Hyder et al. *Journal of Inequalities and Applications* (2022) 2022:95
https://doi.org/10.1186/s13660-022-02831-y

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Enlarged integral inequalities through recent fractional generalized operators

Abd-Allah Hyder[1,2], M.A. Barakat[3,4*] and Ashraf Fathallah[5]

*Correspondence:
barakat14285@yahoo.com
[3]Department of Mathematics, Faculty of Sciences, Al-Azhar University, Assiut, 71524, Egypt
[4]Department of Mathematics, College of Al Wajh, University of Tabuk, Tabuk, 71491, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

This paper is devoted to proving some new fractional inequalities via recent generalized fractional operators. These inequalities are in the Hermite–Hadamard and Minkowski settings. Many previously documented inequalities may clearly be deduced as specific examples from our findings. Moreover, we give some comparative remarks to show the advantage and novelty of the obtained results.

Keywords: Generalized fractional operators; Integral inequalities; Hermite–Hadamard inequalities; Minkowski inequalities

1 Introduction

Fractional integral inequalities are very important in theoretical mathematics and are a substantial tool in dealing with fractional calculus science, which plays a vital role in modeling procedures for a variety of engineering issues [1, 14, 18, 24, 32]. Many fractional models yield better outcomes than identical equivalent models with integer derivatives, as illustrated in [27]. This drives the need for more exact inequalities when working with fractional calculus-based mathematical models. In the existing modification of a certain study, we concentrate on the most prominent Hermite–Hadamard-type inequality [2, 8]. Because of the nature of its definition, convexity is crucial in analyzing inequality for convex functions; for other classes of convex functions and attributes; see [5, 15, 16, 19, 20, 25, 26].

Recently, generalized fractional operators have been used to construct a Hermite–Hadamard-type inequality allowing the ordinary version to be regained in its limit for the generalized fractional parameter, as shown in [1]. In [7] a generalized k-fractional integral inequality is proposed, as well as the Minkowski and Chebyshev integral inequalities that involve the generalized k-fractional integrals. Inequalities of Hermite–Hadamard type under generalized k-fractional integrals were studied in [9]. Guessab and Schmeisser [6] examined the sharp integral inequalities of the Hermite–Hadamard type. Also, Nisar et al. [22] employed the Minkowski and Hermite–Hadamard inequalities to expand the results of Dahmani [4] and proposed a more powerful integral inequality. Hyder et al. [12] utilized some generalized fractional integrals to examine specific fractional-order inequalities in Minkowski and Hermite–Hadamard manners.

Let us start with the traditional Hermite–Hadamard inequality: If $u : B \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $\delta_1, \delta_2 \in B$ with $\delta_1 < \delta_2$,then

$$u\left(\frac{\delta_1 + \delta_2}{2}\right) \le \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} u(\tau)\,d\tau \le \frac{f(\delta_1) + f(\delta_2)}{2}. \tag{1}$$

Additional generalizations and expansions can be found, for instance, in [21, 23, 28, 30]. Moreover, we can begin by recalling some basic fractional notions.

Definition 1.1 ([27]) The Riemann–Liouville fractional left and right integrals of a function H are respectively defined by

$$\mathfrak{J}_{x^+}^{\gamma} H(t) = \frac{1}{\Gamma(\gamma)} \int_x^t (t-\tau)^{\gamma-1} H(\tau)\,d\tau \quad \bigl(t > x, \mathrm{Re}(\gamma) > 0\bigr) \tag{2}$$

and

$$\mathfrak{J}_{y^-}^{\gamma} H(t) = \frac{1}{\Gamma(\gamma)} \int_t^y (\tau-t)^{\gamma-1} H(\tau)\,d\tau \quad \bigl(t < y, \mathrm{Re}(\gamma) > 0\bigr), \tag{3}$$

where Γ is the gamma function.

Definition 1.2 ([13]) The left and right fractional obedient (conformable) integral operators are respectively defined by

$${}^{\lambda}\mathfrak{J}_{x^+}^{\gamma} H(t) = \frac{1}{\Gamma(\gamma)} \int_x^t \left(\frac{(t-x)^{\gamma} - (\tau-x)^{\gamma}}{\gamma}\right)^{\lambda-1} \frac{H(\tau)}{(\tau-x)^{1-\gamma}}\,d\tau, \quad t > x, \tag{4}$$

and

$${}^{\lambda}\mathfrak{J}_{y^-}^{\gamma} H(t) = \frac{1}{\Gamma(\gamma)} \int_t^y \left(\frac{(y-t)^{\gamma} - (y-\tau)^{\gamma}}{\gamma}\right)^{\lambda-1} \frac{H(\tau)}{(y-\tau)^{1-\gamma}}\,d\tau, \quad t < y, \tag{5}$$

where $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$.

Hyder and Barakat [10] enhanced the fractional obedient integral operators and offered more general definitions of the fractional integral operators as follows.

