Al Dohiman and Ould Ahmed Mahmoud *Journal of Inequalities and Applications* (2022) 2022:105
https://doi.org/10.1186/s13660-022-02835-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Class of operators related to a (m,C)-isometric tuple of commuting operators

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Abstract

This paper is concerned with studying a new class of multivariable commuting operators know as left (m,C)-invertible p-tuples and related to a given conjugation transformation C on a Hilbert space $\mathcal{Y}$. Some structural properties of some members of this class are given.

MSC: 47B20; 47B99

Keywords: m-isometry; Left m-invertible operator; Right m-invertible operator; Left (m,C)-invertible operator; Tensor product

1 Introduction

The study of several variables of commuting operators has received great interest on the part of many researchers during recent years, and the reader is referred to the papers [3–6, 10–12, 17, 19–21, 24, 26, 27]. In this framework, our present aim in this paper is to give a new concept of multivariable operators, namely the left (m,C)-invertible p-tuple of operators. It should be noted that some developments on this subject for single variable operators have been carried out in [1, 8, 9, 13–15, 18, 22, 25, 28–30].

First, we introduce some concepts and symbols used in this work.

Let $\mathcal{B}_b[\mathcal{Y}]$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{Y}$. We use $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $\mathbb{C}$ the set of complex numbers. For $p\in\mathbb{N}$, let $\mathbf{A}=(A_1,\ldots,A_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ be a commuting p-tuple of operators ($A_j:\mathcal{Y}\longrightarrow\mathcal{Y}$ is a bounded operator). Let $\mu=(\mu_1,\ldots,\mu_p)\in\mathbb{N}_0^p$ and set $|\mu|:=\sum_{1\le j\le p}|\mu_j|$, $\mu!:=\mu_1!\cdots\mu_p!$. Further, denote by $\mathbf{A}^\mu:=A_1^{\mu_1}A_2^{\mu_2}\cdots A_p^{\mu_d}$ where $A_j^{\mu_j}=\underbrace{A_j.A_j\cdots.A_j}_{\mu_j\text{-times}}$ $(1\le j\le p)$ and $\mathbf{A}^*=(A_1^*,\ldots,A_p^*)$. Recall that an antilinear transformation $C\in\mathcal{B}_b[\mathcal{Y}]$ is a conjugation if C satisfies $\langle Cx\mid Cy\rangle=\overline{\langle x\mid y\rangle}$ $\forall x,y\in\mathcal{Y}$ and $C^2=I_{\mathcal{Y}}$ (see [16]). It should be noted that if C is a conjugation on $\mathcal{Y}$, then

$$\begin{cases}(CAC)^k=CA^kC, & \forall k\in\mathbb{N},\\ (CAC)^*=CA^*C.\end{cases}$$

Let $(A,B) \in \mathcal{B}_b[\mathcal{Y}]^2$ and define the map $\psi_{A,B} : \mathcal{B}_b[\mathcal{Y}] \longrightarrow \mathcal{B}_b[\mathcal{Y}]$ by $\psi_{A,B}(X) = BXA$. Moreover $\psi_{A,B}^{(k)}(I_{\mathcal{Y}}) = B^kA^k$ for all positive integer k, and consider the following quantity,

$$\mathbf{Q}_q(B,A) = \sum_{0\le k\le q} (-1)^{q-k} \binom{q}{k} B^kA^k, \quad q \in \mathbb{N}_0. \tag{1.1}$$

Equation (1.1) was the starting point of some authors to define classes of operators as follows:

(1) If $A \in \mathcal{B}_b[\mathcal{Y}]$ satisfies $\mathbf{Q}_m(A^*,A) = 0$ for some positive integer m,

A is said to be an m-isometric operator ([1]). If A satisfying $\mathbf{Q}_m(A^*,CAC) = 0$ for some positive integer m and a conjugation C, A is said to be an (m,C)-isometric operator ([8]).

(2) Let $A \in \mathcal{B}_b[\mathcal{Y}]$ satisfying $A^{*n}\mathbf{Q}_m(A^*,A)A^n = 0$ for some positive integers m and n, then A is called an n-quasi-m-isometric operator ([23, 27]). If A satisfies

$$A^{*n}\mathbf{Q}_m\big(A^*,CAC\big)A^n = 0,$$

for some positive integers m, n, and a conjugation C, then A is called an n-quasi-(m,C)-isometric operator ([22, 28]).

(3) Let $A \in \mathcal{B}_b[\mathcal{Y}]$ for which there exists $B \in \mathcal{B}_b[\mathcal{Y}]$ such that $\mathbf{Q}_m(B,A) = 0$ for some positive integer m, then A is called left m-invertible ([14, 15, 18, 25]). If A and B satisfy $A^{*n}\mathbf{Q}_m(B,A)A^n = 0$, for some positive integers n and, m, then A is called an n-quasileft m-invertible operator ([13]).

Very recently, the authors of the present paper introduced the concepts of left (m,C)-invertible and right (m,C)-invertible operators. Let $A \in \mathcal{B}_b[\mathcal{Y}]$ for which there exists $B \in \mathcal{B}_b[\mathcal{Y}]$ such that $\mathbf{Q}_m(B,CAC) = 0$ for some positive integer m and conjugation operator C, then A is called a left (m,C)-invertible operator. If A and B satisfy $\mathbf{Q}_m(CAC,B) = 0$, then A is called right (m,C)-invertible ([2]).

Let $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ and $\mathbf{B} = (B_1,\dots,B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ be commuting p-tuples of operators. By the same idea as in [17], we define the map $\psi_{\mathbf{A},\mathbf{B}} : \mathcal{B}_b[\mathcal{Y}] \longrightarrow \mathcal{B}_b[\mathcal{Y}]$ by $\psi_{\mathbf{A},\mathbf{B}}(X) = \sum_{1\le j\le m} B_jXA_j$. It is easy to check that

$$\psi_{\mathbf{A},\mathbf{B}}^{(k)}(I_{\mathcal{Y}}) = \sum_{|\mu|=k} \frac{k!}{\mu!}\mathbf{B}^\mu\mathbf{A}^\mu, \quad k = 0,1,\dots.$$

We set

$$\mathbf{Q}_q(\mathbf{B},\mathbf{A}) := \sum_{0\le k\le q} (-1)^{q-k} \binom{q}{k} \left(\sum_{|\mu|=k} \frac{k!}{\mu!}\mathbf{B}^\mu\mathbf{A}^\mu\right). \tag{1.2}$$

The concept of an m-isometric p-tuple of operators was introduced by Gleason et al. in [17] as follows: A p-tuple of commuting operators $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ is called an m-isometric p-tuple if $\mathbf{A}$ satisfies $\mathbf{Q}_m(\mathbf{A}^*,\mathbf{A}) = 0$ for some positive integer m. However, the concept of an (m,C)-isometric tuple was introduced by Sid Ahmed et al. in [27] as: $\mathbf{A}$ is called an (m,C)-isometric p-tuple if $\mathbf{Q}_m(\mathbf{A}^*,C\mathbf{A}C) = 0$ for some positive integer m and a conjugation C.

Recall that the concepts of left m-invertible and right m-invertible p-tuples of operators was introduced and studied by the second named author in [26]. A p-tuple of commuting operators $\mathbf{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ is called a left m-invertible p-tuple if there exists a commuting p-tuple $\mathbf{B} = (B_1, \dots, B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ such that $\mathbf{Q}_m(\mathbf{B}, \mathbf{A}) = 0$. However, if $\mathbf{Q}_m(\mathbf{A}, \mathbf{B}) = 0$, then $\mathbf{A}$ is called a right m-invertible p-tuple.

In continuation of these studies that have been carried out by many researchers, including our previous works in this field, our aim in this paper is to describe the class of left (m, C)-invertible p-tuples of commuting operators, a generalization of the class of left (m, C)-invertible of single operator on Hilbert spaces.

The outline of the paper is as follows. The second section is concerned with the main themes of the study. Namely, after several examples and interesting remarks, which try to clarify the context, we give some necessary, or even equivalent, conditions in order for a tuple of operators to be a left (m, C)-invertible p-tuple (Theorem 2.13). Section three contains the main results of the paper, namely Theorem 3.7, Theorem 3.9, and Theorem 3.11. In Theorem 3.7 we are interested if the perturbation of a left (m, C)-invertible tuple of operators by a nilpotent tuple remains a (r, C)-invertible p-tuple, where r depends on m and on the order of nilpotency. On the other hand, Theorem 3.9 proves that if $\mathbf{A}$ is a left (m, C)-invertible p-tuple and $\widetilde{\mathbf{A}}$ is a left (n, C)-invertible p-tuple, then $\mathbf{A} * \widetilde{\mathbf{A}}$ is a left $(m + n - 1, C)$-invertible p-tuple under suitable conditions. These results are used to obtain some properties on the tensor product of left (m, C)-invertible p-tuples (Corollary 3.10 and Corollary 3.10). Theorem 3.11 proves that if $\mathbf{A}$ is a left (m, C)-invertible p-tuple and $\widetilde{\mathbf{A}} = (\widetilde{A_1}, \dots, \widetilde{A_p})$ is such that each $\widetilde{A_k}$ is a left (n_k, C)-left invertible for $k = 1, \dots, p$, then $\mathbf{A} \bullet \widetilde{\mathbf{A}}$ is a left $(m + \sum_{1 \le k \le p} n_k - p, C)$-invertible p-tuple.

