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On fractional Simpson type integral inequalities for co-ordinated convex functions

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Abstract

In this study, we prove equality for twice partially differentiable mappings involving the double generalized fractional integral. Using the established identity, we offer some Simpson's type inequalities for partially differentiable co-ordinated convex functions in a rectangle from the plane $\mathbb{R}^2$.

MSC: 26D07; 26D10; 26D15; 26B15; 26B25

Keywords: Simpson's inequality; Integral inequalities; Fractional calculus; Co-ordinated convex functions

1 Introduction

The following inequality is one of the well-known results in the literature called Simpson's inequality.

Theorem 1 *Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a four times continuously differentiable mapping on (κ_1, κ_2) and $\|F^4\|_\infty = \sup |F^4(x)| < \infty$. Then, the following inequality holds*:

$$\left|\frac{1}{6}\left[F(\kappa_1) + F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(x)\,dx\right| \leq \frac{1}{2880}\left\|F^{(4)}\right\|_\infty (\kappa_2-\kappa_1)^4.$$

For recent refinements, counterparts, generalizations, and new Simpson's type inequalities, see [1, 4–8, 11–17, 19–28].

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1 A function $F : \triangle : [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4] \to \mathbb{R}$ is called co-ordinated convex on $\triangle$ for all (x, u), $(y, v) \in \triangle$ and $t, s \in [0, 1]$ if it satisfies the following inequality:

$$\begin{aligned} &F\left(tx + (1-t)y, su + (1-s)v\right) \\ &\quad \leq tsF(x,u) + t(1-s)F(x,v) + s(1-t)F(y,u) + (1-s)(1-t)F(y,v). \end{aligned} \tag{1.1}$$

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The mapping F is co-ordinated concave on $\triangle$, and the inequality (1.1) holds in reversed direction for all $t, s \in [0,1]$ and $(x,u),(y,v) \in \triangle$.

In [5], Dragomir et al. proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 2 *Suppose $F : [\kappa_1,\kappa_2] \to \mathbb{R}$ is an absolutely continuous mapping on $[\kappa_1,\kappa_2]$ whose derivative belongs to $L_p[\kappa_1,\kappa_2]$. Then, the following inequality holds*:

$$\left|\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(x)\,dx\right| \leq \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}} (\kappa_2-\kappa_1)^{\frac{1}{q}} \left\|F'\right\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [19], Sarikaya et al. obtained inequalities for differentiable convex mappings that are connected with Simpson's inequality, and they used the following lemma to prove it.

Lemma 1 *Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° (I° is the interior of I) such that $F' \in L_1[\kappa_1,\kappa_2]$, where $\kappa_1,\kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, then the following equality holds*:

$$\begin{aligned}
&\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2} F(x)\,dx \\
&\quad = \frac{\kappa_2-\kappa_1}{2}\int_0^1 \left[\left(\frac{t}{2}-\frac{1}{3}\right)F'\left(\frac{(1+t)\kappa_1}{2}+\frac{(1-t)\kappa_2}{2}\right)\right. \\
&\qquad \left. + \left(\frac{1}{3}-\frac{t}{2}\right)F'\left(\frac{(1+t)\kappa_1}{2}+\frac{(1-t)\kappa_2}{2}\right)\right] dt.
\end{aligned}$$

The main aim of this article is to set up new Simpson's type inequalities for mappings whose twice partially derivatives in absolute value are co-ordinated convex via generalized fractional integral operators.

2 Generalized fractional integrals operators

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [18].

Let us define a function $\phi : [0,\infty) \to [0,\infty)$ satisfying the following condition:

$$\int_0^1 \frac{\phi(t)}{t}\,dt < \infty.$$

We consider the following left-sided and right-sided generalized fractional integral operators

$$_{a^+}I_\phi f(x) = \int_a^x \frac{\phi(x-t)}{x-t} f(t)\,dt, \quad x > a \tag{2.1}$$

and

$$_{b-}I_{\phi}f(x) = \int_x^b \frac{\phi(t-x)}{t-x} f(t)\,dt, \quad x < b, \tag{2.2}$$

respectively.

Some forms of fractional integrals, namely, Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc. are generalized as the most significant feature of generalized fractional integrals. These important special cases of the integral operators (2.1) and (2.2) are mentioned below:

(1) If we choose $\phi(t) = t$, the operators (2.1) and (2.2) reduce to the Riemann integral.
(2) Considering $\phi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, the operators (2.1) and (2.2) reduce to the Riemann–Liouville fractional integrals $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$, respectively. Here, Γ is the Gamma function.
(3) For $\phi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators (2.1) and (2.2) reduce to the k-Riemann–Liouville fractional integrals $J_{a+,k}^{\alpha}f(x)$ and $J_{b-,k}^{\alpha}f(x)$, respectively. Here, Γ_k is the k-Gamma function.

There are several papers on inequalities for generalized fractional integrals in the literature. In [18], Sarikaya and Ertuğral also proved Hermite–Hadamard inequalities for generalized fractional integrals. In addition, Budak et al. proved Midpoint type inequalities and extensions of Hermite–Hadamard inequalities in the papers [2] and [3], respectively. In [9], Ertuğral and Sarikaya presented some Simpson-type inequalities for these fractional integral operators.

