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Fixed points of weakly K-nonexpansive mappings and a stability result for fixed point iterative process with an application

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Abstract

In this article, we introduce a new type of non-expansive mapping, namely weakly K-nonexpansive mapping, which is weaker than non-expansiveness and stronger than quasi-nonexpansiveness. We prove some weak and strong convergence results using weakly K-nonexpansive mappings. Also, we define weakly (α, K)-nonexpansive mapping and using it prove one stability result for JF-iterative process. Some prominent examples are presented illustrating the facts. A numerical example is given to compare the convergence behavior of some known iterative algorithms for weakly K-nonexpansive mappings. Moreover, we show by example that the class of α-nonexpansive mappings due to Aoyama and Kohsaka and the class of generalized α-nonexpansive mappings due to Pant and Shukla are independent. Finally, our fixed point theorem is applied to obtain a solution of a nonlinear fractional differential equation.

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1 Introduction

Throughout this article, $(\mathcal{B}, \|\cdot\|)$ denotes a real Banach space, and $\mathcal{D}$ is a non-empty, closed and convex subset of $\mathcal{B}$, unless otherwise stated. Let $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a self-mapping, and $\mathrm{Fix}(\Upsilon)$ denotes the set of all fixed points of Υ. Also, we use the notations $u_n \rightharpoonup u$ and $u_n \to u$ for a sequence $\{u_n\}$ converging weakly and strongly to u, respectively.

The self-mapping Υ on $\mathcal{D}$ is said to be *non-expansive* (see [19]) if $\|\Upsilon x - \Upsilon y\| \leq \|x - y\|$ for all $x, y \in \mathcal{D}$ and is said to be *quasi-nonexpansive* (see [19]) if $\mathrm{Fix}(\Upsilon) \neq \emptyset$ and $\|\Upsilon x - \rho\| \leq \|x - \rho\|$ for all $x \in \mathcal{D}$ and $\rho \in \mathrm{Fix}(\Upsilon)$. There are several extensions and generalizations of non-expansive mappings considered by many researchers.

In 1973, Hardy and Rogers [21] introduced the notion of generalized non-expansive mapping as below:

Definition 1.1 ([21]) A mapping $\Upsilon : \mathcal{D} \to \mathcal{D}$ is said to be a *Generalized non-expansive mapping* if for all $x, y \in \mathcal{D}$,

$$\|\Upsilon x - \Upsilon y\| \leq \alpha_1 \|x - y\| + \alpha_2 \|x - \Upsilon x\| + \alpha_3 \|y - \Upsilon y\| + \alpha_4 \|x - \Upsilon y\| + \alpha_4 \|y - \Upsilon x\|, \quad (1.1)$$

where $\alpha_i \geq 0$ with $\sum_{i=1}^{5} \alpha_i \leq 1$. Or equivalently, [11]

$$\|\Upsilon x - \Upsilon y\| \leq \alpha_1 \|x - y\| + \alpha_2 \big(\|x - \Upsilon x\| + \|y - \Upsilon y\|\big) + \alpha_3 \big(\|x - \Upsilon y\| + \|y - \Upsilon x\|\big) \quad (1.2)$$

with $\alpha_i \geq 0$ and $\alpha_1 + 2\alpha_2 + 2\alpha_3 \leq 1$.

It is clear that if $\mathrm{Fix}(\Upsilon) \neq \emptyset$, then Υ is a *quasi-nonexpansive mapping*.

In 2008, Suzuki [44] introduced a new generalization of non-expansive mappings, namely *Condition (C)* as below:

Definition 1.2 ([44]) A mapping $\Upsilon : \mathcal{D} \to \mathcal{D}$ is said to satisfy Condition (C) if for all $x, y \in \mathcal{D}$,

$$\frac{1}{2}\|x - \Upsilon x\| \leq \|x - y\| \quad \Longrightarrow \quad \|\Upsilon x - \Upsilon y\| \leq \|x - y\|. \quad (1.3)$$

It is also clear that a mapping with a fixed point satisfying *Condition (C)* is necessarily a *quasi-nonexpansive mapping*.

After that, in 2011, Aoyama and Kohsaka [9] introduced another class of non-expansive mappings and proved the existence of fixed point of such mappings.

