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Generalized blending type Bernstein operators based on the shape parameter λ

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Abstract

In the present paper, we construct a new class of operators based on new type Bézier bases with a shape parameter λ and positive parameter s. Our operators include some well-known operators, such as classical Bernstein, α-Bernstein, generalized blending type α-Bernstein and λ-Bernstein operators as special case. In this paper, we prove some approximation theorems for these operators. Approximation properties of our operators are illustrated on graphs for variables s, α, λ, and n. It should be mentioned that our operators for $\lambda = 1$ have better approximation than Bernstein and α-Bernstein operators.

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1 Introduction

In 1912, Bernstein constructed Bernstein polynomials to prove Weierstrass Approximation Theorem [28], which says, for any continuous function $f(x)$ on the closed interval $[a,b]$, there exists a sequence of polynomials $p_n(x)$ that converges uniformly to $f(x)$. For a given continuous function $f(x)$ on $[0,1]$, Bernstein operators [6] $B_nC[0,1] \to C[0,1]$ are given by

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1], \tag{2}$$

and

$$\binom{n}{k} = \begin{cases} \frac{n!}{(n-k)!k!}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Many extensions of Bernstein operators have been given in [15, 17, 24], and references to many related works are also cited there. Later, Chen et al. (see [13]) extended Bernstein operators to α-Bernstein operators with a parameter $\alpha \in [0,1]$, which are defined as

$$T_{n,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \tag{4}$$

where $p_{1,0}^{(\alpha)}(x) = 1 - x$, $p_{1,1}^{(\alpha)}(x) = x$, and

$$\begin{aligned} p_{n,k}^{(\alpha)}(x) = \Bigg[&(1-\alpha)\binom{n-2}{k}x + (1-\alpha)\binom{n-2}{k-2}(1-x) \\ &+ \alpha\binom{n}{k}x(1-x)\Bigg]x^{k-1}(1-x)^{n-k-1}, \end{aligned}$$

for $n \geq 2$, $x \in [0,1]$, $f(x) \in C[0,1]$. The α-Bernstein operators and their modifications have been intensively studied by many researchers in recent papers (see [1, 3–5, 11, 21, 23]).

More recently, Aktuğlu et al. (see [3]) introduced and studied generalized blending type α-Bernstein operators by

$$\begin{aligned} L_n^{\alpha,s}(f;x) = \sum_{k=0}^{n} \Bigg\{&(1-\alpha)\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k} \\ &+ (1-\alpha)\binom{n-s}{k}x^{k}(1-x)^{n-s-k+1} \\ &+ \alpha\binom{n}{k}x^{k}(1-x)^{n-k}\Bigg\} f\left(\frac{k}{n}\right) \quad \text{for } n \geq s \end{aligned} \tag{5}$$

and

$$L_n^{\alpha,s}(f;x) = \sum_{k=0}^{n} \binom{n}{k}x^{k}(1-x)^{n-k} f\left(\frac{k}{n}\right) \quad \text{for } n < s, \tag{6}$$

which depend on two parameters α and s, where s is a positive integer, $\alpha \in [0,1]$, $x \in [0,1]$, $f(x) \in C[0,1]$.

One can see that when $s = 1$ and $s = 2$, then operators given by (5) and (6) reduce to ordinary Bernstein operators given by (1) and α-Bernstein operators given by (4), respectively. Aktuğlu and Yashar (see [4]) initiated and investigated some properties of generalized parametric blending type Bernstein operators that depend on four parameters s_1, s_2, a_1, and a_2.

In 2010, Ye et al. [29] introduced and studied new Bézier basis with a shape parameter $\lambda \in [-1,1]$, which is defined by

$$\tilde{b}_{n,k}(\lambda;x) = \begin{cases} b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), & \text{if } k = 0, \\ b_{n,k}(x) + \lambda(\frac{n-2k+1}{n^2-1} b_{n+1,k}(x)) & \\ \quad - \lambda(\frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)), & \text{if } (1 \leq k \leq n-1), \\ b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), & \text{if } k = n. \end{cases} \tag{7}$$

More recently, Cai et al. [10] introduced new λ-Bernstein operators

$$B_{n,\lambda}(f;x)=\sum_{k=0}^{n}\tilde{b}_{n,k}(x)f\left(\frac{k}{n}\right), \tag{8}$$

where $\tilde{b}_{n,k}(x)$ is defined in equation (7). The λ-Bernstein operators become a hot topic for last years and are investigated by many researchers [2, 7–9, 12, 16, 18–20, 22, 25–27].

The main purpose of the present paper is to construct a generalization of blending type Bernstein operators based on new type Bézier bases with a shape parameter λ and positive parameter s. A Korovkin-type approximation theorem will be proven. Moreover, approximation properties will also be discussed. For fixed s, α, λ, n, and specific function, detailed graphs will be given.

2 Construction of the (α,λ,s)-Bernstein operators and some basic results

This section is devoted to the construction and some main properties of the operators $L_{n,\lambda}^{(\alpha,s)}(f;x)$ that include: classical Bernstein, α-Bernstein, generalized blending type α-Bernstein and λ-Bernstein operators given in ([3, 6, 10], and [13]) as a special case. We introduce (α,λ,s)-Bernstein operators as follows:

Definition 1 Let $0\le\alpha\le 1$, $-1\le\lambda\le 1$ and s be a positive integer. Then define

$$L_{n,\lambda}^{(\alpha,s)}(f;x)=\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)f\left(\frac{k}{n}\right),$$

where

$$\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)=\begin{cases}\tilde{b}_{n,k}(\lambda;x), & \text{if } n<s,\\ (1-\alpha)[x\tilde{b}_{n-s,k-s}(\lambda;x)+(1-x)\tilde{b}_{n-s,k}(\lambda;x)] & \\ \quad+\alpha\tilde{b}_{n,k}(\lambda;x), & \text{if } n\ge s\end{cases}$$

and $\tilde{b}_{n,k}(\lambda;x)$ defined in equation (7).