Generalized double fractional integrals are given by Turkay et al. in [26], as follows:

Definition 2 The Generalized double fractional integrals $_{a+,c+}I_{\phi,\psi}$, $_{a+,d-}I_{\phi,\psi}$, $_{b-,c+}I_{\phi,\psi}$, $_{b-,d-}I_{\phi,\psi}$ are defined by

$$_{a+,c+}I_{\phi,\psi}f(x,y) = \int_a^x \int_c^y \frac{\phi(x-t)}{x-t} \frac{\psi(y-s)}{y-s} f(t,s)\,ds\,dt, \quad x > a, y > c, \tag{2.3}$$

$$_{a+,d-}I_{\phi,\psi}f(x,y) = \int_a^x \int_y^d \frac{\phi(x-t)}{x-t} \frac{\psi(s-y)}{s-y} f(t,s)\,ds\,dt, \quad x > a, y < d, \tag{2.4}$$

$$_{b-,c+}I_{\phi,\psi}f(x,y) = \int_x^b \int_c^y \frac{\phi(t-x)}{t-x} \frac{\psi(y-s)}{y-s} f(t,s)\,ds\,dt, \quad x < b, y > c, \tag{2.5}$$

and

$$_{b-,d-}I_{\phi,\psi}f(x,y) = \int_x^b \int_y^d \frac{\phi(t-x)}{t-x} \frac{\psi(s-y)}{s-y} f(t,s)\,ds\,dt, \quad x < b, y < d. \tag{2.6}$$

Here, $f \in L_1([a,b] \times [c,d])$, and the functions $\phi, \psi : [0,\infty) \to [0,\infty)$ satisfy $\int_0^1 \frac{\phi(t)}{t}\,dt < \infty$ and $\int_0^1 \frac{\psi(s)}{s}\,ds < \infty$, respectively.

Using Definition 2, well-known fractional integrals can be obtained by some special choices. For example,

(1) If we choose $\phi(t) = t$ and $\psi(s) = s$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the double Riemann integral.

(2) Considering $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$, then for $\alpha, \beta > 0$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the Riemann–Liouville fractional integrals $J^{\alpha,\beta}_{a+,c+}f(x,y)$, $J^{\alpha,\beta}_{a+,d-}f(x,y)$ $J^{\alpha,\beta}_{b-,c+}f(x,y)$ and $J^{\alpha,\beta}_{b-,d-}f(x,y)$, respectively.

(3) For $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, for $\alpha, \beta, k > 0$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the k-Riemann–Liouville fractional integrals $J^{\alpha,\beta,k}_{a+,c+}f(x,y)$, $J^{\alpha,\beta,k}_{a+,d-}f(x,y)$, $J^{\alpha,\beta,k}_{b-,c+}f(x,y)$ and $J^{\alpha,\beta,k}_{b-,d-}f(x,y)$, respectively ([10]).

In this paper, utilizing generalized double fractional integrals, we extend the results proved in [9] to co-ordinated convex functions.

3 An identity for generalized double fractional integrals

Throughout this paper, for conciseness, we define

$$\Lambda(t) = \int_0^t \frac{\phi(\frac{b-a}{2}u)}{u}\,du, \qquad \nabla(s) = \int_0^s \frac{\psi(\frac{d-c}{2}u)}{u}\,du.$$

In this section, we start the usage of generalized fractional essential operators by subsequent lemma:

Lemma 2 *Let $F : \Delta := [\kappa_1,\kappa_2] \times [\kappa_3,\kappa_4] \to \mathbb{R}$ be a twice partially differentiable mapping on Δ° (Δ° is the interior of Δ). If $\frac{\partial^2 F}{\partial t\,\partial s} \in L(\Delta)$, then we have the following equality for generalized fractional integrals*:

$$\begin{aligned}
&\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\\
&\quad= \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4\Lambda(1)\nabla(1)}\\
&\qquad\times\left\{\int_0^1\int_0^1\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\left(\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right)\frac{\partial^2}{\partial t\,\partial s}\right.\\
&\qquad\times F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt\\
&\qquad+\int_0^1\int_0^1\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\left(\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right)\frac{\partial^2}{\partial t\,\partial s}\\
&\qquad\times F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt\\
&\qquad+\int_0^1\int_0^1\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\left(\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right)\frac{\partial^2}{\partial t\,\partial s}\\
&\qquad\times F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt\\
&\qquad+\int_0^1\int_0^1\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\left(\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right)\frac{\partial^2}{\partial t\,\partial s}\\
&\qquad\times \left.F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt\right\},
\end{aligned}$$

