} \in T^{\frac{pts}{p-t}}_{\infty}(Z)$ for some s large enough such that $2s > 1$ and $ts > 1$. By Lemma E and Lemma G, we have

$$T^{\frac{pts}{p-t}}_{\infty}(Z) = \bigl(T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)\cdot T^{ps}_{qs}(Z)\bigr)^*.$$

Note that if $q \le 2$, then $\frac{2s-1}{2s} + \frac{1}{qs} = \frac{1}{\delta}$ for some $\delta \le 1$. Thus, making some adjustments to the arguments in the proof of (a), we obtain that u belongs to $T^{\frac{pt}{p-t}}_{\infty}(Z)$.

(c) If $p = t$ and $q > 2$, it suffices to prove $u^{1/s} \in T^{\infty}_{\frac{2qs}{q-2}}(Z)$ for some s large enough such that $2s > 1$ and $ts > 1$. An appeal to Lemma G gives that

$$T^{\infty}_{\frac{2qs}{q-2}}(Z) = \bigl(T^1_{\frac{2qs}{2qs-q+2}}(Z)\bigr)^* = \bigl(T^{\frac{ps}{ps-1}}_{\frac{2s}{2s-1}}(Z)\cdot T^{ps}_{qs}(Z)\bigr)^*.$$

Proceeding with the argument as above again, we have that u belongs to $T^{\infty}_{\frac{2q}{q-2}}(Z)$, which finishes the proof of necessity.

3.2 Sufficiency

To prove the sufficiency of Theorem 1.1, we split it into four cases.

(a) If $p > t$, $q > 2$ and $u \in T^{\frac{pt}{p-t}}_{\frac{2q}{q-2}}(Z)$, let $\eta = (1 - n - \frac{2\alpha}{q})\frac{q}{q-2}$. By considering the dilated functions $Rg_\rho(z) = Rg(\rho z)$ $(0 < \rho < 1)$, an approximation argument (see [21, Lemma 7]) shows that according to Lemma H, we have

$$\begin{aligned}
&\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{\eta}\,d\nu(z)\right)^{\frac{q-2}{2q}\frac{pt}{p-t}}d\sigma(\xi)\\
&\lesssim\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}\left|Rg(a_k)\right|^{\frac{2q}{q-2}}\left(1-|a_k|^2\right)^{n+1+\eta}\right)^{\frac{q-2}{2q}\frac{pt}{p-t}}d\sigma(\xi)\\
&=\|u\|^{\frac{pt}{p-t}}_{T^{\frac{pt}{p-t}}_{\frac{2q}{q-2}}(Z)}<\infty,
\end{aligned}$$

which means $Rg\in\mathcal{HT}^{\frac{pt}{p-t}}_{\frac{2q}{q-2},\eta}(\mathbb{B}_n)$. Then, by Lemma A and Holder's inequalities, we have

$$\begin{aligned}
&\|J_gf\|^t_{H^t(\mathbb{B}_n)}\\
&\asymp\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^2\left|Rg(z)\right|^2\left(1-|z|^2\right)^{1-n}\,d\nu(z)\right)^{\frac{t}{2}}d\sigma(\xi)\\
&\lesssim\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^q\left(1-|z|^2\right)^{\alpha}\,d\nu(z)\right)^{\frac{t}{q}}\left(\int_{\Gamma(\xi)}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{\eta}\,d\nu(z)\right)^{\frac{t(q-2)}{2q}}d\sigma(\xi)\\
&\lesssim\left(\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^q\left(1-|z|^2\right)^{\alpha}\,d\nu(z)\right)^{\frac{p}{q}}d\sigma(\xi)\right)^{\frac{t}{p}}\\
&\quad\cdot\left(\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{\eta}\,d\nu(z)\right)^{\frac{t(q-2)}{2q}\frac{p}{p-t}}d\sigma(\xi)\right)^{\frac{p-t}{p}}\\
&\lesssim\|f\|^t_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)}\cdot\|Rg\|^t_{\mathcal{HT}^{\frac{pt}{p-t}}_{\frac{2q}{q-2},\eta}(\mathbb{B}_n)}.
\end{aligned}$$