In order to make the calculations easier, we will use the following representation of $L_{n,\lambda}^{(\alpha,s)}(f;x)$. For any $0\le\alpha\le 1$, $-1\le\lambda\le 1$ and a positive integer s,

$$L_{n,\lambda}^{(\alpha,s)}(f;x)=\begin{cases}B_{n,\lambda}(f;x) & \text{if } n<s,\\ B_{n,\lambda}^{\alpha,s}(f;x), & \text{if } n\ge s.\end{cases} \tag{9}$$

Here, $B_{n,\lambda}(f;x)$ is given by equation (8), and $B_{n,\lambda}^{\alpha,s}(f;x)$ is defined by

$$B_{n,\lambda}^{\alpha,s}(f;x)=(1-\alpha)B_{n,\lambda}^{s,(\star)}(f;x)+\alpha B_{n,\lambda}(f;x), \tag{10}$$

where

$$B_{n,\lambda}^{s,(\star)}(f;x)=\left[x\sum_{k=0}^{n}\tilde{b}_{n-s,k-s}(\lambda;x)+(1-x)\sum_{k=0}^{n}\tilde{b}_{n-s,k}(\lambda;x)\right]f\left(\frac{k}{n}\right).$$

Theorem 1 *For any* $0 \leq \alpha \leq 1$, $-1 \leq \lambda \leq 1$ *and a positive integer* s,

$$L_{n,\lambda}^{(\alpha,s)}(f;x) = \begin{cases} B_{n,\lambda}(f;x), & \text{if } n < s, \\ (1-\alpha)B_{n,\lambda}^{s,(\star)}(f;x) + \alpha B_{n,\lambda}(f;x) & \text{if } n \geq s, \end{cases} \tag{11}$$

where

$$\begin{aligned} B_{n,\lambda}(f;x) &= \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right) \\ &\quad + \lambda \sum_{k=0}^{n-1} \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right] \end{aligned}$$

and

$$\begin{aligned} B_{n,\lambda}^{s,(\star)}(f;x) &= \sum_{k=0}^{n-s} b_{n-s,k}(x)\left[xf\left(\frac{k+s}{n}\right) + (1-x)f\left(\frac{k}{n}\right)\right] \\ &\quad + \lambda x \sum_{k=0}^{n-s-1} \frac{n-s-2k-1}{(n-s)^2-1} b_{n-s+1,k+1}(x) \\ &\quad \times \left[f\left(\frac{k+s+1}{n}\right) - f\left(\frac{k+s}{n}\right)\right] \\ &\quad + \lambda(1-x) \sum_{k=0}^{n-s-1} \frac{n-s-2k-1}{(n-s)^2-1} b_{n-s+1,k+1}(x) \\ &\quad \times \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right]. \end{aligned}$$

Proof Considering the definition of $B_{n,\lambda}(f;x)$, we have

$$\begin{aligned} B_{n,\lambda}(f;x) &= \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right) \\ &= \tilde{b}_{n,0}(\lambda;x) f\left(\frac{0}{n}\right) + \sum_{k=1}^{n-1} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right) \\ &\quad + \tilde{b}_{n,n}(\lambda;x) f\left(\frac{n}{n}\right) \\ &= \left[b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x)\right] f\left(\frac{0}{n}\right) \\ &\quad + \sum_{k=1}^{n-1} \left[b_{n,k}(x) + \lambda \frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \lambda \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right] f\left(\frac{k}{n}\right) \\ &\quad + \left[b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x)\right] f\left(\frac{n}{n}\right) \\ &= b_{n,0}(x) f\left(\frac{0}{n}\right) + \sum_{k=1}^{n-1} b_{n,k}(x) f\left(\frac{k}{n}\right) + b_{n,n}(x) f\left(\frac{n}{n}\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-1}\left[\lambda\frac{n-2k+1}{n^2-1}b_{n+1,k}(x)\right]f\left(\frac{k}{n}\right)-\left[\frac{\lambda}{n+1}b_{n+1,n}(x)\right]f\left(\frac{n}{n}\right)\\
&-\sum_{k=1}^{n-1}\left[\lambda\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\right]f\left(\frac{k}{n}\right)+\left[-\frac{\lambda}{n+1}b_{n+1,1}(x)\right]f\left(\frac{0}{n}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
B_{n,\lambda}(f;x)&=\sum_{k=0}^{n}b_{n,k}(x)f\left(\frac{k}{n}\right)+\sum_{k=1}^{n}\left[\lambda\frac{n-2k+1}{n^2-1}b_{n+1,k}(x)\right]f\left(\frac{k}{n}\right)\\
&\quad-\sum_{k=0}^{n-1}\left[\lambda\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\right]f\left(\frac{k}{n}\right)\\
&=\sum_{k=0}^{n}b_{n,k}(x)f\left(\frac{k}{n}\right)\\
&\quad+\lambda\sum_{k=0}^{n-1}\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\left[f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)\right],
\end{aligned}
$$

which completes the proof of the first part.

Using the same techniques, $B_{n,\lambda}^{s,(\star)}(f;x)$ can be proved in a similar way. □

Lemma 1

(i) (*Linearity*) *The* (α,λ,s)*-Bernstein operators are satisfying the following equality*:

$$L_{n,\lambda}^{(\alpha,s)}(a_1f+a_2g;x)=a_1L_{n,\lambda}^{(\alpha,s)}(f;x)+a_2L_{n,\lambda}^{(\alpha,s)}(g;x),$$

where a_1, a_2 *are real numbers, and* $f(x)$ *and* $g(x)$ *are defined on the closed interval* $[0,1]$.

(ii) (*Monotonicity*) *The* (α,λ,s)*-Bernstein operators are monotone for* $\lambda\in[-1,1]$ *and* $\alpha\in[0,1]$. *Therefore, if* $f(x)\ge g(x)$, *then* $L_{n,\lambda}^{(\alpha,s)}(f;x)\ge L_{n,\lambda}^{(\alpha,s)}(g;x)$ *for* $x\in[0,1]$.

(iii) (*Positivity*) *For a given nonnegative function defined on* $[0,1]$, *the operators* $L_{n,\lambda}^{(\alpha,s)}(f;x)$ *are nonnegative for* $\alpha\in[0,1]$ *and* $\lambda\in[-1,1]$.

(iv) (*End-point interpolation*) *The* (α,λ,s)*-Bernstein operators satisfy the end point interpolation property for* $f(x)$, *that is*

$$L_{n,\lambda}^{(\alpha,s)}(f;0)=f(0);\qquad L_{n,\lambda}^{(\alpha,s)}(f;1)=f(1).$$

Remark 1 The operators $L_{n,\lambda}^{(\alpha,s)}(f;x)$ have the following special cases:

a) If $\alpha=1$ or $s=1$, then $L_{n,\lambda}^{(\alpha,s)}(f;x)$ reduces to the operators given in [10].

b) If $\lambda=0$, $L_{n,\lambda}^{(\alpha,s)}(f;x)$ reduces to the operators given in [3].

c) If $\alpha=1$ and $\lambda=0$, $L_{n,\lambda}^{(\alpha,s)}(f;x)$ reduces to the operators given in [6].

d) If $\lambda=0$ and $s=2$, $L_{n,\lambda}^{(\alpha,s)}(f;x)$ reduces to the operators given in [13].

Lemma 2 (see [10]) *If* $n<s$ *then* $L_{n,\lambda}^{(\alpha,s)}(f;x)=B_{n,\lambda}(f;x)$ *for any* $0\le\alpha\le1$, $-1\le\lambda\le1$ *and*

(i) $B_{n,\lambda}(1;x)=1$;

(ii) $B_{n,\lambda}(t;x)=x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda$;

(iii) $B_{n,\lambda}(t^2;x) = x^2 + \frac{x(1-x)}{n} + \lambda[\frac{2x-4x^2+2x^{n+1}}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}]$.