where

$$\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$$

$$
\begin{aligned}
&= \frac{F(\kappa_1,\kappa_3)+F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)}{36} \\
&\quad + \frac{1}{9}\left[F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\right. \\
&\quad \left.+ 4F\left(\frac{\kappa_2+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right] + \frac{1}{4\Lambda(1)\bigtriangledown(1)}\left[{}_{\kappa_2^-,\kappa_4^-}I_{\phi,\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right. \\
&\quad + {}_{\kappa_2^-,\kappa_3^+}I_{\phi,\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) + {}_{\kappa_1^+,\kappa_4^-}I_{\phi,\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad \left.+ {}_{\kappa_1^+,\kappa_3^+}I_{\phi,\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right] \\
&\quad - \frac{1}{12\Lambda(1)}\left[{}_{\kappa_2^-}I_{\phi}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right) + {}_{\kappa_2^-}I_{\phi}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right) + {}_{\kappa_1^+}I_{\phi}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right. \\
&\quad + {}_{\kappa_1^+}I_{\phi}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right) + 4{}_{\kappa_2^-}I_{\phi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad \left.+ 4{}_{\kappa_1^+}I_{\phi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right] \\
&\quad - \frac{1}{12\bigtriangledown(1)}\left[{}_{\kappa_4^-}I_{\psi}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right) + {}_{\kappa_4^-}I_{\psi}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right) + {}_{\kappa_3^+}I_{\psi}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right. \\
&\quad + {}_{\kappa_3^+}I_{\psi}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right) + 4{}_{\kappa_4^-}I_{\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad \left.+ 4{}_{\kappa_3^+}I_{\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right].
\end{aligned}
$$

Proof It suffices to be aware that

$$
\begin{aligned}
I &= \int_0^1\int_0^1\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\left(\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right)\frac{\partial^2}{\partial t\,\partial s} \\
&\quad \times F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt \\
&\quad + \int_0^1\int_0^1\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\left(\frac{\bigtriangledown(1)}{3}-\frac{\bigtriangledown(s)}{2}\right)\frac{\partial^2}{\partial t\,\partial s} \\
&\quad \times F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt \\
&\quad + \int_0^1\int_0^1\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\left(\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right)\frac{\partial^2}{\partial t\,\partial s} \\
&\quad \times F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt \\
&\quad + \int_0^1\int_0^1\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\left(\frac{\bigtriangledown(1)}{3}-\frac{\bigtriangledown(s)}{2}\right)\frac{\partial^2}{\partial t\,\partial s} \\
&\quad \times F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt \\
&= I_1+I_2+I_3+I_4.
\end{aligned}
$$

Integrating with the aid of using by parts, we obtain

$$\begin{aligned}
I_1 &= \int_0^1 \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) \left(\frac{\nabla(s)}{2} - \frac{\nabla(1)}{3} \right) \frac{\partial^2}{\partial t\, \partial s} \\
&\quad \times F\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2, \frac{1-s}{2}\kappa_3 + \frac{1+s}{2}\kappa_4 \right) ds\, dt \\
&= \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) \left[\frac{2}{\kappa_4 - \kappa_3} \left(\frac{\nabla(1)}{2} - \frac{\nabla(1)}{3} \right) \right. \\
&\quad \times \frac{\partial F}{\partial t} \left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2, \frac{1-s}{2}\kappa_3 + \frac{1+s}{2}\kappa_4 \right) \Big|_0^1 \\
&\quad \left. - \frac{1}{\kappa_4 - \kappa_3} \int_0^1 \frac{\Psi(\frac{\kappa_4 - \kappa_3}{2}s)}{s} \frac{\partial F}{\partial t} \left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2, \frac{1-s}{2}\kappa_3 + \frac{1+s}{2}\kappa_4 \right) ds \right] dt \\
&= \frac{2}{\kappa_4 - \kappa_3} \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) \left(\frac{\nabla(s)}{2} - \frac{\nabla(1)}{3} \right) \frac{\partial F}{\partial t} \left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2, \kappa_4 \right) dt \\
&\quad + \frac{2}{\kappa_4 - \kappa_3} \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) \left(\frac{\nabla(1)}{3} \right) \frac{\partial F}{\partial t} \left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2, \frac{\kappa_4 + \kappa_3}{2} \right) dt \\
&\quad - \frac{1}{\kappa_4 - \kappa_3} \int_0^1 \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) \frac{\psi(\frac{\kappa_4 - \kappa_3}{2}s)}{s} \\
&\quad \times \frac{\partial F}{\partial t} \left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2, \frac{1-s}{2}\kappa_3 + \frac{1+s}{2}\kappa_4 \right) ds\, dt.
\end{aligned}$$