2 Left (m, C)-invertible tuple of commuting operators

This section is concerned with the same themes of the study. Namely, we give some properties and several examples and interesting remarks, which try to clarify the concept.

Let $\mathbf{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\mathbf{B} = (B_1, \dots, B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ and $\mathbf{D} = (D_1, \dots, D_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ be commuting p-tuples of operators, we set

$$\mathbf{Q}_m^{(l)}(\mathbf{B}, \mathbf{A}) := \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\mu| = k} \frac{k!}{\mu!} \mathbf{B}^\mu C \mathbf{A}^\mu C \right), \tag{2.1}$$

and

$$\mathbf{Q}_m^{(r)}(\mathbf{A}, \mathbf{D}) := \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\mu| = k} \frac{k!}{\mu!} C \mathbf{A}^\mu C \mathbf{D}^\mu \right). \tag{2.2}$$

Definition 2.1 Let $\mathbf{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ be a commuting p-tuple of operators. $\mathbf{A}$ is said to be a left (m, C)-invertible p-tuple if there exists a p-tuple of commuting operators $\mathbf{B} = (B_1, \dots, B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ and a conjugation C on $\mathcal{Y}$ such that

$$\sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\mu| = k} \frac{k!}{\mu!} \mathbf{B}^\mu C \mathbf{A}^\mu C \right) = 0 \tag{2.3}$$

or equivalently if $\mathbf{Q}_m^{(l)}(\mathbf{B}, \mathbf{A}) = 0$. However, $\mathbf{A} = (A_1, \dots, A_p)$ is said to be a right-(m, C)-invertible tuple if there exists a p-tuple of commuting operators $\mathbf{D} = (D_1, \dots, D_p) \in \mathcal{B}_b[\mathcal{Y}]^p$

and a conjugation C on $\mathcal{Y}$ such that

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\mu|=k}\frac{k!}{\mu!}C\mathbf{A}^{\mu}C\mathbf{D}^{\mu}\right)=0 \tag{2.4}$$

or equivalently if $\mathbf{Q}_m^{(r)}(\mathbf{A},\mathbf{D})=0$.

Remark 2.2 (1) When $p=1$ this definition coincides with the definition of a left (m,C)-invertible for a single variable operator introduced in [2].

(2) Note that if $A_jC=CA_j$ for all $j=1,\dots,p$, then $\mathbf{A}$ is a left (m,C)-invertible operator p-tuple if and only if $\mathbf{A}$ is a left-m-invertible p-tuple.

Remark 2.3 (1) Since $A_iA_j=A_jA_i$ for $i,j\in\{1,\dots,p\}$ it is easy to see that every permutation of a left (m,C)-invertible p-tuple is also a left (m,C)-invertible p-tuple.

(2) For $\mathbf{A}=(A_1,\dots,A_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ and $\mathbf{B}=(B_1,\dots,B_p)\in\mathcal{B}_b[\mathcal{H}]^p$ such that $A_iA_j=A_jA_i$ and $B_iB_j=B_jB_i$ for $i,j\in\{1,\dots,p\}$, we have

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\mu|=k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{A}^{\mu}\right)=\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\mu|=k}\frac{k!}{\mu!}\mathbf{B}^{\mu}C(C\mathbf{A}C)^{\mu}C\right).$$

From the above identity, it follows that a p-tuple $\mathbf{A}=(A_1,\dots,A_p)$ is a left (m,C)-invertible p-tuple with conjugation C if and only if $C\mathbf{A}C:=(CA_1C,\dots,CA_pC)$ is a left-(m,C)-invertible p-tuple with conjugation C.

We mention this relationship for commuting variables $y=(y_1,\dots,y_p)$ $(y_1+\cdots+y_p)^k=\sum_{|\mu|=k}\binom{k}{\mu}y^{\mu}$. In particular, we have $\sum_{|\mu|=k}\binom{k}{\mu}=p^k$.

Remark 2.4 (1) For $p=2$ and let $\mathbf{A}=(A_1,A_2)\in\mathcal{B}_b[\mathcal{H}]^2$ be a commuting pair of operators, then $\mathbf{A}$ is a left-$(1,C)$-invertible pair for some conjugation C if

$$B_1CA_1C+B_2CA_2C-I_{\mathcal{Y}}=0, \tag{2.5}$$

for some $\mathbf{B}=(B_1,B_2)\in\mathcal{B}_b[\mathcal{H}]^2$. However, it is a left $(2,C)$-invertible pair if

$$B_1^2CA_1^2C+B_2^2CA_2^2C+2B_1B_2CA_1A_2C-2(B_1CA_1C+B_2CA_2C)+I_{\mathcal{Y}}=0, \tag{2.6}$$

for some $\mathbf{B}=(B_1,B_2)\in\mathcal{B}_b[\mathcal{Y}]^2$.

(2) Let $\mathbf{A}=(A_1,\dots,A_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ be a commuting p-tuple of operators. $\mathbf{A}$ is a left $(1,C)$-invertible p-tuple if and only if

$$\sum_{1\leq j\leq p}B_jCA_jC-I_{\mathcal{Y}}=0, \tag{2.7}$$

and it is a left $(2,C)$-invertible p-tuple if and only if

$$I_{\mathcal{Y}}-2\sum_{1\leq j\leq d}B_jCA_jC+\sum_{1\leq j\leq p}B_j^2CA_j^2C+2\sum_{1\leq j<k\leq p}B_jB_kCA_jA_kC=0 \tag{2.8}$$

for some $\mathbf{B}=(B_1,\dots,B_p)\in\mathcal{B}_b[\mathcal{H}]^p$.

Example 2.5 Every (m,C)-isometric p-tuple $\mathbf{A}=(A_1,\dots,A_p)$ is a left (m,C)-invertible p-tuple and its adjoint $\mathbf{A}^*$ is a right (m,C)-invertible p-tuple.

Example 2.6 Let C be a conjugation on $\mathcal{Y}=\mathbb{C}^2$ defined by $C(z_1,z_2)=(\overline{z}_2,\overline{z}_1)$. Consider

$$A_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \sqrt{3}\\ 0 & 1\end{pmatrix}\in\mathcal{B}_b\big[\mathbb{C}^2\big]\quad\text{and}\quad A_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -\sqrt{3}\\ 0 & 1\end{pmatrix}\in\mathcal{B}_b\big[\mathbb{C}^2\big].$$

Then, $\mathbf{A}=(A_1,A_2)$ is a left-$(1,C)$-invertible 2-tuple.

Indeed, observe that $A_1A_2=A_2A_1$ and, moreover, consider

$$B_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ \sqrt{3} & 1\end{pmatrix}\in\mathcal{B}_b\big(\mathbb{C}^2\big)\quad\text{and}\quad B_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ -\sqrt{3} & 1\end{pmatrix}\in\mathcal{B}_b\big(\mathbb{C}^2\big).$$

A direct calculation shows that $B_1CA_1C=\frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ 2\sqrt{3} & 1\end{smallmatrix}\right)$ and $B_2CA_2C=\frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ -2\sqrt{3} & 1\end{smallmatrix}\right)$.

Using these equalities, we now have $Q_1^{(l)}(\mathbf{B},\mathbf{A})=B_1CA_1C+B_2CA_2C-I_{\mathbb{C}^2}=0$, and we are done.

Example 2.7 Let C be a conjugation on $\mathcal{Y}=l^2(\mathbb{C})$ defined by $Ce_k=e_k$, where $(e_k)_k$ is an orthonormal basis. Define $A_1\in\mathcal{B}_b[l^2(\mathbb{C})]$ and $A_2\in\mathcal{B}_b[l^2(\mathbb{C})]$ by

$$A_1e_k=\sqrt{\frac{k+2}{k+1}}e_{k+1}\quad\text{and}\quad A_2e_k=\sqrt{\frac{k+m}{k+1}}e_{k+1}.$$

It was explained in [7] that A_1 is a $(2,C)$-isometric operator and A_2 is a (m,C)-isometric operator. Let $\mathbf{A}=(A_1,0,\dots,0)\in\mathcal{B}_b[l^2(\mathbb{C})]^p$ and $\widetilde{\mathbf{A}}=(0,\dots,A_2)\in\mathcal{B}_b[l^2(\mathbb{C})]^p$.

By elementary calculation we show that $\mathbf{A}$ is a left $(2,C)$-invertible p-tuple with conjugation C and $\widetilde{\mathbf{A}}$ is a left (m,C)-invertible p-tuple with conjugation C.

Example 2.8 Let C be a conjugation on $\mathcal{Y}$ and $A\in\mathcal{B}_b[\mathcal{Y}]$ be a left (m,C)-invertible operator. Then, the operator tuple $\mathbf{A}=(A_1,\dots,A_p)$, where $A_j=A$ for every $j=1,\dots,p$, is a left (m,C)-invertible p-tuple of operators.

In fact, it is clear that $A_iA_j=A_jA_i$ for all $1\le i;j\le p$. Since A is left (m,C)-invertible, then there exists $B\in\mathcal{B}_b[\mathcal{Y}]$ such that $\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}B^kCA^kC=0$.

