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Estimates for p-adic fractional integral operator and its commutators on p-adic Morrey–Herz spaces

Naqash Sarfraz[1*], Muhammad Aslam[2], Mir Zaman[1] and Fahd Jarad[3,4,5*]

*Correspondence:
nsarfraz@math.qau.edu.pk;
fahd@cankaya.edu.tr
[1]Department of Mathematics, University of Kotli Azad Jammu and Kashmir, Kotli, Pakistan
[3]Department of Mathematics, Çankaya University, Ankara, Turkey
Full list of author information is available at the end of the article

Abstract

This research investigates the boundedness of a p-adic fractional integral operator on p-adic Morrey–Herz spaces. In particular, p-adic central bounded mean oscillations ($C\dot{M}O$) and Lipschitz estimate for commutators of the p-adic fractional integral operator are provided as well.

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1 Introduction

According to the well-known Ostrowski theorem [1], any nontrivial valuation on the field of rational numbers $\mathbb{Q}$ is equivalent either to the p-adic valuation $|\cdot|_p$ or to one of real valuations $|\cdot|$, where p is a prime number. The former norm is defined as follows: if any rational number $x \neq 0$ is denoted as $x = p^\gamma s/t$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers s, t are not divisible by p, then $|x|_p = p^\gamma$, $|0|_p = 0$. The norm $|\cdot|_p$ satisfies the strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$. The extended form of any $x \neq 0 \in \mathbb{Q}_p$ (field of p-adic numbers) is given in [2] as

$$x = p^\gamma \sum_{i=0}^{\infty} \alpha_i p^i, \tag{1}$$

where $\alpha_i, \gamma \in \mathbb{Z}$, $\alpha_i \in \frac{\mathbb{Z}}{p\mathbb{Z}_p}$, $\alpha_0 \neq 0$.

In what follows the n-dimensional vector space $\mathbb{Q}_p^n$ over $\mathbb{Q}_p$ is equipped with the following absolute value:

$$|\mathbf{x}|_p = \max_{1\leq r\leq n} |x_r|_p. \tag{2}$$

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Let $B_\gamma(\mathbf{a})$ and $S_\gamma(\mathbf{a})$ represent respectively the ball and sphere of $\mathbb{Q}_p^n$ centered at $\mathbf{a} \in \mathbb{Q}_p^n$ and radius $p^\gamma > 0$:

$$B_\gamma(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \le p^\gamma\right\}, \quad \text{and}$$
$$S_\gamma(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^\gamma\right\}\cdot$$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group with respect to addition, in $\mathbb{Q}_p^n$ there exists a positive Haar measure $d\mathbf{x}$ invariant under shift $d(\mathbf{x} + \mathbf{a}) = d(\mathbf{x})\cdot$ It is worth noting that $d\mathbf{x}$ is normalized by the equality $\int_{B_0(\mathbf{0})} d\mathbf{x} = 1$. It is easy to find $\int_{B_\gamma(\mathbf{a})} d\mathbf{x} = p^{n\gamma}$ and $\int_{S_\gamma(\mathbf{a})} d\mathbf{x} = p^{n\gamma}(1 - p^{-n})$ for any $\mathbf{a} \in \mathbb{Q}_p^n$.

The field of p-adic numbers can be applied in many scientific fields. In physics, the groundbreaking application is the p-adic AdS/CFT [3]. Khrennikov et al. [4] enhanced p-adic wavelet for modeling reaction-diffusion dynamics. Its application in biology includes the models for hierarchical structures of genetic code [5] and protein [6]. Furthermore, p-adic numbers have found a novel application in harmonic analysis and mathematical physics (see, for example, [2, 7–11] and related references).

The topic of fractional calculus is undergoing fast development with more and more appealing applications in the real world (see, for instance, [12–16]). One of the principal options behind the popularity of the area is that fractional-order differentiations and integrations are more beneficial tools in expressing real-world matters than the integer-order ones. Moreover, it is applied in the solution of linear constant coefficient fractional differential equations of any commensurate order and the CRONE control-system design toolbox for the control engineering community (see [17, 18] and related references therein). Recently many researchers have made tremendous contributions to the topic of fractional calculus by developing multiple fractional expressions in diverse publications (see, for instance, [19–25]). Also, its applications have been found in various fields of science and engineering, such as rheology, fluid flow, probability, and electrical networks. In this context, fractional integral operators are an important part of the mathematical analysis as they construct and formulate inequalities which have multiple applications in scientific areas that can be seen in the existing literature [26–28]. The boundedness of fractional integral operator on several function spaces is a key area not only in harmonic analysis but also in differentiation theory, partial differential equation, and potential theory [29–31]. In this link, Taibleson [32] defined the p-adic fractional integral operator as

$$I_\beta^p f(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{f(\mathbf{t})}{|\mathbf{x} - \mathbf{t}|_p^{n-\beta}}\, d\mathbf{t}, \quad 0 < \beta < n.$$

In the past, the above operator has received a considerable attention. The explicit formula of p-adic fractional integral $\mathbb{Q}_p$ along with the development of analytical potential theory on $\mathbb{Q}_p^n$ is acquired in [33] and [34] respectively. The fundamental properties of p-adic fractional integral operator on $\mathbb{Q}_p^n$ can be seen in [32]. Moreover, the boundedness of commutators of p-adic fractional integral operator respectively on central Morrey spaces and generalized Morrey spaces in the p-adic field is obtained in [11] and [35]. Recently, the boundedness of I_β^p on Herz spaces has been reported in [36].