By changing the variables and applying integration on integrals, we obtain

$$\begin{aligned}
I_1 &= \frac{\Lambda(1)\nabla(1)}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F(\kappa_2, \kappa_4) + \frac{2\Lambda(1) \nabla (1)}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_2 + \kappa_1}{2}, \kappa_4 \right) \\
&\quad - \frac{\nabla(1)}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} {}_{\kappa_2^-}I_\phi F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4 \right) + \frac{2\Lambda(1) \nabla (1)}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\kappa_2, \frac{\kappa_4 + \kappa_3}{2} \right) \\
&\quad + \frac{4\Lambda(1) \nabla (1)}{9(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_4 + \kappa_3}{2} \right) \\
&\quad - \frac{2 \nabla (1)}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} {}_{\kappa_2^-}I_\phi F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \\
&\quad - \frac{\Lambda(1)}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} {}_{\kappa_4^-}I_\psi F\left(\kappa_2, \frac{\kappa_4 + \kappa_3}{2} \right) \\
&\quad - \frac{2\Lambda(1)}{3(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} {}_{\kappa_4^-}I_\psi F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right) \\
&\quad + \frac{1}{(\kappa_4 - \kappa_3)(\kappa_2 - \kappa_1)} {}_{\kappa_2^-\kappa_4^-}I_{\phi,\psi} F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2} \right).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
I_2 &= \int_0^1 \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) \left(\frac{\nabla(1)}{3} - \frac{\nabla(s)}{2} \right) \frac{\partial^2}{\partial t\, \partial s} \\
&\quad \times F\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2, \frac{1+s}{2}\kappa_3 + \frac{1-s}{2}\kappa_4 \right) ds\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F(\kappa_2,\kappa_3) + \frac{2\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\frac{\kappa_2+\kappa_1}{2},\kappa_3\right) \\
&\quad - \frac{\nabla(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_2^-}I_\phi F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right) + \frac{2\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\kappa_2,\frac{\kappa_4+\kappa_3}{2}\right) \\
&\quad + \frac{4\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_4+\kappa_3}{2}\right) \\
&\quad - \frac{2\nabla(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_2^-}I_\phi F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad + \frac{\Lambda(1)}{3(\kappa_3-\kappa_4)(\kappa_2-\kappa_1)} {}_{\kappa_3^+}I_\psi F\left(\kappa_2,\frac{\kappa_4+\kappa_3}{2}\right) \\
&\quad + \frac{2\Lambda(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_3^+}I_\psi F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad + \frac{1}{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_2^-,\kappa_3^+}I_{\phi,\psi} F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_0^1\int_0^1 \left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\left(\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right)\frac{\partial^2}{\partial t\,\partial s} \\
&\quad \times F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right) ds\,dt \\
&= \frac{\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F(\kappa_1,\kappa_4) + \frac{2\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\frac{\kappa_2+\kappa_1}{2},\kappa_4\right) \\
&\quad - \frac{\nabla(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_1^+}I_\phi F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right) + \frac{2\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\kappa_1,\frac{\kappa_4+\kappa_3}{2}\right) \\
&\quad + \frac{4\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_4+\kappa_3}{2}\right) \\
&\quad - \frac{2\nabla(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_1^+}I_\phi F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad - \frac{\Lambda(1)}{3(\kappa_3-\kappa_4)(\kappa_2-\kappa_1)} {}_{\kappa_4^-}I_\psi F\left(\kappa_1,\frac{\kappa_4+\kappa_3}{2}\right) \\
&\quad - \frac{2\Lambda(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_4^-}I_\psi F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right) \\
&\quad + \frac{1}{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_1^+,\kappa_4^-}I_{\phi,\psi} F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1\int_0^1 \left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\left(\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right)\frac{\partial^2}{\partial t\,\partial s} \\
&\quad \times F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right) ds\,dt \\
&= \frac{\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F(\kappa_1,\kappa_3) + \frac{2\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\frac{\kappa_2+\kappa_1}{2},\kappa_3\right) \\
&\quad - \frac{\nabla(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_1^+}I_\phi F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right) + \frac{2\Lambda(1)\nabla(1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\kappa_1,\frac{\kappa_4+\kappa_3}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{4\Lambda(1) \triangledown (1)}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} F\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_4+\kappa_3}{2}\right) \\
&- \frac{2 \triangledown (1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_1^+}I_{\phi} F\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) \\
&- \frac{\Lambda(1)}{3(\kappa_3-\kappa_4)(\kappa_2-\kappa_1)} {}_{\kappa_3^+}I_{\psi} F\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_4+\kappa_3}{2}\right) \\
&- \frac{2\Lambda(1)}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_3^+}I_{\psi} F\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right) \\
&+ \frac{1}{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)} {}_{\kappa_1^+,\kappa_3^+}I_{\phi,\psi} F\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right).
\end{aligned}
$$

By adding the above integrals and multiply by $\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{\Lambda(1)\triangledown(1)}$, we can write

$$
\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4\Lambda(1) \triangledown (1)}[I_1+I_2+I_3+I_4] = \Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4),
$$

which completes the proof. □

Corollary 1 *Under the assumption of Lemma 2, with $\phi(t)=t$ and $\psi(s)=s$, then we obtain the following equality for Riemann integrals*:

$$
\begin{aligned}
&\Upsilon(\kappa_1,\kappa_2;\kappa_3,\kappa_4) \\
&\quad= \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4}\int_0^1\int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)\left(\frac{s}{2}-\frac{1}{3}\right) \\
&\qquad\times \frac{\partial^2 F}{\partial t\,\partial s}\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2, \frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right) ds\,dt \\
&\qquad+ \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4}\int_0^1\int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)\left(\frac{1}{3}-\frac{s}{2}\right) \\
&\qquad\times \frac{\partial^2 F}{\partial t\,\partial s}\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2, \frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right) ds\,dt \\
&\qquad+ \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4}\int_0^1\int_0^1\left(\frac{1}{3}-\frac{t}{2}\right)\left(\frac{s}{2}-\frac{1}{3}\right) \\
&\qquad\times \frac{\partial^2 F}{\partial t\,\partial s}\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2, \frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right) ds\,dt \\
&\qquad+ \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4}\int_0^1\int_0^1\left(\frac{1}{3}-\frac{t}{2}\right)\left(\frac{1}{3}-\frac{s}{2}\right) \\
&\qquad\times \frac{\partial^2 F}{\partial t\,\partial s}\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2, \frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right) ds\,dt,
\end{aligned}
$$

where $\Upsilon(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ is defined by