(b) If $p>t$ and $q\le 2$ and $u\in T^{\frac{pt}{p-t}}_{\infty}(Z)$, define

$$U_g(\xi)=\sup_{z\in\Gamma(\xi)}\left|Rg(z)\right|\left(1-|z|^2\right)^{\frac{q-(n+1+\alpha)}{q}},\quad \xi\in\mathbb{S}_n.$$

Using the approximation argument with Lemma I, we obtain

$$\int_{\mathbb{S}_n}\left|U_g(\xi)\right|^{\frac{pt}{p-t}}d\sigma(\xi)\lesssim\int_{\mathbb{S}_n}\sup_{a_k\in\Gamma(\xi)}|u_k|^{\frac{pt}{p-t}}d\sigma(\xi)=\|u\|^{\frac{pt}{p-t}}_{T^{\frac{pt}{p-t}}_{\infty}(Z)}<\infty,$$

which means U_g belongs to $L^{\frac{pt}{p-t}}(\mathbb{S}_n)$. Let $\beta=\alpha+(\frac{2}{q}-1)(n+1+\alpha)$. Then, applying Lemma A, Hölder's inequality, and Lemma B, we have

$$\begin{aligned}
&\|J_gf\|^t_{H^t(\mathbb{B}_n)}\\
&\asymp\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^2\left|Rg(z)\right|^2\left(1-|z|^2\right)^{1-n}\,d\nu(z)\right)^{t/2}d\sigma(\xi)\\
&\lesssim\int_{\mathbb{S}_n}\sup_{z\in\Gamma(\xi)}\left|Rg(z)\right|^t\left(1-|z|^2\right)^{\frac{(1-n-\beta)t}{2}}\cdot\left(\int_{\Gamma(\xi)}\left|f(z)\right|^2\left(1-|z|^2\right)^{\beta}\,d\nu(z)\right)^{t/2}d\sigma(\xi)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^2\left(1-|z|^2\right)^{\beta}\,dv(z)\right)^{p/2}d\sigma(\xi)\right)^{t/p} \\
&\quad\cdot\left(\int_{\mathbb{S}_n}\sup_{z\in\Gamma(\xi)}\left|Rg(z)\right|^{\frac{pt}{p-t}}\left(1-|z|^2\right)^{\frac{q-(n+1+\alpha)}{q}\frac{pt}{p-t}}\,d\sigma(\xi)\right)^{\frac{p-t}{p}} \\
&= \|f\|^t_{\mathcal{HT}^p_{2,\beta}(\mathbb{B}_n)}\cdot\|\mathcal{U}_g\|^t_{L^{\frac{pt}{p-t}}(\mathbb{S}_n)}\lesssim\|f\|^t_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)}\cdot\|\mathcal{U}_g\|^t_{L^{\frac{pt}{p-t}}(\mathbb{S}_n)}.
\end{aligned}
$$

(c) If $p=t$, $q>2$ and $u\in T^{\infty}_{\frac{2q}{q-2}}(Z)$, by Lemma J, we can obtain

$$
\sup_{w\in\mathbb{B}_n}\frac{1}{(1-|w|^2)^n}\int_{Q(w)}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{\frac{q-(n+1+\alpha)}{q}\frac{2q}{q-2}-1}dv(z)\lesssim\|u\|^{\frac{2q}{q-2}}_{T^{\infty}_{\frac{2q}{q-2}}(Z)}<\infty,
$$

which means $Rg\in\mathcal{CT}_{\frac{2q}{q-2},\eta}(\mathbb{B}_n)$, where $\eta=(1-n-\frac{2\alpha}{q})\frac{q}{q-2}$. Applying the embedding theorem for Hardy spaces, we obtain that for any $\xi\in\mathbb{S}_n$,