Definition 1.3 ([9]) A mapping $\Upsilon : \mathcal{D} \to \mathcal{D}$ is said to be an α-nonexpansive mapping if

$$\|\Upsilon x - \Upsilon y\|^2 \leq \alpha \big(\|x - \Upsilon y\|^2 + \|y - \Upsilon x\|^2\big) + (1 - 2\alpha)\|x - y\|^2 \quad (1.4)$$

for all $x, y \in \mathcal{D}$ and for some $0 \leq \alpha < 1$.

Furthermore, in 2017, Pant and Shukla [34] introduced a larger class of mappings, which contains both *Suzuki-type mappings* and *α-nonexpansive mappings,* and established some convergence theorem.

Definition 1.4 ([34]) A mapping $\Upsilon : \mathcal{D} \to \mathcal{D}$ is said to be a generalized α-nonexpansive mapping if $\frac{1}{2}\|x - \Upsilon x\| \leq \|x - y\|$ implies

$$\|\Upsilon x - \Upsilon y\| \leq \alpha \big(\|x - \Upsilon y\| + \|y - \Upsilon x\|\big) + (1 - 2\alpha)\|x - y\| \quad (1.5)$$

for all $x, y \in \mathcal{D}$ and for some $0 \leq \alpha < 1$.

It can be easily prove that for both *α-nonexpansive mappings* and *generalized α-nonexpansive mappings*, if $\mathrm{Fix}(\Upsilon) \neq \emptyset$, then they are *quasi-nonexpansive mappings*. Thus, all the classes of mappings defined in (1.2), (1.3), (1.4), and (1.5) are wider than the class of *non-expansive mappings* and are narrower than the *quasi-nonexpansive mappings*.

Very recently, in 2020, Ali et al. [6] showed that the Suzuki *Condition (C)* and the *generalized non-expansive mapping* are independent.

On the other hand, the iterative processes have great importance in modern fixed point theory. To find a fixed point of a self-mapping defined on a metric type space, we often use the Picard iteration. On a distance space $\mathcal{X}$ for a mapping Υ, the Picard iterative process is defined by $u_{n+1} = \Upsilon u_n$ with an initial guess $u_1 \in \mathcal{X}$. Most of the researchers working on fixed point theory use this iterative process to obtain fixed points of a mapping [1, 14, 18, 41].

In 1953, Mann [26] first initiated an iterative process to approximate the fixed point for non-expansive mappings with an initial guess $u_1 \in \mathcal{D}$ as:

$$u_{n+1} = (1-\tau_n)u_n + \tau_n \Upsilon u_n, \tag{1.6}$$

where $\{\tau_n\}$ is a sequence in $(0,1)$.

After that, Ishikawa [24] in 1974 introduced a two step iterative process with the help of two constant sequences $\{\tau_n\}$ and $\{\xi_n\}$ in $(0,1)$ with an initial guess $u_1 \in \mathcal{D}$ as:

$$\begin{cases} u_{n+1} = (1-\tau_n)u_n + \tau_n \Upsilon v_n, \\ v_n = (1-\xi_n)u_n + \xi_n \Upsilon u_n, \end{cases} \tag{1.7}$$

which the convergence is faster than the Mann iterative process.

In the last few years, several researchers obtained various iterative process to approximate fixed points of various classes of mappings. Among them are the iterations introduced by Noor [28], Agarwal et al. [2], Thakur et al. [45], and Piri et al. [36], as well as Picard-S iteration [35], M-iteration [48], M*-iteration [47], K-iteration [22], etc.

Very recently, in 2020, Ali et al. [6] have introduced a new iterative process called *JF*-iterative process with an initial guess $u_1 \in \mathcal{D}$, which is as follows:

$$\begin{cases} u_{n+1} = \Upsilon((1-\tau_n)v_n + \tau_n \Upsilon v_n), \\ v_n = \Upsilon(w_n), \\ w_n = \Upsilon((1-\xi_n)u_n + \xi_n \Upsilon u_n), \end{cases} \tag{1.8}$$

where $\{\xi_n\}$ and $\{\tau_n\}$ are two sequences in $(0,1)$.

Considering *generalized non-expansive mappings*, they proved in [6] that the iterative process given by (1.8) converges faster than the Mann iteration, Ishikawa iteration, Noor iteration, S-iteration, Picard-S iteration, and Thakur et al. iteration.