$$
\begin{aligned}
&\Upsilon(\kappa_1,\kappa_2;\kappa_3,\kappa_4) \\
&\quad= \frac{F(\kappa_1,\frac{\kappa_3+\kappa_4}{2})+F(\kappa_2,\frac{\kappa_3+\kappa_4}{2})+4F(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2})+F(\frac{\kappa_1+\kappa_2}{2},\kappa_3)+F(\frac{\kappa_1+\kappa_2}{2},\kappa_4)}{9}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{F(\kappa_1,\kappa_3)+F(\kappa_2,\kappa_3)+F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_4)}{36} \\
&-\frac{1}{6(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\left[F(x,\kappa_3)+4F\left(x,\frac{\kappa_3+\kappa_4}{2}\right)+F(x,\kappa_4)\right]dx \\
&-\frac{1}{6(\kappa_4-\kappa_3)}\int_{\kappa_3}^{\kappa_4}\left[F(\kappa_1,y)+4F\left(\frac{\kappa_1+\kappa_2}{2},y\right)+F(x,\kappa_4)\right]dy \\
&+\frac{1}{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(x,y)\,dy\,dx.
\end{aligned}
$$

Corollary 2 *Under the assumption of Lemma 2, with $\phi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$, then we obtain the following equality for Riemann–Liouville fractional integrals*:

$$
\begin{aligned}
&\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4) \\
&\quad=\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4} \\
&\quad\times\left\{\int_0^1\int_0^1\left(\frac{t^\alpha}{2}-\frac{1}{3}\right)\left(\frac{s^\beta}{2}-\frac{1}{3}\right)\right. \\
&\quad\times\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt \\
&\quad+\int_0^1\int_0^1\left(\frac{t^\alpha}{2}-\frac{1}{3}\right)\left(\frac{1}{3}-\frac{s^\beta}{2}\right) \\
&\quad\times\frac{\partial^2}{\partial t\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt \\
&\quad+\int_0^1\int_0^1\left(\frac{1}{3}-\frac{t^\alpha}{2}\right)\left(\frac{s^\beta}{2}-\frac{1}{3}\right) \\
&\quad\times\frac{\partial^2}{\partial t\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt \\
&\quad+\int_0^1\int_0^1\left(\frac{1}{3}-\frac{t^\alpha}{2}\right)\left(\frac{1}{3}-\frac{s^\beta}{2}\right) \\
&\quad\left.\times\frac{\partial^2}{\partial t\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4) \\
&\quad=\frac{F(\kappa_1,\kappa_3)+F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)}{36} \\
&\quad+\frac{1}{9}\left[F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\right. \\
&\quad\left.+4F\left(\frac{\kappa_2+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right] \\
&\quad+\frac{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^\alpha(\kappa_4-\kappa_3)^\beta}\left[J^{\alpha,\beta}_{\kappa_2^-,\kappa_4^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+J^{\alpha,\beta}_{\kappa_2^-,\kappa_3^+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\\
&+J^{\alpha,\beta}_{\kappa_1^+,\kappa_4^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+J^{\alpha,\beta}_{\kappa_1^+,\kappa_3^+\phi,\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\Bigg]\\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{6(\kappa_2-\kappa_1)^\alpha}\Bigg[J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\\
&+J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+4J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+4J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\Bigg]\\
&-\frac{2^{\beta-1}\Gamma(\beta+1)}{6(\kappa_4-\kappa_3)^\beta}\Bigg[J^{\beta}_{\kappa_4^-}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\beta}_{\kappa_4^-}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\beta}_{\kappa_3^+}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\\
&+J^{\beta}_{\kappa_3^+}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+4J^{\beta}_{\kappa_4^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+4J^{\beta}_{\kappa_3^+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\Bigg].
\end{aligned}
$$

Corollary 3 *Under the assumption of Lemma 2, with $\phi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, then we obtain the following equality for k-Riemann–Liouville fractional integrals*:

$$
\begin{aligned}
&\$(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\\
&=\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4}\\
&\quad\times\Bigg\{\int_0^1\int_0^1\left(\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right)\left(\frac{s^{\frac{\beta}{k}}}{2}-\frac{1}{3}\right)\\
&\quad\times\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt\\
&\quad+\int_0^1\int_0^1\left(\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right)\left(\frac{1}{3}-\frac{s^{\frac{\beta}{k}}}{2}\right)\\
&\quad\times\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt\\
&\quad+\int_0^1\int_0^1\left(\frac{1}{3}-\frac{t^{\frac{\alpha}{k}}}{2}\right)\left(\frac{s^{\frac{\beta}{k}}}{2}-\frac{1}{3}\right)\\
&\quad\times\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)ds\,dt\\
&\quad+\int_0^1\int_0^1\left(\frac{1}{3}-\frac{t^{\frac{\alpha}{k}}}{2}\right)\left(\frac{1}{3}-\frac{s^{\frac{\beta}{k}}}{2}\right)\\
&\quad\times\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)ds\,dt\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\$(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\\
&=\frac{F(\kappa_1,\kappa_3)+F(\kappa_2,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_1,\kappa_4)}{36}\\
&\quad+\frac{1}{9}\Bigg[F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+4F\left(\frac{\kappa_2+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\Bigg]\\
&+\frac{2^{\frac{\alpha+\beta}{k}-2}\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(\kappa_2-\kappa_1)^{\alpha}(\kappa_4-\kappa_3)^{\beta}}\Bigg[J^{\alpha,\beta,k}_{\kappa_2^-,\kappa_4^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\\
&+J^{\alpha,\beta,k}_{\kappa_2^-,\kappa_3^+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\\
&+J^{\alpha,\beta,k}_{\kappa_1^+,\kappa_4^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+J^{\alpha,\beta,k}_{\kappa_1^+,\kappa_3^+\phi,\psi}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\Bigg]\\
&-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{6(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\Bigg[J^{\alpha,k}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+J^{\alpha,k}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+J^{\alpha,k}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\\
&+J^{\alpha,k}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+4J^{\alpha,k}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+4J^{\alpha,k}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\Bigg]\\
&-\frac{2^{\frac{\beta}{k}-1}\Gamma_k(\beta+k)}{6(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\Bigg[J^{\beta,k}_{\kappa_4^-}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\beta,k}_{\kappa_4^-}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\beta,k}_{\kappa_3^+}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\\
&+J^{\beta,k}_{\kappa_3^+}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+4J^{\beta,k}_{\kappa_4^-}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+4J^{\beta,k}_{\kappa_3^+}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\Bigg].
\end{aligned}
$$