Theorem 2 *If $n \geq s$, for any $0 \leq \alpha \leq 1$ and $-1 \leq \lambda \leq 1$, we have the following identities*:

$$
\begin{aligned}
&\text{(i)}\quad L_{n,\lambda}^{(\alpha,s)}(1;x) = 1,\\
&\text{(ii)}\quad L_{n,\lambda}^{(\alpha,s)}(t;x) = x + (1-\alpha)\lambda\left[\frac{1-2x+x^{n-s+1}-(1-x)^{n-s+1}}{n(n-s-1)}\right]\\
&\qquad + \alpha\lambda\left[\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\right],\\
&\text{(iii)}\quad L_{n,\lambda}^{(\alpha,s))}\left(t^2;x\right) = x^2 + \frac{[n+(1-\alpha)s(s-1)]x(1-x)}{n^2}\\
&\qquad + \frac{\alpha\lambda}{n}\left[\frac{2x-4x^2+2x^{n+1}}{(n-1)}\right]\\
&\qquad + \frac{(1-\alpha)\lambda}{n}\left[\frac{2x-4x^2+2x^{n-s+1}}{(n-s-1)}\right]\\
&\qquad + \frac{\alpha\lambda}{n^2}\left[\frac{x^{n+1}+(1-x)^{n+1}-1}{(n-1)}\right]\\
&\qquad + \frac{(1-\alpha)\lambda}{n^2}\left[\frac{x^{n-s+1}+(1-x)^{n-s+1}-1}{(n-s-1)}\right]\\
&\qquad + \left[\frac{2sx(x^{n-s+1}-(1-x)^{n-s+1})}{(n-s-1)}\right].
\end{aligned}
$$

Proof Recall that for $n \geq s$,

$$L_{n,\lambda}^{(\alpha,s)}(f;x) = (1-\alpha)B_{n,\lambda}^{s,(\star)}(f;x) + \alpha B_{n,\lambda}(f;x).$$

Now,
(i) $L_{n,\lambda}^{(\alpha,s)}(1;x) = (1-\alpha)B_{n,\lambda}^{s,(\star)}(1;x) + \alpha B_{n,\lambda}(1;x) = 1$.
(ii) Since $B_{n,\lambda}(t;x)$ is given in Lemma 2, we only need to find $B_{n,\lambda}^{s,(\star)}(t;x)$.

$$
\begin{aligned}
B_{n,\lambda}^{s,(\star)}(t;x) &= \sum_{k=0}^{n-s} b_{n-s,k}(x)\left[x\left(\frac{k+s}{n}\right) + (1-x)\left(\frac{k}{n}\right)\right]\\
&\quad + \lambda x \sum_{k=0}^{n-s-1} \frac{n-s-2k-1}{(n-s)^2-1} b_{n-s+1,k+1}(x)\left[\left(\frac{k+s+1}{n}\right) - \left(\frac{k+s}{n}\right)\right]\\
&\quad + \lambda(1-x) \sum_{k=0}^{n-s-1} \frac{n-s-2k-1}{(n-s)^2-1} b_{n-s+1,k+1}(x)\left[\left(\frac{k+1}{n}\right) - \left(\frac{k}{n}\right)\right]\\
&= \sum_{k=0}^{n-s} b_{n-s,k}(x)\left(\frac{k}{n} + \frac{sx}{n}\right)\\
&\quad + \lambda \sum_{k=0}^{n-s-1} \frac{n-s-2k-1}{(n-s)^2-1} b_{n-s+1,k+1}(x)\frac{1}{n}
\end{aligned}
$$

$$= \frac{n-s}{n}x + \frac{sx}{n} + \frac{\lambda}{n(n-s+1)} \sum_{k=0}^{n-s-1} b_{n-s+1,k+1}(x)$$
$$- \frac{2\lambda}{n((n-s)^2-1)} \sum_{k=0}^{n-s-1} kb_{n-s+1,k+1}(x).$$

Define $\Delta_1(n,s;x)$ and $\Delta_2(n,s;x)$ as

$$\Delta_1(n,s;x) = \sum_{k=0}^{n-s-1} b_{n-s+1,k+1}(x) \tag{12}$$

and

$$\Delta_2(n,s;x) = \sum_{k=0}^{n-s-1} kb_{n-s+1,k+1}(x). \tag{13}$$

Now, we need to calculate $\Delta_1(n,s;x)$ and $\Delta_2(n,s;x)$

$$\begin{aligned}\Delta_1(n,s;x) &= \sum_{k=0}^{n-s-1} b_{n-s+1,k+1}(x) = \sum_{k=1}^{n-s} b_{n-s+1,k}(x) \\ &= \sum_{k=0}^{n-s+1} b_{n-s+1,k}(x) - b_{n-s+1,0}(x) - b_{n-s+1,n-s+1}(x) \\ &= 1-(1-x)^{n-s+1} - x^{n-s+1}\end{aligned}$$

and

$$\begin{aligned}\Delta_2(n,s;x) &= \sum_{k=0}^{n-s-1} kb_{n-s+1,k+1}(x) \\ &= \sum_{k=0}^{n-s-1} \frac{k(n-s+1)!}{(n-s-k)!(k+1)!}x^{k+1}(1-x)^{n-s-k} \\ &= \sum_{k=0}^{n-s-1} \frac{(n-s+1)!}{(n-s-k)!k!}x^{k+1}(1-x)^{n-s-k} - \sum_{k=0}^{n-s-1} b_{n-s+1,k+1}(x) \\ &= (n-s+1)x \sum_{k=0}^{n-s-1} b_{n-s,k}(x) - \Delta_1(n,s;x) \\ &= (n-s+1)x\left(\sum_{k=0}^{n-s} b_{n-s,k}(x) - b_{n-s,n-s}(x)\right) - \Delta_1(n,s;x) \\ &= (n-s+1)x\left(1-x^{n-s}\right) - \left[1-(1-x)^{n-s+1} - x^{n-s+1}\right].\end{aligned}$$

Hence, we get

$$\begin{aligned}B_{n,\lambda}^{s,(\star)}(t;x) &= x + \lambda\left[\frac{1}{n(n-s+1)}\Delta_1(n,s;x) - \frac{2}{n((n-s)^2-1)}\Delta_2(n,s;x)\right] \\ &= x + \lambda\left[\frac{1-(1-x)^{n-s+1} - x^{n-s+1}}{n(n-s+1)}\right]\end{aligned}$$

$$-2\lambda\left[\frac{(n-s+1)x(1-x^{n-s})-[1-(1-x)^{n-s+1}-x^{n-s+1}]}{n((n-s)^2-1)}\right]$$
$$=x+\lambda\left[\frac{1-2x+x^{n-s+1}-(1-x)^{n-s+1}}{n(n-s-1)}\right].$$