$$
\begin{aligned}
\int_{\Gamma(\xi)}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{\eta}dv(z)&\lesssim\int_{\mathbb{B}_n}\frac{\chi_{\Gamma(\xi)}(z)}{|1-\langle z,\xi\rangle|^n}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{n+\eta}dv(z) \\
&\lesssim\|Rg\|^{\frac{2q}{q-2}}_{\mathcal{CT}_{\frac{2q}{q-2},\eta}(\mathbb{B}_n)}\sup_{0<\rho<1}\left\|\frac{\chi_{\Gamma(\xi)}(\cdot)}{(1-\langle\cdot,\xi\rangle)^n}\right\|_{L^1(\rho\mathbb{S}_n)} \\
&\lesssim\|Rg\|^{\frac{2q}{q-2}}_{\mathcal{CT}_{\frac{2q}{q-2},\eta}(\mathbb{B}_n)},
\end{aligned}
$$

where $\chi_{\Gamma(\xi)}$ is the characteristic function of $\Gamma(\xi)$. Then, Lemma A and Hölder's inequality give that

$$
\begin{aligned}
&\|J_gf\|^t_{H^t(\mathbb{B}_n)} \\
&\quad\asymp\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^2\left|Rg(z)\right|^2\left(1-|z|^2\right)^{1-n}dv(z)\right)^{\frac{t}{2}}d\sigma(\xi) \\
&\quad\lesssim\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)}\left|f(z)\right|^q\left(1-|z|^2\right)^{\alpha}dv(z)\right)^{\frac{t}{q}}\left(\int_{\Gamma(\xi)}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{\eta}dv(z)\right)^{\frac{t(q-2)}{2q}}d\sigma(\xi) \\
&\quad\lesssim\sup_{\xi\in\mathbb{S}_n}\int_{\Gamma(\xi)}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{\eta}dv(z)\cdot\|f\|^t_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)} \\
&\quad\lesssim\|Rg\|^{\frac{2q}{q-2}}_{\mathcal{CT}_{\frac{2q}{q-2},\eta}(\mathbb{B}_n)}\cdot\|f\|^t_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)}.
\end{aligned}
$$

(d) First, the case for $p=t$, $q\le 2$ is particularly simple. Indeed, in this case, Lemma B implies that $\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)\subset\mathcal{HT}^p_{2,\beta}(\mathbb{B}_n)$, where $\beta=\alpha+(\frac{2}{q}-1)(n+1+\alpha)$. Since $u_k(1-|a_k|^2)^{n(\frac{1}{t}-\frac{1}{p})}\in l^{\infty}$, we can obtain

$$
\sup_{z\in\mathbb{B}_n}\left|Rg(z)\right|\left(1-|z|^2\right)^{\frac{q-(n+1+\alpha)}{q}+n(\frac{1}{t}-\frac{1}{p})}<\infty.
$$

Then, we have

$$
\|J_gf\|^t_{H^t(\mathbb{B}_n)}\lesssim\|f\|^t_{\mathcal{HT}^t_{2,\beta}(\mathbb{B}_n)}\cdot\sup_{z\in\mathbb{B}_n}\left|Rg(z)\right|^t\left(1-|z|^2\right)^{\frac{q-(n+1+\alpha)}{q}\cdot t}\lesssim\|f\|^t_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)}.
$$

