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Approximate solutions for set optimization with an order cone that has nonempty quasirelative interiors

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Abstract

In a real normed linear space, when the quasirelative interior is not empty, a class of order relation is introduced with Minkowski difference. Two classes of nonlinear functions are introduced, and their properties are discussed. A class of approximately efficient solutions and approximate weakly efficient solutions are introduced for set optimization. With nonlinear functions, optimality conditions are established for approximate solutions. Some examples are given to illustrate our main results.

MSC: 90C33; 90C46; 90C59

Keywords: Set optimization; Quasirelative interior; Approximate weakly efficient solution; Optimality condition

1 Introduction

Set-valued optimization is the promotion of vector optimization that has applications in many aspects, such as engineering, cybernetics, military, finance, etc. The research on set-valued optimization has achieved fruitful results (see [6, 9, 13, 15]). The study of set optimization has been investigated by some authors (see [11, 12, 14, 21, 23]). Kuroiwa [14] considered the criteria of solutions of set optimization, showed some examples with respect to the criteria, introduced some type of semicontinuities for set-valued maps, and showed existence theorems of solutions. Karaman and Soyertem [11] used the Minkowski difference to define new order relations on the set family, and discussed the relations among these orders.

In the research of vector-optimization problems, different kinds of solutions defined by the ordering cone of the image space play an important role, mainly including efficient solutions, weakly efficient solutions, various properly efficient solutions, and corresponding approximate solutions. Among them, the weakly efficient solution defined by the nonempty interior based on an ordered cone has good properties. However, in many cases, the interior of the order cone is an empty set. For example, for any $1 < p < +\infty$, the normed space l^p, partially ordered by the positive cone, is an important space in applications, however, the positive cone has an empty interior. Therefore, it is a very worthwhile topic to use the generalized interior such as the relative topological interior, quasiinterior, and quasirelative interior to define the corresponding weakly efficient solution for the vector-optimization problem. In this research, there have been many results, such as Adan and Novo [1] who used a relative topological interior to propose corresponding weakly efficient solutions, etc. Therefore, when the interior of the topology is an empty set, using a generalized interior to propose different approximate solutions and study some properties is also a very meaningful direction.

As is known, the efficient solution set may be empty, but approximate solutions always exist under weak assumptions. Therefore, it is very meaningful to introduce the concept of approximate solutions (see [5, 16, 22, 26]). Loridan [17] presented some properties of ϵ-solutions for vector-minimization problems where the function to be optimized takes its values in the Euclidean space $\mathbb{R}^p$. Qiu and Yang [19] studied the approximate solutions for the vector-optimization problem with set-valued functions, and discussed the relationships between approximate solutions and weak efficient solutions. Based on the set order relation introduced by Karaman et al. [2], Gupta et al. [7] defined a new concept of approximate weakly minimal solution for constrained set-optimization problems. Jahn and Ha [10] studied a new set-optimization problem, which is to minimize a set-valued map that takes a value in a real linear space; this set-valued map also has a preorder induced by a convex cone, and used the Minkowski difference to define a new order, and some of their properties were obtained. Zhao et al. [24] established a new nonlinear separation theorem to study the vector-optimization problem in which the topological interior or even the relative topological interior of the ordered cone may be empty.

Optimality conditions are important parts of vector-optimization problems and important foundations for establishing modern optimization algorithms [18, 20, 25]. Tung et al. [20] considered the set-optimization problem with mixed constraints, and investigated necessary and sufficient Karush–Kuhn–Tucker optimality conditions for strict minimal solutions. Zhao et al. [25] proposed a projected subgradient method for solving constrained nondifferentiable, quasiconvex, multiobjective optimization problems, and presented numerical results to illustrate their findings.

When the interior of the ordered cone is empty, but the quasirelative interior is nonempty, how do we introduce approximate solutions of set optimization? How do we establish the optimality conditions?

The paper is organized as follows. Section 2 gives some preliminaries. In Sect. 3, we give definitions of two kinds of nonlinear functions and investigate their properties. In Sect. 4, we give several kinds of approximate solutions for set optimization and study their properties. In Sect. 5, we establish optimality conditions for approximate solutions. Finally, Sect. 6 draws some conclusions from the paper.

2 Preliminaries

Let X be a linear space, Y be a real normed linear space, and K be a proper pointed convex cone in Y. Denote by $\mathcal{P}(Y)$ and $\mathcal{B}(Y)$ the families of nonempty subsets and nonempty bounded subsets of Y, respectively. Let A be a nonempty subset of Y, $\operatorname{int} A$ and $\operatorname{cl} A$ denote the interior and closure of A, respectively. The generated cone of A is defined as

$$\operatorname{cone} A = \{\alpha a | \alpha \geq 0, a \in A\}.$$

Suppose that A is a nonempty convex subset of Y, the quasirelative interior [24] of A is defined as

$$\mathrm{qri}A := \{y \in A : \mathrm{clcone}(A - y) \text{ is a linear subspeace of } Y\}.$$

Remark 2.1 ([3, 24]) Let $A \subset Y$ be a convex subset with a nonempty interior, then $\mathrm{int}A = \mathrm{qri}A$.

Definition 2.1 ([11]) Let $A, B \in \mathcal{P}(Y)$. The Minkowski difference of A and B is defined as

$$A \dot{-} B = \{y \in Y : y + B \subset A\} = \bigcap_{b \in B} (A - b).$$

Lemma 2.1 ([11]) *Let $A, B \in \mathcal{P}(Y)$ and $c \in Y$. Then,*
(i) $(c + A) \dot{-} B = c + (A \dot{-} B)$;
(ii) $A \dot{-} (c + B) = -c + (A \dot{-} B)$.

Lemma 2.2 ([11]) *If $A \in \mathcal{B}(Y)$, then $A \dot{-} A = \{0_Y\}$.*

Lemma 2.3 ([4]) *Let C be a nonempty convex subset of Y and $\alpha \in \mathbb{R}$. Then,*
(i) $\mathrm{qri}(\alpha C) = \alpha \mathrm{qri}C$;
(ii) $t\mathrm{qri}C + (1 - t)C \subset \mathrm{qri}C$, $\forall t \in (0, 1]$.

Remark 2.2 From Lemma 2.3, it follows that if C is a nonempty convex cone of Y, then $C + \mathrm{qri}C \subset \mathrm{qri}C$.

In the rest of the paper, we assume that K and S are nonempty pointed convex cones of Y, $\mathrm{int}K \neq \emptyset$ and $\mathrm{qri}S \neq \emptyset$, respectively.

Definition 2.2 ([7]) Let $A, B \in \mathcal{P}(Y)$. If $(A \dot{-} B) \cap (-S) \neq \emptyset$, then we note that A is less than or equal to B with respect to S, denoted by $A \preceq_S^m B$.

Remark 2.3 The order relation that $A \preceq_S^m B$ in [7] is equivalent to that $A \preceq_S^{m_2} B$ in [11].

In the following, by using $\mathrm{qri}S$, we introduce a new type of relationship.

Definition 2.3 Let $A, B \in \mathcal{P}(Y)$. If $(A \dot{-} B) \cap (-\mathrm{qri}S) \neq \emptyset$, then we note that A is less than B with respect to $\mathrm{qri}S$, denoted by $A \prec_{\mathrm{qri}S}^m B$.

Definition 2.4 ([8]) Let $A, B \in \mathcal{P}(Y)$. If $(\bigcap_{a \in A} (a + K)) \cap B \neq \emptyset$, then we note that A is less than or equal to B with respect to K under the h relation, denoted by $A \preceq_K^h B$.

Definition 2.5 ([8]) Let $A, B \in \mathcal{P}(Y)$. If $A = B$, then $A \preceq_K^p B$; if $A \neq B$, then $A \preceq_K^p B \Longleftrightarrow A \preceq_K^h B$, then we note that A is less than or equal to B with respect to K under the p relation.