Now, a natural question arises: How can we approximate the fixed point of such mappings using a certain iterative scheme if a mapping does not belong to any of non-expansive, generalized non-expansive, Condition (C), α-nonexpansive and generalized α-nonexpansive classes? In this paper, we answer this question only partially. Indeed, inspired by the papers [33] and [40], we introduce a new class of non-expansive mappings, namely *weakly K-nonexpansive mappings*, which is defined as follows:

Definition 1.5 A mapping $\Upsilon : \mathcal{D} \to \mathcal{D}$ is said to be a weakly K-nonexpansive mapping if there exists $K \geq 0$ such that

$$\|\Upsilon x - \Upsilon y\| \leq \|x-y\| + K\|x-\Upsilon x\|.\|y-\Upsilon y\|. \tag{1.9}$$

It is to be noted that a weakly K-nonexpansive mapping does not guarantee the existence of fixed point. In particular, if $K = 0$, then (1.9) reduces to *non-expansive mapping*, and if $\mathrm{Fix}(\Upsilon) \neq \emptyset$, then (1.9) reduces to *quasi-nonexpansive mapping*. Thus, the class of *weakly K-nonexpansive mappings* is larger than the that of *non-expansive mappings* and smaller than the class of *quasi-nonexpansive mappings*.

Here, using weakly K-nonexpansive mappings Υ with $\mathrm{Fix}(\Upsilon) \neq \emptyset$, we establish a convergence theorem for the *JF*-iterative process to approximate fixed point for such mappings, and finally we compare its convergence rate by providing a numerical example with some other known iterative process.

In 1967, Ostrowski [32] was the first who studied the stability of iterative procedures in a metric space for the Picard iteration.

Definition 1.6 ([10]) Let $(\mathcal{X}, d)$ be a metric space and $\Upsilon : \mathcal{X} \to \mathcal{X}$ be a mapping. Let $u_1 \in \mathcal{X}$ and $u_{n+1} = f(\Upsilon, u_n)$ be a general iterative process involving the mapping Υ. Suppose that $\{u_n\}_n$ converges to a fixed point $\rho \in \mathcal{X}$ of Υ. Let $\{x_n\}_n \subset \mathcal{X}$ be any sequence and let $\epsilon_n := d(x_{n+1}, f(\Upsilon, x_n))$ for all $n \in \mathbb{N}$. Then the iterative process $u_{n+1} = f(\Upsilon, u_n)$ is Υ-stable (or stable with respect to the mapping Υ) if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} x_n = \rho$.

The stability of different iterative procedures of certain contractive mappings have been studied by several researchers (See, [20, 37–39, 46], and [10, 16]).

In 1995, Osilike [31] proposed a new type of contractive mapping in a normed linear space $\mathcal{X}$ as: for all $x, y \in \mathcal{X}$, there exists $\alpha \in [0, 1)$ and $K \geq 0$ such that

$$\|\Upsilon x - \Upsilon y\| \leq \alpha \|x - y\| + K\|x - \Upsilon x\|. \tag{1.10}$$

Using this contractive condition, he proved that the Picard and Ishikawa iterating sequences are Υ-stable. Thereafter, in 2003, Imoru et al. [23] generalized the contractive mapping due to Osilike by replacing the constant K by a certain function as follows and proved some stability results for the Picard and Mann iterative processes (see, also [29] for the Ishikawa iterative process):

For all $x, y \in \mathcal{X}$, there exists $\alpha \in [0, 1)$ and a monotone increasing and continuous function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(0) = 0$ such that

$$\|\Upsilon x - \Upsilon y\| \leq \alpha \|x - y\| + \psi\big(\|x - \Upsilon x\|\big). \tag{1.11}$$

Now, the question is: Does there exist a larger class of contractive mappings than that of (1.11) so that the stability results can be improved? We have also answered this question partially. Indeed, we employ another type of non-expansivity, namely the weakly (α, K)-nonexpansive mappings defined as follows, and show by an example that there are such mappings, which do not satisfy (1.11), but they are weakly (α, K)-nonexpansive mappings.