Definition 1.3 The general improved fractional left and right integral operators of a function H are respectively given by

$${}^{\lambda}\mathfrak{J}_{x^+}^{\gamma} H(t) = \frac{1}{\Gamma(\gamma)} \int_x^t h^{\lambda-1}(t-x, \tau-x, \gamma) \frac{H(\tau)}{\vartheta(\tau-x, \gamma)}\,d\tau, \quad t > x, \tag{6}$$

and

$${}^{\lambda}\mathfrak{J}_{y^-}^{\gamma} H(t) = \frac{1}{\Gamma(\gamma)} \int_t^y h^{\lambda-1}(y-t, y-\tau, \gamma) \frac{H(\tau)}{\vartheta(y-\tau, \gamma)}\,d\tau, \quad t < y, \tag{7}$$

where

$$h(t, \tau, \gamma) = \int_{\tau}^{t} \frac{d\tau^*}{\vartheta(\tau^*, \gamma)}, \tag{8}$$

and $\vartheta : \mathbb{R}_+ \times (0,1] \to \mathbb{R}_+$ is a continuous function fulfilling the conditions:
- $\vartheta(t,1) = 1 \ \forall t \in \mathbb{R}_+$,
- $\vartheta(t,\gamma) \neq 0 \ \forall (t,\gamma) \in \mathbb{R}_+ \times (0,1]$,
- $\vartheta(\cdot,\gamma_1) \neq \vartheta(\cdot,\gamma_2) \ \forall \gamma_1,\gamma_2 \in [0,1]$.

In 2020, Hyder and Soliman [11] introduced the new generalized theta-obedient integral

$$\mathfrak{I}^{\gamma}_{\theta,q}H(t) = \int_0^t \frac{(\tau - q(1+\theta_q(\tau,\gamma))H(\tau)}{(\tau - q)\theta_q(\tau,\gamma)}\,d\tau, \quad t \in \mathbb{R}_+, \tag{9}$$

where $\tau \neq q \in \mathbb{R}$, and $\theta_q : \mathbb{R}_+ \times (0,1] \to \mathbb{R}$ is a continuous function satisfying the following conditions:
- $\theta_q(t,1) = 1 \ \forall t \in \mathbb{R}_+$,
- $\theta_q(t,\gamma) \neq 0 \ \forall (t,\gamma) \in \mathbb{R}_+ \times (0,1]$,
- $\theta_q(\cdot,\gamma_1) \neq \theta_q(\cdot,\gamma_2) \ \forall \gamma_1,\gamma_2 \in [0,1]$,
- $\theta_0(t,\gamma) = \vartheta(t,\gamma) \ \forall (t,\gamma) \in \mathbb{R}_+ \times (0,1]$.

Using the Cauchy formula for iterated integrals, we can iterate the integral (9) n times and obtain the following result:

$$\begin{aligned} {}^{n}\mathfrak{I}^{\gamma}_{\theta,q}H(t) &= \int_0^t \frac{(\tau_1 - q(1+\theta_q(\tau_1,\gamma))\,d\tau_1}{(\tau_1 - q)\theta_q(\tau_1,\gamma)} \int_0^{\tau_1} \frac{(\tau_2 - q(1+\theta_q(\tau_2,\gamma))\,d\tau_2}{(\tau_2 - q)\theta_q(\tau_2,\gamma)} \\ &\quad \cdots \int_0^{\tau_{n-1}} \frac{(\tau_n - q(1+\theta_q(\tau_n,\gamma))H(\tau_n)}{(\tau_n - q)\theta_q(\tau_n,\gamma)}\,d\tau_n \\ &= \frac{1}{\Gamma(n)} \int_0^t g_q^{n-1}(t,\tau,\gamma) \frac{(\tau - q(1+\theta_q(\tau,\gamma))H(\tau)}{(\tau - q)\theta_q(\tau,\gamma)}\,d\tau, \end{aligned} \tag{10}$$

where

$$g_q(t,\tau,\gamma) = \int_\tau^t \frac{(u - q(1+\theta_q(u,\gamma))}{(u-q)\theta_q(u,\gamma)}\,du. \tag{11}$$

Replacing the natural number n by a complex number λ, we define the generalized fractional theta-obedient integral as follows.

Definition 1.4 The generalized fractional theta-obedient integral of a function H is defined by

$${}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}H(t) = \frac{1}{\Gamma(\lambda)} \int_0^t g_q^{\lambda-1}(t,\tau,\gamma) \frac{(\tau - q(1+\theta_q(\tau,\gamma))H(\tau)}{(\tau - q)\theta_q(\tau,\gamma)}\,d\tau, \tag{12}$$

where $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, and g_q is defined by (11).