4 Simpson type inequalities for generalized double fractional integrals

In this section, we present some Simpson-type inequalities for generalized fractional integrals.

Theorem 3 *We assume that the conditions of Lemma 2 hold. If the mapping $|\frac{\partial^2 F}{\partial t\,\partial s}|$ is convex on $\triangle$, then we have the following inequality for generalized fractional integrals*:

$$
\begin{aligned}
&\left|\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right|\\
&\quad\le\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4\Lambda(1)\triangledown(1)}\left(\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\triangledown(s)}{2}-\frac{\triangledown(1)}{3}\right|ds\,dt\right)\\
&\quad\times\left[\left|\frac{\partial^2F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|+\left|\frac{\partial^2F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|\left|\frac{\partial^2F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|\right],
\end{aligned}
$$

where $\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ is defined as in Lemma 2.

Proof By taking modulus in Lemma 2, we obtain

$$
\begin{aligned}
&\left|\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right| \qquad (4.1)\\
&\quad\le\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4\Lambda(1)\triangledown(1)}\\
&\quad\times\Bigg\{\int_0^1\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|\left|\left(\frac{\triangledown(s)}{2}-\frac{\triangledown(1)}{3}\right)\right|\\
&\quad\times\left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right|ds\,dt\\
&\quad+\int_0^1\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|\left|\left(\frac{\triangledown(1)}{3}-\frac{\triangledown(s)}{2}\right)\right|
\end{aligned}
$$

$$\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right| ds\,dt$$
$$+\int_0^1\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|\left|\left(\frac{\triangledown(s)}{2}-\frac{\triangledown(1)}{3}\right)\right|$$
$$\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right| ds\,dt$$
$$+\int_0^1\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|\left|\left(\frac{\triangledown(1)}{3}-\frac{\triangledown(s)}{2}\right)\right|$$
$$\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right| ds\,dt\Bigg\}.$$

Since the mapping $|\frac{\partial F}{\partial t\,\partial s}|$ is co-ordinated convex on $\triangle$, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|\left|\left(\frac{\triangledown(s)}{2}-\frac{\triangledown(1)}{3}\right)\right| \\
&\quad\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right| ds\,dt \\
&\le \int_0^1\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|\left|\left(\frac{\triangledown(s)}{2}-\frac{\triangledown(1)}{3}\right)\right|\left(\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|\right. \\
&\quad+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right| \\
&\quad\left.+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|\right) ds\,dt.
\end{aligned} \tag{4.2}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|\left|\left(\frac{\triangledown(1)}{3}-\frac{\triangledown(s)}{2}\right)\right| \\
&\quad\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right| ds\,dt \\
&\le \int_0^1\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|\left|\left(\frac{\triangledown(1)}{3}-\frac{\triangledown(s)}{2}\right)\right|\left(\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|\right. \\
&\quad+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right| \\
&\quad\left.+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|\right) ds\,dt,
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\int_0^1\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|\left|\left(\frac{\triangledown(s)}{2}-\frac{\triangledown(1)}{3}\right)\right| \\
&\quad\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right| ds\,dt \\
&\le \int_0^1\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|\left|\left(\frac{\triangledown(s)}{2}-\frac{\triangledown(1)}{3}\right)\right|\left(\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|\right. \\
&\quad+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|
\end{aligned} \tag{4.4}$$

$$+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|\right)ds\,dt$$

and

$$\begin{aligned}
&\int_0^1\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|\left|\left(\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right)\right| \\
&\quad\times\left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right|ds\,dt \\
&\quad\le\int_0^1\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|\left|\left(\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right)\right|\left(\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|\right. \\
&\quad\times\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right| \\
&\quad\left.+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|\right)ds\,dt.
\end{aligned}\tag{4.5}$$

Using the inequalities (4.2)–(4.5) in (4.1), the proof is completed. □

Corollary 4 *If we take $\phi(t)=t$ and $\psi(s)=s$ in Theorem 3, then Theorem 3 reduces to* [15, *Theorem* 3].