Definition 2.6 ([8]) Let $A, B \in \mathcal{P}(Y)$. If $(\bigcap_{a \in A} (a + K) + \mathrm{int}K) \cap B \neq \emptyset$, then we note that A is weakly less than B with respect to K under the p relation, denoted by $A \ll_K^p B$.

Proposition 2.1 *Let $A, B \in \mathcal{P}(Y)$. If $A = B$, then $A \not\ll_K^p B$.*

Proof Suppose to the contrary that $A \ll_K^p B$, then $(\bigcap_{a\in A}(a+K)+\mathrm{int}K)\cap A \neq \emptyset$. There exists $a_1 \in A$ such that $a_1 \in \bigcap_{a\in A}(a+K)+\mathrm{int}K$, therefore $a_1 \in a_1+K+\mathrm{int}K \subset a_1+\mathrm{int}K$. Then, we obtain $0 \in \mathrm{intK}$, which leads to a contradiction. Therefore, $A \not\ll_K^p B$. □

Proposition 2.2 *If $A \preceq_K^h B$, then $A \preceq_K^p B$.*

Proof It follows from the definition that the conclusion is true. □

Remark 2.4 The converse of Proposition 2.2 does not hold, as is shown in the following examples.

Example 1 Let $A = B = \{(x,y)|x^2+y^2 \leq 1\}$ and $K = \mathbb{R}_+^2$. It is clear that $A \preceq_K^p B$. On the other hand, $(\bigcap_{a\in A}(a+K))\cap B = \emptyset$. Hence, $A \not\preceq_K^h B$.

Example 2 Let $Y = l^2$, $K = l_+^2 = \{y = (y_n)_{n\in\mathbb{N}^+} \in Y|y_n \geq 0, n \in \mathbb{N}^+\}$ and $A = B = \{y = (y_n)_{n\in\mathbb{N}^+} \in K|\sum_{n=1}^{\infty} y_n \leq 1, n \in \mathbb{N}^+\}$. It is clear that $A \preceq_K^p B$. In the following, we prove $(\bigcap_{a\in A}(a+K))\cap B = \emptyset$. Suppose to the contrary that $(\bigcap_{a\in A}(a+K))\cap B \neq \emptyset$, then there exists $a_1 \in B = A$ such that $a_1 \in a+K$, $\forall a \in A$, that is $a_1 - a \in K$ for any $a \in A$, this obviously does not hold. Hence, $A \not\preceq_K^h B$.

Proposition 2.3 *If $A \ll_K^p B$, then $A \preceq_K^h B$ and $A \preceq_K^p B$.*

Proof Let $A \ll_K^p$B, then $(\bigcap_{a\in A}(a+K)+\mathrm{int}K) \cap B \neq \emptyset$. Since $\bigcap_{a\in A}(a+K)+\mathrm{int}K \subset \bigcap_{a\in A}(a+K)$, we obtain $A \preceq_K^h B$. We obtain $A \neq B$ from Proposition 2.1, therefore $A \preceq_K^p B$. □

3 Nonlinear functional

Let $\hat{e} \in \mathrm{qri}S$ and $\eta \in Y$, we construct the function $I_{\hat{e}}^m(\cdot,\cdot) : \mathcal{P}(Y)\times\mathcal{P}(Y) \to \bar{\mathbb{R}}$ as

$$I_{\hat{e}}^m(A,B) = \inf\{t \in \mathbb{R} \mid A \preceq_S^m t\hat{e}+B-\eta\}, \quad \forall A,B \in \mathcal{P}(Y).$$

Let $\hat{k} \in \mathrm{int}K$, $\xi \in Y$, we construct the function $I_{\hat{k}}^p(\cdot,\cdot) : \mathcal{P}(Y)\times\mathcal{P}(Y) \to \bar{\mathbb{R}}$ as

$$I_{\hat{k}}^p(A,B) = \inf\{t \in \mathbb{R} \mid A \preceq_K^p t\hat{k}+B-\xi\}, \quad \forall A,B \in \mathcal{P}(Y).$$

Proposition 3.1 *Let $A,B \in \mathcal{P}(Y)$. If $A = r\hat{k}+B-\xi$, then $I_{\hat{k}}^p(A,B) = r$.*

Proof If $A = r\hat{k}+B-\xi$, then it is clear that $A \preceq_K^p r\hat{k}+B-\xi$, hence $I_{\hat{k}}^p(A,B) \leq r$. In the following, we prove $A \not\preceq_K^p (r-\varepsilon)\hat{k}+B-\xi$, $\forall\varepsilon > 0$. Otherwise, there exists $\varepsilon_1 > 0$ such that $A \preceq_K^p (r-\varepsilon_1)\hat{k}+B-\xi$, hence $A \preceq_K^h (r-\varepsilon_1)\hat{k}+B-\xi$, that is $\bigcap_{a\in A}(a+K)\cap((r-\varepsilon_1)\hat{k}+B-\xi) \neq \emptyset$. Therefore, $\bigcap_{a\in A}(a+K)\cap(A-\varepsilon_1\hat{k}) \neq \emptyset$. Hence, there exists $a_1 \in A$ such that $a_1 - \varepsilon_1\hat{k} \in a+K$, $\forall a \in A$, thus $-\varepsilon_1\hat{k} \in K$, hence $\hat{k} \in -K$, which leads to a contradiction. Then, $I_{\hat{k}}^p(A,B) \geq r$. Hence, we have $I_{\hat{k}}^p(A,B) = r$. □

Proposition 3.2 *Let $A,B \in \mathcal{P}(Y)$. If $A \preceq_K^h r\hat{k}+B-\xi$, then $A \preceq_K^h (r+\varepsilon)\hat{k}+B-\xi$, $\forall\varepsilon > 0$.*

Proof If $A \preceq_K^h r\hat{k} + B - \xi$, then $\bigcap_{a\in A} (a + K) \cap (r\hat{k} + B - \xi) \neq \emptyset$. Hence, there exists $b_1 \in B$ such that $r\hat{k} + b_1 - \xi \in \bigcap_{a\in A} (a + K)$. Hence, $(r + \varepsilon)\hat{k} + b_1 - \xi \in \bigcap_{a\in A} (a + K) + \varepsilon\hat{k} \subset \bigcap_{a\in A} (a + K) + \mathrm{int}K \subset \bigcap_{a\in A} (a + K)$, $\forall\varepsilon > 0$. Therefore, $\bigcap_{a\in A} (a + K) \cap ((r+\varepsilon)\hat{k} + B - \xi) \neq \emptyset$, that is $A \preceq_K^h (r + \varepsilon)\hat{k} + B - \xi$. □

Remark 3.1 The following example indicates that the above proposition is not necessarily true for $\preceq_K^p$.

Example 3 Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = B = \{(x, y)|x^2 + y^2 \leq 1\}$, $r = 1$, $\xi = (1, 1)$ and $\hat{k} = (1, 1)$. It is clear that $A \preceq_K^p r\hat{k} + B - \xi$. When $0 < \varepsilon < \sqrt{2} - 1$, we obtain that $\bigcap_{a\in A} (a + K) \cap ((r + \varepsilon)\hat{k} + B - \xi) = \emptyset$. Hence, $A \not\preceq_K^p (r + \varepsilon)\hat{k} + B - \xi$.