Furthermore, suppose that $b\colon \mathbb{Q}_p^n \to \mathbb{R}$ and $f\colon \mathbb{Q}_p^n \to \mathbb{R}$ are measurable mappings, then the commutators of fractional integral operator can be defined as follows:

$$I_\beta^{p,b} f(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{(b(x) - b(t))f(\mathbf{t})}{|\mathbf{x}-\mathbf{t}|_p^{n-\beta}}\, d\mathbf{t}.$$

Motivated by the above results, we obtain the boundedness of p-adic fractional integral operator on Morrey–Herz spaces. Furthermore, $C\dot{M}O$ and Lipschitz estimates for commutators of p-adic fractional integral operator on Morrey–Herz spaces are also studied. Before turning to our main results, let us specify that χ_k is the characteristic function of the sphere and a letter C denotes a constant with values that may differ at its occurrence. Also, we use the following generalized Hölder inequality in the article. Let $q > 1$ be a real number such that $1/q + 1/q' = 1$, and let $f \in L^q(\mathbb{Q}_p^n)$ and $g \in L^{q'}(\mathbb{Q}_p^n)$, then fg is integrable on $\mathbb{Q}_p^n$ and

$$\int_{\mathbb{Q}_p^n} \bigl|f(\mathbf{x})g(\mathbf{x})\bigr|\, d\mathbf{x} \le \|f\|_{L^q(\mathbb{Q}_p^n)} \|g\|_{L^{q'}(\mathbb{Q}_p^n)}.$$

Definition 1.1 ([37]) Let $1 < p < \infty$. The p-adic space ($C\dot{M}O$) is defined as follows:

$$\|f\|_{C\dot{M}O^p(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \biggl(\frac{1}{|B_\gamma|_H}\int_{B_\gamma} \bigl|f(\mathbf{t}) - f_{B_\gamma}\bigr|^p\, d\mathbf{t}\biggr)^{1/p} < \infty, \tag{3}$$

where $f_{B_\gamma} = \frac{1}{|B_\gamma|_H}\int_{B_\gamma} f(\mathbf{t})\, d\mathbf{t}$, $|B_\gamma|_H$ denotes the Haar measure of B_γ.

Definition 1.2 ([38]) Let $\alpha \in \mathbb{R}$, $\lambda \ge 0$, $0 < p < \infty$, and $0 < q < \infty$. The homogeneous p-adic Morrey–Herz space is defined by

$$M\dot{K}_{q,p}^{\alpha,\lambda}(\mathbb{Q}_p^n) = \bigl\{f \in L_{\mathrm{loc}}^q\bigl(\mathbb{Q}_p^n \setminus \{0\}\bigr) : \|f\|_{M\dot{K}_{q,p}^{\alpha,\lambda}(\mathbb{Q}_p^n)} < \infty\bigr\},$$

where

$$\|f\|_{M\dot{K}_{q,p}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Biggl(\sum_{k=-\infty}^{k_0} p^{k\alpha q}\|f\chi_k\|_{L^p(\mathbb{Q}_p^n)}^q\Biggr)^{1/q}.$$

The above space is reduced to Herz space in p-adic field if $M\dot{K}_{q,p}^{\alpha,0}(\mathbb{Q}_p^n) = \dot{K}_p^{\alpha,q}(\mathbb{Q}_p^n)$, see [39] for details.

Definition 1.3 ([38]) Let $\gamma \in \mathbb{R}^+$. The Lipschitz space $\Lambda_\gamma(\mathbb{Q}_p^n)$ is the space of all measurable function f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} = \sup_{\mathbf{x},\mathbf{h}\in\mathbb{Q}_p^n,\mathbf{h}\neq 0} \frac{|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|}{|\mathbf{h}|_p^\gamma} < \infty.$$

Now, we move towards the first key result of the article.

2 Estimates for I_β^p

In the following section we obtain the boundedness of p-adic fractional integral operator on Morrey–Herz spaces.

Theorem 2.1 *Let $0<\beta<n$, $1<p_1,p_2<\infty$, $p_1'<p_2<\infty$, $0<q_1,q_2<\infty$, and $\frac{1}{p_1}-\frac{1}{p_2}=\frac{\beta}{n}$. Suppose that $-n/p_2<\alpha<n/p_1'$ and $0<\lambda<\alpha$, then*

$$\|I_\beta^p f\|_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2}(\mathbb{Q}_p^n)}\leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}$$

for all $f\in M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)$.