Definition 1.7 A mapping $\Upsilon : \mathcal{D} \to \mathcal{D}$ is said to be a weakly (α, K)-nonexpansive mapping if $\exists\ \alpha \in (0, 1)$ and $K \geq 0$ such that

$$\|\Upsilon x - \Upsilon y\| \leq \alpha.\|x - y\| + K\|x - \Upsilon x\|.\|y - \Upsilon y\|. \tag{1.12}$$

Using weakly (α, K)-nonexpansive mapping (1.12), we prove stability results for the *JF*-iterative process (1.8).

2 Preliminaries

In this section, we recall some basic definitions, preliminary facts, and Lemmas, which we have used in our main results.

Definition 2.1 ([19]) A Banach space $\mathcal{B}$ is said to be *strictly convex* if for all $x, y \in \mathcal{B}$ with $x \neq y$ and $\|x\| = \|y\| = 1$ implies $\|\frac{x+y}{2}\| < 2$, and $\mathcal{B}$ is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, $\exists$ a $0 < \delta < 1$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$; $\forall x, y \in \mathcal{B}$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$.

A mapping $\Upsilon : \mathcal{D} \to \mathcal{B}$ is said to be *demiclosed at* $y \in \mathcal{B}$ (see [19]) if for every sequence $\{u_n\} \subset \mathcal{D}$ with $u_n \rightharpoonup x$ for some $x \in \mathcal{D}$ and $\Upsilon u_n \to y$ implies that $\Upsilon x = y$.

A Banach space $\mathcal{B}$ is said to satisfy *Opial's property*(see [30]) if for any arbitrary sequence $\{u_n\} \subset \mathcal{B}$ with $u_n \rightharpoonup x \in \mathcal{B}$ such that for all $y \in \mathcal{B} \setminus \{x\}$,

$$\liminf_{n\to+\infty}\|u_n - x\| < \liminf_{n\to+\infty}\|u_n - y\|.$$

Let $\{u_n\}$ be a bounded sequence in $\mathcal{B}$. Then, for every $x \in \mathcal{D}$, we define (see [19]):

- Asymptotic radius of $\{u_n\}$ relative to x by

$$r\bigl(x, \{u_n\}\bigr) := \limsup_{n\to+\infty}\|u_n - x\|.$$

- Asymptotic radius of $\{u_n\}$ relative to $\mathcal{D}$ by

$$r\bigl(\mathcal{D}, \{u_n\}\bigr) := \inf_{x\in\mathcal{D}} r\bigl(x, \{u_n\}\bigr).$$

- Asymptotic centre of $\{u_n\}$ relative to $\mathcal{D}$ by

$$A\bigl(\mathcal{D}, \{u_n\}\bigr) := \bigl\{x \in \mathcal{D} : r\bigl(x, \{u_n\}\bigr) = r\bigl(\mathcal{D}, \{u_n\}\bigr)\bigr\}.$$

Moreover, if $\mathcal{B}$ is uniformly convex, then it is well known that $A(\mathcal{D}, \{u_n\})$ is a singleton set.

A mapping $\Upsilon : \mathcal{D} \to \mathcal{D}$ is said to satisfy *Condition (I)* (see [43]) if there exists a non-decreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$, for all $t > 0$ such that $\|x - \Upsilon x\| \geq \varphi(d(x, \mathrm{Fix}(\Upsilon)))$, for all $x \in \mathcal{D}$; where $d(x, \mathrm{Fix}(\Upsilon)) := \inf_{\rho\in\mathrm{Fix}(\Upsilon)}\|x - \rho\|$.

Lemma 2.2 ([42]) *Let $\mathcal{B}$ be a uniformly convex Banach space and $0 < r \leq s_n \leq t < 1$ for all $n \in \mathbb{N}$. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences in $\mathcal{B}$ satisfying $\limsup_{n\to+\infty}\|a_n\| \leq s$, $\limsup_{n\to+\infty}\|b_n\| \leq s$ and $\limsup_{n\to+\infty}\|s_n a_n + (1 - s_n)b_n\| = s$ for some $s \geq 0$. Then $\lim_{n\to+\infty}\|a_n - b_n\| = 0$.*

Lemma 2.3 ([15]) *Let μ be a real number with $0 \leq \mu < 1$ and $\{\epsilon_n\}$ be a sequence of positive reals such that $\lim_{n\to+\infty}\epsilon_n = 0$. Then, for any sequence of positive reals $\{x_n\}$ satisfying $x_{n+1} \leq \epsilon_n + \mu x_n$, we have $\lim_{n\to+\infty} x_n = 0$.*

3 Some basic discussions

In this section, we discuss the nature of our weakly K-nonexpansive and weakly (α, K)-nonexpansive mappings, compare them with the other previously defined mappings, and prove some basic properties of our newly defined non-expansive type mappings, which we used in our main results. Pant & Shukla [34] proved that every mapping satisfying *Condition (C)* (1.3) is a *generalized α-nonexpansive mapping* (1.5), but the reverse implication is not true.