Proposition 3.3 *Let $A, B \in \mathcal{P}(Y)$, $r \in \mathbb{R}$ and $\hat{e} \in \mathrm{qri}S$. Then, the following statements hold*:

(i) *If $I_{\hat{e}}^m(A, B) < r$, then $A \prec_{\mathrm{qri}S}^m r\hat{e} + B - \eta$.*

(ii) *Let $A \dot{-} B$ be compact. If $I_{\hat{e}}^m(A, B) = r$, then $A \preceq_{\mathrm{cl}S}^m r\hat{e} + B - \eta$ and $A \not\preceq_S^m (r - \lambda)\hat{e} + B - \eta$, $\forall\lambda > 0$.*

(iii) *If $A \preceq_S^m r\hat{e} + B - \eta$ and $A \not\preceq_S^m (r - \lambda)\hat{e} + B - \eta$, $\forall\lambda > 0$, then $I_{\hat{e}}^m(A, B) = r$.*

Proof (i) Assume that $I_{\hat{e}}^m(A, B) = \inf\{t \in \mathbb{R} \mid A \preceq_S^m t\hat{e} + B - \eta\} = \alpha < r$ and let $\mu = r - \alpha$. Then, there exists $t_1 \in \mathbb{R}$ such that $A \preceq_S^m t_1\hat{e} + B - \eta$ and $\alpha \leq t_1 < \alpha + \mu = r$, that is $(A \dot{-} (t_1\hat{e} + B - \eta)) \cap (-S) \neq \emptyset$. Hence, there exists $s_1 \in -S$ such that $s_1 \in A \dot{-} B - t_1\hat{e} + \eta$. Hence,

$$s_1 - (r - t_1)\hat{e} \in A \dot{-} B - r\hat{e} + \eta$$

and

$$s_1 - (r - t_1)\hat{e} \in -S - \mathrm{qri}S \subset -\mathrm{qri}S.$$

Hence, we obtain $(A \dot{-} (r\hat{e} + B - \eta)) \cap (-\mathrm{qri}S) \neq \emptyset$, that is $A \prec_{\mathrm{qri}S}^m r\hat{e} + B - \eta$.

(ii) Let $I_{\hat{e}}^m(A, B) = r$. Since $I_{\hat{e}}^m(A, B) < r + \frac{1}{n}$ for all $n \in \mathbb{N}^+$, we have $A \prec_{\mathrm{qri}S}^m (r + \frac{1}{n})\hat{e} + B - \eta$ from (i). Hence, we obtain $A \preceq_S^m (r + \frac{1}{n})\hat{e} + B - \eta$ for all $n \in \mathbb{N}^+$, that is

$$\left(A \dot{-} B - \left(r + \frac{1}{n}\right)\hat{e} + \eta\right) \cap (-S) \neq \emptyset.$$

Hence, there exists $x_n \in A \dot{-} B$ such that

$$x_n - \left(r + \frac{1}{n}\right)\hat{e} + \eta \in -S, \quad \forall n \in \mathbb{N}^+. \tag{3.1}$$

Since $A \dot{-} B$ is compact, it follows that there exists a convergent subsequence of $\{x_n\}$. Without loss of generality, we let $x_n \to x_0 \in A \dot{-} B$. From (3.1), we obtain

$$x_n - \left(r + \frac{1}{n}\right)\hat{e} + \eta \to x_0 - r\hat{e} + \eta \in -\mathrm{cl}S.$$

Hence, we obtain $(A \dot{-} (r\hat{e} + B) + \eta) \cap (-\mathrm{cl}S) \neq \emptyset$, that is $A \preceq_{\mathrm{cl}S}^m r\hat{e} + B - \eta$. Since $I_{\hat{e}}^m(A, B) = r$, we obtain $A \not\preceq_S^m (r - \lambda)\hat{e} + B - \eta$ for any $\lambda > 0$ from the definition.

(iii) It follows that $I_{\hat{e}}^m(A, B) = r$ from the definition. □

The converse of Proposition 3.3(i) is not true. The following example justifies this.

Example 4 Let $Y = l^2$, $\|y\|_2 = (\sum_{n=1}^{\infty} y_i^2)^{\frac{1}{2}}$, $\forall y = (y_n)_{n\in\mathbb{N}}$ and $S = l_+^2 = \{y = (y_n)_{n\in\mathbb{N}} \in Y | y_n \geq 0, n \in \mathbb{N}\}$. We obtain $\mathrm{qri}S = \{y \in Y | y_n > 0, n \in \mathbb{N}\}$ from [24]. Let $a_1 = (-\frac{1}{4}, -\frac{1}{4^2}, -\frac{1}{4^3}, \ldots, -\frac{1}{4^n}, \ldots)$, $a_2 = (-\frac{1}{5}, -\frac{1}{5^2}, -\frac{1}{5^3}, \ldots - \frac{1}{5^n}, \ldots)$, $A = \{a_1, a_2\}$, $\hat{e} = (\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots, \frac{1}{3^n}, \ldots)$, $B = \{0_Y\}$, $\eta = 0_Y$. We obtain $(A \dot{-} B) \cap (-\mathrm{qri}S) \neq \emptyset$, hence, $A \prec_{\mathrm{qri}S}^m B + r_0\hat{e}$ for $r_0 = 0$.

Next, we verify $I_{\hat{e}}^m(A,B) \geq r_0 = 0$. Suppose to the contrary that there exists $t_0 < 0$ such that $A \preceq_S^m t_0\hat{e} + B$, then $A \preceq_S^m t_0\hat{e}$. We obtain $a_1 - t_0\hat{e} \in -S$ or $a_2 - t_0\hat{e} \in -S$. If $a_1 - t_0\hat{e} \in -S$, then $-\frac{1}{4^n} - t_0\frac{1}{3^n} \leq 0$, for $n \to \infty$ we obtain $t_0 \geq 0$; if $a_2 - t_0\hat{e} \in -S$, similarly, we obtain $t_0 \geq 0$.

Proposition 3.4 *Let $A, B \in \mathcal{P}(Y)$, $r \in \mathbb{R}$ and $\hat{k} \in \mathrm{int}K$. Then, the following statements are true*:

(i) *If $A \ll_K^p r\hat{k} + B - \xi$, then $I_{\hat{k}}^p(A,B) < r$.*

(ii) *Let B be compact, then $I_{\hat{k}}^p(A,B) = r$ if and only if $A \preceq_{\mathrm{cl}K}^p r\hat{k} + B - \xi$ and $A \npreceq_K^p (r-\varepsilon)\hat{k} + B - \xi$, $\forall \varepsilon > 0$.*

Proof (i) If $A \ll_K^p r\hat{k} + B - \xi$, then $(\bigcap_{a\in A}(a+K) + \mathrm{int}K) \cap (r\hat{k} + B - \xi) \neq \emptyset$, hence, there exists $b_1 \in B$ such that $r\hat{k} + b_1 - \xi \in \bigcap_{a\in A}(a+K) + \mathrm{int}K$. Hence, there is $y_1 \in \bigcap_{a\in A}(a+K)$ such that $r\hat{k} + b_1 - \xi \in y_1 + \mathrm{int}K$, then there exists $\epsilon_1 > 0$ such that $r\hat{k} + b_1 - \xi - \epsilon_1\hat{k} \in y_1 + \mathrm{int}K$, that is $(\bigcap_{a\in A}(a+K) + \mathrm{int}K) \cap ((r-\epsilon_1)\hat{k} + B - \xi) \neq \emptyset$. Then, we obtain $(\bigcap_{a\in A}(a+K)) \cap ((r-\epsilon_1)\hat{k} + B - \xi) \neq \emptyset$, i.e., $A \preceq_K^h (r-\epsilon_1)\hat{k} + B - \xi$. We obtain $A \preceq_K^p (r-\epsilon_1)\hat{k} + B - \xi$ from Proposition 2.2. Hence, $I_{\hat{k}}^p(A,B) \leq r - \epsilon_1 < r$.