Proof of Theorem 2.1 Since $f\in M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)$ and $i\in\mathbb{Z}$, we can write

$$f(\mathbf{x})=\sum_{i=-\infty}^{\infty}f(\mathbf{x})\chi_i(\mathbf{x})=\sum_{i=-\infty}^{\infty}f_i(\mathbf{x}).$$

We proceed as

$$\begin{aligned}\|I_\beta^p(f)\|_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2}(\mathbb{Q}_p^n)}&=\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{\alpha q_2k}\|I_\beta^p(f)\chi_k\|^{q_2}_{L^{p_2}(\mathbb{Q}_p^n)}\right)^{1/q_2}\\&\leq\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{\alpha q_2k}\sum_{i=-\infty}^{k}\|I_\beta^p(f_i)\chi_k\|^{q_2}_{L^{p_2}(\mathbb{Q}_p^n)}\right)^{1/q_2}\\&\quad+\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{\alpha q_2k}\sum_{i=k+1}^{\infty}\|I_\beta^p(f_i)\chi_k\|^{q_2}_{L^{p_2}(\mathbb{Q}_p^n)}\right)^{1/q_2}\\&=I+II.\end{aligned}\tag{4}$$

Now, we estimate I, noticing that for each $\mathbf{x}\in S_k$ and $\mathbf{t}\in S_i$ with $i\leq k$. Now, since $\frac{\beta}{n}=\frac{1}{p_1}-\frac{1}{p_2}$ and applying Hölder's inequality, we have

$$\begin{aligned}\|I_\beta^p(f_i)\chi_k\|^{p_2}_{L^{p_2}(\mathbb{Q}_p^n)}&=\int_{S_k}\left|\int_{S_i}\frac{f(\mathbf{t})}{|\mathbf{x}-\mathbf{t}|_p^{n-\beta}}\,d\mathbf{t}\right|^{p_2}d\mathbf{x}\\&\leq\int_{S_k}\int_{S_i}\left|\frac{f(\mathbf{t})}{|\mathbf{x}-\mathbf{t}|_p^{n-\beta}}\,d\mathbf{t}\right|^{p_2}d\mathbf{x}\\&\leq p^{kp_2(\beta-n)}\int_{S_k}\left(\int_{S_i}|f(\mathbf{t})|\,d\mathbf{t}\right)^{p_2}d\mathbf{x}\\&\leq p^{kp_2(\beta-n)}p^{kn}\left(\left(\int_{S_i}|f(\mathbf{t})|^{p_1}\,d\mathbf{t}\right)^{1/p_1}\left(\int_{S_i}d\mathbf{t}\right)^{1/p_1'}\right)^{p_2}\\&\leq Cp^{(i-k)n/p_1'}\|f_i\|^{p_2}_{L^{p_1}(\mathbb{Q}_p^n)}.\end{aligned}\tag{5}$$

By means of Jensen's inequality and (5), we have

$$
\begin{aligned}
I &= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_2}\sum_{i=-\infty}^{k}\left\|I_\beta^p(f_i)\chi_k\right\|_{L^{p_2}(\mathbb{Q}_p^n)}^{q_2}\right)^{1/q_2}\\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1}\sum_{i=-\infty}^{k}\left\|I_\beta^p(f_i)\chi_k\right\|_{L^{p_2}(\mathbb{Q}_p^n)}^{q_1}\right)^{1/q_1}\\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1}\sum_{i=-\infty}^{k}\left(p^{(i-k)n/p_1'}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}\right)^{q_1}\right)^{1/q_1}.
\end{aligned}
$$

From now on, we divide our proof in two cases. When $1 < q_1 < \infty$ and $\alpha < n/p_1'$. Applying Hölder's inequality, we get

$$
\begin{aligned}
I^{q_1} &= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k} p^{i\alpha}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} p^{(i-k)(n/p_1'-\alpha)}\right)^{q_1}\\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k} p^{i\alpha q_1}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(i-k)(n/p_1'-\alpha)q_1/2}\right)\\
&\quad\times\left(\sum_{i=-\infty}^{k} p^{(i-k)(n/p_1'-\alpha)q_1/2}\right)^{q_1/q_1'}\\
&= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sum_{i=-\infty}^{k_0} p^{i\alpha q_1}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1}\sum_{k=i}^{\infty} p^{(i-k)(n/p_1'-\alpha)q_1/2}\\
&\le C\|f\|_{M\dot{K}_{q_1,p_1}^{\alpha,\lambda}(\mathbb{Q}_p^n)}^{q_1}.
\end{aligned}
$$

When $0 < q_1 \le 1$, we tackle the case as follows:

$$
\begin{aligned}
I^{q_1} &\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sum_{k=-\infty}^{k_0} p^{k\alpha q_1}\sum_{i=-\infty}^{k}\left(p^{(i-k)n/p_1'}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}\right)^{q_1}\\
&= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k} p^{i\alpha}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} p^{(i-k)(n/p_1'-\alpha)}\right)^{q_1}\\
&= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{k} p^{i\alpha q_1}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(i-k)(n/p_1'-\alpha)q_1}\\
&= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sum_{i=-\infty}^{k_0} p^{i\alpha q_1}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1}\sum_{k=i}^{\infty} p^{(i-k)(n/p_1'-\alpha)q_1}\\
&\le C\|f\|_{M\dot{K}_{q_1,p_1}^{\alpha,\lambda}(\mathbb{Q}_p^n)}^{q_1}.
\end{aligned}
$$