In this paper, we employ recently developed generalized fractional operators to construct novel fractional inequalities for integrable nonnegative functions. These inequalities concern the Hermite–Hadamard and Minkowski inequalities. Our outcomes can be compared by the previous results established in [3, 12, 22]. The inequalities obtained in these references can be derived as particular cases. Also, we show in this work that the inequality of [22, Theorem 2.5] is incorrect. Finally, this paper is organized as follows: Sect. 2 contains the main results, and Sect. 3 provides concluding remarks.

2 Main results

In this section, we establish generalized fractional inequalities in the Hermite–Hadamard and Minkowski settings using newly discovered fractional integral operators. To support this claim, we offer the following theorems.

Theorem 2.1 *Let* $\lambda, \gamma > 0$ *and* $s \geq 1$, *and let* H, B *be two functions on* $[0, \infty)$ *such that for all* $t > 0$, $H(t), B(t) > 0, {}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} H^s(t) < \infty$, *and* ${}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} B^s(t) < \infty$. *If* $0 < j \leq \frac{H(\tau)}{B(\tau)} \leq J$, $\tau \in [0, t]$, *and* $\tau \geq q(1 + \theta_q(\tau, \gamma))$, *then we have the following inequality*:

$$\left({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} H^s(t)\right)^{1/s} + \left({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} B^s(t)\right)^{1/s} \leq \frac{1 + (j+2)J}{(j+1)(J+1)} \left({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} (H+B)^s(t)\right)^{1/s}. \tag{13}$$

Proof According to the condition $\frac{H(\tau)}{B(\tau)} \leq J$, $\tau \in [0, t]$, $t > 0$, we get

$$(J+1)^s H^s(\tau) \leq J^s (H+B)^s(\tau). \tag{14}$$

Therefore we have

$$\begin{aligned} &\frac{(J+1)^s}{\Gamma(\lambda)} g_q^{\lambda-1}(t, \tau, \gamma) \frac{(\tau - q(1 + \theta_q(\tau, \gamma)))H^s(\tau)}{(\tau - q)\theta_q(\tau, \gamma)} \\ &\quad \leq \frac{J^s}{\Gamma(\lambda)} g_q^{\lambda-1}(t, \tau, \gamma) \frac{(\tau - q(1 + \theta_q(\tau, \gamma)))(H+B)^s(\tau)}{(\tau - q)\theta_q(\tau, \gamma)}. \end{aligned} \tag{15}$$

Using (12), we can integrate inequality (15) from 0 to t with respect to τ:

$${}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} H^s(t) \leq \frac{J^s}{(J+1)^s} {}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} (H+B)^s(t). \tag{16}$$

Hence

$$\left({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} H^s(t)\right)^{1/s} \leq \frac{J}{J+1} \left({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} (H+B)^s(t)\right)^{1/s}. \tag{17}$$

Now, according to the condition $\frac{H(\tau)}{B(\tau)} \geq j$, we have

$$\left(1 + \frac{1}{j}\right)^s B^s(\tau) \leq \left(\frac{1}{j}\right)^s (H+B)^s(\tau) \tag{18}$$

and

$$\begin{aligned} &\frac{1}{\Gamma(\lambda)} \left(1 + \frac{1}{j}\right)^s g_q^{\lambda-1}(t, \tau, \gamma) \frac{(\tau - q(1 + \theta_q(\tau, \gamma)))B^s(\tau)}{(\tau - q)\theta_q(\tau, \gamma)} \\ &\quad \leq \frac{1}{\Gamma(\lambda)} \left(\frac{1}{j}\right)^s g_q^{\lambda-1}(t, \tau, \gamma) \frac{(\tau - q(1 + \theta_q(\tau, \gamma)))(H+B)^s(\tau)}{(\tau - q)\theta_q(\tau, \gamma)}. \end{aligned} \tag{19}$$

Using (14), we can integrate inequality (19) from 0 to t with respect to τ:

$$\left({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} B^s(t)\right)^{1/s} \leq \frac{1}{j+1} \left({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q} (H+B)^s(t)\right)^{1/s}. \tag{20}$$

Thus by adding inequalities (17) and (20) we obtain the required inequality (13). □

Theorem 2.2 *Let $\gamma > 0$, $s \geq 1$, and $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, and let H, B be two functions on $[0,\infty)$ such that for all $t > 0$, $H(t), B(t) > 0$, ${}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}H^s(t) < \infty$, and ${}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}B^s(t) < \infty$. If $0 < j \leq \frac{H(\tau)}{B(\tau)} \leq J$, $\tau \in [0,t]$, and $\tau \geq q(1 + \theta_q(\tau,\gamma))$, then we have the following inequality*:

$$\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}H^s(t)\right)^{2/s} + \left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}B^s(t)\right)^{2/s} \geq \left(\frac{(J+1)(j+1)}{J} - 2\right)\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}(H+B)^s(t)\right)^{1/s}. \tag{21}$$

Proof By multiplying the two inequalities (17) and (20) we get

$$\left(\frac{(J+1)(j+1)}{J}\right)\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}H^s(t)\right)^{1/s}\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}B^s(t)\right)^{1/s} \leq \left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}(H+B)^s(t)\right)^{2/s}. \tag{22}$$