(ii) Let $I_{\hat{k}}^p(A,B) = r$. Case 1: if $A = r\hat{k} + B - \xi$, then $A \preceq_{\mathrm{cl}K}^p r\hat{k} + B - \xi$; Case 2: if $A \neq r\hat{k} + B - \xi$, from the definition of infimum, it follows that there exists $t_n \in [r, r + \frac{1}{n})$ such that $A \preceq_K^p t_n\hat{k} + B - \xi$ for $n \in \mathbb{N}$. We further assert that $A \neq t_n\hat{k} + B - \xi$ for any $n \in \mathbb{N}$. Otherwise, if there is $n_1 \in \mathbb{N}$ such that $A = t_{n_1}\hat{k} + B - \xi$, which together with $A \neq r\hat{k} + B - \xi$ gives that $t_{n_1} \neq r$. On the other hand, from Proposition 3.1, it follows that $I_{\hat{k}}^p(A,B) = t_{n_1}$. This contradicts $I_{\hat{k}}^p(A,B) = r$. Therefore, $A \preceq_K^h t_n\hat{k} + B - \xi$. We obtain $A \preceq_K^h (r + \frac{1}{n})\hat{k} + B - \xi$ from Proposition 3.2, $\forall n \in \mathbb{N}$. Hence, there exists $x_n \in B$ such that

$$\left(r + \frac{1}{n}\right)\hat{k} + x_n - \xi \in \bigcap_{a\in A}(a+K), \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Since B is compact, there exists a convergent subsequence of $\{x_n\}$. Without loss of generality, let $x_n \to x_0 \in B$. From (3.2), we have

$$\left(r + \frac{1}{n}\right)\hat{k} + x_n - \xi \to r\hat{k} + x_0 - \xi \in \bigcap_{a\in A}(a + \mathrm{cl}K).$$

Hence, we obtain $\bigcap_{a\in A}(a + \mathrm{cl}K) \cap (r\hat{k} + B - \xi) \neq \emptyset$, hence, $A \preceq_{\mathrm{cl}K}^h r\hat{k} + B - \xi$. Therefore, $A \preceq_{\mathrm{cl}K}^p r\hat{k} + B - \xi$. We obtain $A \npreceq_K^p (r-\varepsilon)\hat{k} + B - \xi$ for $\varepsilon > 0$ from the definition of $I_{\hat{k}}^p(A,B) = r$.

Conversely, let $A \preceq_{\mathrm{cl}K}^p r\hat{k} + B - \xi$. Case 1: if $A = r\hat{k} + B - \xi$, then $A \preceq_K^p r\hat{k} + B - \xi$, then $I_{\hat{k}}^p(A,B) \leq r$; Case 2: if $A \neq r\hat{k} + B - \xi$, then $A \preceq_{\mathrm{cl}K}^h r\hat{k} + B - \xi$, there exists $y_1 \in \bigcap_{a\in A}(a + \mathrm{cl}K) \cap (r\hat{k} + B - \xi)$. Then, $y_1 \in a + \mathrm{cl}K$ for all $a \in A$ and $y_1 \in r\hat{k} + B - \xi$. Hence, $y_1 + \frac{1}{n}\hat{k} \in a + \mathrm{cl}K + \mathrm{int}K \subset a + \mathrm{int}K$ for $n \in \mathbb{N}$, then $y_1 + \frac{1}{n}\hat{k} \in \bigcap_{a\in A}(a + \mathrm{int}K)$, so $y_1 + \frac{1}{n}\hat{k} \in$

$\bigcap_{a\in A}(a+\mathrm{int}K)\cap[(r+\frac{1}{n})\hat{k}+B-\xi]$. Hence,

$$\bigcap_{a\in A}(a+\mathrm{int}K)\cap\left[\left(r+\frac{1}{n}\right)\hat{k}+B-\xi\right]\neq\emptyset,\quad\forall n\in\mathbb{N}.$$

We obtain $A\preceq^h_{\mathrm{int}K}(r+\frac{1}{n})\hat{k}+B-\xi$, thus $A\preceq^h_K(r+\frac{1}{n})\hat{k}+B-\xi$, we obtain $A\preceq^p_K(r+\frac{1}{n})\hat{k}+B-\xi$ from Proposition 2.2. Hence, we have $I^p_{\hat{k}}(A,B)\leq r+\frac{1}{n}$. Then, $I^p_{\hat{k}}(A,B)\leq r$ for $n\to\infty$. From the assumption we have $I^p_{\hat{k}}(A,B)\geq r$. Hence, we have $I^p_{\hat{k}}(A,B)=r$. □

The following example shows that the converse of Proposition 3.4(i) is false.

Example 5 Let $A=\{(x,y)|x^2+y^2\leq 1\}$, $B=A$, $K=\mathbb{R}^2_+$, $\hat{k}=(1,1)$ and $\xi=(1,1)$. We can obtain $A=t\hat{k}+B-\xi$ for $t=1$, and $\bigcap_{a\in A}(a+K)\cap(t\hat{k}+B-\xi)=\emptyset$ for $t<1$, respectively. Then, $I^p_{\hat{k}}(A,B)=1$. However, when $1<r\leq\sqrt{2}$, we have $(\bigcap_{a\in A}(a+K)+\mathrm{int}K)\cap(r\hat{k}+B-\xi)=\emptyset$. Hence, $A\not\ll^p_K r\hat{k}+B-\xi$.

Definition 3.1 A function $T(\cdot):\mathcal{P}(Y)\to\bar{\mathbb{R}}$ is called m-increasing (p-increasing) on $\mathcal{P}(Y)$ if $A,B\in\mathcal{P}(Y)$ and $A\preceq^m_S B(A\preceq^p_K B)$ implies $T(A)\leq T(B)$.

Proposition 3.5 *Let $A\in\mathcal{P}(Y)$. Then, $I^m_{\hat{e}}(\cdot,A)$ is m-increasing on $\mathcal{P}(Y)$.*

Proof It can be proved similarly to [11, Proposition 15]. □

Corollary 3.1 *Let $A\in\mathcal{B}(Y)$, then $I^m_{\hat{e}}(A-\eta,A)=0$.*

Proof $I^m_{\hat{e}}(A-\eta,A)=\inf\{t\in\mathbb{R}|A-\eta\preceq^m_S A+t\hat{e}-\eta\}=\inf\{t\in\mathbb{R}|A\preceq^m_S A+t\hat{e}\}=\inf\{t\in\mathbb{R}|A\dot{-}A\preceq^m_S t\hat{e}\}=\inf\{t\in\mathbb{R}|t\hat{e}\in S\}=0$. □

Proposition 3.6 *Let $A\in\mathcal{P}(Y)$. Then, $I^p_{\hat{k}}(\cdot,A)$ is p-increasing on $\mathcal{P}(Y)$.*

Proof Let $D,E\in\mathcal{P}(Y)$ and $D\preceq^p_K E$. Case 1: if $I^p_{\hat{k}}(E,A)=\alpha\in\mathbb{R}$, then $E\preceq^p_K(\alpha+\epsilon)\hat{k}+A-\xi$, $\forall\epsilon>0$. Since $D\preceq^p_K E$ and $\preceq^p_K$ is transitive, we have $D\preceq^p_K(\alpha+\epsilon)\hat{k}+A-\xi$. Hence, $I^p_{\hat{k}}(D,A)\leq\alpha+\epsilon$. We obtain $I^p_{\hat{k}}(D,A)\leq\alpha=I^p_{\hat{k}}(E,A)$ for $\epsilon\to 0^+$.

Case 2: If $I^p_{\hat{k}}(E,A)=-\infty$, then $E\preceq^p_K t\hat{k}+A-\xi,\forall t\in\mathbb{R}$. Since $D\preceq^p_K E$ and $\preceq^p_K$ is transitive, we have $D\preceq^p_K t\hat{k}+A-\xi,\forall t\in\mathbb{R}$. Then, $I^p_{\hat{k}}(D,A)=-\infty$.