Ali et al. [6] proved via some examples that the *generalized non-expansive mapping* (1.2) due to Hardy-Rogers and the *Condition (C)* (1.3) are independent.

Also, it has already been proved that *generalized α-nonexpansive mapping* (1.5) is not necessarily a *α-nonexpansive mapping* (1.4) (see [34], Example 3.4).

First, we will prove that the class of *α-nonexpansive mappings* (1.4) and the class of *generalized α-nonexpansive mapping* (1.5) are independent. For this purpose, we consider the following example:

Example 3.1 Let $\mathcal{B} = \mathbb{R}$ and $\mathcal{D} = \{1, 2, 4\}$. Define $\Upsilon : \mathcal{D} \to \mathcal{D}$ by

$$\begin{cases} 1 \mapsto 2, \\ 2 \mapsto 4, \\ 4 \mapsto 4. \end{cases}$$

Then, Υ is an α-nonexpansive mapping but not generalized α-nonexpansive mapping.

To prove that Υ is an α-nonexpansive mapping, take $\alpha = \frac{3}{7}$.
Case 1: If $x = 2$ and $y = 1$, then

$$\|\Upsilon x - \Upsilon y\|^2 = 4 = \alpha\big(\|x - \Upsilon y\|^2 + \|y - \Upsilon x\|^2\big) + (1 - 2\alpha)\|x - y\|^2.$$

Case 2: If $(x, y) \in \{(1, 1), (2, 2), (4, 4), (2, 4)\}$ then, $\|\Upsilon x - \Upsilon y\|^2 = 0$ and since $\alpha = \frac{3}{7} > 0$ and $1 - 2\alpha = \frac{1}{7} > 0$, so, $\alpha(\|x - \Upsilon y\|^2 + \|y - \Upsilon x\|^2) + (1 - 2\alpha)\|x - y\|^2 \geq 0$. Thus, $\|\Upsilon x - \Upsilon y\|^2 \leq \alpha(\|x - \Upsilon y\|^2 + \|y - \Upsilon x\|^2) + (1 - 2\alpha)\|x - y\|^2$ holds.

Case 3: If $x = 1$ and $y = 4$ then,

$$\|\Upsilon x - \Upsilon y\|^2 = 4 < \frac{48}{7} = \alpha\big(\|x - \Upsilon y\|^2 + \|y - \Upsilon x\|^2\big) + (1 - 2\alpha)\|x - y\|^2.$$

Hence, Υ is an α-nonexpansive mapping.

Now, take $x = 2$ and $y = 1$. Then, $\frac{1}{2}\|x - \Upsilon x\| = 1 = \|x - y\|$. Suppose that there exists $\alpha \in [0, 1)$ such that $\|\Upsilon x - \Upsilon y\| \leq \alpha(\|x - \Upsilon y\| + \|y - \Upsilon x\|) + (1 - 2\alpha)\|x - y\|$. Then, $2 \leq 3\alpha + (1 - 2\alpha)$ implies $\alpha \geq 1$, a contradiction. Hence, Υ is not a generalized α-nonexpansive mapping.

Also, note that Υ is a *weakly K-nonexpansive mapping* for $K = 1$.

Example 3.2 Let $\mathcal{B} := \mathbb{R}$ and $\mathcal{D} = [0, 4]$. Define $\Upsilon : \mathcal{D} \to \mathcal{D}$ by

$$\Upsilon x = \begin{cases} 0, & \text{if } 0 \leq x < 4, \\ 3, & \text{if } x = 4. \end{cases} \tag{3.1}$$

Then, Υ is a weakly K-nonexpansive mapping, whether it is neither a generalized α-nonexpansive mapping nor satisfies the Suzuki Condition (C).

Take $K = 1$. If $x, y \in [0,4)$, then (1.9) holds trivially, as $\|\Upsilon x - \Upsilon y\| = 0$.