Now we evaluate II for each $\mathbf{x} \in S_k$ and $\mathbf{t} \in S_i$ with $k+1 \leq i$. By means of Hölder's inequality, we arrive at

$$\begin{aligned}
\|I_\beta^p(f_i)\chi_k\|_{L^{p_2}(\mathbb{Q}_p^n)}^{p_2} &= \int_{S_k} \left| \int_{S_i} \frac{f(\mathbf{t})}{|\mathbf{x}-\mathbf{t}|_p^{n-\beta}}\, d\mathbf{t} \right|^{p_2} d\mathbf{x} \\
&\leq \int_{S_k} \left(\int_{S_i} p^{i(\beta-n)} |f(\mathbf{t})|\, d\mathbf{t} \right)^{p_2} d\mathbf{x} \\
&\leq C p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{p_2}.
\end{aligned} \tag{6}$$

Now, by Jensen's inequality and (6), we have

$$II \leq C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \sum_{i=k+1}^{\infty} \left(p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1}.$$

In the rest of the proof we consider a couple of cases. When $1 < q_1 < \infty$, since $\alpha > -n/p_2$ and by the application of Hölder's inequality, we have

$$\begin{aligned}
(II)^{q_1} &\leq C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{i=k+1}^{\infty} p^{i\alpha} p^{(k-i)(n/p_2+\alpha)} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{i=k+1}^{\infty} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} \right) \\
&\quad \times \left(\sum_{i=k+1}^{\infty} p^{(k-i)(n/p_2+\alpha)q_1'/2} \right)^{q_1/q_1'} \\
&= C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{i=k+1}^{\infty} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} \right) \\
&= C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k+1}^{k_0+1} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} \\
&\quad + C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} \\
&= II_1 + II_2.
\end{aligned}$$

Because of $\alpha > -n/p_2$, the estimation of II_1 is given by

$$\begin{aligned}
II_1 &= C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \sum_{k=-\infty}^{i-1} p^{(k-i)(n/p_2+\alpha)q_1/2} \\
&= C \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \\
&\leq C \|f\|_{M\dot{K}_{q_1,p_1}^{\alpha,\lambda}(\mathbb{Q}_p^n)}^{q_1}.
\end{aligned}$$

Next, since $\lambda < \alpha/2$, we have

$$\begin{aligned}
II_2 &= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \\
&\leq C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} p^{i\lambda q_1} \\
&\quad \times p^{-i\lambda q_1} \left(\sum_{l=-\infty}^{i} p^{\alpha l q_1} \|f_l\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \right) \\
&\leq C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} p^{i\lambda q_1} \|f\|_{M\dot{K}_{q_1,p_1}^{\alpha,\lambda}(\mathbb{Q}_p^n)}^{q_1} \\
&= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1/2} \right) \left(\sum_{i=k_0+2}^{\infty} p^{i\lambda q_1(\lambda-\alpha/2)} \right) \|f\|_{M\dot{K}_{q_1,p_1}^{\alpha,\lambda}(\mathbb{Q}_p^n)}^{q_1} \\
&= C\|f\|_{M\dot{K}_{q_1,p_1}^{\alpha,\lambda}(\mathbb{Q}_p^n)}^{q_1}.
\end{aligned}$$

If $0 < q_1 \leq 1$, we have

$$\begin{aligned}
(II)^{q_1} &\leq C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{\infty} \sum_{i=k+1}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \\
&= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k+1}^{k_0+1} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(k-i)(n/p_2+\alpha)q_1} \\
&\quad + C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} p^{(k-i)(n/p_2+\alpha)q_1} \\
&= II_1' + II_2'.
\end{aligned}$$

Since $\alpha > -n/p_2$, the estimate of II_1' is done as follows:

$$\begin{aligned}
II_1' &= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \sum_{k=-\infty}^{i-1} p^{(k-i)(n/p_2+\alpha)q_1} \\
&= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \\
&\leq C\|f\|_{M\dot{K}_{q_1,p_1}^{\alpha,\lambda}(\mathbb{Q}_p^n)}^{q_1}.
\end{aligned}$$

Now $\alpha > 0$ and $\lambda < \alpha$, the estimation of II_2' is carried out as follows:

$$\begin{aligned}
II_2' &= C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}^{q_1} \\
&\leq C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1} p^{i\lambda q_1}
\end{aligned}$$

$$
\begin{aligned}
&\times p^{-i\lambda q_1}\left(\sum_{l=-\infty}^{i} p^{\alpha l q_1}\|f_l\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)}\right)\\
&\leq C\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}\sum_{i=k_0+2}^{\infty} p^{i\alpha q_1}p^{(k-i)(n/p_2+\alpha)q_1}p^{i\lambda q_1}\|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}\\
&= C\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1}\right)\left(\sum_{i=k_0+2}^{\infty} p^{i\lambda q_1(\lambda-\alpha)}\right)\|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}\\
&= C\|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

By combining the estimates of I and II, the proof of Theorem 2.1 is completed. □

3 $C\dot{M}O$ and Lipschitz estimates for $I^{p,b}_{\beta}$

In the present section we establish the $C\dot{M}O(\mathbb{Q}_p^n)$ and Lipschitz estimates for commutators of p-adic fractional integral operator on Morrey–Herz spaces. However, before doing that, we need an important result which is as follows.