Case 3: If $I^p_{\hat{k}}(E,A)=+\infty$, then it is clear that $I^p_{\hat{k}}(D,A)\leq I^p_{\hat{k}}(E,A)$. □

Proposition 3.7 *Let $A,B\in P(Y)$ and $I^m_{\hat{e}}(A,B)$ be finite. If $A\dot{-}B$ is compact and S is closed, then $A\preceq^m_S B-\eta$ if and only if $I^m_{\hat{e}}(A,B)\leq 0$.*

Proof It can be proved similarly to [11, Proposition 19(i)]. □

Proposition 3.8 *Let $A\in\mathcal{B}(Y)$, then $I^p_{\hat{k}}(A-\xi,A)=0$.*

Proof $I^p_{\hat{k}}(A-\xi,A)=\inf\{t\in\mathbb{R}|A-\xi\preceq^p_K A+t\hat{k}-\xi\}=\inf\{t\in\mathbb{R}|A\preceq^p_K A+t\hat{k}\}$. When $t=0$, $A\preceq^p_K A$ is obviously true, that is $I^p_{\hat{k}}(A-\xi,A)\leq 0$. Now, we prove that $A\not\preceq^p_K A+t\hat{k}$, $\forall t<0$. Suppose to the contrary that $A\preceq^p_K A+t_1\hat{k}$ for some $t_1<0$. Since $A\neq A+t_1\hat{k}$, we obtain

$A \preceq_K^h A + t_1\hat{k}$, that is $\bigcap_{a\in A}(a+K)\cap(t_1\hat{k}+A)\neq\emptyset$. Then, there is $a_1\in A$ such that $t_1\hat{k}+a_1\in\bigcap_{a\in A}(a+K)$, that is $t_1\hat{k}+a_1\in a+K$, $\forall a\in A$. Let $a=a_1$, then $t_1\hat{k}\in K$. Since $t_1<0$, we have $\hat{k}\in -K$. Therefore $\mathrm{int}K\cap(-K)\neq\emptyset$, which contradicts that K is a pointed convex cone. Hence, $A\not\preceq_K^p A+t\hat{k}$ for $t<0$, i.e., $I_{\hat{k}}^p(A-\xi,A)\geq 0$. Hence, $I_{\hat{k}}^p(A-\xi,A)=0$. □

Corollary 3.2 *Let $A,B\in\mathcal{P}(Y)$. If B is compact, K is closed and $I_{\hat{k}}^p(A,B)$ is finite, then $I_{\hat{k}}^p(A,B)=\min\{t\in\mathbb{R}|A\preceq_K^p t\hat{k}+B-\xi\}$.*

Proof We have $\{t\in\mathbb{R}|A\preceq_K^p t\hat{k}+B-\xi\}=\{t\in\mathbb{R}|I_{\hat{k}}^p(A,B)\leq t\}=[I_{\hat{k}}^p(A,B),\infty)$ from Proposition 3.4(ii). Hence, $I_{\hat{k}}^p(A,B)=\min\{t\in\mathbb{R}|A\preceq_K^p t\hat{k}+B-\xi\}$. □

4 Approximately efficient solutions and approximate weakly efficient solutions for set-optimization problems

We consider a set-valued mapping $F: X\rightrightarrows Y$ and a nonempty set $T\subset X$ where $F(x)\in\mathcal{P}(Y)$ for each $x\in X$. We deal with the constrained set-optimization problem (P) defined by

$$\text{(P)}\quad \min F(x),\qquad \text{s.t. } x\in T.$$

Definition 4.1 ([7]) An element $x_0\in T$ is called an m-efficient solution of (P), if $F(x)\not\preceq_S^m F(x_0)$ or $F(x)=F(x_0)$ for any $x\in T$. Let m-$\mathrm{E}_S(F,T)$ denote the set of m-efficient solutions of (P).

In what follows, we introduce a new class of approximate solutions.

Definition 4.2 Let $\eta\in Y$. An element $x_0\in T$ is called an η-m-efficient solution of (P), if $F(x)+\eta\not\preceq_S^m F(x_0)$ or $F(x)+\eta=F(x_0)$ for $x\in T$. Let η-m-$E_S(F,T)$ denote the set of η-m-efficient solutions of (P).

In the following, by using qri$S$, we introduce another new class of weakly efficient solutions.

Definition 4.3 Let $\eta\in Y$. An element $x_0\in T$ is called

(i) an m-weakly efficient solution of (P) with respect to qri$S$, if $F(x)\not\prec_{\mathrm{qri}S}^m F(x_0)$ for $x\in T$;

(ii) an η-m-weakly efficient solution of (P) with respect to qri$S$, if $F(x)+\eta\not\prec_{\mathrm{qri}S}^m F(x_0)$ for $x\in T$.

Let m-$W_{\mathrm{qri}S}(F,T)$ and η-m-$W_{\mathrm{qri}S}(F,T)$ denote the sets of m-weakly efficient solutions and η-m-weakly efficient solutions of (P) with respect to qri$S$, respectively.

Remark 4.1 If $\eta=0$, then η-m-weakly efficient solutions reduce to m-weakly efficient solutions and η-m-efficient solutions reduce to m-efficient solutions of (P), respectively.

Theorem 4.1 *Let $\eta\in S$. Then, the following statements are true*:

(i) *Let $F(x_0)\in\mathcal{B}(Y)$ and $x_0\in m$-$E_S(F,T)$, then $x_0\in\eta$-m-$E_S(F,T)$.*

(ii) *Let S be closed, $F(x_0) \in \mathcal{B}(Y)$ and $x_0 \in \eta\text{-}m\text{-}E_S(F,T)$, then $x_0 \in \eta\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$.*
(iii) $m\text{-}W_{\mathrm{qri}S}(F,T) \subset \bigcap_{\eta \in S\setminus\{0\}} \eta\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$.
(iv) *Let S be closed, then $\bigcap_{\eta \in S\setminus\{0\}} \eta\text{-}m\text{-}W_{\mathrm{qri}S}(F,T) \subset m\text{-}W_{\mathrm{qri}S}(F,T)$.*

Proof (i) ① If $\eta = 0$, then the conclusion holds obviously.

② If $\eta \in S\setminus\{0\}$, suppose that $x_0 \in m\text{-}E_S(F,T)$. $\forall x \in T$, we consider two cases.

Case 1: when $F(x) = F(x_0)$, we prove $((F(x)+\eta) \dot{-} F(x_0)) \cap (-S) = \emptyset$. In fact

$$\big(F(x)+\eta\big) \dot{-} F(x_0) = \eta + F(x) \dot{-} F(x_0) = \eta + 0_Y = \eta.$$

Since $\eta \in S\setminus\{0\}$ and S is a proper pointed convex cone, then $\eta \notin -S$. Hence, we obtain $((F(x)+\eta) \dot{-} F(x_0)) \cap (-S) = \emptyset$, thus $F(x)+\eta \npreceq_S^m F(x_0)$;

Case 2: when $F(x) \neq F(x_0)$. Since $x_0 \in m\text{-}E_S(F,T)$, we obtain $F(x) \npreceq_S^m F(x_0)$, that is

$$\big(F(x) \dot{-} F(x_0)\big) \cap (-S) = \emptyset. \tag{4.1}$$

If $F(x)+\eta = F(x_0)$, then $x_0 \in \eta\text{-}m\text{-}E_S(F,T)$ holds obviously. If $F(x)+\eta \neq F(x_0)$, then we prove $((F(x)+\eta) \dot{-} F(x_0)) \cap (-S) = \emptyset$. Suppose to the contrary that there exists x_1 such that $((F(x_1)+\eta) \dot{-} F(x_0)) \cap (-S) \neq \emptyset$, then there exist $s_1 \in -S$ and $y_0 \in F(x_1) \dot{-} F(x_0)$ such that $s_1 = \eta + y_0$. Then, $y_0 = s_1 - \eta \in -S - S \subset -S$. Thus, $y_0 \in (F(x_1) \dot{-} F(x_0)) \cap (-S)$, which contradicts (4.1).

Therefore, we obtain $x_0 \in \eta\text{-}m\text{-}E_S(F,T)$.