If $x = 4$ and $y \in [0,4)$, then $\|\Upsilon x - \Upsilon y\| = 3 < 4 = \|x - y\| + K\|y - \Upsilon y\|.\|x - \Upsilon x\|$. Therefore, Υ is a *weakly K-nonexpansive mapping.*

Take $x = 4$ and $y = 3$. Then $\frac{1}{2}\|x - \Upsilon x\| = \frac{1}{2} < 1 = \|x - y\|$. Suppose that there exists $\alpha \in [0,1)$ such that $\|\Upsilon x - \Upsilon y\| \leq \alpha(\|x - \Upsilon y\| + \|y - \Upsilon x\|) + (1 - 2\alpha)\|x - y\|$. Then $3 \leq 4\alpha + (1 - 2\alpha)$ implies $\alpha \geq 1$, a contradiction. Therefore, Υ is not a *generalized α-nonexpansive mapping* and contrapositively Υ does not satisfy *Condition (C).*

Example 3.3 Let $\mathcal{B} := \mathbb{R}^2$. Define a norm on $\mathbb{R}^2$ by $\|x\| = \|(x_1, x_2)\| := |x_1| + |x_2|$. Then $(\mathcal{B}, \|\cdot\|)$ is a Banach space. Consider a subset of $\mathcal{D} \subset \mathbb{R}^2$ defined as:

$\mathcal{D} := \{(0,0), (2,0), (4,0), (0,4), (4,5), (5,4)\}$. Define a map $\Upsilon : \mathcal{D} \to \mathcal{D}$ by

$$\Upsilon x := \begin{cases} (0,0), & \text{if } x \in \{(0,0), (2,0)\}, \\ (2,0), & \text{if } x = (4,0), \\ (4,0), & \text{if } x \in \{(4,5), (0,4)\}, \\ (0,4), & \text{if } x = (5,4). \end{cases} \tag{3.2}$$

Then, Υ is a weakly K-nonexpansive mapping but satisfies none of the (1.2), (1.3), (1.4), and (1.5).

It can be easily verify that Υ is a weakly K-nonexpansive mapping for $K = 1$.

Take $x = (4,5)$ and $y = (5,4)$. Suppose that there exists $\alpha \in [0,1)$ satisfying $\|\Upsilon x - \Upsilon y\|^2 \leq \alpha(\|x - \Upsilon y\|^2 + \|y - \Upsilon x\|^2) + (1 - 2\alpha)\|x - y\|^2$. Then, $64 \leq \alpha(25 + 25) + 4(1 - 2\alpha)$ implies $\alpha \geq \frac{30}{21} > 1$, a contradiction. Therefore, Υ is not an *α-nonexpansive mapping* (1.4).

Taking the same x and y, suppose that there exists $\alpha_1, \alpha_2, \alpha_3 \in [0,1)$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 \leq 1$ such that (1.2) holds. Then, $8 \leq 2\alpha_1 + 10\alpha_2 + 10\alpha_3 \leq 2 + 6\alpha_2 + 6\alpha_3 \leq 5 - 3\alpha_1$ implies $\alpha_1 \leq -1$, a contradiction. Therefore, Υ is not a *generalized non-expansive mapping* (1.2).

Next take $x = (0,4)$ and $y = (5,4)$. Then $\frac{1}{2}\|x - \Upsilon x\| = 4 < 5 = \|x - y\|$. Now, suppose that there exists $\alpha \in [0,1)$ satisfying

$$\|\Upsilon x - \Upsilon y\| \leq \alpha\big(\|x - \Upsilon y\| + \|y - \Upsilon x\|\big) + (1 - 2\alpha)\|x - y\|.$$

Then, $8 \leq \alpha(0 + 5) + 5(1 - 2\alpha)$ implies $\alpha \leq -\frac{3}{5}$, a contradiction. Therefore, Υ is not a *generalized α-nonexpansive mapping* (1.5). Then, contrapositively, Υ does not satisfy *Condition (C)* (1.3). Moreover, Υ is not a *non-expansive mapping.*