Corollary 5 *In Theorem 3, if we use $\phi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$, then we obtain the following inequality for Riemann–Loiuville fractional integrals*:

$$\begin{aligned}
&\left|\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right| \\
&\quad\le\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4} \\
&\quad\times\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{1}{\alpha}+1}+\frac{1}{2(\alpha+1)}-\frac{1}{3}\right)\left(\frac{\beta}{\beta+1}\left(\frac{2}{3}\right)^{\frac{1}{\beta}+1}+\frac{1}{2(\beta+1)}-\frac{1}{3}\right) \\
&\quad\times\left[\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|\right],
\end{aligned}$$

where $|\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4)|$ is defined as in Corollary 2.

Corollary 6 *If we take $\phi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ in Theorem 3, we obtain the following inequality for k-Riemann–Louville fractional integrals*:

$$\begin{aligned}
&\left|\$(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right| \\
&\quad\le\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4}\left(\frac{\alpha}{\alpha+k}\left(\frac{2}{3}\right)^{\frac{k}{\alpha}+1}+\frac{k}{2(\alpha+k)}-\frac{1}{3}\right) \\
&\quad\times\left(\frac{\beta}{\beta+k}\left(\frac{2}{3}\right)^{\frac{k}{\beta}+1}+\frac{k}{2(\beta+k)}-\frac{1}{3}\right) \\
&\quad\times\left[\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|\right],
\end{aligned}$$

where $\$(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ is defined as in Corollary 3.

Theorem 4 *Suppose that the assumptions of Lemma 2 hold. If the mapping $|\frac{\partial^2 F}{\partial t\,\partial s}|^q$ is co-ordinated convex on Δ, then we have the following inequality for generalized fractional integrals*:

$$\begin{aligned}
&\left|\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right| \\
&\quad\le \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{4\Lambda(1)\bigtriangledown(1)} \\
&\qquad\times\Bigg[\left(\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\left|\left(\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right)\right|^p ds\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\left|\frac{\bigtriangledown(1)}{3}-\frac{\bigtriangledown(s)}{2}\right|^p ds\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right|^p ds\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{\bigtriangledown(1)}{3}-\frac{\bigtriangledown(s)}{2}\right|^p ds\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}}\Bigg],
\end{aligned}$$

where $\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ is defined as in Lemma 2 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof From Hölder's inequality and co-ordinated convexity of $|\frac{\partial^2 F}{\partial t\,\partial s}|^q$, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|\left|\left(\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right)\right| \\
&\qquad\times\left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right| ds\,dt \\
&\quad\le\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right|^p ds\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial t\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right|^q dtds\right)^{\frac{1}{q}} \\
&\quad\le\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right|^p ds\right)^{\frac{1}{p}} \\
&\qquad\times\int_0^1\int_0^1\left(\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right.
\end{aligned} \tag{4.6}$$

$$+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\Bigg)^{\frac{1}{q}}\,ds\,dt$$
$$=\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right|^p ds\right)^{\frac{1}{p}}$$
$$\times\left(\frac{|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}}.$$

Similarly, we obtain

$$\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right| \tag{4.7}$$
$$\times\left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right|ds\,dt$$
$$\le\left(\int_0^1\left|\left(\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\left|\left(\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right)\right|^p ds\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}},$$

$$\int_0^1\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right| \tag{4.8}$$
$$\times\left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right|ds\,dt$$
$$\le\left(\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\left|\left(\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right)\right|^p ds\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}},$$

and

$$\int_0^1\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right| \tag{4.9}$$
$$\times\left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right|ds\,dt$$
$$\le\left(\int_0^1\left|\left(\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\left(\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right)\right|^p ds\right)^{\frac{1}{p}}$$
$$\times\left(\frac{9|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)|^q+3|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)|^q+|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)|^q}{16}\right)^{\frac{1}{q}}.$$

Using the inequalities (4.6)–(4.9) in (4.1), we obtain the required result. □

Corollary 7 *If we take $\phi(t)=t$ and $\psi(s)=s$ in Theorem 3, we have the following Simpson inequality for Riemann integrals*:

$$\left|\Upsilon(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right|$$

$$
\begin{aligned}
&\leq \frac{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}{144}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{2}{p}}\left(\frac{1}{16}\right)^{\frac{1}{q}} \\
&\quad\times\Bigg[\left(\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where $\Upsilon(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ is defined as in Corollary 1.

Corollary 8 *If we take $\phi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{t^\beta}{\Gamma(\beta)}$ in Theorem 3, we obtain the following Simpson inequality for Riemann–Liouville fractional integrals:*

$$
\begin{aligned}
&\left|\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right| \\
&\leq \left(\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{s^\alpha}{2}-\frac{1}{3}\right|^p ds\right)^{\frac{1}{p}}\left(\frac{1}{16}\right)^{\frac{1}{q}} \\
&\quad\times\Bigg[\left(\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where $\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ is defined as in Corollary 2.