Consider $\mathbf{B}=(B,\dots,B)\in\mathcal{B}_b[\mathcal{Y}]^p$ and applying the multinomial expansion, we obtain

$$\begin{aligned}
&\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\left(\sum_{|\mu|=j}\frac{j!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right)\\
&\quad=\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\left(\sum_{|\mu|=j}\frac{j!}{\mu!}B^{|\mu|}CA^{|\mu|}C\right)\\
&\quad=\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}B^jCA^jC\\
&\quad=0.
\end{aligned}$$

Hence, $\mathbf{Q}_m^{(l)}(\mathbf{B},\mathbf{A})=0$ and therefore, $\mathbf{B}$ is a left-(m,C)-inverse of $\mathbf{A}$.

Remark 2.9 It should be noted that the question about left (m,C)-invertibility for a p-tuple of commuting operators is nontrivial. There exists a p-tuple of commuting operators $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ such that each A_j is a left (m,C)-invertible for all $j = 1,\dots,p$, however, $\mathbf{A} = (A_1,\dots,A_p)$ is not a left (m,C)-invertible p-tuple. (We refer the reader to [27, Example 2.4].)

Lemma 2.10 *Let $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ and $\mathbf{B} = (B_1,\dots,B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ be commuting p-tuples of operators. Then, the following identity holds*

$$\sum_{|\mu|=n+1} \binom{n+1}{\mu} \mathbf{B}^\mu C\mathbf{A}^\mu C = \sum_{|\mu|=n} \binom{n}{\mu} \Big(\sum_{1\le j\le p} B_j \mathbf{B}^\mu C\mathbf{A}^\mu C(CA_jC) \Big), \tag{2.9}$$

for all $n \in \mathbb{N}_0$ and where $\binom{n}{\mu} = \frac{n!}{\mu!}$.

Proof

$$\begin{aligned}
\sum_{|\mu|=n+1} \binom{n+1}{\mu} \mathbf{B}^\mu C\mathbf{A}^\mu C &= \sum_{|\mu|=n+1} \frac{(n+1)!}{\mu!} \mathbf{B}^\mu C\mathbf{A}^\mu C \\
&= \sum_{|\mu|=n+1} \frac{n!(n+1)}{\mu!} \mathbf{B}^\mu C\mathbf{A}^\mu C \\
&= \sum_{|\mu|=n+1} \frac{n!(\mu_1+\cdots+\mu_p)}{\mu_1!\cdots\mu_p!} \mathbf{B}^\mu C\mathbf{A}^\mu C \\
&= \sum_{1\le j\le p} \sum_{|\mu|=n+1} \frac{n!(\mu_j)}{\mu_1!\cdots\mu_p!} \mathbf{B}^\mu C\mathbf{A}^\mu C \\
&= \sum_{1\le j\le p} \sum_{|\mu|=n+1} \frac{n!}{\mu_1!\cdots(\mu_j-1)!\cdots\mu_p!} \mathbf{B}^\mu C\mathbf{A}^\mu C \\
&= \sum_{1\le j\le p} \sum_{|\mu|=n} \frac{n!}{\mu_1!\cdots(\mu_j)!\cdots\mu_p!} B_j\mathbf{B}^\mu C\mathbf{A}^\mu A_j C \\
&= \sum_{1\le j\le p} \sum_{|\mu|=n} \frac{n!}{\mu_1!\cdots\mu_j!\cdots\mu_p!} B_j\mathbf{B}^\mu C\mathbf{A}^\mu C(CA_jC) \\
&= \sum_{1\le j\le p} \sum_{|\mu|=n} \binom{n}{\mu} B_j\mathbf{B}^\mu C\mathbf{A}^\mu C(CA_jC).
\end{aligned}$$

□

Proposition 2.11 *Let $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ and $\mathbf{B} = (B_1,\dots,B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ be commuting tuples of operators and C be a conjugation on $\mathcal{Y}$.*

(1) *The maps $\mathbf{Q}_m^{(l)}, \mathbf{Q}_m^{(r)} : \mathcal{B}_b[\mathcal{Y}]^p \times \mathcal{B}_b[\mathcal{Y}]^p \longrightarrow \mathcal{B}_b[\mathcal{Y}]$ satisfy the recursive relations*

$$\mathbf{Q}_{m+1}^{(l)}(\mathbf{B},\mathbf{A}) = \sum_{1\le j\le p} B_j \mathbf{Q}_m^{(l)}(\mathbf{B},\mathbf{A})(CA_jC) - \mathbf{Q}_m^{(l)}(\mathbf{B},\mathbf{A}), \tag{2.10}$$

$$\mathbf{Q}_{m+1}^{(r)}(\mathbf{A},\mathbf{D}) = \sum_{1\le j\le p} (CA_jC)\mathbf{Q}_m^{(r)}(\mathbf{A},\mathbf{D})(D_j) - \mathbf{Q}_m^{(r)}(\mathbf{A},\mathbf{D}). \tag{2.11}$$

(2) *If $\mathbf{A}$ is a left (m,C)-invertible p-tuple, then $\mathbf{A}$ is a left (n,C)-invertible operator p-tuple for all $n \ge m$.*

(3) *If* $\mathbf{A}$ *is a right* (m,C)*-invertible* p*-tuple, then* $\mathbf{A}$ *is a right* (n,C)*-invertible operator* p*-tuple for all* $n \geq m$.

Proof (1) According to Eq. (2.1) and Lemma 2.10, we have

$$\begin{aligned}
&\mathbf{Q}_{m+1}^{(l)}(\mathbf{B},\mathbf{A}) \\
&= \sum_{0\leq k\leq m+1} (-1)^{m+1-k}\binom{m+1}{k}\left(\sum_{|\mu|=k}\frac{k!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right) \\
&= (-1)^{m+1}I_{\mathcal{Y}} - \sum_{1\leq k\leq m}(-1)^{m-k}\left[\binom{m}{k}+\binom{m}{k-1}\right]\left(\sum_{|\mu|=k}\frac{k!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right) \\
&\quad + \sum_{|\mu|=m+1}\frac{(m+1)!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C \\
&= -\mathbf{Q}_{m}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{0\leq k\leq m-1}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\mu|=k+1}\frac{(k+1)!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right) \\
&\quad + \left(\sum_{|\mu|=m+1}\frac{(m+1)!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right) \\
&= -\mathbf{Q}_{m}^{(l)}(\mathbf{B},\mathbf{A}) \\
&\quad + \sum_{1\leq j\leq p}\sum_{0\leq k\leq m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\mu|=k}\binom{m}{k}\frac{k!}{\mu!}B_j\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C(CA_jC) \\
&\quad + \sum_{1\leq j\leq p}\sum_{|\mu|=m}\frac{m!}{\mu!}B_j\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C(CA_jC) \\
&= -\mathbf{Q}_{m}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{1\leq j\leq p}B_j\left(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\mu|=k}\frac{k!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right)(CA_jC) \\
&= -\mathbf{Q}_{m}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{1\leq j\leq p}B_j\mathbf{Q}_{m}^{(l)}(\mathbf{B},\mathbf{A})(CA_jC).
\end{aligned}$$

The statement in (2) follows immediately from (2.10). □

Proposition 2.12 *Let* $\mathbf{A} = (A_1,\ldots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ *be a commuting* p*-tuple and* C *be a conjugation operator on* $\mathcal{Y}$.

(1) *If* $\mathbf{A}$ *is a left* $(2,C)$*-invertible* p*-tuple with its left* $(2,C)$*-inverse* p*-tuple* $\mathbf{B} = (B_1,\ldots,B_p)$, *then the following identities hold*

$$\sum_{|\mu|=n}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C = (1-n)I_{\mathcal{Y}} + n\left(\sum_{1\leq j\leq p}B_jCA_jC\right), \quad \forall n \in \mathbb{N}_0, \tag{2.12}$$

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{|\mu|=n}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right) = \mathbf{Q}_{1}^{(l)}(\mathbf{B},\mathbf{A}). \tag{2.13}$$

(2) *If* $\mathbf{A}$ *is a right* $(2,C)$*-invertible* p*-tuple with its right* $(2,C)$*-inverse* p*-tuple* $\mathbf{D}=(D_1,\dots,D_p)$*, then the following identities hold*

$$\sum_{|\mu|=n}\frac{n!}{\mu!}C\mathbf{A}^{\mu}C\mathbf{D}^{\mu}=(1-n)I_{\mathcal{Y}}+n\Bigl(\sum_{1\le j\le p}CA_jCD_j\Bigr),\quad \forall n\in\mathbb{N}_0, \tag{2.14}$$

$$\lim_{n\to\infty}\frac{1}{n}\Bigl(\sum_{|\mu|=n}\frac{n!}{\mu!}C\mathbf{A}^{\mu}C\mathbf{D}^{\mu}\Bigr)=\mathbf{Q}_1^{(r)}(\mathbf{A},\mathbf{D}). \tag{2.15}$$

Proof (1) Eq. (2.12) is proved by induction. When $n=0$ or $n=1$ the statement is trivially true. Assume that the statement is true for some integer n and prove it for $n+1$. Indeed, according to Lemma 2.10, we have

$$\sum_{|\mu|=n+1}\frac{(n+1)!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C=\sum_{1\le k\le p}B_k\Bigl(\sum_{|\mu|=n}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\Bigr)CA_kC.$$