By the Minkowski inequality we obtain

$$\begin{aligned}\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}(H+B)^s(t)\right)^{2/s} &\leq \left(\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}H^s(t)\right)^{1/s} + \left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}B^s(t)\right)^{1/s}\right)^2 \\ &= \left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}H^s(t)\right)^{2/s} + \left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}B^s(t)\right)^{2/s} \\ &\quad + 2\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}H^s(t)\right)^{1/s}\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}B^s(t)\right)^{1/s}.\end{aligned} \tag{23}$$

Thus we can derive the desired inequality (21) using inequalities (22) and (23). □

Lemma 2.1 ([4]) *Let H be a concave function on $[a,b]$. Then we have the following inequalities*:

$$H(a) + H(b) \leq H(a+b-t) + H(t) \leq 2H\left(\frac{a+b}{2}\right). \tag{24}$$

Theorem 2.3 *Let $\lambda, \gamma > 0$, $\lambda \in \mathbb{C}$, and $c, d > 1$, and let H, B be two functions on $[0,\infty)$ such that $H(t), B(t) > 0$ for $t > 0$. If the functions H^c, B^d are concave on $[0,\infty)$, then we have the following inequality*:

$$\begin{aligned}&\frac{1}{2^{c+d}}\left(H(0) + H\left(\gamma g_q(t,0,\gamma)\right)\right)^c\left(B(0) + B\left(\gamma g_q(t,0,\gamma)\right)\right)^d\left({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}\left(\gamma^{\lambda-1}g_q^{\lambda-1}(t,0,\gamma)\right)\right)^2 \\ &\quad \leq {}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}\left(\gamma^{\lambda-1}g_q^{\lambda-1}(t,0,\gamma)H^c\left(\gamma g_q(t,0,\gamma)\right)\right){}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q} \\ &\qquad \times \left(\gamma^{\lambda-1}g_q^{\lambda-1}(t,0,\gamma)B^d\left(\gamma g_q(t,0,\gamma)\right)\right).\end{aligned} \tag{25}$$

Proof By Lemma 2.1 and the concavity of the functions H^c, B^d, for $t > 0, \gamma > 0$, and $\tau \in [0,t]$, we have

$$\begin{aligned}H^c(0) + H^c\left(\gamma g_q(t,0,\gamma)\right) &\leq H^c\left(\gamma g_q(t,\tau,\gamma)\right) + H^c\left(\gamma g_q(\tau,0,\gamma)\right) \\ &\leq 2H^c\left(\frac{\gamma}{2}g_q(t,0,\gamma)\right),\end{aligned} \tag{26}$$

$$\begin{aligned}B^d(0) + B^d\left(\gamma g_q(t,0,\gamma)\right) &\leq B^d\left(\gamma g_q(t,\tau,\gamma)\right) + B^d\left(\gamma g_q(\tau,0,\gamma)\right) \\ &\leq 2B^d\left(\frac{\gamma}{2}g_q(t,0,\gamma)\right).\end{aligned} \tag{27}$$

Multiplying inequalities (26) and (27) by $\frac{\tau-q(1+\theta_q(\tau,\gamma))}{\Gamma(\lambda)(\tau-q)\theta_q(\tau,\gamma)}(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma))^{\lambda-1}$ and integrating the resulting inequalities from 0 to t, we get

$$\begin{aligned}
&\frac{H^c(0)+H^c(\gamma g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{\tau-q(1+\theta_q(\tau,\gamma))\,d\tau}{(\tau-q)\theta_q(\tau,\gamma)}\\
&\quad\le\frac{1}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))H^c(\gamma g_q(t,\tau,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\\
&\qquad+\frac{1}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))H^c(\gamma g_q(\tau,0,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\\
&\quad\le\frac{2H^c(\frac{\gamma}{2}g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))\,d\tau}{(\tau-q)\theta_q(\tau,\gamma)},
\end{aligned}\tag{28}$$

and

$$\begin{aligned}
&\frac{B^d(0)+B^d(\gamma g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))\,d\tau}{(\tau-q)\theta_q(\tau,\gamma)}\\
&\quad\le\frac{1}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))B^d(\gamma g_q(t,\tau,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\\
&\qquad+\frac{1}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))B^d(\gamma g_q(\tau,0,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\\
&\quad\le\frac{2B^d(\frac{\gamma}{2}g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))\,d\tau}{(\tau-q)\theta_q(\tau,\gamma)}.
\end{aligned}\tag{29}$$