Corollary 9 *If we take $\phi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{t^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ in Theorem 3, we obtain the following Simpson inequality for k-Riemann–Liouville fractional integrals:*

$$
\begin{aligned}
&\left|\$(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right| \\
&\leq \left(\left|\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right| dt\right)^{\frac{1}{p}}\left(\left|\frac{s^{\frac{\beta}{k}}}{2}-\frac{1}{3}\right| ds\right)^{\frac{1}{p}} \\
&\quad\times\Bigg[\left(\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(9\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q+3\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right)^{\frac{1}{q}}\Bigg],$$

where $\$(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ *is defined as in Corollary* 3.

Theorem 5 *Suppose that the assumptions of Lemma* 2 *hold. If the mapping* $|\frac{\partial^2 F}{\partial t\,\partial s}|^q$, $q>1$, *is co-ordinated convex on* Δ, *then we have the following inequality for generalized fractional integrals*:

$$\begin{aligned}
&\left|\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\right| \\
&\quad\le \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{\Lambda(1)\bigtriangledown(1)}\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right|ds\right)^{1-\frac{1}{q}} \\
&\qquad\times\Bigg\{\Bigg(\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right| \\
&\qquad\times\Bigg[\left(\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right. \\
&\qquad\left.+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right]ds\,dt\Bigg)^{\frac{1}{q}} \\
&\qquad+\Bigg(\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right| \\
&\qquad\times\Bigg[\left(\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right. \\
&\qquad\left.+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right]ds\,dt\Bigg)^{\frac{1}{q}} \\
&\qquad+\Bigg(\int_0^1\int_0^1\Bigg(\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right| \\
&\qquad\times\Bigg[\left(\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right. \\
&\qquad\left.+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right]ds\,dt\Bigg)^{\frac{1}{q}} \\
&\qquad+\Bigg(\int_0^1\int_0^1\Bigg(\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|\frac{\bigtriangledown(1)}{3}-\frac{\bigtriangledown(s)}{2}\right| \\
&\qquad\times\Bigg[\left(\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right. \\
&\qquad\left.+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right]ds\,dt\Bigg)^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

where $\Re(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ *is defined as in Lemma* 2.

Proof Using the power mean inequality and coordinate-convexity of $|\frac{\partial^2 F}{\partial t\,\partial s}|^q$, we have

$$\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\bigtriangledown(s)}{2}-\frac{\bigtriangledown(1)}{3}\right| \tag{4.10}$$

$$
\begin{aligned}
&\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right| ds\,dt \\
&\leq \left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right|ds\right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right|\right. \\
&\times \left.\left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right|^q dtds\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right|ds\right)^{1-\frac{1}{q}} \\
&\times (\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right| \\
&\times \left[\left(\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right.\right. \\
&+\left.\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right] ds\,dt\Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right| \qquad (4.11)\\
&\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right| ds\,dt \\
&\leq \left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right|ds\right)^{1-\frac{1}{q}} \\
&\times (\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right| \\
&\times \left[\left(\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right.\right. \\
&+\left.\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\right] ds\,dt\Bigg)^{\frac{1}{q}}, \\
&\int_0^1\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right| \qquad (4.12)\\
&\times \left|\frac{\partial^2}{\partial t\,\partial s}F\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1-s}{2}\kappa_3+\frac{1+s}{2}\kappa_4\right)\right| ds\,dt \\
&\leq \left(\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right|ds\right)^{1-\frac{1}{q}} \\
&\times (\int_0^1\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right| \\
&\times \left[\left(\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 F}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q\right.\right.
\end{aligned}
$$

$$+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 \digamma}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 \digamma}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\Bigg]\,ds\,dt\Bigg)^{\frac{1}{q}},$$

and

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right| \\
&\quad\times\left|\frac{\partial^2}{\partial t\,\partial s}\digamma\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2,\frac{1+s}{2}\kappa_3+\frac{1-s}{2}\kappa_4\right)\right|\,ds\,dt \\
&\leq\left(\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{\nabla(1)}{3}-\frac{\nabla(s)}{2}\right|\,ds\right)^{1-\frac{1}{q}} \\
&\quad\times\Bigg(\int_0^1\int_0^1\Bigg(\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|\frac{\nabla(s)}{2}-\frac{\nabla(1)}{3}\right| \\
&\quad\times\Bigg[\Bigg(\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 \digamma}{\partial t\,\partial s}(\kappa_1,\kappa_3)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 \digamma}{\partial t\,\partial s}(\kappa_1,\kappa_4)\right|^q \\
&\quad+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 \digamma}{\partial t\,\partial s}(\kappa_2,\kappa_3)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 \digamma}{\partial t\,\partial s}(\kappa_2,\kappa_4)\right|^q\Bigg]\,ds\,dt\Bigg)^{\frac{1}{q}}.
\end{aligned}\tag{4.13}$$

By substituting the inequalities (4.10)–(4.12) in (4.1), we obtain the desired result. □

Remark 1 By special choices of the functions ϕ and ψ in Theorem 5, one can obtain several new Simpson-type inequalities. These are left to the reader.

5 Concluding remarks

In this paper, we present several generalized fractional Simpson-type inequalities for functions whose partial derivatives in absolute value are co-ordinated convex functions. We also show that the results given here are a strong generalization of some already published ones. In the forthcoming papers, researchers can use the techniques of this work to obtain similar inequalities for other kinds of co-ordinated convexity.

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The authors declare that they have no competing interests.

Author contributions
SK: computation, writing original draft. HB: supervision, writing-review, and editing. All authors read and approved the final manuscript.

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