Finally, using above equality and Lemma 2, we have

$$\begin{aligned}L_{n,\lambda}^{(\alpha,s)}(t;x)&=x+\lambda\frac{1-2x+x^{n-s+1}-(1-x)^{n-s+1}}{n(n-s-1)}\\&\quad+\alpha\lambda\left[\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\right]\\&\quad-\left[\frac{1-2x+x^{n-s+1}-(1-x)^{n-s+1}}{n(n-s-1)}\right].\end{aligned}$$

(iii) Direct calculations yield that

$$\begin{aligned}B_{n,\lambda}^{s,(\star)}(t^2;x)&=\sum_{k=0}^{n-s}b_{n-s,k}(x)\left[x\left(\frac{k+s}{n}\right)^2+(1-x)\left(\frac{k}{n}\right)^2\right]\\&\quad+\lambda x\sum_{k=0}^{n-s-1}\frac{n-s-2k-1}{(n-s)^2-1}b_{n-s+1,k+1}(x)\left[\left(\frac{k+s+1}{n}\right)^2-\left(\frac{k+s}{n}\right)^2\right]\\&\quad+\lambda(1-x)\sum_{k=0}^{n-s-1}\frac{n-s-2k-1}{(n-s)^2-1}b_{n-s+1,k+1}(x)\left[\left(\frac{k+1}{n}\right)^2-\left(\frac{k}{n}\right)^2\right]\\&=\sum_{k=0}^{n-s}b_{n-s,k}(x)\left[\frac{k^2}{n^2}+\frac{2sxk}{n^2}+\frac{s^2x}{n^2}\right]\\&\quad+\lambda\sum_{k=0}^{n-s-1}\frac{n-s-2k-1}{(n-s)^2-1}b_{n-s+1,k+1}(x)\left[\frac{2k}{n^2}+\frac{2sx+1}{n^2}\right]\\&=x^2+\frac{(s(s-1)+n)x(1-x)}{n^2}\\&\quad+\frac{2\lambda}{n^2(n-s+1)}\Delta_2(n,s;x)+\frac{(2sx+1)\lambda}{n^2(n-s+1)}\Delta_1(n,s;x)\\&\quad+\frac{-4\lambda}{n^2((n-s)^2-1)}\Delta_3(n,s;x)+\frac{-2(2sx+1)\lambda}{n^2((n-s)^2-1)}\Delta_2(n,s;x),\end{aligned}$$

where

$$\Delta_3(n,s;x)=\sum_{k=0}^{n-s-1}k^2b_{n-s+1,k+1}(x).\tag{14}$$

After similar calculations, we may write

$$\begin{aligned}\Delta_3(n,s;x)&=(n-s+1)(n-s)x^2\left[\sum_{k=0}^{n-s-1}b_{n-s-1,k}(x)-b_{n-s-1,n-s-1}(x)\right]\\&\quad+(n-s+1)x\left[\sum_{k=0}^{n-s}b_{n-s,k}(x)-b_{n-s,n-s}(x)\right]-2\Delta_2(n,s;x)-\Delta_1(n,s;x),\end{aligned}$$

which implies that

$$\begin{aligned} B_{n,\lambda}^{s,(\star)}\big(t^2;x\big) = x^2 &+ \frac{(s(s-1)+n)x(1-x)}{n^2} + \frac{\lambda}{n}\left[\frac{2x-4x^2+2x^{n-s+1}}{n-s-1}\right] \\ &+ \frac{\lambda}{n^2}\left[\frac{-1+x^{n-s+1}+(1-x)^{n-s+1}}{n-s-1}\right] \\ &+ \frac{\lambda}{n^2}\left[\frac{2sx(x^{n-s+1}-(1-x)^{n-s+1})}{n-s-1}\right]. \end{aligned}$$

Finally, using above equality and Lemma 2, we have

$$\begin{aligned} L_{n,\lambda}^{(\alpha,s)}\big(t^2;x\big) &= (1-\alpha)B_{n,\lambda}^{s,(\star)}(t;x) + \alpha B_{n,\lambda}(t;x) \\ &= x^2 + \frac{(s(s-1)+n)x(1-x)}{n^2} + \frac{\alpha\lambda}{n}\left[\frac{2x-4x^2+2x^{n+1}}{n-1}\right] \\ &\quad + \frac{(1-\alpha)\lambda}{n}\left[\frac{2x-4x^2+2x^{n-s+1}}{n-s-1}\right] + \frac{\alpha\lambda}{n^2}\left[\frac{x^{n+1}+(1-x)^{n+1}-1}{n-1}\right] \\ &\quad + \frac{(1-\alpha)\lambda}{n^2}\left[\frac{-1+x^{n-s+1}+(1-x)^{n-s+1}+2sx(x^{n-s+1}-(1-x)^{n-s+1})}{n-s-1}\right]. \qquad \square \end{aligned}$$

Corollary 1 *For fixed $\alpha, x \in [0,1]$, $\lambda \in [-1,1]$ and $n \geq s$, where s is a positive integer, we have*

$$\text{(i)} \quad L_{n,\lambda}^{(\alpha,s)}(t-x;x) = (1-\alpha)\lambda\left[\frac{1-2x+x^{n-s+1}-(1-x)^{n-s+1}}{n(n-s-1)}\right] + \alpha\lambda\left[\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\right],$$

$$\begin{aligned} \text{(ii)} \quad L_{n,\lambda}^{(\alpha,s)}\big((t-x)^2;x\big) &= \frac{[n+(1-\alpha)s(s-1)]x(1-x)}{n^2} \\ &\quad + 2\alpha\lambda\left[\frac{(1-x)^{n+1}}{n(n-1)}\right] + 2(1-\alpha)\lambda\left[\frac{(1-x)^{n-s+1}}{n(n-s-1)}\right] \\ &\quad + \alpha\lambda\left[\frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}\right] \\ &\quad + (1-\alpha)\lambda\left[\frac{x^{n-s+1}+(1-x)^{n-s+1}-1}{n^2(n-s-1)}\right] \\ &\quad + 2(1-\alpha)\lambda sx\left[\frac{x^{n-s+1}-(1-x)^{n-s+1}}{n^2(n-s-1)}\right]. \end{aligned}$$

Lemma 3 *For $f \in C[0,1]$, $\alpha, x \in [0,1]$, $\lambda \in [-1,1]$ and a positive integer s, we have the following inequality*:

$$\big\|L_{n,\lambda}^{(\alpha,s)}(f;x)\big\| \leq \|f\|, \tag{15}$$

where $\|\cdot\|$ represents the uniform norm on $C[0,1]$.