Setting $g_q(t,\tau,\gamma)=g_q(\eta,0,\gamma)$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))\,d\tau}{(\tau-q)\theta_q(\tau,\gamma)}\\
&\quad={}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1}\big),
\end{aligned}\tag{30}$$

$$\begin{aligned}
&\frac{1}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))H^c(\gamma g_q(t,\tau,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\\
&\quad={}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1}H^c\big(\gamma g_q(t,0,\gamma)\big),
\end{aligned}\tag{31}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\lambda)}\int_0^t\big(\gamma g_q(t,\tau,\gamma)g_q(\tau,0,\gamma)\big)^{\lambda-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))B^d(\gamma g_q(t,\tau,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\\
&\quad={}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1}B^d\big(\gamma g_q(t,0,\gamma)\big).
\end{aligned}\tag{32}$$

Therefore by (28), (30), and (31) we get

$$\begin{aligned}
&\big(H^c(0)+H^c\big(\gamma g_q(t,0,\gamma)\big)\big({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1}\big)\big)\\
&\quad\le 2^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1}H^c\big(\gamma g_q(t,0,\gamma)\big)\\
&\quad\le 2H^c\left(\frac{\gamma}{2}g_q(t,0,\gamma)\right)\big({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1}\big)\big),
\end{aligned}\tag{33}$$

and by (29), (30), and (32) we get

$$\begin{aligned}
&(B^d(0) + B^d(\gamma g_q(t,0,\gamma))({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}))\\
&\quad\leq 2^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}B^d(\gamma g_q(t,0,\gamma))\\
&\quad\leq 2B^d\left(\frac{\gamma}{2}g_q(t,0,\gamma)\right)({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1})).
\end{aligned}\tag{34}$$

Hence by multiplying both inequalities (33) and (34) it follows that

$$\begin{aligned}
&(H^c(0) + H^c(\gamma g_q(t,0,\gamma))(B^d(0) + B^d(\gamma g_q(t,0,\gamma))({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}))^2\\
&\quad\leq 4({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}H^c(\gamma g_q(t,0,\gamma)))\\
&\qquad\times({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}B^d(\gamma g_q(t,0,\gamma))).
\end{aligned}\tag{35}$$

Since $H(t)B(t) > 0$ for $t > 0$, for $\gamma > 0, c \geq 0$, and $d \geq 0$, we have

$$\left(\frac{H^c(0) + H^c(\gamma g_q(t,0,\gamma)}{2}\right)^{1/c} \geq \frac{H(0) + H(\gamma g_q(t,0,\gamma)}{2},\tag{36}$$

and

$$\left(\frac{B^d(0) + B^d(\gamma g_q(t,0,\gamma)}{2}\right)^{1/d} \geq \frac{B(0) + B(\gamma g_q(t,0,\gamma)}{2}.\tag{37}$$

Therefore

$$\begin{aligned}
&\left(\frac{H^c(0) + H^c(\gamma g_q(t,0,\gamma)}{2}\right)({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}))\\
&\quad\geq\left(\frac{H(0) + H(\gamma g_q(t,0,\gamma)}{2}\right)^c({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1})),
\end{aligned}\tag{38}$$

and

$$\begin{aligned}
&\left(\frac{B^d(0) + B^d(\gamma g_q(t,0,\gamma)}{2}\right)({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}))\\
&\quad\geq\left(\frac{B(0) + B(\gamma g_q(t,0,\gamma)}{2}\right)^d({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1})).
\end{aligned}\tag{39}$$

Using inequalities (38) and (39), we obtain

$$\begin{aligned}
&(H^c(0) + H^c(\gamma g_q(t,0,\gamma))(B^d(0) + B^d(\gamma g_q(t,0,\gamma))({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}))^2\\
&\quad\geq 2^{2-c-d}(H(0) + H(\gamma g_q(t,0,\gamma))^c(B(0) + B(\gamma g_q(t,0,\gamma))^d\\
&\qquad\times({}^{\lambda}\mathfrak{I}^{\gamma}_{\theta,q}((\gamma g_q(t,0,\gamma))^{\lambda-1}))^2.
\end{aligned}\tag{40}$$

Hence we derive the needed inequality (25) by combining inequalities (35) and (40). □

Theorem 2.4 *Let $\lambda, \gamma, \nu > 0$, $\lambda, \nu \in \mathbb{C}$, and $c, d > 1$, and let H, B be two functions on $[0, \infty)$ such that $H(t), B(t) > 0$ for $t > 0$. If H^c, B^d are concave on $[0, \infty)$, then we have the following inequality*:

$$\begin{aligned}
&\frac{1}{2^{c+d-2}}\big(H(0)+H(t)\big)^c\big(B(0)+B(t)\big)^d\big({}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\gamma^{\nu-1}g_q^{\nu-1}(t,0,\gamma)\big)\big)^2 \\
&\quad\le\bigg[\frac{\gamma^{\nu-\lambda}\Gamma(\nu)}{\Gamma(\lambda)}^{\nu}\mathfrak{J}^{\gamma}\big(\gamma^{\lambda-1}g_q^{\lambda-1}(t,0,\gamma)H^c\big(\gamma g_q(t,0,\gamma)\big)\big) \\
&\qquad+{}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\gamma^{\nu-1}g_q^{\nu-1}(t,0,\gamma)H^c\big(\gamma g_q(t,0,\gamma)\big)\big)\bigg] \\
&\qquad\times\bigg[\frac{\gamma^{\nu-\lambda}\Gamma(\nu)}{\Gamma(\lambda)}^{\nu}\mathfrak{J}^{\gamma}\big(\gamma^{\lambda-1}g_q^{\lambda-1}(t,0,\gamma)B^d\big(\gamma g_q(t,0,\gamma)\big)\big) \\
&\qquad+{}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\gamma^{\nu-1}g_q^{\nu-1}(t,0,\gamma)B^d\big(\gamma g_q(t,0,\gamma)\big)\big)\bigg].
\end{aligned}\tag{41}$$

Proof Multiplying inequalities (26) and (27) by $\frac{\tau-q(1+\theta_q(\tau,\gamma))}{\Gamma(\lambda)(\tau-q)\theta_q(\tau,\gamma)}g_q^{\lambda-1}(t,\tau,\gamma)(\gamma g_q(\tau,0,\gamma))^{\nu-1}$ and then integrating the resulting inequalities with respect to τ from 0 to t, we obtain

$$\begin{aligned}
&\frac{H^c(0)+H^c(\gamma g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{\tau-q(1+\theta_q(\tau,\gamma))}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau \\
&\quad\le\frac{1}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))H^c(\gamma g_q(t,\tau,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\,d\tau \\
&\qquad+\frac{1}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))H^c(\gamma g_q(\tau,0,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau \\
&\quad\le\frac{2H^c(\frac{\gamma}{2}g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{\tau-q(1+\theta_q(\tau,\gamma))}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau,
\end{aligned}\tag{42}$$

and

$$\begin{aligned}
&\frac{B^d(0)+B^d(\gamma g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{\tau-q(1+\theta_q(\tau,\gamma))}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau \\
&\quad\le\frac{1}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))B^d(\gamma g_q(t,\tau,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau\,d\tau \\
&\qquad+\frac{1}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{(\tau-q(1+\theta_q(\tau,\gamma)))B^d(\gamma g_q(\tau,0,\gamma)}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau \\
&\quad\le\frac{2B^d(\frac{\gamma}{2}g_q(t,0,\gamma))}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{\tau-q(1+\theta_q(\tau,\gamma))}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau.
\end{aligned}\tag{43}$$

Setting $g_q(t,\tau,\gamma)=g_q(\eta,0,\gamma)$,we have

$$\begin{aligned}
&\frac{1}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{\tau-q(1+\theta_q(\tau,\gamma))}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau \\
&\quad={}^{\lambda}\mathfrak{J}^{\gamma}_{\theta,q}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1}\big),
\end{aligned}\tag{44}$$

$$\frac{1}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}H^c(\gamma g_q(t,\tau,\gamma)\frac{\tau-q(1+\theta_q(\tau,\gamma))}{(\tau-q)\theta_q(\tau,\gamma)}\,d\tau$$

$$= \frac{\gamma^{\nu-\lambda}\Gamma(\nu)^{\nu}}{\Gamma(\lambda)} \mathfrak{J}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1} H^{c}\big(\gamma g_q(t,0,\gamma)\big), \tag{45}$$

and

$$\begin{aligned} &\frac{1}{\Gamma(\lambda)}\int_0^t g_q^{\lambda-1}(t,\tau,\gamma)\big(\gamma g_q(\tau,0,\gamma)\big)^{\nu-1}\frac{\tau-q(1+\theta_q(\tau,\gamma))}{(\tau-q)\theta_q(\tau,\gamma)} B^{d}(\gamma g_q(t,\tau,\gamma)\,d\tau \\ &\quad = \frac{\gamma^{\nu-\lambda}\Gamma(\nu)^{\nu}}{\Gamma(\lambda)} \mathfrak{J}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1} B^{d}\big(\gamma g_q(t,0,\gamma)\big). \end{aligned} \tag{46}$$

Therefore, by (41), (44), and (45) we get

$$\begin{aligned} &\big(H^{c}(0) + H^{c}\big(\gamma g_q(t,0,\gamma)\big)\big({}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1}\big)\big) \\ &\quad \le \frac{\gamma^{\nu-\lambda}\Gamma(\nu)^{\nu}}{\Gamma(\lambda)} \mathfrak{J}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1} H^{c}\big(\gamma g_q(t,0,\gamma)\big) \\ &\qquad + {}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1} H^{c}\big(\gamma g_q(t,0,\gamma)\big). \end{aligned} \tag{47}$$

Also, by (43), (44), and (46) we get

$$\begin{aligned} &\big(B^{d}(0) + B^{d}\big(\gamma g_q(t,0,\gamma)\big)\big({}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1}\big)\big) \\ &\quad \le \frac{\gamma^{\nu-\lambda}\Gamma(\nu)^{\nu}}{\Gamma(\lambda)} \mathfrak{J}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1} B^{d}\big(\gamma g_q(t,0,\gamma)\big) \\ &\qquad + {}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1} B^{d}\big(\gamma g_q(t,0,\gamma)\big). \end{aligned} \tag{48}$$