Next, for the remaining case $p < t$, there exists some r such that $p < r < t$ and denote that $\eta = (\frac{r}{p} - 1)n - 1 + \frac{r(n+1+\alpha)}{q}$. Then, according to Lemma D and Lemma C, we have

$$\begin{aligned}
\|J_g f\|_{H^t(\mathbb{B}_n)}^t &\lesssim \|R(J_g f)\|_{A^r_{r-n-1+\frac{nr}{t}}(\mathbb{B}_n)}^t \\
&= \left(\int_{\mathbb{B}_n} |f(z)|^r |Rg(z)|^r (1-|z|^2)^{r-n-1+\frac{nr}{t}}\, dv(z)\right)^{t/r} \\
&\lesssim \left(\int_{\mathbb{B}_n} |f(z)|^r (1-|z|^2)^{\eta}\, dv(z)\right)^{t/r} \cdot \sup_{z\in\mathbb{B}_n} |Rg(z)|^t (1-|z|^2)^{\frac{q-(n+1+\alpha)}{q}\cdot t + nt(\frac{1}{t}-\frac{1}{p})} \\
&\lesssim \|f\|_{A^r_\eta(\mathbb{B}_n)}^t \lesssim \|f\|_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)}^t.
\end{aligned}$$

Theorem 1.1 is now proven.

4 Proof of Theorem 1.2

4.1 Necessity

Suppose $J_g : \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \to H^t(\mathbb{B}_n)$ is compact. It is obvious that (a) holds by Theorem 1.1, so we only need to prove (b), (c), and (d). Denote

$$E = \left\{\lambda = \{\lambda_k\} \in T^p_q(Z) : \|\lambda\|_{T^p_q(Z)} = 1\right\}$$

to be the unit sphere of $T^p_q(Z)$, and let

$$S_Z(\lambda)(z) = \sum_{k=1}^{\infty} \lambda_k \frac{(1-|a_k|^2)^\theta}{(1-\langle z, a_k\rangle)^{\theta+\frac{n+1+\alpha}{q}}}, \quad z \in \mathbb{B}_n$$

be the bounded operator defined in Lemma F, where $Z = \{a_k\}$ is an r-lattice and r is small enough. Since $S_Z(E)$ is a bounded set and J_g is compact, the set $J_g \circ S_Z(E)$ is relatively compact in $H^t(\mathbb{B}_n)$. It is well known that a relatively compact set must be a totally bounded set, and then for any $\varepsilon > 0$, there exist a finite number of functions $h_1, \dots, h_N$, such that $J_g \circ S_Z(E) \subset \bigcup_{i=1}^N B(h_i, \frac{\varepsilon}{2})$, where $B(h, \frac{\varepsilon}{2}) := \{f \in J_g \circ S_Z(E) : \|f - h\|_{H^t(\mathbb{B}_n)} < \frac{\varepsilon}{2}\}$. Observing that $\sup_{i=1,\dots,N} \|h_i\|_{H^t(\mathbb{B}_n)} < \infty$, for the above $\varepsilon > 0$, there exists $\rho_0 \in (0,1)$ such that

$$\sup_{i=1,\dots,N} \left(\int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho)}} |Rh_i(z)|^2 (1-|z|^2)^{1-n}\, dv(z)\right)^{t/2} d\sigma(\xi)\right)^{1/t} < \frac{\varepsilon}{2}$$

whenever $\rho > \rho_0$. Thus, for any $\lambda \in E$, there exists some $i_0 \in \{1, \dots, N\}$ such that $J_g \circ S_Z(\lambda) \in B(h_{i_0}, \frac{\varepsilon}{2})$, and we can deduce that

$$\begin{aligned}
&\left(\int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho)}} |Rg(z)S_Z(\lambda)(z)|^2 (1-|z|^2)^{1-n}\, dv(z)\right)^{t/2} d\sigma(\xi)\right)^{1/t} \\
&\quad\lesssim \left(\int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho)}} |Rg(z)S_Z(\lambda)(z) - Rh_{i_0}(z)|^2 (1-|z|^2)^{1-n}\, dv(z)\right)^{t/2} d\sigma(\xi)\right)^{1/t} \\
&\qquad + \left(\int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho)}} |Rh_{i_0}(z)|^2 (1-|z|^2)^{1-n}\, dv(z)\right)^{t/2} d\sigma(\xi)\right)^{1/t} \\
&\quad\lesssim \|J_g \circ S_Z(\lambda) - h_{i_0}\|_{H^t(\mathbb{B}_n)} + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