Proof

$$\begin{aligned}\left\|L_{n,\lambda}^{(\alpha,s)}(f;x)\right\| &\leq \sum_{k=0}^{n} \tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|f\left(\frac{k}{n}\right)\right| \\ &\leq \|f\| L_{n,\lambda}^{(\alpha,s)}(1;x) = \|f\|. \end{aligned}$$ □

Corollary 2 *For $n \in \{1,2,\ldots\}$, we have the following inequalities*:

$$L_{n,\lambda}^{(\alpha,s)}\big((t-x);x\big) \leq \Psi_{n,s,1}(x;\alpha)$$

and

$$L_{n,\lambda}^{(\alpha,s)}\big((t-x)^2;x\big) \leq \Psi_{n,s,2}(x;\alpha),$$

where

$$\Psi_{n,s,1}(x;\alpha) = \begin{cases} \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)} & \text{if } n < s, \\ (1-\alpha)[\frac{1+2x+x^{n-s+1}+(1-x)^{n-s+1}}{n(n-s-1)}] & \\ \quad + \alpha[\frac{1+2x+x^{n+1}+(1-x)^{n+1}}{n(n-1)}] & \text{if } n \geq s, \end{cases}$$

$$\Psi_{n,s,2}(x;\alpha) = \begin{cases} \frac{x(1-x)}{n} + \frac{2(1-x)^{n+1}}{n(n-1)} & \\ \quad + \frac{x^{n+1}+(1-x)^{n+1}+1}{n^2(n-1)} & \text{if } n < s, \\ \frac{[n+(1-\alpha)s(s-1)]x(1-x)}{n^2} + \frac{2\alpha(1-x)^{n+1}}{n(n-1)} & \\ \quad + \frac{2(1-\alpha)(1-x)^{n-s+1}}{n(n-s-1)} + \alpha[\frac{x^{n+1}+(1-x)^{n+1}+1}{n^2(n-1)}] & \\ \quad + (1-\alpha)[\frac{x^{n-s+1}+(1-x)^{n-s+1}+1}{n^2(n-s-1)}] & \\ \quad + (1-\alpha)2sx[\frac{x^{n-s+1}+(1-x)^{n-s+1}}{n^2(n-s-1)}] & \text{if } n \geq s. \end{cases}$$

3 Approximation properties of (α,λ,s)-Bernstein operators

This section is devoted to the approximation properties of the operators $L_{n,\lambda}^{(\alpha,s)}(f;x)$. In this section, we will prove a Korovkin-type approximation theorem and approximation theorems by means of modulus of continuity and the Lipschitz function.

Theorem 3 *If $f \in C[0,1]$, $\alpha, x \in [0,1]$, $\lambda \in [-1,1]$, and s is a positive integer, then $L_{n,\lambda}^{(\alpha,s)}(f;x)$ converge uniformly to $f(x)$ on the closed interval* $[0,1]$.

Proof By the Korovkin Theorem, it is enough to show that $L_{n,\lambda}^{(\alpha,s)}(e_i;x)$ converges uniformly to $e_i(x)$, where $e_i(x) = x^i$, i = 0,1,2. From Lemma 2 and Theorem 2, $L_{n,\lambda}^{(\alpha,s)}(1;x) = 1$ for positive integer s and

$$L_{n,\lambda}^{(\alpha,s)}(t;x) = \begin{cases} x + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda & \text{if } n < s, \\ x + (1-\alpha)\lambda[\frac{1-2x+x^{n-s+1}-(1-x)^{n-s+1}}{n(n-s-1)}] & \\ \quad + \alpha\lambda[\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}] & \text{if } n \geq s \end{cases}$$

and

$$L_{n,\lambda}^{(\alpha,s)}\left(t^2;x\right)=\begin{cases} x^2+\frac{x(1-x)}{n}+\lambda[\frac{2x-4x^2+2x^{n+1}}{n(n-1)}+\frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}] & \text{if } n<s,\\ x^2+\frac{[n+(1-\alpha)s(s-1)]x(1-x)}{n^2}+\alpha\lambda[\frac{2x-4x^2+2x^{n+1}}{n(n-1)}] & \\ \quad+(1-\alpha)\lambda[\frac{2x-4x^2+2x^{n-s+1}}{n(n-s-1)}] & \\ \quad+\alpha\lambda[\frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}] & \\ \quad+(1-\alpha)\lambda[\frac{x^{n-s+1}+(1-x)^{n-s+1}-1}{n^2(n-s-1)}] & \\ \quad+2sx(1-\alpha)\lambda[\frac{x^{n-s+1}-(1-x)^{n-s+1}}{n^2(n-s-1)}] & \text{if } n\geq s. \end{cases}$$

It is easily to see that for each case, $L_{n,\lambda}^{(\alpha,s)}(t;x)$ and $L_{n,\lambda}^{(\alpha,s)}(t^2;x)$ converge uniformly to $e_1(x)=x$ and $e_2(x)=x^2$, respectively. This completes the proof. □

We use modulus of continuity to give quantitative error estimates for (α,λ,s)-Bernstein operators. We denote the usual modulus of continuity for $f\in C[0,1]$ as

$$w(f;\delta)=\sup_{0<h\leq\delta}\sup_{x,x+h\in[0,1]}\left|f(x+h)-f(x)\right|.$$

Theorem 4 *For any $f\in C[0,1]$, s positive integer, $x,\alpha\in[0,1]$ and $\lambda\in[-1,1]$, we have*

$$\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right|\leq 2\omega\bigl(f;\Psi_{n,s,2}(x;\alpha)\bigr),$$

where ω is the usual modulus of the continuity.

Proof Since $L_{n,\lambda}^{(\alpha,s)}(1;x)=1$ and $\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\geq 0$ on $[0,1]$, we can write

$$\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right|\leq\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|f\left(\frac{k}{n}\right)-f(x)\right|.$$

If we use the following properties of the modulus of continuity

$$\left|f(t)-f(x)\right|\leq\omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2}+1\right)$$

and

$$\omega(f;\gamma\delta)\leq(1+\gamma)\omega(f;\delta),$$

where γ is a positive constant, we obtain

$$\begin{aligned}\left|f\left(\frac{k}{n}\right)-f(x)\right|&\leq\omega\left(f;\frac{1}{\delta}\left|\frac{k}{n}-x\right|\delta\right)\\&\leq\left(1+\frac{1}{\delta}\left|\frac{k}{n}-x\right|\right)\omega(f;\delta).\end{aligned}$$

Consequently, we can write

$$\begin{aligned}\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| &\le \sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\le \sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left(1+\frac{1}{\delta}\left|\frac{k}{n}-x\right|\right)\omega(f;\delta) \\ &= \left(1+\frac{1}{\delta}\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|\frac{k}{n}-x\right|\right)\omega(f;\delta).\end{aligned}$$

If we apply the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|\frac{k}{n}-x\right| &\le \left[\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left(\frac{k}{n}-x\right)^2\right]^{\frac{1}{2}} \\ &= \left[L_{n,\lambda}^{(\alpha,s)}\left((t-x)^2;x\right)\right]^{\frac{1}{2}} \le \left[\Psi_{n,s,2}(x;\alpha)\right]^{\frac{1}{2}}.\end{aligned}$$