By multiplying inequalities (47) and (48) we get

$$\begin{aligned} &\big(H^{c}(0) + H^{c}\big(\gamma g_q(t,0,\gamma)\big)\big)\big(B^{d}(0) + B^{d}\big(\gamma g_q(t,0,\gamma)\big)\big)\big({}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1}\big)\big)^{2} \\ &\quad \le \bigg[\frac{\gamma^{\nu-\lambda}\Gamma(\nu)^{\nu}}{\Gamma(\lambda)} \mathfrak{J}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1} H^{c}\big(\gamma g_q(t,0,\gamma)\big) \\ &\qquad + {}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1} H^{c}\big(\gamma g_q(t,0,\gamma)\big)\bigg] \end{aligned} \tag{49}$$

$$\begin{aligned} &\quad \le \bigg[\frac{\gamma^{\nu-\lambda}\Gamma(\nu)^{\nu}}{\Gamma(\lambda)} \mathfrak{J}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1} B^{d}\big(\gamma g_q(t,0,\gamma)\big) \\ &\qquad + {}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\nu-1} B^{d}\big(\gamma g_q(t,0,\gamma)\big)\bigg]. \end{aligned} \tag{50}$$

According to inequalities (38) and (39), we obtain

$$\begin{aligned} &\big(H^{c}(0) + H^{c}\big(\gamma g_q(t,0,\gamma)\big)\big)\big(B^{d}(0) + B^{d}\big(\gamma g_q(t,0,\gamma)\big)\big)\big({}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)\big)^{\lambda-1}\big)\big)^{2} \\ &\quad \ge 2^{2-k-l}\big(H(0) + H\big(\gamma g_q(t,0,\gamma)\big)\big)^{c}\big(B(0) + B\big(\gamma g_q(t,0,\gamma)\big)\big)^{d} \\ &\qquad \times \big({}^{\lambda}\mathfrak{J}_{\theta,q}^{\gamma}\big(\big(\gamma g_q(t,0,\gamma)\big)^{\lambda-1}\big)\big)^{2}. \end{aligned} \tag{51}$$

Hence we derive the needed inequality (41) by combining inequalities (49) and (51). □

3 Comparative cases

In this study, we establish the Hermite–Hadamard and Minkowski inequalities in the context of newly generalized fractional integral operators.

We examine some specific cases arising from our findings in this section by presenting some particular examples of our fractional integral operator described by equation (12) as the following cases.

Case I. If we put $q = 0$ in (12), then we get the general improved fractional operator defined in [10]. When this operator is applied to the Hermite–Hadamard and Minkowski inequalities, the obtained results are reduced to the following corollaries.

Corollary 3.1 *Let $\lambda, \gamma > 0$, $\lambda \in \mathbb{C}$, and $c, d > 1$. Let H, B be two functions on $[0, \infty)$ such that $H(t), B(t) > 0$ for $t > 0$. If the functions H^c, B^d are concave on $[0, \infty)$, then we have the following inequality*:

$$\begin{aligned} &\frac{1}{2^{c+d}}\big(H(0)+H\big(\gamma g(t,0,\gamma)\big)\big)^c\big(B(0)+B\big(\gamma g(t,0,\gamma)\big)\big)^d\big({}^{\lambda}\mathfrak{J}_{\theta}^{\gamma}\big(\gamma^{\lambda-1}g^{\lambda-1}(t,0,\gamma)\big)\big)^2 \\ &\quad\le {}^{\lambda}\mathfrak{J}_{\theta}^{\gamma}\big(\gamma^{\lambda-1}g^{\lambda-1}(t,0,\gamma)H^c\big(\gamma g(t,0,\gamma)\big)\big){}^{\lambda}\mathfrak{J}_{\theta}^{\gamma}\big(\gamma^{\lambda-1}g^{\lambda-1}(t,0,\gamma)B^d\big(\gamma g(t,0,\gamma)\big)\big). \end{aligned} \tag{52}$$