Lemma 3.1 ([40]) *Let $b\in C\dot{M}O^1(\mathbb{Q}_p^n)$ and let $k,i\in\mathbb{Z}$. Then*

$$
\left|b(\mathbf{t})-b_{B_i}\right|\leq\left|b(\mathbf{t})-b_{B_k}\right|+p^n|k-i|\|b\|_{C\dot{M}O^1(\mathbb{Q}_p^n)}.
$$

Theorem 3.2 *Let $<\beta<n$, $b\in C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)$, $1<p_1,p_2<\infty$, $p_1'<p_2<\infty$, $0<q_1,q_2<\infty$, and $\frac{1}{p_1}-\frac{1}{p_2}=\frac{\beta}{n}$. Suppose $-n/p_2<\alpha<n/p_1'$ and $0<\lambda<\alpha$, then*

$$
\left\|I^{p,b}_{\beta}f\right\|_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2}(\mathbb{Q}_p^n)}\leq C\|b\|_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}
$$

for all $f\in M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)$.

Theorem 3.3 *Let $0<\beta<n$, $b\in\Lambda_{\gamma}(\mathbb{Q}_p^n)$ $(0<\gamma<1)$, $1<p_1$, $p_2<\infty$, $p_1'<p_2<\infty$, $0<q_1$, $q_2<\infty$, and $\frac{1}{p_1}-\frac{1}{p_2}=\frac{\beta+\gamma}{n}$. Suppose $-n/p_2<\alpha<n/p_1'$ and $0<\lambda<\alpha$, then*

$$
\left\|I^{p,b}_{\beta}f\right\|_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2}(\mathbb{Q}_p^n)}\leq C\|b\|_{\Lambda_{\gamma}(\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}
$$

for all $f\in M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)$.

Proof of Theorem 3.2 For $i\in\mathbb{Z}$, we can write

$$
f(\mathbf{x})=\sum_{i=-\infty}^{\infty}f(\mathbf{x})\chi_i(\mathbf{x})=\sum_{i=-\infty}^{\infty}f_i(\mathbf{x}).
$$

Thus, we begin

$$\begin{aligned}
\left\| I_\beta^{p,b}(f)\right\|_{M\dot{K}_{q_2,p_2}^{\alpha,\lambda}(\mathbb{Q}_p^n)} &= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{\alpha q_2 k}\left\| I_\beta^{p,b}(f)\chi_k\right\|_{L^{p_2}(\mathbb{Q}_p^n)}^{q_2}\right)^{1/q_2} \\
&\le C\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{\alpha q_2 k}\sum_{i=-\infty}^{k}\left\| I_\beta^{p,b}(f_i)\chi_k\right\|_{L^{p_2}(\mathbb{Q}_p^n)}^{q_2}\right)^{1/q_2} \\
&\quad + C\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{\alpha q_2 k}\sum_{i=k+1}^{\infty}\left\| I_\beta^{p,b}(f_i)\chi_k\right\|_{L^{p_2}(\mathbb{Q}_p^n)}^{q_2}\right)^{1/q_2} \\
&= L + LL.
\end{aligned} \tag{7}$$

The estimate of L is carried out as follows. For each $\mathbf{x}\in S_k$ and $\mathbf{t}\in S_i$ with $i\le k$, we have

$$\begin{aligned}
\left\| I_\beta^{p,b}(f_i)\chi_k\right\|_{L^{p_2}(\mathbb{Q}_p^n)}^{p_2} &= \int_{S_k}\left|\int_{S_i}\frac{f(\mathbf{t})}{|\mathbf{x}-\mathbf{t}|_p^{n-\beta}}\big(b(\mathbf{x})-b(\mathbf{t})\big)\,d\mathbf{t}\right|^{p_2} d\mathbf{x} \\
&\le Cp^{kp_2(\beta-n)}\int_{S_k}\left(\int_{S_i}\left|f(\mathbf{t})\big(b(\mathbf{x})-b(\mathbf{t})\big)\right|\,d\mathbf{t}\right)^{p_2} d\mathbf{x} \\
&\le Cp^{kp_2(\beta-n)}\int_{S_k}\left(\int_{S_i}\left|f(\mathbf{t})\big(b(\mathbf{x})-b_{B_k}\big)\right|\,d\mathbf{t}\right)^{p_2} d\mathbf{x} \\
&\quad + Cp^{kp_2(\beta-n)}\int_{S_k}\left(\int_{S_i}\left|f(\mathbf{t})\big(b(\mathbf{t})-b_{B_k}\big)\right|\,d\mathbf{t}\right)^{p_2} d\mathbf{x} \\
&= L_1 + L_2.
\end{aligned} \tag{8}$$

By noticing that $\frac{1}{p_1}-\frac{1}{p_2}=\frac{\beta}{n}$ and applying Hölder's inequality, we get

$$\begin{aligned}
L_1 &\le Cp^{kp_2(\beta-n)}\int_{B_k}\left|b(\mathbf{x})-b_{B_k}\right|^{p_2} \\
&\quad\times\left(\left(\int_{S_i}\left|f(\mathbf{t})\right|^{p_1} d\mathbf{t}\right)^{1/p_1}\left(\int_{S_i} d\mathbf{t}\right)^{1/p_1'}\right)^{p_2} d\mathbf{x} \\
&\le C\|b\|_{CMO^{p_2}(\mathbb{Q}_p^n)}^{p_2}\left\{p^{(i-k)n/p_1'}\|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}\right\}^{p_2}.
\end{aligned} \tag{9}$$

Next we move towards the estimation of L_2, for this we use Lemma 3.1.