So, we have

$$\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| \le \left(1+\frac{[\Psi_{n,s,2}(x;\alpha)]^{\frac{1}{2}}}{\delta}\right)\omega(f;\delta).$$

Choosing $\delta=[\Psi_{n,s,2}(x;\alpha)]^{\frac{1}{2}}$, we complete the proof. □

Theorem 5 *If $f\in C'[0,1]$, s positive integer, $x,\alpha\in[0,1]$ and $\lambda\in[-1,1]$, then*

$$\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| \le \left|\Psi_{n,s,2}(x;\alpha)\right|\left|f'(x)\right| + 2\sqrt{\Psi_{n,s,2}(x;\alpha)}\,\omega\left(f';\sqrt{\Psi_{n,s,2}(x;\alpha)}\right).$$

Proof We have the following equality by applying the mean value theorem of differential calculus:

$$f\left(\frac{k}{n}\right)-f(x)=\left(\frac{k}{n}-x\right)f'(x)+\left(\frac{k}{n}-x\right)\left[f'(c)-f'(x)\right],$$

where $c=c_{n,k}(x)\in(x,\frac{k}{n})$. If we multiply both sides of the above equality by $\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)$ and sum from 0 to n, we get

$$\begin{aligned}\sum_{k=0}^{n}\left[f\left(\frac{k}{n}\right)-f(x)\right]\tilde{b}_{n,k}^{\alpha,s}(\lambda;x) &= \sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left[\frac{k}{n}-x\right]f'(x) \\ &\quad+\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left[f\left(\frac{k}{n}\right)-f(x)\right]\left(f'(c)-f'(x)\right).\end{aligned}$$

Equivalently,

$$\begin{aligned}L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x) &= L_{n,\lambda}^{(\alpha,s)}(t-x;x)f'(x) \\ &\quad+\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left[f\left(\frac{k}{n}\right)-f(x)\right]\left(f'(c)-f'(x)\right).\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| &\leq \left|L_{n,\lambda}^{(\alpha,s)}(t-x;x)\right|\left|f'(x)\right| \\ &\quad + \sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|f\left(\frac{k}{n}\right)-f(x)\right|\left|f'(c)-f'(x)\right|.\end{aligned}$$

Here, we observe that

$$\begin{aligned}\left|f'(c)-f'(x)\right| &= \left(1+\frac{1}{\delta}|c-x|\right)\omega\left(f';\delta\right) \\ &\leq \left(1+\frac{1}{\delta}\left|\frac{k}{n}-x\right|\right)\omega\left(f';\delta\right),\end{aligned}$$

where δ is any positive number, which does not depend on k, and ω is the usual modulus.

Consequently, we get

$$\begin{aligned}\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| &\leq \left|L_{n,\lambda}^{(\alpha,s)}(t-x;x)\right|\left|f'(x)\right| + \sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|\frac{k}{n}-x\right|\omega\left(f';\delta\right) \\ &\quad + \frac{1}{\delta}\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left(\frac{k}{n}-x\right)^{2}\omega\left(f';\delta\right) \\ &\leq \left|L_{n,\lambda}^{(\alpha,s)}(t-x;x)\right|\left|f'(x)\right| + \sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left|\frac{k}{n}-x\right|\omega\left(f';\delta\right) \\ &\quad + \frac{1}{\delta}L_{n,\lambda}^{(\alpha,s)}\left((t-x)^{2};x\right)\omega\left(f';\delta\right).\end{aligned}$$

Using the Cauchy–Schwarz inequality

$$\begin{aligned}\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| &\leq \left|L_{n,\lambda}^{(\alpha,s)}(t-x;x)\right|\left|f'(x)\right| \\ &\quad + \left[\sum_{k=0}^{n}\tilde{b}_{n,k}^{\alpha,s}(\lambda;x)\left(\frac{k}{n}-x\right)^{2}\right]^{\frac{1}{2}}\omega\left(f';\delta\right) \\ &\quad + \frac{1}{\delta}L_{n,\lambda}^{(\alpha,s)}\left((t-x)^{2};x\right)\omega\left(f';\delta\right) \\ &= \left|L_{n,\lambda}^{(\alpha,s)}(t-x;x)\right|\left|f'(x)\right| + \left[L_{n,\lambda}^{(\alpha,s)}\left((t-x)^{2};x\right)\right]^{\frac{1}{2}}\omega\left(f';\delta\right) \\ &\quad + \frac{1}{\delta}L_{n,\lambda}^{(\alpha,s)}\left((t-x)^{2};x\right)\omega\left(f';\delta\right) \\ &= \left|L_{n,\lambda}^{(\alpha,s)}(t-x;x)\right|\left|f'(x)\right| \\ &\quad + \omega\left(f';\delta\right)\left(1+\frac{[L_{n,\lambda}^{(\alpha,s)}((t-x)^{2};x)]^{\frac{1}{2}}}{\delta}\right)\left[L_{n,\lambda}^{(\alpha,s)}\left((t-x)^{2};x\right)\right]^{\frac{1}{2}} \\ &\leq \left|\Psi_{n,s,1}(x;\alpha)\right|\left|f'(x)\right| \\ &\quad + \omega\left(f';\delta\right)\left(1+\frac{\sqrt{\Psi_{n,s,2}(x;\alpha)}}{\delta}\right)\sqrt{\Psi_{n,s,2}(x;\alpha)}\end{aligned}$$

and choosing $\delta=\sqrt{\Psi_{n,s,2}(x;\alpha)}$, we complete the proof. □

The Petree K-functional is given by

$$K_2(f;\delta) := \inf_{g\in C^2[0,1]} \bigl\{ \|f-g\| + \delta \bigl\|g''\bigr\| \bigr\} \quad \bigl(\delta > 0 \text{ and } f \in [0,1]\bigr),$$

where $C^2[0,1] = \{g \in C[0,1] : g', g'' \in C[0,1]\}$. It is given in [14] that there exists $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}),$$

where the second order modulus of continuity of smoothness for $f \in C[0,1]$ is defined as

$$\omega_2(f;\delta) = \sup_{0<h\le\delta} \sup_{x,x+h,x+2h\in[0,1]} \bigl|f(x+2h) - 2f(x+h) + f(x)\bigr|.$$

Now, we can prove the following theorem:

Theorem 6 *If $f \in C[0,1]$, $\alpha, x \in [0,1]$, $\lambda \in [-1,1]$, and s is arbitrary positive integer, then*

$$\bigl|L_{n,\lambda}^{(\alpha,s)}(f;x) - f(x)\bigr| \le C\omega_2\left(f; \frac{1}{2}\sqrt{\Psi_{n,s,2}(x;\alpha) + \bigl(\Psi_{n,s,1}(x;\alpha)\bigr)^2}\right) + \omega\bigl(f;\Psi_{n,s,1}(x;\alpha)\bigr),$$

where C is a positive constant.