(ii) Let $x_0 \in \eta\text{-}m\text{-}E_S(F,T)$. $\forall x \in T$, we consider two cases. Case 1: when $F(x)+\eta \npreceq_S^m F(x_0)$, that is $((F(x)+\eta) \dot{-} F(x_0)) \cap (-S) = \emptyset$. It follows from $\mathrm{qri}S \subset S$ that

$$\big(\big(F(x)+\eta\big) \dot{-} F(x_0)\big) \cap (-\mathrm{qri}S) = \emptyset.$$

Case 2: when $F(x)+\eta = F(x_0)$, we obtain $(F(x)+\eta) \dot{-} F(x_0) = \{0\}$ from $F(x_0) \in \mathcal{B}(Y)$. On the other hand, we obtain $0 \notin \mathrm{qri}S$ from [24, Lemma 4.1], then

$$\big(\big(F(x)+\eta\big) \dot{-} F(x_0)\big) \cap (-\mathrm{qri}S) = \emptyset.$$

Thus, $F(x)+\eta \nprec_{\mathrm{qri}S}^m F(x_0)$, $\forall x \in T$. Hence, $x_0 \in \eta\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$.

(iii) Let $x_0 \in m\text{-}W_{\mathrm{qri}S}(F,T)$, then $F(x) \nprec_{\mathrm{qri}S}^m F(x_0)$ for $\forall x \in T$, that is

$$\big(F(x) \dot{-} F(x_0)\big) \cap (-\mathrm{qri}S) = \emptyset, \quad \forall x \in T. \tag{4.2}$$

Suppose to the contrary that $x_0 \notin \bigcap_{\eta \in S\setminus\{0\}} \eta\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$, then $x_0 \notin \eta_1\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$ for some $\eta_1 \in S\setminus\{0\}$. Then, there exists $x_1 \in T$ such that $((\eta_1 + F(x_1)) \dot{-} F(x_0)) \cap (-\mathrm{qri}S) \neq \emptyset$. Then, there exists $s_1 \in -\mathrm{qri}S$ and $y_0 \in F(x_1) \dot{-} F(x_0)$ such that $s_1 = \eta_1 + y_0$. Hence, $y_0 = s_1 - \eta_1 \in -\mathrm{qri}S - S \subset -\mathrm{qri}S$, we obtain $y_0 \in (F(x_1) \dot{-} F(x_0)) \cap (-\mathrm{qri}S)$, which contradicts (4.2).

(iv) Let $x_0 \in T$ and $x_0 \notin m\text{-}W_{\mathrm{qri}S}(F,T)$. Then, there exists $x_2 \in T$ such that $(F(x_2) \dot{-} F(x_0)) \cap (-\mathrm{qri}S) \neq \emptyset$. Hence, there is $s_2 \in \mathrm{qri}S$ such that $-s_2 \in F(x_2) \dot{-} F(x_0)$, and $-\frac{s_2}{2} = -s_2 + \frac{s_2}{2} \in F(x_2) \dot{-} F(x_0) + \frac{s_2}{2}$. It follows that $\frac{s_2}{2} \in \mathrm{qri}S$ from the fact that S is a convex cone.

Therefore,

$$\left(F(x_2) \dot{-} F(x_0) + \frac{s_2}{2}\right) \cap (-\mathrm{qri}S) \neq \emptyset.$$

Hence, $x_0 \notin \frac{s_2}{2}\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$. We obtain $0 \notin \mathrm{qri}S$ from [24, Lemma 4.1], then $s_2 \neq 0$. Therefore, $\frac{s_2}{2} \in S\backslash\{0\}$. Hence, $x_0 \notin \bigcap_{\eta\in S\backslash\{0\}} \eta\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$. □

Remark 4.2 In the following, let $e \in S\backslash\{0\}$, we consider a special class of approximate weakly efficient solutions, where $\eta = \epsilon e$, $\epsilon > 0$. From Theorem 4.1(iii), we obtain $m\text{-}W_{\mathrm{qri}S}(F,T) \subset \bigcap_{\epsilon>0} \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$. However, the converse inclusion may not hold. The following example illustrates the case.

Example 6 Let $X = \mathbb{R}$, $T = [-1,1]$, $Y = l^2$, $e = (\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots)$, $\|y\|_2 = (\sum_{i=1}^{\infty} y_i^2)^{\frac{1}{2}}$, $\forall y = (y_n)_{n\in\mathbb{N}}$, and $S = l_+^2 = \{y = (y_n)_{n\in\mathbb{N}} \in Y | y_n \geq 0, n \in \mathbb{N}\}$. We obtain $\mathrm{qri}S = \{y \in Y | y_n > 0, n \in \mathbb{N}\}$ from [24]. Consider $F : T \rightrightarrows Y$ defined as

$$F(x) = \begin{cases} \{(\frac{x}{2}, \frac{x}{2^2}, \frac{x}{2^3}, \ldots, \frac{x}{2^n}, \ldots)\}, & 0 \leq x \leq 1, \\ \{(-\frac{1}{3}, -\frac{1}{3^2}, -\frac{1}{3^3}, \ldots, -\frac{1}{3^n}, \ldots)\}, & -1 \leq x < 0. \end{cases}$$

A direct calculation gives that $m\text{-}W_{\mathrm{qri}S}(F,T) = [-1,0)$, thus $[-1,0) \subset \bigcap_{\epsilon>0} \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$.

Let us verify $0 \in \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$ for $\forall \epsilon > 0$. In fact, $F(x) \dot{-} F(0) + \epsilon e = F(x) + \epsilon e$. We consider two cases. Case 1: when $x \in [0,1]$, $F(x) + \epsilon e = (\frac{x+\epsilon}{2}, \frac{x+\epsilon}{2^2}, \ldots, \frac{x+\epsilon}{2^n}, \ldots) \notin -\mathrm{qri}K$; Case 2: when $x \in [-1,0)$, $F(x) + \epsilon e = (-\frac{1}{3} + \frac{\epsilon}{2}, -\frac{1}{3^2} + \frac{\epsilon}{2^2}, \ldots, -\frac{1}{3^n} + \frac{\epsilon}{2^n}, \ldots)$. Since

$$-\frac{1}{3^n} + \frac{\epsilon}{2^n} = \frac{1}{2^n}\left(-\left(\frac{2}{3}\right)^n + \epsilon\right) \to 0^+,$$

therefore $F(x) + \epsilon e \notin -\mathrm{qri}S$. It follows from the above two cases that $(F(x) \dot{-} F(0) + \epsilon e) \cap (-\mathrm{qri}S) = \emptyset$ for $\forall x \in T$. Thus, $0 \in \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$.

It follows from Theorem 4.1(iii) that

$$[-1,0] \subset \bigcap_{\epsilon>0} \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T). \tag{4.3}$$

In the following, we prove

$$x_0 \notin x_0 e\text{-}m\text{-}W_{\mathrm{qri}K}(F,T), \quad \forall x_0 \in (0,1]. \tag{4.4}$$

In fact, take $x_1 = -1 \in T$, $(F(x_1) + x_0 e) \dot{-} F(x_0) = F(-1) \in -\mathrm{qri}S$. Hence, $x_0 \notin \epsilon_0 e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$, where $\epsilon_0 = x_0 > 0$. Hence, we obtain $x_0 \notin \cap_{\epsilon>0} \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T)$. From $\bigcap_{\epsilon>0} \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T) \subset [-1,1]$ and (4.4) we obtain $\bigcap_{\epsilon>0} \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T) \subset [-1,0]$, which together with (4.3) gives

$$\bigcap_{\epsilon>0} \epsilon e\text{-}m\text{-}W_{\mathrm{qri}S}(F,T) = [-1,0].$$

Next, we introduce the efficient solution and approximately efficient solution under the $\preceq_K^p$ and $\ll_K^p$ order relation, respectively.