From the induction hypothesis, we have

$$\begin{aligned}
&\sum_{|\mu|=n+1}\frac{(n+1)!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\\
&\quad=\sum_{1\le k\le p}B_k\Bigl((1-n)I_{\mathcal{Y}}+n\sum_{1\le j\le p}B_jCA_jC\Bigr)CA_kC\\
&\quad=(1-n)\sum_{1\le k\le p}B_kCA_kC+n\sum_{1\le j,k\le p}B_kB_jCA_kA_jC\\
&\quad=(1-n)\sum_{1\le k\le p}B_kCA_kC+n\sum_{1\le j\le p}B_j^2CA_j^2C\\
&\qquad+2n\Bigl(\sum_{1\le j<k\le p}B_jB_kCA_jA_kC\Bigr).
\end{aligned}$$

Since $\mathbf{A}$ is a left $(2,C)$-invertible p-tuple with its left $(2,C)$ inverse $\mathbf{B}$, we have by (2.8)

$$\begin{aligned}
&\sum_{|\mu|=n+1}\frac{(n+1)!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\\
&\quad=(1-n)\sum_{1\le k\le p}B_kCA_kC+n\Bigl(-I_{\mathcal{Y}}+2\sum_{1\le j\le p}B_jCA_jC\Bigr)\\
&\quad=-nI_{\mathcal{Y}}+(n+1)\Bigl(\sum_{1\le k\le p}B_kCA_kC\Bigr).
\end{aligned}$$

This shows the claim is true in the case of $n+1$. The identity (2.13) follows from the first one by taking $n\to\infty$.

The results and techniques for proving (2.14) and (2.15) are very similar. □

Theorem 2.13 *Let* $\mathbf{A}=(A_1,\dots,A_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ *and* $\mathbf{B}=(B_1,\dots,B_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ *be commuting* p*-tuples of operators. The following statements hold*

(i)

$$\sum_{|\mu|=n}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C=\sum_{0\leq j\leq n}\binom{n}{j}\mathbf{Q}_j^{(l)}(\mathbf{B},\mathbf{A}),\tag{2.16}$$

for every $n\in\mathbb{N}_0$.

(ii) $\mathbf{A}$ *is a left* (m,C)*-invertible* p*-tuple if and only if there exists a* p*-tuple of operators* $\mathbf{B}=(B_1,\dots,B_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ *such that*

$$\sum_{|\mu|=n}\frac{n!}{\mu}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C=\sum_{0\leq j\leq m-1}\binom{n}{j}\mathbf{Q}_j^{(l)}(\mathbf{B},\mathbf{A});\quad \forall n\in\mathbb{N}_0.\tag{2.17}$$

(iii) *If* $\mathbf{A}$ *is a left* (m,C)*-invertible* p*-tuple with its left* $(2,C)$*-inverse* $\underset{\sim}{B}=(B_1,\dots,B_p)$, *then*

$$\mathbf{Q}_{m-1}^{(l)}(\mathbf{B},\mathbf{A})=\lim_{n\longrightarrow\infty}\frac{1}{\binom{n}{(m-1)}}\left(\sum_{|\mu|=n}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\right).\tag{2.18}$$

Proof (i) Observe that when $n=0$ or $n=1$ the identity (2.16) is valued. Assume the statement (2.16) is true for n. We shall deduce it at step $n+1$. By virtue of (2.1) and (2.16) we obtain

$$\begin{aligned}
\sum_{|\mu|=n+1}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C&=\mathbf{Q}_{n+1}^{(l)}(\mathbf{B},\mathbf{A})-\sum_{0\leq j\leq n}(-1)^{n+1-j}\binom{n+1}{j}\sum_{|\mu|=j}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C\\
&=\mathbf{Q}_{n+1}^{(l)}(\mathbf{B},\mathbf{A})-\sum_{0\leq j\leq n}(-1)^{n+1-j}\binom{n+1}{j}\sum_{0\leq k\leq j}\binom{j}{k}\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\\
&=\mathbf{Q}_{n+1}^{(l)}(\mathbf{B},\mathbf{A})-\sum_{0\leq k\leq n}\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\sum_{k\leq j\leq n}(-1)^{n+1-j}\binom{n+1}{j}n^{(j)}\\
&=\mathbf{Q}_{n+1}^{(l)}(\mathbf{B},\mathbf{A})-\sum_{0\leq k\leq n}\binom{n+1}{k}\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\\
&\quad\times\underbrace{\left(\sum_{k\leq j\leq n}(-1)^{n+1-j}\binom{n+1-j}{j-k}\right)}_{=-1}\\
&=\sum_{0\leq k\leq n+1}\binom{n+1}{k}\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}).
\end{aligned}$$

This shows the claim is true in the case of n + 1.

(ii) If we assume that $\mathbf{A}$ is a left (m,C)-invertible p-tuple with its left (m,C)-inverse p-tuple $\mathbf{B}$, then $\mathbf{Q}_q^{(l)}(\mathbf{B},\mathbf{A})=0$ for all $q\geq m$ (by Proposition 2.11). Therefore, (2.17) follows from (2.16).

On the other hand, if (2.17) holds for all $n\geq 1$, then $\mathbf{Q}_q^{(l)}(\mathbf{B},\mathbf{A})=0$ for $q\geq m$ by (2.16). Therefore, $\mathbf{A}$ is a left (m,C)-invertible p-tuple.

(iii) By (2.16) if $\mathbf{A}$ is a left (m,C)-invertible p-tuple, we obtain

$$\sum_{|\mu|=n}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C=\sum_{0\leq j\leq m-2}\binom{n}{j}\mathbf{Q}_j^{(l)}(\mathbf{B},\mathbf{A})+\binom{n}{m-1}\mathbf{Q}_{m-1}^{(l)}(\mathbf{B},\mathbf{A}).$$

and moreover,

$$\frac{1}{\binom{n}{m-1}}\sum_{|\mu|=n}\frac{n!}{\mu!}\mathbf{B}^{\mu}C\mathbf{A}^{\mu}C=\sum_{0\leq j\leq m-2}\frac{1}{\binom{n}{m-1}}\binom{n}{j}\mathbf{Q}_j^{(l)}(\mathbf{B},\mathbf{A})+\mathbf{Q}_{m-1}^{(l)}(\mathbf{B},\mathbf{A}).$$

By taking $n\to\infty$ we obtain the desired result. □

3 Perturbation, product, and tensor product

This section is devoted to the study of some questions related to the perturbation, product and tensor product of a left (m,C)-invertible p-tuple of operators. In order to examine these questions we introduce the following powerful lemmas.

Lemma 3.1 *Let* $\mu=(\mu_1,\dots,\mu_p)\in\mathbb{N}_0^p$, $k\in\mathbb{N}$ *and* $n\in\mathbb{N}$ *be such that* $|\mu|+k=n+1$. *For* $1\leq r\leq p$, *let* $1_r=(0,\dots,\underbrace{1}_{r},\dots,0)\in\mathbb{N}^p$. *Then,*

$$\binom{n+1}{\mu,k}=\sum_{1\leq r\leq p}\binom{n}{\mu-1_r,k}+\binom{n}{\mu,k-1},\tag{3.1}$$

where $\binom{n}{\mu,k}=\frac{n!}{\mu!k!}$.

Proof The proof is similar to the proof of [12, Lemma 2.3], hence we omit it. □

Lemma 3.2 *Let* $\mathbf{A}=(A_1,\dots,A_p)\in\mathcal{B}_b[\mathcal{Y}]^p$, $\mathbf{B}=(B_1,\dots,B_p)\in\mathcal{B}_b[\mathcal{Y}]^p$, *and* $\mathbf{N}=(N_1,\dots,N_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ *be commuting tuples operators such that* $[B_j,N_k]=0$ *for all* $(j,k)\in\{1,\dots,p\}^2$. *Then, the following identities hold*:

$$\mathbf{Q}_n^{(l)}(\mathbf{B}+\mathbf{N},\mathbf{A})=\sum_{|\mu|+k=n}\binom{n}{\mu,k}\mathbf{N}^{\mu}\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C,\tag{3.2}$$

$$\mathbf{Q}_n^{(r)}(\mathbf{A},\mathbf{B}+\mathbf{N})=\sum_{|\mu|+k=n}\binom{n}{\mu,k}C\mathbf{A}^{\mu}C\mathbf{Q}_k^{(r)}(\mathbf{A},\mathbf{B})\mathbf{N}^{\mu}.\tag{3.3}$$

Proof We prove the identity (3.2) by induction on n. When $n=1$, we have

$$\begin{aligned}&\sum_{|\mu|+k=1}\binom{1}{\mu,k}(\mathbf{N})^{\mu}\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\\&\quad=\sum_{1\leq j\leq p}N_jCA_jC+\sum_{1\leq j\leq p}B_jCA_jC-I\\&\quad=\sum_{1\leq j\leq p}(B_j+N_j)CA_jC-I\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le j\le p} (B_j + N_j)\mathbf{Q}_0^{(l)}(\mathbf{B}+\mathbf{N},\mathbf{A})(CT_jC) - \mathbf{Q}_0^{(l)}(\mathbf{B}+\mathbf{N},\mathbf{A}) \\
&= \mathbf{Q}_1^{(l)}(\mathbf{B}+\mathbf{N},\mathbf{A}) \quad \text{(by Proposition 2.11)}.
\end{aligned}
$$