Proof We denote $\varepsilon_{n,\lambda}^{(\alpha,s)}(x) = L_{n,\lambda}^{(\alpha,s)}(t;x)$ and define the auxiliary operator

$$\widetilde{L_{n,\lambda}^{(\alpha,s)}}(f;x) = L_{n,\lambda}^{(\alpha,s)}(f;x) + f(x) - f\bigl(\varepsilon_{n,\lambda}^{(\alpha,s)}(x)\bigr). \tag{16}$$

Using Lemma 2 and Theorem 2, one can easily see that

$$\widetilde{L_{n,\lambda}^{(\alpha,s)}}(1;x) = L_{n,\lambda}^{(\alpha,s)}(1;x) = 1,$$

and

$$\widetilde{L_{n,\lambda}^{(\alpha,s)}}(t;x) = L_{n,\lambda}^{(\alpha,s)}(t;x) + x - \varepsilon_{n,\lambda}^{(\alpha,s)}(x) = x.$$

So, we have $\widetilde{L_{n,\lambda}^{(\alpha,s)}}(t-x;x) = 0$.

Let $g \in C^2[0,1]$. By the Taylor expansion, we have the following equality:

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du. \tag{17}$$

Applying the operators $\widetilde{L_{n,\lambda}^{(\alpha,s)}}$ on both sides of (17), we get

$$\begin{aligned}\widetilde{L_{n,\lambda}^{(\alpha,s)}}(g;x) &= g(x) + \widetilde{L_{n,\lambda}^{(\alpha,s)}}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\ &= g(x) + L_{n,\lambda}^{(\alpha,s)}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\ &\quad - \int_x^{\varepsilon_{n,\lambda}^{(\alpha,s)}(x)} \left(\varepsilon_{n,\lambda}^{(\alpha,s)}(x)-u\right)g''(u)\,du.\end{aligned}$$

Hence,

$$\begin{aligned}\widetilde{L_{n,\lambda}^{(\alpha,s)}}(g;x) - g(x) &= L_{n,\lambda}^{(\alpha,s)}\left(\int_x^t (t-u)g''(u)\,du;x\right)\\ &\quad - \int_x^{\varepsilon_{n,\lambda}^{(\alpha,s)}(x)} \left(\varepsilon_{n,\lambda}^{(\alpha,s)}(x)-u\right)g''(u)\,du.\end{aligned}$$

Using the above equation, we get the following inequality:

$$\begin{aligned}\left|\widetilde{L_{n,\lambda}^{(\alpha,s)}}(g;x) - g(x)\right| &\leq \left|L_{n,\lambda}^{(\alpha,s)}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right|\\ &\quad + \left|\int_x^{\varepsilon_{n,\lambda}^{(\alpha,s)}(x)} \left(\varepsilon_{n,\lambda}^{(\alpha,s)}(x)-u\right)g''(u)\,du\right|\\ &\leq L_{n,\lambda}^{(\alpha,s)}\left(\left|\int_x^t (t-u)g''(u)\,du\right|;x\right)\\ &\quad + \int_x^{\varepsilon_{n,\lambda}^{(\alpha,s)}(x)} \left|\varepsilon_{n,\lambda}^{(\alpha,s)}(x)-u\right|\left|g''(u)\right|\,du,\end{aligned}$$

which implies that

$$\begin{aligned}\left|\widetilde{L_{n,\lambda}^{(\alpha,s)}}(g;x) - g(x)\right| &\leq L_{n,\lambda}^{(\alpha,s)}\left(\left|\int_x^t \left|(t-u)\right|\,du\right|;x\right)\left\|g''\right\| \qquad (18)\\ &\quad + \int_x^{\varepsilon_{n,\lambda}^{(\alpha,s)}(x)} \left|\varepsilon_{n,\lambda}^{(\alpha,s)}(x)-u\right|\,du\left\|g''\right\|\\ &\leq L_{n,\lambda}^{(\alpha,s)}\left((t-x)^2;x\right)\left\|g''\right\| + \left(\varepsilon_{n,\lambda}^{(\alpha,s)}(x)-u\right)^2\left\|g''\right\|\\ &\leq \left[\Psi_{n,s,2}(x;\alpha) + \left(\Psi_{n,s,1}(x;\alpha)\right)^2\right]\left\|g''\right\|. \qquad (19)\end{aligned}$$

So, we have

$$\left|\widetilde{L_{n,\lambda}^{(\alpha,s)}}(g;x) - g(x)\right| \leq \left[\Psi_{n,2}(x;\alpha) + \left(\Psi_{n,1}(x;\alpha)\right)^2\right]\left\|g''\right\|.$$

On the other hand, from Lemma 3 and the auxiliary operators (16), we get

$$\begin{aligned}\left|\widetilde{L_{n,\lambda}^{(\alpha,s)}}(f;x)\right| &\leq \left|L_{n,\lambda}^{(\alpha,s)}(f;x)\right| + \left|f(x)\right| + \left|f\left(\varepsilon_{n,\lambda}^{(\alpha,s)}(x)\right)\right|\\ &\leq \|f\|\left|L_{n,\lambda}^{(\alpha,s)}(f;x)\right| + 2\|f\| \leq 3\|f\| \qquad (20)\end{aligned}$$

for all $f \in C[0,1]$ and $x \in [0,1]$.

Now for $f \in C[0,1]$ and $g \in C^2[0,1]$, using (18) and (20), we obtain that

$$\begin{aligned}\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| &\leq \left|\widetilde{L_{n,\lambda}^{(\alpha,s)}}(f;x)-\widetilde{L_{n,\lambda}^{(\alpha,s)}}(g;x)\right| + \left|\widetilde{L_{n,\lambda}^{(\alpha,s)}}(g;x)-g(x)\right| \\ &\quad + \left|g(x)-f(x)\right| + \left|f\left(\varepsilon_{n,\lambda}^{(\alpha,s)}(x)\right)-f(x)\right| \\ &\leq 4\|f-g\| + \left[\Psi_{n,s,2}(x;\alpha) + \left(\Psi_{n,s,1}(x;\alpha)\right)^2\right]\|g''\| \\ &\quad + \omega\left(f;\Psi_{n,s,1}(x;\alpha)\right).\end{aligned}$$

Therefore, taking infimum on the right-hand side over all $g \in C^2[0,1]$, we get

$$\begin{aligned}\left|L_{n,\lambda}^{(\alpha,s)}(f;x)-f(x)\right| &\leq 4K_2\left(f;\frac{\Psi_{n,s,2}(x;\alpha)+(\Psi_{n,s,1}(x;\alpha))^2}{4}\right) \\ &\quad + \omega\left(f;\Psi_{n,s,1}(x;\alpha)\right).\end{aligned}$$