$$\begin{aligned}
L_2 &\le Cp^{kp_2(\beta-n)}\int_{S_k}\left(\int_{S_i}\left|f(\mathbf{t})\big(b(\mathbf{t})-b_{B_i}\big)\right|\,d\mathbf{t}\right)^{p_2} d\mathbf{x} \\
&\quad + C\|b\|_{CMO^1(\mathbb{Q}_p^n)}^{p_2} p^{kp_2(\beta-n)}\int_{S_k}\left((k-i)\int_{S_i}\left|f(\mathbf{t})\right|\,d\mathbf{t}\right)^{p_2} d\mathbf{x} \\
&= L_{21} + L_{22}.
\end{aligned} \tag{10}$$

We make use of Hölder's inequality along with $\frac{1}{p_1} - \frac{1}{p_2} = \frac{\beta}{n}$ to evaluate L_{21}.

$$\begin{aligned}
L_{21} &\le Cp^{kp_2(\beta-n)} \int_{S_k} \left(\left(\left(\int_{S_i} \left| b(\mathbf{t}) - b_{B_i} \right|^{p_1'} d\mathbf{t} \right)^{1/p_1'} \right.\right.\\
&\quad \left.\left. \times \left(\int_{S_i} \left| f(\mathbf{t}) \right|^{p_1} d\mathbf{t} \right)^{1/p_1} \right)^{p_2} d\mathbf{x}\right.\\
&\le C\|b\|^{p_2}_{CMO^{p_1'}(\mathbb{Q}_p^n)} p^{(i-k)np_2/p_1'} \|f_i\|^{p_2}_{L^{p_1}(\mathbb{Q}_p^n)}\\
&= C\|b\|^{p_2}_{CMO^{p_1'}(\mathbb{Q}_p^n)} \left(p^{(i-k)n/p_1'} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{p_2}.
\end{aligned}$$

In a similar way we can estimate L_{22}. Since $\frac{1}{p_1} - \frac{1}{p_2} = \frac{\beta}{n}$, using Hölder's inequality, we obtain

$$\begin{aligned}
L_{22} &\le C\|b\|^{p_2}_{CMO^1(\mathbb{Q}_p^n)} p^{kp_2(\beta-n)}\\
&\quad \times \int_{S_k} \left((k-i) \left(\int_{S_i} \left| f(\mathbf{t}) \right|^{p_1} d\mathbf{t} \right)^{1/p_1} \left(\int_{S_i} d\mathbf{t} \right)^{1/p_1'} \right)^{p_2} d\mathbf{x}\\
&= C\|b\|^{p_2}_{CMO^1(\mathbb{Q}_p^n)} \left((k-i) p^{(i-k)n/p_1'} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{p_2}.
\end{aligned}$$

Combining the above values and by means of Jensen's inequality, we have

$$\begin{aligned}
L &\le C\|b\|_{CMO^{p_2}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=-\infty}^{k} p^{(i-k)n/p_1'} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1}\\
&\quad + C\|b\|_{CMO^{p_1'}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=-\infty}^{k} p^{(i-k)n/p_1'} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1}\\
&\quad + C\|b\|_{CMO^1(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=-\infty}^{k} (k-i) p^{(i-k)n/p_1'} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1}\\
&\le C\|b\|_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=-\infty}^{k} (k-i) p^{(i-k)n/p_1'} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1}.
\end{aligned}$$

From here on, we consider a couple of cases. When $1 < q_1 < \infty$, applying Hölder's inequality from the outset, we have

$$\begin{aligned}
L^{q_1} &= C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{i=-\infty}^{k} p^{i\alpha} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right.\\
&\quad \left. \times (k-i) p^{(i-k)(n/p_1'-\alpha)} \right)^{q_1}\\
&\le C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=-\infty}^{k} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} p^{(i-k)(n/p_1'-\alpha)q_1/2}\\
&\quad \times \left(\sum_{i=-\infty}^{k} (k-i)^{q_1'} p^{(i-k)(n/p_1'-\alpha)q_1'/2} \right)^{q_1/q_1'}
\end{aligned}$$

$$
\begin{aligned}
&= C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\quad \times \sum_{k=i}^{k_0} p^{(i-k)(n/p_1'-\alpha)q_1/2} \\
&= C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

We consider the other case, when $0 < q_1 \leq 1$, since $\alpha < n/p_1'$, we handle the case as

$$
\begin{aligned}
L^{q_1} &= C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \Bigg(\sum_{i=-\infty}^{k} (k-i) \\
&\quad \times p^{(i-k)n/p_1'} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \Bigg)^{q_1} \\
&\leq C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=-\infty}^{k} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\quad \times (k-i)^{q_1} p^{(i-k)(n/p_1'-\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\quad \times \sum_{k=i}^{k_0} (k-i)^{q_1} p^{(i-k)(n/p_1'-\alpha)q_1} \\
&= C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Next we turn our attention towards estimating LL. For each $\mathbf{x} \in S_k$ and $\mathbf{t} \in S_i$ with $k+1 \leq i$, we proceed as follows:

$$
\begin{aligned}
\big\|I_\beta^{p,b}(f_i)\chi_k\big\|^{p_2}_{L^{p_2}(\mathbb{Q}_p^n)} &= \int_{S_k} \left| \int_{S_i} \frac{f(\mathbf{t})}{|\mathbf{x}-\mathbf{t}|_p^{n-\beta}} \big(b(\mathbf{x}) - b(\mathbf{t})\big)\, d\mathbf{t} \right|^{p_2} d\mathbf{x} \\
&\leq \int_{S_k} \left(\int_{S_i} p^{i(\beta-n)} \big| f(\mathbf{t})\big(b(\mathbf{x}) - b(\mathbf{t})\big)\big|\, d\mathbf{t} \right)^{p_2} d\mathbf{x} \\
&\leq C \int_{S_k} \left(\int_{S_i} p^{i(\beta-n)} \big| f(\mathbf{t})\big(b(\mathbf{x}) - b_{B_k}\big)\big|\, d\mathbf{t} \right)^{p_2} d\mathbf{x} \\
&\quad + C \int_{S_k} \left(\int_{S_i} p^{i(\beta-n)} \big| f(\mathbf{t})\big(b(\mathbf{t}) - b_{B_k}\big)\big|\, d\mathbf{t} \right)^{p_2} d\mathbf{x} \\
&= LL_1 + LL_2.
\end{aligned} \tag{11}
$$

We apply Hölder's inequality for evaluating LL_1. Since $\frac{1}{p_1} - \frac{1}{p_2} = \frac{\beta}{n}$, we obtain

$$
\begin{aligned}
LL_1 &\leq C \int_{B_k} \big|b(\mathbf{x}) - b_{B_k}\big|^{p_2} \left(p^{i(\beta-n)} \left(\int_{S_i} \big|f(\mathbf{t})\big|^{p_1}\, d\mathbf{t} \right)^{1/p_1} \left(\int_{S_i} d\mathbf{t} \right)^{1/p_1'} \right)^{p_2} d\mathbf{x} \\
&\leq C\|b\|^{p_2}_{CMO^{p_2}(\mathbb{Q}_p^n)} \big(p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)}\big)^{p_2}.
\end{aligned} \tag{12}
$$

The estimation of LL_2 is done by using Lemma 3.1.

$$\begin{aligned} LL_2 &\leq C \int_{S_k} \left(\int_{S_i} p^{i(\beta-n)} \left| f(\mathbf{t}) \big(b(\mathbf{t}) - b_{B_i} \big) \right| d\mathbf{t} \right)^{p_2} d\mathbf{x} \\ &\quad + C \|b\|^{p_2}_{CMO^1(\mathbb{Q}_p^n)} \int_{S_k} \left((i-k) \int_{S_i} p^{i(\beta-n)} \left| f(\mathbf{t}) \right| d\mathbf{t} \right)^{p_2} d\mathbf{x} \\ &= LL_{21} + LL_{22}. \end{aligned} \tag{13}$$

Implying Hölder's inequality to evaluate LL_{21}, we have

$$\begin{aligned} LL_{21} &\leq C \int_{S_k} \left(p^{i(\beta-n)} \left(\int_{S_i} \left| b(\mathbf{t}) - b_{B_i} \right|^{p_1'} d\mathbf{t} \right)^{1/p_1'} \right. \\ &\quad \times \left. \left(\int_{S_i} \left| f(\mathbf{t}) \right|^{p_1} d\mathbf{t} \right)^{1/p_1} \right)^{p_2} d\mathbf{x} \\ &\leq C \|b\|^{p_2}_{CMO^{p_1'}(\mathbb{Q}_p^n)} \big(p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \big)^{p_2}. \end{aligned}$$

Now, we estimate LL_{22}, for this Hölder's inequality is helpful, therefore we arrive at

$$\begin{aligned} LL_{22} &\leq C \|b\|^{p_2}_{CMO^1(\mathbb{Q}_p^n)} \\ &\quad \times \int_{S_k} \left((i-k) p^{i(\beta-n)} \left(\int_{S_i} \left| f(\mathbf{t}) \right|^{p_1} d\mathbf{t} \right)^{1/p_1} \left(\int_{S_i} d\mathbf{t} \right)^{1/p_1'} \right)^{p_2} d\mathbf{x} \\ &= C \|b\|^{p_2}_{CMO^1(\mathbb{Q}_p^n)} \big((i-k) p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \big)^{p_2}. \end{aligned} \tag{14}$$

Combining the above values and by means of Jensen's inequality, we have

$$\begin{aligned} LL &\leq C \|b\|_{C\dot{M}O^{p_2}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=k+1}^{\infty} p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1} \\ &\quad + C \|b\|_{C\dot{M}O^{p_1'}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=k+1}^{\infty} p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1} \\ &\quad + C \|b\|_{C\dot{M}O^1(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=k+1}^{\infty} (i-k) \right. \right. \\ &\quad \left. \left. \times p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1} \\ &\leq C \|b\|_{C\dot{M}O^{\max\{p_1', p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \left(\sum_{i=k+1}^{\infty} (i-k) \right. \right. \\ &\quad \left. \left. \times p^{(k-i)n/p_2} \|f_i\|_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \right)^{1/q_1}. \end{aligned}$$