Assume that (3.2) holds for n. According to Proposition 2.11, it holds that

$$
\begin{aligned}
&\mathbf{Q}_{n+1}^{(l)}(\mathbf{B}+\mathbf{N},\mathbf{A}) \\
&\quad= \sum_{1\le r\le p} (B_l + N_l)\mathbf{Q}_n^{(l)}(\mathbf{B}+\mathbf{N},\mathbf{A})(CA_jC) - \mathbf{Q}_n^{(l)}(\mathbf{B}+\mathbf{N},\mathbf{A}) \\
&\quad= \sum_{1\le r\le p} (B_l + N_l)\Bigg(\sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{N}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu\Bigg) \\
&\qquad - \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{N}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu C \\
&\quad= \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{N}^\mu \Bigg(\sum_{1\le r\le p} (B_r + N_r)\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})(CA_rC) - \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\Bigg)C\mathbf{A}^\mu C \\
&\quad= \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{N}^\mu \Bigg(\sum_{1\le r\le p} B_r\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})(CA_rC) + \sum_{1\le r\le p} N_r\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}) \\
&\qquad - \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\Bigg)\big(C\mathbf{A}^\mu C\big) \\
&\quad= \Bigg(\sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{N}^\mu \mathbf{Q}_{k+1}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{|\mu|+k=n} \binom{n}{\mu,k} \sum_{1\le r\le p} \mathbf{N}^\mu N_r \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\Bigg) \\
&\qquad \times \big(C\mathbf{A}^\mu C\big) \\
&\quad= \Bigg(\sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{N}^\mu \mathbf{Q}_{k+1}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{|\mu|+k=n} \binom{n}{\mu,k} \\
&\qquad \times \sum_{1\le r\le p} N_1^{\mu_1}\cdots N_r^{\mu_r+1}\cdots N_p^{\mu_p}\mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\Bigg)\big(C\mathbf{A}^\mu C\big) \\
&\quad= \Bigg(\sum_{|\mu|+k=n+1} \Bigg(\binom{n}{\mu,k-1} + \sum_{1\le r\le p}\binom{n}{\mu-1_r,k}\Bigg)\Bigg)\mathbf{N}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\big(C\mathbf{A}^\mu C\big) \\
&\quad= \sum_{|\mu|+k=n+1} \binom{n+1}{\mu,k} \mathbf{N}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A})\big(C\mathbf{A}^\mu C\big).
\end{aligned}
$$

□

Remark 3.3 When $p = 1$, Lemma 3.2 coincides with [18, Lemma 1].

Let $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ and $\mathbf{B} = (B_1,\dots,B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$. We set

$$
\mathbf{A} * \mathbf{B} = (A_1B_1,\dots,A_1B_p,\dots,A_2B_1,\dots,A_2B_p,\dots,A_pB_1,\dots,A_pB_p).
$$

Lemma 3.4 *Let* $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\mathbf{B} = (B_1,\dots,B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\widetilde{\mathbf{A}} = (\widetilde{A_1},\dots,\widetilde{A_p}) \in \mathbf{B}_b[\mathcal{Y}]^p$, *and* $\widetilde{\mathbf{B}} = (\widetilde{B}_1,\dots,\widetilde{B}_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ *be commuting tuples of operators such that*

$$[B_j,\widetilde{B}_r] = [A_j,\widetilde{A}_r] = [\widetilde{B}_j, CA_rC] = 0 \quad \text{for all } j,r \in \{1,\dots,p\},$$

then

$$\mathbf{Q}_n^{(l)}(\mathbf{B}*\widetilde{\mathbf{B}},\mathbf{A}*\widetilde{\mathbf{A}}) = \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{B}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}) C\mathbf{A}^\mu C \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \tag{3.4}$$

$$= \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}) \widetilde{\mathbf{B}}^\mu \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}})(C\widetilde{\mathbf{A}}C). \tag{3.5}$$

Proof For $n = 1$ we have,

$$\begin{aligned}
&\sum_{|\mu|+k=1} \binom{1}{\mu,k} \mathbf{B}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}) C\mathbf{A}^\mu C \mathbf{Q}_{1-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&\quad = \sum_{1\le j\le p} B_jCA_jC\mathbf{Q}_1^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) + \mathbf{Q}_1^{(l)}(\mathbf{B},\mathbf{A})\mathbf{Q}_0^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&\quad = \Big(\sum_{1\le j\le p} B_jCA_jC\Big)\Big(\sum_{1\le j\le p} \widetilde{B}_jC\widetilde{A}_jC - I_{\mathcal{Y}}\Big) + \sum_{1\le j\le p} B_jCA_jC - I_{\mathcal{Y}} \\
&\quad = \sum_{1\le j,k\le p} B_jCA_jC.\widetilde{B}_kC\widetilde{A_k}C - I_{\mathcal{Y}} \\
&\quad = \sum_{1\le j,k\le p} B_j.\widetilde{B}_kCA_j\widetilde{A_k}C - I_{\mathcal{Y}} \\
&\quad = \mathbf{Q}_1^{(l)}(\mathbf{B}*\widetilde{\mathbf{B}},\mathbf{A}*\widetilde{\mathbf{A}}).
\end{aligned}$$

Assume that (3.4) is true for n and prove it for $n+1$. In fact, from Proposition 2.11, we have

$$\begin{aligned}
&\mathbf{Q}_{n+1}^{(l)}(\mathbf{B}*\widetilde{\mathbf{B}},\mathbf{A}*\widetilde{\mathbf{A}}) \\
&\quad = \sum_{1\le j,r\le p} (B_j\widetilde{B}_r)\mathbf{Q}_n^{(l)}(\mathbf{B}*\widetilde{\mathbf{B}},\mathbf{A}*\widetilde{\mathbf{A}})(CA_j\widetilde{A}_rC) - \mathbf{Q}_n^{(l)}(\mathbf{B}*\widetilde{\mathbf{B}},\mathbf{A}*\widetilde{\mathbf{A}}) \\
&\quad = \sum_{1\le j,r\le p} (B_j\widetilde{B}_r)\Big[\sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{B}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}) C\mathbf{A}^\mu C \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}})\Big](CA_j\widehat{A}_rC) \\
&\qquad - \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{B}^\mu \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}) C\mathbf{A}^\mu C \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}).
\end{aligned}$$

Under the assumptions $[B_j,\widetilde{B}_r] = [A_j,\widetilde{A}_r] = [\widetilde{B}_j, CA_rC] = 0$ for all $j,r \in \{1,\dots,p\}$, we obtain

$$\begin{aligned}
&\mathbf{Q}_{n+1}^{(l)}(\mathbf{B}*\widetilde{\mathbf{B}},\mathbf{A}*\widetilde{\mathbf{A}}) \\
&\quad = \sum_{|\mu|+k=n} \binom{n}{\mu,k} \Big[\sum_{1\le j,r\le p} \mathbf{B}^\mu B_j \mathbf{Q}_k^{(l)}(\mathbf{B},\mathbf{A}) \big(C\mathbf{A}^\mu A_jC\big)\widetilde{B}_r \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}})(C\widetilde{A}_rC\Big]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&= \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \big(B_j \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A})(C A_j C)\big) C \mathbf{A}^{\mu} C \Big(\sum_{1 \le r \le p} \widetilde{B}_r \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}})(C \widetilde{A}_r C) \Big) \\
&\quad - \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&= \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \big[\mathbf{Q}_{k+1}^{(l)}(\mathbf{B}, \mathbf{A}) + \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A})\big] C \mathbf{A}^{\mu} C \big[\mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) + \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}})\big] \\
&\quad - \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&= \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_{k+1}^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_{k+1}^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|+k=n} \binom{n}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}).
\end{aligned}
$$