whenever $\rho > \rho_0$, which is the same as

$$\int_{\mathbb{S}_n}\left(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho)}}\left|\sum_{k=1}^{\infty}\lambda_k\frac{(1-|a_k|^2)^{\theta}}{(1-\langle z,a_k\rangle)^{\theta+\frac{n+1+\alpha}{q}}}\right|^2\left|Rg(z)\right|^2\left(1-|z|^2\right)^{1-n}\,dv(z)\right)^{t/2}d\sigma(\xi)$$
$$\lesssim \varepsilon^t\|\lambda\|^t_{T^p_q(Z)}$$

for any $\lambda \in T^p_q(Z)$ and $\rho > \rho_0$. Let $r_k(x)$ be the Rademacher functions. Replacing λ_k by $\lambda_k r_k(x)$, and utilizing the same method as in the proof of the corresponding case in Theorem 1.1, we obtain that

$$\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|\lambda_k|^2\left|Rg(a_k)\right|^2\left(1-|a_k|^2\right)^{\frac{2q-2(n+1+\alpha)}{q}}\cdot\chi_{\{|z|\geq\rho\}}(a_k)\right)^{t/2}d\sigma(\xi)\lesssim\varepsilon^t\|\lambda\|^t_{T^p_q(Z)}$$

for $\rho > \rho_0' := \inf\{|a_k| : D(a_k,\delta)\subset\{|z|\geq\rho_0\}\}$, where $\chi_{\{|z|\geq\rho\}}$ is the characteristic function. Denote

$$u_\rho=\{u_{\rho,k}\}=\left\{\left|Rg(a_k)\right|\left(1-|a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}}\cdot\chi_{\{|z|\geq\rho\}}(a_k)\right\}.$$

Then, we have

$$\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|\lambda_k|^2|u_{\rho,k}|^2\right)^{t/2}d\sigma(\xi)\lesssim\varepsilon^t\|\lambda\|^t_{T^p_q(Z)}\quad\text{for any }\rho>\rho_0'. \tag{3}$$

(b) If $p > t$ and $q \leq 2$, applying the duality and factorization of sequence tent spaces as in the proof of Theorem 1.1, we can obtain the desired result. To this end, it is sufficient to prove that for some s large enough such that $2s > 1$ and $ts > 1$, $\|u_\rho^{1/s}\|_{T_\infty^{\frac{pts}{p-t}}(Z)}\lesssim\varepsilon^t$ whenever $\rho > \rho_0'$, i.e.,

$$\sup_{\rho>\rho_0'}\left(\int_{\mathbb{S}_n}\sup_{a_k\in\Gamma(\xi)\setminus\overline{D(0,\rho)}}\left|Rg(a_k)\right|^{\frac{pt}{p-t}}\left(1-|a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}\frac{pt}{p-t}}d\sigma(\xi)\right)^{\frac{p-t}{pts}}\lesssim\varepsilon^t.$$

By Lemma E and Lemma G, we have

$$T_\infty^{\frac{pts}{p-t}}(Z)=\left(T_\delta^{\frac{pts}{pts-p+t}}(Z)\right)^*=\left(T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)\cdot T^{ps}_{qs}(Z)\right)^*.$$

Note that if $q \leq 2$, then $\frac{2s-1}{2s}+\frac{1}{qs}=\frac{1}{\delta}$ for some $\delta\leq 1$. Take $\nu=\{\nu_k\}\in T_\delta^{\frac{pts}{pts-p+t}}(Z)$ and factor it as $\nu_k = l_k\cdot\lambda_k^{1/s}$, where $l=\{l_k\}\in T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)$, $\lambda=\{\lambda_k\}\in T^p_q(Z)$. Then, using Hölder's inequalities, we obtain