Definition 4.4 Let $x_0 \in T$ and $\xi \in K$. An element $x_0 \in T$ is called

(i) an efficient solution of (P) under order relation $\preceq_K^p$, if $F(x) \npreceq_K^p F(x_0)$ or $F(x) = F(x_0)$ for $\forall x \in T$;

(ii) a ξ-efficient solution of (P) under order relation $\preceq_K^p$, if $F(x) + \xi \npreceq_K^p F(x_0)$ or $F(x) + \xi = F(x_0)$ for $\forall x \in T$;

(iii) a weakly efficient solution of (P) under order relation $\ll_K^p$, if $F(x) \not\ll_K^p F(x_0)$ for $\forall x \in T$;

(iv) a ξ-weakly efficient solution of (P) under order relation $\ll_K^p$, if $F(x) + \xi \not\ll_K^p F(x_0)$ for $\forall x \in T$.

Let $E_K^p(F,T)$, ξ-$E_K^p(F,T)$, $W_K^p(F,T)$, and ξ-$W_K^p(F,T)$ denote the sets of efficient solutions, ξ-efficient solutions, weakly efficient solutions, and ξ-weakly efficient solutions of (P), respectively.

Theorem 4.2 *Let $\xi \in K$, then the following statements are true*:

(i) $E_K^p(F,T) \subset W_K^p(F,T)$;

(ii) $E_K^p(F,T) \subset \xi$-$E_K^p(F,T)$;

(iii) $W_K^p(F,T) = \bigcap_{\xi \in K\setminus\{0\}} \xi$-$W_K^p(F,T)$.

Proof (i) Let $x_0 \in T$ and $x_0 \notin W_K^p(F,T)$, then there exists $x_1 \in T$ such that $F(x_1) \ll_K^p F(x_0)$. It follows from Propositions 2.1 and 2.3 that $F(x_1) \neq F(x_0)$ and $F(x_1) \preceq_K^p F(x_0)$. Hence, $x_0 \notin E_K^p(F,T)$.

(ii) If $\xi = 0$, then the conclusion is obviously true. If $\xi \neq 0$, let $x_0 \in E_K^p(F,T)$. $\forall x \in T$, we consider two cases. Case 1: if $F(x) + \xi = F(x_0)$, then $x_0 \in \xi$-$E_K^p(F,T)$ is obvious; Case 2: if $F(x) + \xi \neq F(x_0)$, in the following, we prove $\bigcap_{y \in F(x)}(y + \xi + K) \cap F(x_0) = \emptyset$.

(1) When $F(x) \neq F(x_0)$, it follows from $x_0 \in E_K^p(F,T)$ that $F(x) \npreceq_K^p F(x_0)$. Hence, $\bigcap_{y\in F(x)}(y+K) \cap F(x_0) = \emptyset$. From $y + \xi + K \subset y + K$ we obtain $\bigcap_{y\in F(x)}(y + \xi + K) \cap F(x_0) = \emptyset$. Hence, $F(x) + \xi \npreceq_K^p F(x_0)$;

(2) When $F(x) = F(x_0)$, in the following, we prove $\bigcap_{y\in F(x_0)}(y + \xi + K) \cap F(x_0) = \emptyset$. Otherwise, there exists $y_0 \in F(x_0)$ such that $y_0 \in \bigcap_{y\in F(x_0)}(y + \xi + K)$, then $y_0 \in y_0 + \xi + K$. Therefore, $-\xi \in K$. It follows that $\xi \in K \cap (-K) = \{0\}$. Hence $\xi = 0$, which contradicts $\xi \neq 0$. Hence, $F(x) + \xi \npreceq_K^p F(x_0)$.

Summarizing the above discussion, we obtain $x_0 \in \xi$-$E_K^p(F,T)$.

(iii) "$\subset$" Let $x_0 \in W_K^p(F,T)$, then $(\bigcap_{y\in F(x)}(y + K) + \mathrm{int}K) \cap F(x_0) = \emptyset$, $\forall x \in T$. For each $\xi \in K\setminus\{0\}$, $\bigcap_{y\in F(x)}(y + \xi + K) = \bigcap_{y\in F(x)}(y + K) + \xi$. Therefore, $\bigcap_{y\in F(x)}(y + \xi + K) + \mathrm{int}K = \bigcap_{y\in F(x)}(y + K) + \xi + \mathrm{int}K \subset \bigcap_{y\in F(x)}(y + K) + \mathrm{int}K$. Hence, $(\bigcap_{y\in F(x)}(y + \xi + K) + \mathrm{int}K) \cap F(x_0) = \emptyset$, that is $F(x) + \xi \not\ll_K^p F(x_0)$. We obtain $x_0 \in \xi$-$W_K^p(F,T)$.

"$\supset$" Let $x_0 \in T$ and $x_0 \notin W_K^p(F,T)$. Then, there exists $x_1 \in T$ such that $F(x_1) \ll_K^p F(x_0)$, that is $(\bigcap_{y\in F(x_1)}(y + K) + \mathrm{int}K) \cap F(x_0) \neq \emptyset$. Hence, there is $y_0 \in F(x_0)$ such that $y_0 \in \bigcap_{y\in F(x_1)}(y + K) + \mathrm{int}K$, then there exists $y_1 \in \bigcap_{y\in F(x_1)}(y + K)$ such that $y_0 \in y_1 + \mathrm{int}K$, therefore $y_0 \in \mathrm{int}(y_1 + \mathrm{int}K)$. Therefore, there exists $\xi_1 \in K\setminus\{0\}$ such that $y_0 - \xi_1 \in y_1 + \mathrm{int}K$, i.e., $y_0 \in y_1 + \xi_1 + \mathrm{int}K \subset \bigcap_{y\in F(x_1)}(y + K + \xi_1) + \mathrm{int}K$. We obtain $(\bigcap_{y\in F(x_1)}(y + K + \xi_1) + \mathrm{int}K) \cap F(x_0) \neq \emptyset$, that is $F(x_1) + \xi_1 \ll_K^p F(x_0)$. Therefore, $x_0 \notin \xi_1$-$W_K^p(F,T)$. Hence, $x_0 \notin \bigcap_{\xi\in K\setminus\{0\}} \xi$-$W_K^p(F,T)$. $\square$

Remark 4.3 Similar to the proof of Theorem 4.2(i), we can prove $\xi\text{-}E_K^p(F,T) \subset \xi\text{-}W_K^p(F,T)$.

5 Optimality conditions

Consider the set-optimization problem

$$(P)\begin{cases} \min F(x) \\ \text{s.t.} \quad x \in T. \end{cases}$$

Theorem 5.1 *Let $F(x)$ be compact for any $x \in T$ and S be closed, $x_0 \in T$. Then, x_0 is an η-m-efficient solution of (P) if and only if there exists a function $\phi : \mathcal{P}(Y) \to \bar{\mathbb{R}}$ that is m-increasing on $\mathcal{P}(Y)$ and satisfies the following statements*:

(i) $\phi(F(x_0) - \eta) = 0$,

(ii) $\phi(F(x)) > 0$ *for all* $x \in T \backslash \{x \in T : F(x) + \eta = F(x_0)\}$.

Proof Assume that $x_0 \in T$ is an η-m-efficient solution of (P). Define $\phi : \mathcal{P}(Y) \to \bar{\mathbb{R}}$ by $\phi(\cdot) = I_{\hat{e}}^m(\cdot, F(x_0))$. Then, $\phi(\cdot)$ is m-increasing on $\mathcal{P}(Y)$ from Proposition 3.5. We have $\phi(F(x_0) - \eta) = 0$ from Corollary 3.1. Since x_0 is an η-m-efficient solution, we obtain $F(x) + \eta \not\preceq_S^m F(x_0)$ for all $x \in T \backslash \{x \in T : F(x) + \eta = F(x_0)\}$ from the definition. It follows from Proposition 3.7 that $I_{\hat{e}}^m(F(x), F(x_0)) > 0$. Hence, $T(F(x)) > 0$.