By observing that

$$
\begin{aligned}
&\sum_{|\mu|+k=n+1} \binom{n+1}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&= \mathbf{Q}_{n+1}^{(l)}(\mathbf{B}, \mathbf{A}) + \sum_{|\mu|+k=n+1} \binom{n+1}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|=n+1} \binom{n+1}{\mu} \mathbf{B}^{\mu} C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&= \mathbf{Q}_{n+1}^{(l)}(\mathbf{B}, \mathbf{A}) + \sum_{|\mu|+k=n+1} \Big(\sum_{1 \le r \le p} \binom{n}{\mu - 1_r, k} + \binom{n}{\mu, k-1} \Big) \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \\
&\quad \times \mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|=n+1} \binom{n+1}{\mu} \mathbf{B}^{\mu} C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&= \mathbf{Q}_{n+1}^{(l)}(\mathbf{B}, \mathbf{A}) + \sum_{|\mu|+k=n+1} \binom{n}{\mu, k-1} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|=n+1} \binom{n+1}{\mu} \mathbf{B}^{\mu} C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) + \\
&\quad + \sum_{|\mu|+k=n+1} \sum_{1 \le r \le p} \binom{n}{\mu - 1_r, k} \mathbf{B}^{\mu} \mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{Q}_{n+1}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{B}^{\mu}\mathbf{Q}_{k+1}^{(l)}(\mathbf{B},\mathbf{A})(CAC)\mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|=n} \binom{n}{\mu} \Big(\sum_{1\le r\le p} B_r\big(\mathbf{B}^{\mu}\big(CA^{\mu}C\big)(CA_rC)\big)\mathbf{Q}_{n+1}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|+k=n} \binom{n}{\mu} \mathbf{B}^{\mu}\Big(\sum_{1\le r\le p} B_r\mathbf{Q}_{k}^{(l)}(\mathbf{B},\mathbf{A})(CA_rC)\Big)\big(CA^{\mu}C\big)\mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&= \mathbf{Q}_{n+1}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{|\mu|+k=n} \binom{n}{\mu,k} \mathbf{B}^{\mu}\mathbf{Q}_{k+1}^{(l)}(\mathbf{B},\mathbf{A})(CAC)\mathbf{Q}_{n-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|=n} \binom{n}{\mu} \Big(\sum_{1\le r\le p} B_r\big(\mathbf{B}^{\mu}\big(CA^{\mu}C\big)(CA_rC)\big)\mathbf{Q}_{n+1}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&\quad + \sum_{|\mu|+k=n} \binom{n}{\mu} \mathbf{B}^{\mu}\big(\mathbf{Q}_{k+1}^{(l)}(\mathbf{B},\mathbf{A}) + \mathbf{Q}_{k}^{(l)}(\mathbf{B},\mathbf{A})\big)\big(CA^{\mu}C\big)\Big)\mathbf{Q}_{n+1-k}^{(l)}(\widetilde{\mathbf{B}},\widetilde{\mathbf{A}}) \\
&= \mathbf{Q}_{n+1}^{(l)}(\mathbf{B}*\widetilde{\mathbf{B}},\mathbf{A}*\widetilde{\mathbf{A}}).
\end{aligned}
$$

□

Remark 3.5 When $p = 1$, Lemma 3.4 coincides with [18, Lemma 12].

For $\mathbf{A} = (A_1,\dots,A_p) \in \mathbf{B}_b[\mathcal{Y}]^p$ and $\mathbf{B} = (B_1,\dots,B_p) \in \mathbf{B}_b[\mathcal{Y}]^p$, we set

$$\mathbf{A} \bullet \mathbf{B} = (A_1B_1, A_2B_2,\dots,A_pB_p).$$

Lemma 3.6 *Let* $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\mathbf{B} = (B_1,\dots,B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\widetilde{\mathbf{A}} = (\widetilde{A_1},\dots,\widetilde{A_p}) \in \mathbf{B}_b[\mathcal{Y}]^p$, *and* $\widetilde{\mathbf{B}} = (\widetilde{B_1},\dots,\widetilde{B_p}) \in \mathcal{B}_b[\mathcal{Y}]^p$ *be commuting tuples of operators such that*

$$[B_j,\widetilde{B_r}] = [A_j,\widetilde{A_r}] = [\widetilde{B_j}, CA_rC] = 0 \quad \textit{for all } j,r \in \{1,\dots,p\},$$

then

$$\mathbf{Q}_{n}^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}}) = \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})\big(CA^{\mu}C\big)\prod_{i=1}^{p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B_i},\widetilde{A_i}), \tag{3.6}$$

for all $n \in \mathbb{N}$.

Proof We will prove (3.6) by mathematical induction. For $n = 1$, we have

$$
\begin{aligned}
\mathbf{Q}_{1}^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}}) &= \sum_{0\le k\le 1}\sum_{|\mu|=k}\binom{1}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}_{1-k}^{(l)}(\mathbf{B},\mathbf{A})\big(CA^{\mu}C\big)\prod_{i=1}^{p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B_i},\widetilde{A_i}) \\
&= \mathbf{Q}_{1}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{1\le j\le p} B_j(CA_jC)\mathbf{Q}_{1}^{(l)}(\widetilde{B_j},\widetilde{A_j}) \\
&= \mathbf{Q}_{1}^{(l)}(\mathbf{B},\mathbf{A}) + \sum_{1\le j\le p} B_j(CA_jC)(\widetilde{B_j}C\widetilde{A_j}C - I) \\
&= \sum_{1\le j\le p} B_jCA_jC\widetilde{B_j}C\widetilde{A_j}C - I.
\end{aligned}
$$

Therefore, by Proposition 2.11, we obtain that

$$\begin{aligned}\mathbf{Q}_1^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}}) &= \sum_{1\le j\le p}(B_j\widetilde{B}_j\mathbf{Q}_0^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}})(CA_j\widetilde{A}_jC) - \mathbf{Q}_0^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}})\\ &= \sum_{1\le j\le p} B_jCA_jC\widetilde{B}_jC\widetilde{A}_jC - I.\end{aligned}$$

Hence, (3.6) is true for $n = 1$. Assume it is true for n and prove it for $n + 1$.

Following the conditions

$$[B_j,\widetilde{B}_r] = [A_j,\widetilde{A}_r] = [\widetilde{B}_j, CA_rC] = 0 \quad \text{for all } j,r \in \{1,\dots,p\},$$

and Proposition 2.11 we obtain

$$\begin{aligned}
&Q_{n+1}^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}})\\
&\quad= \sum_{1\le j\le p}(B_j\widetilde{B}_j)\mathbf{Q}_n^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}})(CA_j\widetilde{A}_jC) - \mathbf{Q}_n^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}},\mathbf{A}\bullet\widetilde{\mathbf{A}})\\
&\quad= \sum_{1\le j\le p}(B_j\widetilde{B}_j)\Bigg(\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu C\prod_{1\le i\le p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i)\Bigg)\\
&\qquad\times(CA_j\widetilde{A}_jC) - \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu C\prod_{1\le i\le p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i)\\
&\quad= \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\beta!}\Bigg(\sum_{1\le j\le p}B_j\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu A_jC\widetilde{B}_j\prod_{i=1}^{p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i)\Bigg)\\
&\qquad\times(C\widetilde{A}_jC) - \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu C\prod_{i=1}^{p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i)\\
&\quad= \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\sum_{1\le j\le p}B_j\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu A_jC\mathbf{Q}_{\mu_1}^{(l)}(\widetilde{B}_1,\widetilde{A_1})\\
&\qquad\cdots\underbrace{\widetilde{B}_j\mathbf{Q}_{\mu_j}^{(l)}(\widetilde{B}_j,\widetilde{A}_j)C\widetilde{A}_jC\cdots\mathbf{Q}_{\mu_p}^{(l)}(\widetilde{B}_p,\widetilde{A_p})}\\
&\qquad- \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu C\prod_{i=1}^{p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i)\\
&\quad= \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\sum_{1\le j\le p}B_j\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu A_jC\mathbf{Q}_{\mu_1}^{(l)}(\widetilde{B}_1,\widetilde{A_1})\\
&\qquad\cdots\underbrace{\big(\mathbf{Q}_{\mu_j+1}^{(l)}(\widetilde{B}_j,\widetilde{A}_j) + \mathbf{Q}_{\mu_j}^{(l)}(\widetilde{B}_j,\widetilde{A}_j)\big)}\cdots\mathbf{Q}_{\mu_p}^{(l)}(\widetilde{B}_p,\widetilde{A_p})\\
&\qquad- \sum_{0\le k\le n}\sum_{|\beta|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})C\mathbf{A}^\mu C\prod_{i=1}^{p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i)\\
&\quad= \sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\underbrace{\sum_{1\le j\le p}B_j\mathbf{B}^\mu\mathbf{Q}_{n-k}^{(l)}(\mathbf{B},\mathbf{A})(CA_jC)}\,C\mathbf{A}^\mu C\prod_{i=1}^{p}\mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\sum_{1\le j\le p}B_j\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}A_jC\mathbf{Q}^{(l)}_{\mu_1}(\widetilde{B}_1,\widetilde{A_1})\cdots\mathbf{Q}^{(l)}_{\mu_{j+1}}(\widetilde{B}_j,\widetilde{A}_j)\\
&\cdots\mathbf{Q}^{(l)}_{\mu_p}(\widetilde{B}_p,\widetilde{A}_p)-\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
=&\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\big(\mathbf{Q}^{(l)}_{n+1-k}(\mathbf{B},\mathbf{A})+\mathbf{Q}^{(l)}_{n-k}(\mathbf{B},\mathbf{A})\big)C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
&+\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\sum_{1\le j\le p}B_j\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}A_jC\mathbf{Q}^{(l)}_{\mu_1}(\widetilde{B}_1,\widetilde{A}_i)\cdots\mathbf{Q}^{(l)}_{\mu_{j+1}}(\widetilde{B}_j,\widetilde{A}_j)\\
&\cdots\mathbf{Q}^{(l)}_{\mu_p}(\widetilde{B}_p,\widetilde{A}_p)-\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
=&\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n+1-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
&+\sum_{0\le k\le n}\sum_{|\mu|=k+1}\binom{n}{k}\frac{(k+1)!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\sum_{0\le k\le n+1}\sum_{|\mu|=k}\binom{n+1}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n+1-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(i)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
=&\mathbf{Q}^{(l)}_{n+1}(\mathbf{B},\mathbf{A})+\sum_{1\le k\le n}\sum_{|\mu|=k}\binom{n+1}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n+1-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
&+\sum_{|\mu|=n+1}\frac{(n+1)!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{0}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
=&\mathbf{Q}^{(l)}_{n+1}(\mathbf{B},\mathbf{A})+\sum_{1\le k\le n}\sum_{|\mu|=k}\left(\binom{n}{k}+\binom{n}{k-1}\right)\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n+1-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\\
&\times\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)+\sum_{|\mu|=n+1}\frac{(n+1)!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{0}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\times\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
=&\sum_{0\le k\le n}\sum_{|\mu|=k}\binom{n}{k}\frac{k!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n+1-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=1}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i)\\
&+\sum_{0\le k\le n}\sum_{|\mu|=k+1}\binom{n}{k}\frac{(k+1)!}{\mu!}\mathbf{B}^{\mu}\mathbf{Q}^{(l)}_{n-k}(\mathbf{B},\mathbf{A})C\mathbf{A}^{\mu}C\prod_{i=11}^{p}\mathbf{Q}^{(l)}_{\mu_i}(\widetilde{B}_i,\widetilde{A}_i).
\end{aligned}
$$

Hence, we obtain this result. □

Let $\mathbf{N}=(N_1,\dots,N_p)\in\mathcal{B}_b[\mathcal{Y}]^p$ be a commuting p-tuple, we say that $\mathbf{N}$ is q-nilpotent if $\mathbf{N}^{\mu}=N_1^{\mu_1}...N_p^{\mu_p}=0$ for all $\mu=(\mu_1,\dots,\mu_p)\in\mathbb{N}_0^p$ with $\mu_1+\cdots+\mu_p=q$ ([19]).