From this point on, we consider two cases. When $1 < q_1 < \infty$ and by Hölder's inequality, we get

$$
\begin{aligned}
(LL)^{q_1} &\le C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{i=k+1}^{\infty} p^{i\alpha} p^{(k-i)(n/p_2+\alpha)} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \right)^{q_1} \\
&\le C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left(\sum_{i=k+1}^{\infty} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \right. \\
&\quad \left. \times p^{(k-i)(n/p_2+\alpha)q_1/2} \right) \left(\sum_{i=k+1}^{\infty} p^{(k-i)(n/p_2+\alpha)q_1'/2} \right)^{q_1/q_1'} \\
&= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k+1}^{\infty} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} p^{(k-i)(n/p_2+\alpha)q_1/2} \\
&= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k+1}^{k_0+1} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} p^{(k-i)(n/p_2+\alpha)q_1/2} \\
&\quad + C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} p^{(k-i)(n/p_2+\alpha)q_1/2} \\
&= LL_1' + LL_2'.
\end{aligned}
$$

Since $\alpha > -n/p_2$, the evaluation of LL_1' is obtained by

$$
\begin{aligned}
LL_1' &= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\quad \times \sum_{k=-\infty}^{i-1} p^{(k-i)(n/p_2+\alpha)q_1/2} \\
&= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\le C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Next, by noticing that $\lambda < \alpha/2$, we have

$$
\begin{aligned}
LL_2' &= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} \\
&\quad \times p^{(k-i)(n/p_2+\alpha)q_1/2} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\le C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1/2} p^{i\lambda q_1} \\
&\quad \times p^{-i\lambda q_1} \left(\sum_{l=-\infty}^{i} p^{\alpha l q_1} \|f_l\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} \\
&\quad \times p^{(k-i)(n/p_2+\alpha)q_1/2} p^{i\lambda q_1} \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)} \\
&= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1/2}\right)\left(\sum_{i=k_0+2}^{\infty} p^{i\lambda q_1(\lambda-\alpha/2)}\right) \\
&\quad \times \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)} \\
&= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

For the second case $0 < q_1 \leq 1$, we are down to

$$
\begin{aligned}
(LL)^{q_1} &\leq C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{\infty} \sum_{i=k+1}^{\infty} p^{i\alpha q_1} \\
&\quad \times p^{(k-i)(n/p_2+\alpha)q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&= C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k+1}^{k_0+1} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\quad \times p^{(k-i)(n/p_2+\alpha)q_1} + C\|b\|^{q_1}_{CMO^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \\
&\quad \times \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} p^{(k-i)(n/p_2+\alpha)q_1} \\
&= LL_1'' + LL_2''.
\end{aligned}
$$

Because $\alpha > -n/p_2$, we estimate LL_1'' as

$$
\begin{aligned}
LL_1'' &= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\quad \times \sum_{k=-\infty}^{i-1} p^{(k-i)(n/p_2+\alpha)q_1} \\
&= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{i=-\infty}^{k_0+1} p^{i\alpha q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \\
&\leq C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Since $\alpha > 0$ and $\lambda < \alpha$, the estimation of LL_2'' is given by

$$
\begin{aligned}
LL_2'' &= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{i=k_0+2}^{\infty} p^{i\alpha q_1} \\
&\quad \times p^{(k-i)(n/p_2+\alpha)q_1} \|f_i\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)}
\end{aligned}
$$

$$\leq C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0+2}^{\infty} p^{i\alpha q_1} p^{(k-i)(n/p_2+\alpha)q_1} p^{i\lambda q_1}$$
$$\times p^{-i\lambda q_1} \left(\sum_{l=-\infty}^{i} p^{\alpha l q_1} \|f_l\|^{q_1}_{L^{p_1}(\mathbb{Q}_p^n)} \right)$$
$$= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda q_1} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q_1} \right)$$
$$\times \left(\sum_{i=k_0+2}^{\infty} p^{i\lambda q_1(\lambda-\alpha)} \right) \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}$$
$$= C\|b\|^{q_1}_{C\dot{M}O^{\max\{p_1',p_2\}}(\mathbb{Q}_p^n)} \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1}(\mathbb{Q}_p^n)}.$$

Combining the estimates of L and LL, we are done with the proof of Theorem 3.2. □

Proof of Theorem 3.3 When $b \in \Lambda_\gamma(\mathbb{Q}_p^n)$, the inequality

$$\left| I^{p,b}_{\beta} f(\mathbf{x}) \right| \leq C\|b\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} \left| I^{p,b}_{\beta+\gamma} f(x) \right|$$

can be easily obtained under the assumptions of Theorem 3.3, and the proof of Theorem 3.3 is a consequence of Theorem 2.1. □

4 Application

As an application, we can characterize p-adic Morrey–Herz space in terms of p-adic wavelets as they are used to construct solutions of p-adic Navier–Stokes equation and semi-linear evolutionary pseudo-differential equations, see for instance [41, 42].

5 Conclusion

Herein, the boundedness of a p-adic fractional integral operator on p-adic Morrey–Herz spaces is studied. In addition, the property of boundedness for commutators of the p-adic fractional integral operator on Morrey–Herz spaces in the p-adic field is also obtained when the symbol function is from $C\dot{M}O$ and Lipschitz spaces.

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Author details
[1]Department of Mathematics, University of Kotli Azad Jammu and Kashmir, Kotli, Pakistan. [2]Department of Mathematics, College of Sciences, King Khalid University, Abha, Saudi Arabia. [3]Department of Mathematics, Çankaya University, Ankara, Turkey. [4]Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia. [5]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

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