$$\begin{aligned}\left|\sum_k \nu_k u_{\rho,k}^{1/s}\left(1-|a_k|^2\right)^n\right| &\lesssim \int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|l_k|\cdot|\lambda_k|^{1/s}\cdot|u_{\rho,k}|^{1/s}\right)d\sigma(\xi)\\ &\lesssim \int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|l_k|^{\frac{2s}{2s-1}}\right)^{\frac{2s-1}{2s}}\left(\sum_{a_k\in\Gamma(\xi)}|\lambda_k|^2|u_{\rho,k}|^2\right)^{\frac{1}{2s}}d\sigma(\xi)\end{aligned}$$

$$\lesssim \|l\|_{T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)}\left(\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)}|\lambda_k|^2|u_{\rho,k}|^2\right)^{\frac{t}{2}}d\sigma(\xi)\right)^{\frac{1}{ts}}.$$

Combining this with (3), we establish that

$$\left|\sum_k v_k u_{\rho,k}^{1/s}\left(1-|a_k|^2\right)^n\right| \lesssim \|l\|_{T^{\frac{ts}{ts-1}}_{\frac{2s}{2s-1}}(Z)}\varepsilon^{1/s}\|\lambda\|^{1/s}_{T^p_q(Z)}$$

whenever $\rho > \rho_0'$. Considering all possible factorizations yields

$$\left|\sum_k v_k u_{\rho,k}^{1/s}\left(1-|a_k|^2\right)^n\right| \lesssim \varepsilon^{1/s}\|v\|_{T_{\delta}^{\frac{pts}{pts-p+t}}(Z)}$$

whenever $\rho > \rho_0'$. By the duality of tent spaces of sequences given in Lemma E, we have $\|u_\rho^{1/s}\|_{T_\infty^{\frac{pts}{p-t}}(Z)} \lesssim \varepsilon^t$ whenever $\rho > \rho_0'$.

(c) If $p = t$ and $q > 2$, observing that

$$\lim_{|w|\to 1^-}\frac{1}{(1-|w|^2)^n}\sum_{a_k\in Q(w)}\left(\left|Rg(a_k)\right|\left(1-|a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}}\right)^{\frac{2q}{q-2}}\left(1-|a_k|^2\right)^n = 0,$$

is equivalent to

$$\lim_{\rho\to 1^-}\sup_{w\in\mathbb{B}_n}\frac{1}{(1-|w|^2)^n}\sum_{a_k\in Q(w)}|u_{\rho,k}|^{2q/(q-2)}\left(1-|a_k|^2\right)^n = 0,$$

it suffices to prove for some s large enough such that $2s > 1$ and $ts > 1$, $\|u_\rho^{1/s}\|_{T^\infty_{2q/(q-2)}(Z)} \lesssim \varepsilon^t$ whenever $\rho > \rho_0'$. An appeal to Lemma G gives that

$$T^\infty_{\frac{2qs}{q-2}}(Z) = \left(T^1_{\frac{2qs}{2qs-q+2}}(Z)\right)^* = \left(T^{\frac{ps}{ps-1}}_{\frac{2s}{2s-1}}(Z)\cdot T^{ps}_{qs}(Z)\right)^*.$$

Proceeding with the similar argument as above, we can obtain the desired result.

(d) If $p = t$, $q \le 2$ or $p < t$, note that $|F_a(z)| \to 0$ uniformly on any compact subsets of $\mathbb{B}_n$, as $|a| \to 1^-$, where F_a are defined in (1). The compactness of J_g implies that

$$\lim_{|a|\to 1^-}\|J_g F_a\|_{H^t} = 0.$$

By the standard pointwise estimate for the derivative of $H^t(\mathbb{B}_n)$ functions, and replacing z by a, we obtain

$$\lim_{|a|\to 1^-}\left|Rg(a)\right|\left(1-|a|^2\right)^{\frac{q-(n+1+\alpha)}{q}+n(\frac{1}{t}-\frac{1}{p})} = 0,$$

which is the same as

$$\lim_{k\to\infty}\left|Rg(a_k)\right|\left(1-|a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}+n(\frac{1}{t}-\frac{1}{p})} = 0.$$

Then, the proof of necessity is complete.