Theorem 3.7 *Let* $\mathbf{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\mathbf{B} = (B_1, \dots, B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, *and* $\mathbf{N} = (N_1, \dots, N_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ *be commuting tuples of operators. Assume that* $[B_k, N_j] = 0$ *for all* $(k,j) \in \{1, \dots, p\}^2$ *and* $\mathbf{N}$ *is a nilpotent tuple of order* q. *If* $\mathbf{B}$ *is a left* (m, C)*-inverse of* $\mathbf{A}$, *then* $\mathbf{B} + \mathbf{N} = (B_1 + N_1, \dots, B_p + N_p)$ *is a left* $(m + q - 1, C)$*-inverse of* $\mathbf{A}$.

Proof According to Lemma 3.2, we have

$$\mathbf{Q}^{(l)}_{m+q-1}(\mathbf{B} + \mathbf{N}, \mathbf{A}) = \sum_{|\mu|+k=m+q-1} \binom{m+q-1}{\mu, k} \mathbf{N}^{\mu} \mathbf{Q}^{(l)}_k(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C.$$

If $|\mu| \geq q$, then $\mathbf{N}^{\mu} = 0$. If $|\mu| \leq q - 1$, then $k \geq m$ and, hence, $\mathbf{Q}^{(l)}_k(\mathbf{B}, \mathbf{A}) = 0$.

Hence, $\mathbf{Q}^{(l)}_{m+q-1}(\mathbf{B} + \mathbf{N}, \mathbf{A}) = 0$ and therefore, $\mathbf{B} + \mathbf{N}$ is a left-$(m + q - 1, C)$-inverse of $\mathbf{A}$. □

For $\mathbf{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ and $\mathbf{B} = (B_1, \dots, B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$. Set

$$\mathbf{A} \otimes \mathbf{B} = (A_1 \otimes B_1, \dots, A_p \otimes B_p) \in \mathcal{B}_b[\mathcal{Y} \overline{\otimes} \mathcal{Y}]^p$$

the tensor product of $\mathbf{A}$ and $\mathbf{B}$. It should be noted that the following corollary is an interesting consequence of Theorem 3.7.

Corollary 3.8 *Let* $\mathbf{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]$ *be a left* (m, C)*-invertible* p*-tuple of commuting operators with its left* (m, C)*-inverse* $\mathbf{B} = (B_1, \dots, B_p)$ *and let* $\mathbf{N} = (N_1, \dots, N_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ *be a* q*-nilpotent* p*-tuple of commuting operators. Then,*

$$\mathbf{B} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{N} := (B_1 \otimes I + I \otimes N_1, \dots, B_p \otimes I + I \otimes N_p) \in \mathcal{B}_b[\mathcal{Y} \overline{\otimes} \mathcal{Y}]^p$$

is a left $(m + q - 1, C \otimes C)$*-inverse* p*-tuple.*

Proof By observing that $(B_j \otimes I)(I \otimes N_k) = (I \otimes N_k)(B_j \otimes I)$ for all $j, k \in \{1, \dots, p\}$ and moreover $\mathbf{B} \otimes \mathbf{I} \in \mathcal{B}_b[\mathcal{H} \overline{\otimes} \mathcal{Y}]^p$ is a left $(m, C \otimes C)$-inverse p-tuple of $\mathbf{A} \otimes \mathbf{I} \in \mathcal{B}_b[\mathcal{H} \overline{\otimes} \mathcal{Y}]^p$. $\mathbf{I} \otimes \mathbf{N} \in \mathcal{B}_p[\mathcal{Y} \overline{\otimes} \mathcal{Y}]^p$ is a nilpotent p-tuple of order q. Hence, $\mathbf{B} \otimes \mathbf{I}$ and $\mathbf{I} \otimes \mathbf{N}$ satisfy the conditions of Theorem 3.7. Therefore, $\mathbf{B} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{N}$ is a left $(m + q - 1, C \otimes C)$-inverse p-tuple. □

Theorem 3.9 *Let* $\mathbf{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\mathbf{B} = (B_1, \dots, B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\widetilde{\mathbf{A}} = (\widetilde{A_1}, \dots, \widetilde{A_p}) \in \mathcal{B}_b[\mathcal{Y}]^p$, *and* $\widetilde{\mathbf{B}} = (\widetilde{B_1}, \dots, \widetilde{B_p}) \in \mathcal{B}_b[\mathcal{Y}]^p$ *be commuting tuples of operators such that*

$$[B_j, \widetilde{B}_r] = [A_j, \widetilde{A}_r] = [\widetilde{B}_j, CA_rC] = 0 \quad \textit{for all } j, r \in \{1, \dots, p\}.$$

If $\mathbf{B}$ *is a left* (m, C)*-inverse of* $\mathbf{A}$ *and* $\widetilde{\mathbf{B}}$ *is a left* (n, C)*-inverse of* $\widetilde{\mathbf{A}}$, *then* $\mathbf{B} * \widetilde{\mathbf{B}}$ *is a* $(m + n - 1, C)$*-left inverse of* $\mathbf{A} * \widetilde{\mathbf{A}}$.

Proof In view of Lemma 3.4, we have

$$\begin{aligned} &\mathbf{Q}^{(l)}_{m+n-1}(\mathbf{B} * \widetilde{\mathbf{B}}, \mathbf{A} * \widetilde{\mathbf{A}}) \\ &\quad = \sum_{|\mu|+k=m+n-1} \binom{n+m-1}{\mu, k} \mathbf{B}^{\mu} \mathbf{Q}^{(l)}_k(\mathbf{B}, \mathbf{A}) C \mathbf{A}^{\mu} C \mathbf{Q}^{(l)}_{m+n-1-k}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}). \end{aligned}$$

If $k \geq m$, then $\mathbf{Q}_k^{(l)}(\mathbf{B}, \mathbf{A}) = 0$ and if $k < m$ then $m + n - 1 - k > n - 1$ and so

$$\mathbf{Q}_{m+n-1-k}^{(l)}(\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}) = 0.$$

Therefore, $\mathbf{Q}_{m+n-1}^{(l)}(\mathbf{B} * \widetilde{\mathbf{B}}, \mathbf{A} * \widetilde{\mathbf{A}}) = 0$. □

Corollary 3.10 *Let $\mathcal{A} = (A_1, \dots, A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ be a left (m, C)-invertible p-tuple and $\widetilde{\mathbf{A}} = (\widetilde{A_1}, \dots, \widetilde{A_p}) \in \mathcal{B}_b[\mathcal{Y}]^p$ be a left (n, D)-invertible k-tuple. If $\mathbf{B} = (B_1, \dots, B_p)$ is a left (m, C)-inverse p-tuple of $\mathbf{A}$ and $\widetilde{\mathbf{B}} = (\widetilde{B}_1, \dots, \widetilde{B}_p)$ is a left (n, D)-inverse p-tuple of $\widetilde{\mathbf{A}}$. Then,*

$$\mathbf{A} \otimes^* \widetilde{\mathbf{A}} = (A_1 \otimes \widetilde{A_1}, \dots, A_1 \otimes \widetilde{A_p}, \dots, A_p \otimes \widetilde{A_1}, \dots, A_p \otimes \widetilde{A_p})$$

is a left $(m + n - 1, C \otimes D)$-invertible p^2-tuple with its left $(m + n - 1, C \otimes C)$-inverse p^2-tuple

$$\mathbf{B} \otimes^* \widetilde{\mathbf{B}} = (B_1 \otimes \widetilde{B}_1, \dots, B_1 \otimes \widetilde{B}_p, \dots, B_p \otimes \widetilde{B}_1, \dots, B_p \otimes \widetilde{B}_p),$$

where C and D are conjugations on $\mathcal{Y}$, respectively.