□

4 Conclusion remarks

In the present research paper, we introduce the operators $L_{n,\lambda}^{(\alpha,s)}(f;x)$. Our operators $L_{n,\lambda}^{(\alpha,s)}(f;x)$ are based on new type Bézier bases with a shape parameter λ and positive parameter s. Moreover, operators $L_{n,\lambda}^{(\alpha,s)}(f;x)$ include classical Bernstein, α-Bernstein, generalized blending type α-Bernstein and λ-Bernstein operators as a special case. It should be mentioned that for $\lambda = 0$ and $s = 2$, our operators reduce to the operators defined by Chen et al. [13]. In this paper, some approximation properties of $L_{n,\lambda}^{(\alpha,s)}(f;x)$ are proved and also are illustrated by graphical representations (see Fig. 1, Fig. 2, Fig. 3, Fig. 4). Our operators $L_{n,\lambda}^{(\alpha,s)}(f;x)$ have better approximation for $\lambda = 1$ (see Fig. 2). Therefore, our operators have better approximation in comparison with the operators suggested and studied by Chen et al. (see Fig. 2). Finally, it should be mentioned that since our operators have better approximation for $\lambda = 1$, it gives better approximation than the other operators that can be obtained from our operators for $\lambda = 0$. For example, taking $\lambda = 0$ and $\alpha = 1$, our operators reduce to Bernstein operators, and our operators for $\alpha = 1$ and $\lambda = 1$ give better approximation than Bernstein operators.

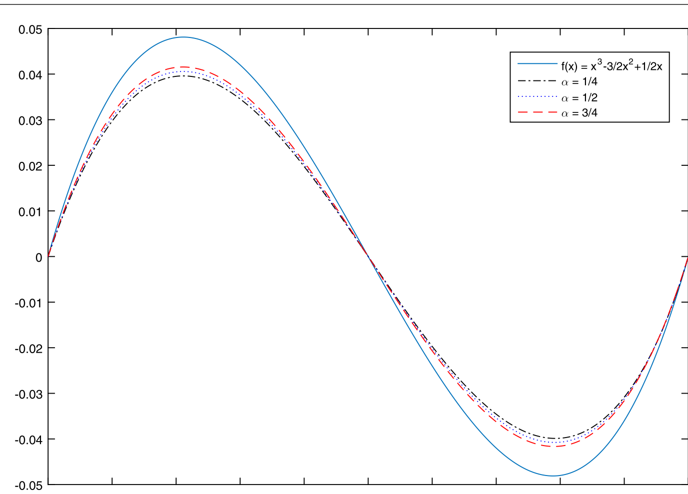

Figure 1 Approximation of $L_{n,\lambda}^{(\alpha,s)}(f;x)$ to $f(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ for $n = 10$, $s = 2$, $\lambda = 1$ and $\alpha = 0.25, 0.5, 0.75$

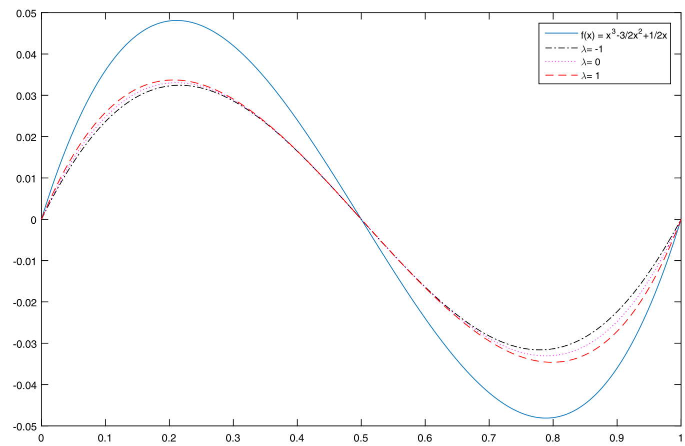

Figure 2 Approximation of $L_{n,\lambda}^{(\alpha,s)}(f;x)$ to $f(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ for $n = 10$, $s = 2$, $\alpha = \frac{1}{4}$ and $\lambda = -1, 0, 1$

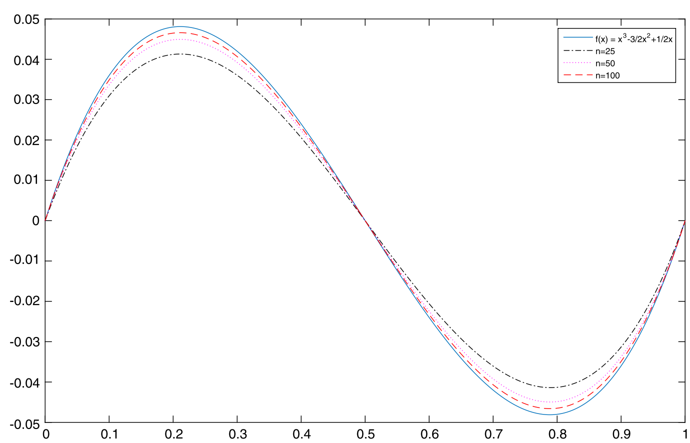

Figure 3 Approximation of $L_{n,\lambda}^{(\alpha,s)}(f;x)$ to $f(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ for $\lambda = -1$, $s = 2$, and $\alpha = \frac{1}{2}$ $n = 25, 50, 100$

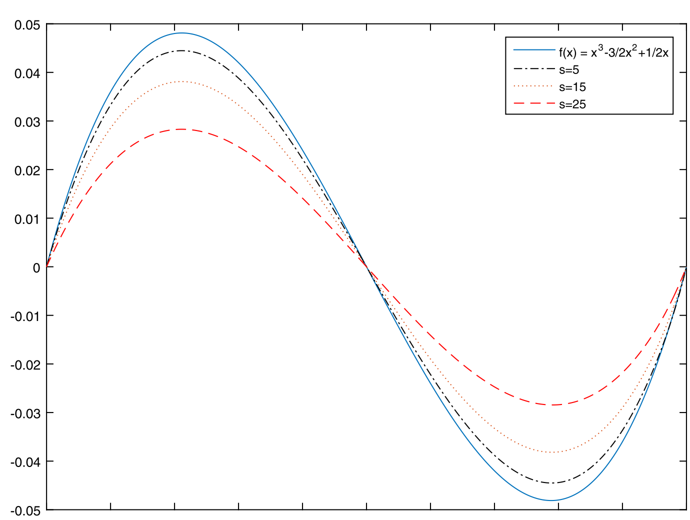

Figure 4 Approximation of $L_{n,\lambda}^{(\alpha,s)}(f;x)$ to $f(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ for $\lambda = 1$, $n = 10$, and $\alpha = \frac{1}{2}$ $s = 5, 15, 25$

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Declarations

Competing interests
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Author contributions
Contributed in proper investigation and Methodology HA and EB. Studied and prepared the manuscript EB and MSA. Analyzed the results and made necessary improvements HA and HG. Writing, reviewing and editing the paper EB and HG. Project administration by HA and HG, software and sketching of graphs MSA. All authors read and approved the final manuscript.

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