4.2 Sufficiency

To prove the compactness of $J_g : \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \to H^t(\mathbb{B}_n)$, let $\{f_k\}_{k=1}^\infty \subset \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)$ and satisfy $\sup_k \|f_k\|_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)} < \infty$. Then, $\{f_k\}$ is uniformly bounded on compact subsets of $\mathbb{B}_n$, and hence $\{f_k\}$ forms a normal family by Montel's theorem. Therefore, we can extract a subsequence $\{f_{n_k}\}_{k=1}^\infty$ that converges uniformly on compact subsets of $\mathbb{B}_n$ to a holomorphic function f. Fatou's Lemma shows that $f \in \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)$. Denote $h_k = f_{n_k} - f$, then $h_k \in \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)$. We just need to prove that $\lim_{k\to\infty} \|J_g h_k\|_{H^t} = 0$, which can yield that $J_g : \mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n) \to H^t(\mathbb{B}_n)$ is compact.

(a) If $p > t$ and $q > 2$ with

$$\int_{\mathbb{S}_n} \Big(\sup_{a_k \in \Gamma(\xi)} \big|Rg(a_k)\big|^{\frac{2q}{q-2}} \big(1 - |a_k|^2\big)^{\frac{q-(n+1+\alpha)}{q}\cdot\frac{2q}{q-2}} \Big)^{\frac{pt}{p-t}\cdot\frac{q-2}{2q}} \, d\sigma(\xi) < \infty,$$

according to the proof of Theorem 1.1, we have $Rg \in \mathcal{HT}^{\frac{pt}{p-t}}_{\frac{2q}{q-2},\eta}(\mathbb{B}_n)$, where $\eta = (1 - n - \frac{2\alpha}{q})\frac{q}{q-2}$. Thus, by the dominated convergence theorem, for any $\varepsilon > 0$, there exists $\rho_0 \in (0,1)$ such that

$$\sup_{\rho\ge\rho_0} \bigg(\int_{\mathbb{S}_n} \bigg(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho_0)}} \big|Rg(z)\big|^{\frac{2q}{q-2}} \big(1 - |z|^2\big)^{\eta} \, d\nu(z) \bigg)^{\frac{pt}{p-t}\frac{q-2}{2q}} d\sigma(\xi) \bigg)^{\frac{p-t}{pt}} < \varepsilon.$$

Observing that $|h_k(z)| \to 0$ uniformly on any compact subsets of $\mathbb{B}_n$, we can choose k_0 large enough such that $|h_k(z)| < \varepsilon$ for any $k \ge k_0$ and $|z| \le \rho_0$, and then we have

$$\begin{aligned}
&\|J_g h_k\|^t_{H^t(\mathbb{B}_n)} \\
&\quad\asymp \int_{\mathbb{S}_n} \bigg(\int_{\Gamma(\xi)\cap\{|z|\le\rho_0\}} \big|h_k(z)\big|^2 \big|Rg(z)\big|^2 \big(1 - |z|^2\big)^{1-n} \, d\nu(z) \bigg)^{t/2} d\sigma(\xi) \\
&\qquad + \int_{\mathbb{S}_n} \bigg(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho_0)}} \big|h_k(z)\big|^2 \big|Rg(z)\big|^2 \big(1 - |z|^2\big)^{1-n} \, d\nu(z) \bigg)^{t/2} d\sigma(\xi) \\
&\quad\lesssim \varepsilon^t + \|h_k\|^t_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)} \bigg(\int_{\mathbb{S}_n} \bigg(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho_0)}} \big|Rg(z)\big|^{\frac{2q}{q-2}} \big(1 - |z|^2\big)^{\eta} \, d\nu(z) \bigg)^{\frac{pt}{p-t}\frac{q-2}{2q}} d\sigma(\xi) \bigg)^{\frac{p-t}{p}} \\
&\quad\lesssim \varepsilon^t.
\end{aligned}$$