Corollary 3.2 *Let $\lambda, \gamma, \nu > 0$, $\lambda, \nu \in \mathbb{C}$, and $c, d > 1$, and let H, B be two functions on $[0, \infty)$ such that $H(t), B(t) > 0$ for $t > 0$. If H^c, B^d are concave functions on $[0, \infty)$, then we have the following inequality*:

$$\begin{aligned} &\frac{1}{2^{c+d-2}}\big(H(0)+H(t)\big)^c\big(B(0)+B(t)\big)^d\big({}^{\lambda}\mathfrak{J}_{\theta}^{\gamma}\big(\gamma^{\nu-1}g^{\nu-1}(t,0,\gamma)\big)\big)^2 \\ &\quad\le \left[\frac{\gamma^{\nu-\lambda}\Gamma(\nu)}{\Gamma(\lambda)}^{\nu}\mathfrak{J}^{\gamma}\big(\gamma^{\lambda-1}g^{\lambda-1}(t,0,\gamma)H^c\big(\gamma g(t,0,\gamma)\big)\big)\right. \\ &\qquad\left.+{}^{\lambda}\mathfrak{J}_{\theta}^{\gamma}\big(\gamma^{\nu-1}g^{\nu-1}(t,0,\gamma)H^c\big(\gamma g(t,0,\gamma)\big)\big)\right] \\ &\qquad\times\left[\frac{\gamma^{\nu-\lambda}\Gamma(\nu)}{\Gamma(\lambda)}^{\nu}\mathfrak{J}^{\gamma}\big(\gamma^{\lambda-1}g^{\lambda-1}(t,0,\gamma)B^d\big(\gamma g(t,0,\gamma)\big)\big)\right. \\ &\qquad\left.+{}^{\lambda}\mathfrak{J}_{\theta}^{\gamma}\big(\gamma^{\nu-1}g^{\nu-1}(t,0,\gamma)B^d\big(\gamma g(t,0,\gamma)\big)\big)\right]. \end{aligned} \tag{53}$$

Case II. If we put $\gamma = 1$ and $g(t,\tau,\gamma) = \ln t - \ln\tau$ in Corollary 3.2, then we obtain the integral inequalities due to Hadamard fractional integral operator as follows.

Corollary 3.3 *Let $\lambda, \nu > 0$, $\lambda, \nu \in \mathbb{C}$, and $k, l > 1$, and let H, B be two functions on $[0, \infty)$ such that $H(t), B(t) > 0$ for $t > 0$. If H^c, B^d are concave functions on $[0, \infty)$, then we have the following inequality*:

$$\begin{aligned} &\frac{1}{2^{k+l-2}}\big(H(0)+H(t)\big)^c\big(B(0)+B(t)\big)^d\big({}^{\lambda}\mathfrak{J}^1\big((\ln t)^{\nu-1}\big)\big)^2 \\ &\quad\le\left[\frac{\Gamma(\nu)}{\Gamma(\lambda)}^{\nu}\mathfrak{J}^1\big((\ln t)^{\lambda-1}H^c(\ln t)\big)+^{\lambda}\mathfrak{J}^1\big((\ln t)^{\nu-1}H^c(\ln t)\right] \\ &\qquad\times\left[\frac{\Gamma(\nu)}{\Gamma(\lambda)}^{\nu}\mathfrak{J}^1\big((\ln t)^{\lambda-1}B^d(\ln t)\big)+^{\lambda}\mathfrak{J}^1\big((\ln t)^{\nu-1}B^d(\ln t)\big)\right]. \end{aligned} \tag{54}$$

Case III. If we put $g(t,\tau,\gamma) = \frac{t^\gamma - \tau^\gamma}{\gamma}$ in Corollaries 3.1, 3.2, we will obtain the integral inequalities due to Nizar et al. [22].

Case IV. If $g(t,\tau,\gamma) = \frac{t^\gamma - \tau^\gamma}{\gamma}$ and $\gamma = 1$, then we obtain all the fractional inequalities introduced by Dahmani [4].

Case V. If we put $g(t,\tau,\gamma) = \frac{t^\gamma - \tau^\gamma}{\gamma}$ and $\gamma = \lambda = 1$, then all the outcomes will reduced to the traditional inequalities introduced in [3].

4 Conclusions

In the context of the generalized fractional theta-obedient integral, a variety of research directions related to integral inequalities can be examined in equation (12). More research on the Hermite–Hadamard inequality with differentiable h-convex functions [31], Hermite–Hadamard inequality for s-convex functions [29], the binary Brunn–Minkowski inequality [17], and other topics are predicted under this operator.

Acknowledgements
The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Groups Program under grant RGP.2/15/43.

Funding
This research was funded by King Khalid University, grant RGP.2/15/43.

Availability of data and materials
The data that support the findings of this study are available from the authors upon request.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Author contributions
The manuscript was written by AH, MAB, and AF. AH was also responsible of acquiring finance. AH and MAB worked together to accomplish the formal analysis, writing the review, and editing. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha, 61413, Saudi Arabia. [2]Department of Engineering Mathematics and Physics, Faculty of Engineering, Al-Azhar University, Cairo, 11371, Egypt. [3]Department of Mathematics, Faculty of Sciences, Al-Azhar University, Assiut, 71524, Egypt. [4]Department of Mathematics, College of Al Wajh, University of Tabuk, Tabuk, 71491, Saudi Arabia. [5]Department of Mathematics, Faculty of Engineering, Misr International University, Cairo, 11828, Egypt.

Publisher's Note

Received: 5 November 2021 Accepted: 13 May 2022 Published online: 18 July 2022

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