Proof Since $\mathbf{A} = (A_1, \dots, A_p)$ is a left (m, C)-invertible p-tuple and $\widetilde{\mathbf{A}} = (\widetilde{A_1}, \dots, \widetilde{A_p})$ is a left (n, D)-isometric tuple of operators, it follows that $\mathbf{A} \otimes \mathbf{I} = (A_1 \otimes I, \dots, A_p \otimes I)$ is a left $(m, C \otimes D)$-invertible p-tuple with its left (m, C)-inverse p-tuple $\mathbf{B} \otimes \mathbf{I} = (B_1 \otimes I_1, \dots, B_p \otimes I)$ and $\mathbf{I} \otimes \widetilde{\mathbf{A}} = (I \otimes \widetilde{A_1}, \cdots, I \otimes \widetilde{A_p})$ is a left $(n, C \otimes D)$-invertible p-tuple with its left $(n, C \otimes D)$-inverse p-tuple $\mathbf{I} \otimes \widetilde{\mathbf{B}} = (I \otimes \widetilde{B}_1, \dots, I \otimes \widetilde{B}_p)$-$p$- tuple. However,

$$[B_j \otimes I, I \otimes \widetilde{B}_r] = [A_j \otimes I, I \otimes \widetilde{A}_r] = \big[I \otimes \widetilde{B}_r, (C \otimes D)(A_j \otimes I)(C \otimes D)\big] = 0,$$

for $1 \leq j \leq p$ and $1 \leq r \leq p$.

Since

$$\mathbf{A} \otimes^* \widetilde{\mathbf{A}} = (R_{11}, \dots, R_{1p}, R_{21}, \dots, R_{2p}, \dots R_{p1}, \dots, R_{pp}),$$

where

$$R_{jr} = (A_j \otimes)(I \otimes \widetilde{A}_r) \quad \text{for all } j = 1, \dots, p \text{ and } r = 1, \dots, p,$$

and

$$\mathbf{B} \otimes^* \widetilde{\mathbf{B}} = (S_{11}, \dots, S_{1p}, S_{21}, \dots, S_{2p}, \dots S_{p1}, \dots, S_{p^2}),$$

where

$$S_{jr} = (B_j \otimes I)(I \otimes \widetilde{B}_r) \quad \text{for all } j = 1, \dots, p \text{ and } r = 1, \dots, p.$$

According to Theorem 3.9 we deduce that $\mathbf{A} \otimes^* \widetilde{\mathbf{A}}$ is a left $(m + n - 1, C \otimes D)$ invertible p^2-tuple with its left $(m + n - 1, C \otimes D)$-inverse p^2-tuple $\mathbf{B} \otimes^* \widetilde{\mathbf{B}}$. □

Theorem 3.11 *Let* $\mathbf{A} = (A_1,\dots,A_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\mathbf{B} = (B_1,\dots,B_p) \in \mathcal{B}_b[\mathcal{Y}]^p$, $\widetilde{\mathbf{A}} = (\widetilde{A_1},\dots,\widetilde{A_p}) \in \mathcal{B}_b[\mathcal{Y}]^p$, *and* $\widetilde{\mathbf{B}} = (\widetilde{B}_1,\dots,\widetilde{B}_p) \in \mathcal{B}_b[\mathcal{Y}]^p$ *be commuting tuples of operators that satisfy the following conditions*

$$[B_j, \widetilde{B}_r] = [A_j, \widetilde{A}_r] = [\widetilde{B}_j, CA_rC] = 0 \quad \textit{for all } j,r \in \{1,\dots,p\}.$$

If $\mathbf{B}$ *is a left* (m,C)*-inverse of* $\mathbf{A}$ *and* $\widetilde{B}_k$ *is a left* (n_k,C)*-inverse of* $\widetilde{A_k}$ *for* $k = 1,\dots,p$, *then* $\mathbf{B} \bullet \widetilde{\mathbf{B}}$ *is a left* $(m + \sum_{1\le k\le p} n_k - p, C)$*-inverse of* $\mathbf{A} \bullet \widetilde{\mathbf{A}}$.

Proof Set $d = m + n - p$, where $n = n_1 + \cdots + n_p$. According to Lemma 3.6, we have

$$\mathbf{Q}_d^{(l)}(\mathbf{B}\bullet\widetilde{\mathbf{B}}, \mathbf{A}\bullet\widetilde{\mathbf{A}}) = \sum_{0\le k\le d}\sum_{|\mu|=k}\binom{d}{k}\frac{k!}{\mu!}\mathbf{B}^\mu \mathbf{Q}_{d-k}^{(l)}(\mathbf{B},\mathbf{A})\bigl(C\mathbf{A}^\mu C\bigr)\prod_{i=1}^p \mathbf{Q}_{\mu_i}^{(l)}(\widetilde{B}_i,\widetilde{A}_i).$$

When $k \in \{0,\dots,n-p\}$ we have $d-k \ge m$ and therefore $\mathbf{Q}_{d-k}^{(l)}(\mathbf{B},\mathbf{A}) = 0$.

When $k > n-p$ and $|\mu| = k$, then there exists $i_0 \in \{1,\dots,p\}$ such that $\mu_{i_0} \ge n_{i_0}$ and, hence, $\mathbf{Q}_{\mu_{i_0}}^{(l)}(\widetilde{B_{i_0}},\widetilde{A_{i_0}}) = 0$. □

The following Corollary is a useful application of Theorem 3.11.

Corollary 3.12 *Let* $\mathbf{A} = (A_1,\dots,A_p)$, $\widetilde{\mathbf{A}} = (\widetilde{A_1},\dots,\widetilde{A_p})$, $\mathbf{B} = (B_1,\dots,B_p)$ *and* $\widetilde{\mathbf{B}} = (\widetilde{B}_1,\dots,\widetilde{B}_p)$ *be commuting* p*-tuples of operators. Assume that* $\mathbf{B}$ *is a left* (m,C)*-inverse* p*-tuple of* $\mathbf{A}$ *and* $\widetilde{B}_k$ *is a left* (n_k,C)*-inverse of* $\widetilde{A_k}$ *for* $k = 1,\dots,p$. *Then*, $\mathbf{B}\otimes\widetilde{\mathbf{B}} = (B_1\otimes\widetilde{B}_1,\dots,B_p\otimes\widetilde{B}_p)$ *is a left* $(m + \sum_{1\le k\le p} n_k - p, C\otimes C)$*-inverse* p*-tuple of* $\mathbf{A}\otimes\widetilde{\mathbf{A}} = (A_1\otimes\widetilde{A_1},\dots,A_p\otimes\widetilde{A_p})$.

Proof We will use the elementary identities,

$$\begin{aligned}\mathbf{B}\otimes\widetilde{\mathbf{B}} &= (B_1\otimes\widetilde{B}_1,\dots,B_p\otimes\widetilde{B}_p)\\ &= \bigl((B_1\otimes I)(I\otimes\widetilde{B}_1),\dots,(B_p\otimes I)(I\otimes\widetilde{B}_p)\bigr)\\ &= (\mathbf{B}\otimes I)\bullet(I\otimes\widetilde{\mathbf{B}})\end{aligned}$$

and similarly

$$\mathbf{A}\otimes\widetilde{\mathbf{A}} = (\mathbf{A}\otimes I)\bullet(I\otimes\widetilde{\mathbf{A}}).$$

Since $\mathbf{B}$ is a left (m,C)-inverse p-tuple of $\mathbf{A}$ and $\widetilde{B}_k$ is a left (n_k,C)-inverse of $\widetilde{A_k}$ for $k = 1,\dots,p$, it is easily seen that $\mathbf{B}\otimes I$ is a left $(m, C\otimes C)$-left inverse p-tuple of $\mathbf{A}\otimes I$ and $I\otimes\widetilde{B}_k$ is a left $(n_k, C\otimes C)$-left inverse of $I\otimes\widetilde{A_k}$ for each $k = 1,\dots,p$.

In addition, it is obvious that

$$[B_j\otimes I, I\otimes\widetilde{B}_r] = [A_j\otimes I, I\otimes\widetilde{A}_r] = \bigl[I\otimes\widetilde{B}_j, (C\otimes C)(A_r\otimes I)(C\otimes C)\bigr] = 0,$$

for all $j,r \in \{1,\dots,p\}$. By applying Theorem 3.11 we deduce that $\mathbf{B}\otimes\widetilde{\mathbf{B}} = (\mathbf{B}\otimes I)\bullet(I\otimes\widetilde{\mathbf{B}})$ is a left $(m + \sum_{1\le k\le p} n_k - p, C\otimes C)$-inverse p-tuple of $\mathbf{A}\otimes\widetilde{\mathbf{A}} = (\mathbf{A}\otimes I)\bullet(I\otimes\widetilde{\mathbf{A}})$. The proof is achieved. □

Acknowledgements
The authors extend their appreciation to the Deanship of Scientific Research at Jouf University for funding this work through research grant No (DSR2020-05-2547). The authors are extremely grateful to the editor and the anonymous reviewers for their valuable comments and suggestions, which have helped improve the quality of our manuscript.

Funding
This research was supported by the Deanship of Scientific Research at Jouf University through research grant No (DSR2020-05-2547).

Availability of data and materials
Data sharing is not applicable to this paper.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
AA and SM contributed their efforts jointly to this manuscript. In summary, AA initiated the investigation, proposed some preliminary ideas, and conducted some detailed investigations. SM analyzed the proposed ideas, made some calculations, and gave many comments. All authors read and approved the final manuscript.

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Received: 13 January 2022 Accepted: 13 July 2022 Published online: 08 August 2022

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