(b) If $p > t$ and $q \le 2$, the assumption

$$\lim_{\rho\to 1^-} \int_{\mathbb{S}_n} \Big(\sup_{a_k \in \Gamma(\xi)\setminus\overline{D(0,\rho)}} \big|Rg(a_k)\big| \big(1 - |a_k|^2\big)^{\frac{q-(n+1+\alpha)}{q}} \Big)^{\frac{pt}{p-t}} d\sigma(\xi) = 0$$

implies that for any $\varepsilon > 0$, there exists $\rho_0 \in (0,1)$ such that

$$\sup_{\rho\ge\rho_0} \bigg(\int_{\mathbb{S}_n} \bigg(\int_{\Gamma(\xi)\setminus\overline{D(0,\rho_0)}} \big|Rg(z)\big| \big(1 - |z|^2\big)^{\frac{q-(n+1+\alpha)}{q}} \, d\nu(z) \bigg)^{\frac{pt}{p-t}} d\sigma(\xi) \bigg)^{\frac{p-t}{pt}} < \varepsilon.$$

Choose k_0 such that $\sup_{k\geq k_0,|z|\leq\rho_0}|h_k(z)|<\varepsilon$. By a similar argument as the previous case, we have

$$\begin{aligned}&\|J_g h_k\|^t_{H^t(\mathbb{B}_n)}\\&\quad\lesssim\varepsilon^t+\|h_k\|^t_{\mathcal{HT}^p_{q,\alpha}(\mathbb{B}_n)}\left(\int_{\mathbb{S}_n}\left(\sup_{z\in\Gamma(\xi)\setminus\overline{D(0,\rho_0)}}\left|Rg(z)\right|\left(1-|z|^2\right)^{\frac{q-(n+1+\alpha)}{q}}\right)^{\frac{pt}{p-t}}d\sigma(\xi)\right)^{\frac{p-t}{p}}\\&\quad\lesssim\varepsilon^t.\end{aligned}$$

(c) If $p=t$ and $q>2$, the assumption

$$\lim_{|w|\to1^-}\frac{1}{(1-|w|^2)^n}\sum_{a_k\in Q(w)}\left(\left|Rg(a_k)\right|\left(1-|a_k|^2\right)^{\frac{q-(n+1+\alpha)}{q}}\right)^{\frac{2q}{q-2}}\left(1-|a_k|^2\right)^n=0$$

implies that

$$\lim_{\rho\to1^-}\sup_{w\in\mathbb{B}_n}\frac{1}{(1-|w|^2)^n}\int_{Q(w)\setminus\overline{D(0,\rho)}}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{n+\eta}dv(z)=0,$$

where $\eta=(1-n-\frac{2\alpha}{q})\frac{q}{q-2}$. Thus, for any $\varepsilon>0$, there exists $\rho_0\in(0,1)$ such that

$$\sup_{w\in\mathbb{B}_n,\rho\geq\rho_0}\frac{1}{(1-|w|^2)^n}\int_{Q(w)\setminus\overline{D(0,\rho)}}\left|Rg(z)\right|^{\frac{2q}{q-2}}\left(1-|z|^2\right)^{n+\eta}dv(z)<\varepsilon.$$

Then, we can obtain $\|J_g h_k\|^t_{H^t}\lesssim\varepsilon$ by a similar technique as the proof of Theorem 1.1.

(d) If $p=t$ and $q\leq2$ or $p<t$, the assumption implies that

$$\lim_{|z|\to1^-}\left|Rg(z)\right|\left(1-|z|^2\right)^{\frac{q-(n+1+\alpha)}{q}+n(\frac{1}{t}-\frac{1}{p})}=0.$$

Then, we can complete the proof of Theorem 1.2 by following the standard modifying arguments as in the proof of Theorem 1.1.

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