Conversely, let (i) and (ii) be true for some $\phi : \mathcal{P}(Y) \to \bar{\mathbb{R}}$ that is m-increasing on $\mathcal{P}(Y)$. Suppose to the contrary that x_0 is not an η-m-efficient solution of (P). Then, there exists $x_1 \in T$ such that $F(x_1) + \eta \preceq_S^m F(x_0)$ and $F(x_1) + \eta \neq F(x_0)$, we obtain $I_{\hat{e}}^m(F(x_1), F(x_0)) \leq 0$ from Proposition 3.7. Hence, $\phi(F(x_1)) \leq 0$. This contradicts (ii). Therefore, x_0 is an η-m-efficient solution of (P). □

In what follows, we give an example to illustrate the necessity of the above theorem.

Example 7 Let $X = \mathbb{R}$, $T = [0,1]$, $Y = l^2$ and $S = l_+^2 = \{y = (y_n)_{n\in\mathbb{N}} \in Y | y_n \geq 0, n \in .\mathbb{N}\}$. Take $\hat{e} = (1, 1, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots)$, $\eta = (1, 2, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots)$. Let $F : X \rightrightarrows Y$ be defined as

$$F(x) = \{\lambda(0, x, 0, 0, \ldots) + (1 - \lambda)(1, x, 0, 0, \ldots), \lambda \in [0,1]\}, \quad x \in X.$$

A direct calculation gives that $F(x) \dot{-} F(0) = \{(0, x, 0, 0, \ldots)\}$ for $x \in T$. Hence, we have $(F(x) \dot{-} F(0) + \eta) \cap (-S) = \emptyset$, thus $F(x) + \eta \not\preceq_S^m F(0)$, and $F(x) + \eta \neq F(0)$ for $\forall x \in T$. Hence, $x_0 = 0$ is an η-m-efficient solution.

Define a function $\phi : \mathcal{P}(Y) \to \bar{\mathbb{R}}$ by

$$\phi(A) = I_{\hat{e}}^m(A, F(0)) = \inf\{t \in \mathbb{R} \mid A \preceq_S^m t\hat{e} + F(0) - \eta\}, \quad \forall A \in \mathcal{P}(Y).$$

Then,

$$\begin{aligned} \phi(F(0) - \eta) &= I_{\hat{e}}^m(F(0) - \eta, F(0)) \\ &= \inf\{t \in \mathbb{R} | F(0) \dot{-} F(0) \preceq_S^m t\hat{e}\} \\ &= \inf\left\{t \in \mathbb{R} | (0, 0, 0, \ldots) \leq t\left(1, 1, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right)\right\} \\ &= \inf\{t \in \mathbb{R} | t \geq 0\} \\ &= 0, \end{aligned}$$

$$\begin{aligned}\forall x \in T, \quad \phi\big(F(x)\big) &= I_{\hat{e}}^{m}\big(F(x), F(0)\big)\\ &= \inf\big\{t \in \mathbb{R} | F(x) \dot{-} F(0) \preceq_S^m t\hat{e} - \eta\big\}\\ &= \inf\left\{t \in \mathbb{R} | (0, x, 0, \ldots) \le \left(t-1, t-2, \frac{t-1}{3}, \frac{t-1}{4}, \ldots, \frac{t-1}{n}, \ldots\right)\right\}\\ &= \inf\{t \in \mathbb{R} | t \ge 2 + x\}\\ &= 2 + x.\end{aligned}$$

Thus, $\phi(F(x)) > 0$ for all $x \in T \backslash \{x \in T : F(x) + \eta = F(0)\}$.

Remark 5.1 According to Remark 2.1 and Theorem 4.1(i), Theorem 5.1 generalizes and improves [11, Theorem 2] in the following two aspects:

(i) From a nonempty interior to a nonempty quasirelative interior of an order cone.
(ii) From efficient solutions to approximately efficient solutions.

Theorem 5.2 *Let $T \subset X$, $F : T \to 2^Y$ and K be closed, $x_0 \in T$ and $F(x_0) \in \mathcal{P}(Y)$. Suppose that there exists a functional $\varphi : \mathcal{P}(Y) \to \bar{\mathbb{R}}$ satisfying*

(i) *φ is p-increasing on $\mathcal{P}(Y)$;*
(ii) *$\varphi(F(x_0) - \xi) = 0$;*
(iii) *for any $x \in T \backslash \{x \in T : F(x) + \xi = F(x_0)\}$, $\varphi(F(x)) > 0$.*

Then, x_0 is a ξ-efficient solution of (P).

Proof If x_0 is not a ξ-efficient solution of (P), then there exists $x_1 \in T$ such that $F(x_1) + \xi \preceq_K^p F(x_0)$ and $F(x_1) + \xi \neq F(x_0)$. Then, $\varphi(F(x_1)) \le \varphi(F(x_0) - \xi) = 0$, this is in contradiction with (iii). Hence, $x_0 \in T$ is a ξ-efficient solution of (P). □

Remark 5.2 We partly consider the converse proposition of Theorem 5.2. Let x_0 be a ξ-efficient solution of (P), although we can construct a functional φ similar to the proof of necessity of Theorem 5.1, however, φ may not satisfy (i)–(iii) of Theorem 5.2. The following example illustrates the case.

Example 8 Let $X = \mathbb{R}$, $T = [0, 2]$ and $K = \mathbb{R}_+^2$. Define the set-valued map $F \rightrightarrows Y$ as

$$F(x) = \begin{cases}\{(y_1, y_2) \in \mathbb{R}^2 | (y_1 - 1)^2 + (y_2 - 1)^2 \le 1\}, & x \in [0, 1),\\ \{(y_1, y_2) \in \mathbb{R}^2 | (y_1 - \frac{6-\sqrt{2}}{4})^2 + (y_2 - \frac{6-\sqrt{2}}{4})^2 \le 1\}, & x \in [1, 2].\end{cases}$$

Take $x_0 = 2$, $\hat{k} = (1, 1)$ and $\xi = (\frac{2-\sqrt{2}}{8}, \frac{2-\sqrt{2}}{8})$. Since $\bigcap_{y \in F(x)} (y + K + \xi) \cap F(x_0) = \emptyset$, we obtain $F(x) + \xi \not\preceq_K^p F(x_0)$ for any $x \in T$. Hence, x_0 is a ξ-efficient solution. For the functional $\varphi : \mathcal{P}(Y) \to \bar{\mathbb{R}}$ defined as $\varphi(A) = I_{\hat{k}}^p(A, F(x_0))$, $\forall A \in \mathcal{P}(Y)$, we can verify that φ satisfies (i)–(ii). On the other hand, when $x \in [0, 1)$, $\varphi(F(x)) = I_{\hat{k}}^p(F(x), F(x_0)) = \inf\{t \in \mathbb{R} | F(x) \preceq_K^p t\hat{k} + F(x_0) - \xi\}$. If $t = -\frac{2-\sqrt{2}}{8}$, then $F(x) + \xi = t\hat{k} + F(x_0)$, hence $F(x) + \xi \preceq_K^p t\hat{k} + F(x_0)$; if $t < -\frac{2-\sqrt{2}}{8}$, then $F(x) + \xi \not\preceq_K^p t\hat{k} + F(x_0)$. Therefore, $\varphi(F(x)) = -\frac{2-\sqrt{2}}{8} < 0$, thus φ does not satisfy (iii).

6 Conclusions

In this paper, when the ordered cone S has nonempty quasirelative interiors or ordered cone K has nonempty interiors, we introduce several kinds of order relations $\preceq_K^p$, $\ll_K^p$, $\preceq_S^m$, and $\prec^{m}_{\mathrm{qri}K}$, respectively. We introduce two kinds of nonlinear functions, whose properties are discussed. With order relations $\preceq^{p}_{K}$, $\ll^{p}_{K}$, $\preceq^{m}_{S}$, and $\prec^{m}_{\mathrm{qri}K}$, we introduce several classes of approximately efficient solutions and approximate weakly efficient solutions, respectively, for set optimization, and study the relationship among them. The optimality conditions for approximate solutions of set optimization are established.

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Author contribution
PZ and YX together made the major analysis and the original draft preparation. BH analyzed all the results and made necessary improvements. PZ is the major contributor in writing the paper. All authors read and approved the final manuscript.

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