Proposition 3.4 *For a weakly K-nonexpansive mapping* $\Upsilon : \mathcal{D} \to \mathcal{D}$ *we have,*

$$\|x - \Upsilon y\| \leq \|x - y\| + \|x - \Upsilon x\|\big(1 + K\|y - \Upsilon y\|\big) \quad \textit{for all } x, y \in \mathcal{D}. \tag{3.3}$$

Proof Simply using the triangle inequality, we have

$$\begin{aligned} \|x - \Upsilon y\| &\leq \|x - \Upsilon x\| + \|\Upsilon x - \Upsilon y\| \\ &\leq \|x - \Upsilon x\| + \|x - y\| + K\|x - \Upsilon x\|.\|y - \Upsilon y\| \\ &= \|x - y\| + \|x - \Upsilon x\|\big(1 + K\|y - \Upsilon y\|\big), \end{aligned}$$

for all $x, y \in \mathcal{D}$. □

Lemma 3.5 *Let $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a weakly K-nonexpansive mapping, where $\mathcal{D}$ is a closed subset of a Banach space $\mathcal{B}$. Then,* $\mathrm{Fix}(\Upsilon)$ *is closed. Moreover, if $\mathcal{B}$ is strictly convex, and $\mathcal{D}$ is convex, then* $\mathrm{Fix}(\Upsilon)$ *is convex.*

Proof To show that $\mathrm{Fix}(\Upsilon)$ is closed, let us assume that $\rho \in \overline{\mathrm{Fix}(\Upsilon)}$. Then, there exists a sequence $\{\rho_n\} \subset \mathrm{Fix}(\Upsilon)$ such that $\rho_n \xrightarrow{n\to+\infty} \rho$.

Now, using (3.3), we have

$$\begin{aligned}\|\rho_n - \Upsilon\rho\| &\le \|\rho_n - \rho\| + \|\rho_n - \Upsilon\rho_n\|\bigl(1 + K\|\rho - \Upsilon\rho\|\bigr)\\ &= \|\rho_n - \rho\|, \quad \text{since } \rho_n = \Upsilon\rho_n.\end{aligned}$$

Taking the limit on both sides, we have $\lim_{n\to+\infty}\|\rho_n - \Upsilon\rho\| \le \lim_{n\to+\infty}\|\rho_n - \rho\| = 0$, which implies that $\rho_n \xrightarrow{n\to+\infty} \Upsilon\rho$, and hence, $\rho = \Upsilon\rho$, *i.e.*, $\rho \in \mathrm{Fix}(\Upsilon)$, and consequently, $\mathrm{Fix}(\Upsilon)$ is closed.

Now, we will show that $\mathrm{Fix}(\Upsilon)$ is convex. For this aim, let $\rho_1, \rho_2 \in \mathrm{Fix}(\Upsilon)$ with $\rho_1 \neq \rho_2$ and let $0 < \mu < 1$. Put $\rho := \mu\rho_1 + (1-\mu)\rho_2$. We claim that $\rho \in \mathrm{Fix}(\Upsilon)$.

Using (3.3), we have,

$$\|\rho_1 - \Upsilon\rho\| \le \|\rho_1 - \rho\| + \|\rho_1 - \Upsilon\rho_1\|\bigl(1 + K\|\rho - \Upsilon\rho\|\bigr) = \|\rho_1 - \rho\|. \tag{3.4}$$

Similarly,

$$\|\rho_2 - \Upsilon\rho\| \le \|\rho_2 - \rho\|. \tag{3.5}$$

Now,

$$\begin{aligned}\|\rho_1 - \rho_2\| &\le \|\rho_1 - \Upsilon\rho\| + \|\rho_2 - \Upsilon\rho\|\\ &\le \|\rho_1 - \rho\| + \|\rho_2 - \rho\|\\ &= \|\rho_1 - \rho_2\|, \quad \text{putting the value of } \rho\end{aligned}$$

implies that $\|\rho_1 - \Upsilon\rho\| + \|\Upsilon\rho - \rho_2\| = \|\rho_1 - \rho_2\|$. Since $\mathcal{B}$ is strictly convex, there exists a constant $\kappa > 0$ such that $\rho_1 - \Upsilon\rho = \kappa(\Upsilon\rho - \rho_2)$. Then, $\Upsilon\rho = \delta\rho_1 + (1-\delta)\rho_2$, where $\delta = \frac{1}{1+\kappa} \in (0,1)$. Now, using (3.4) and (3.5), we get

$$(1-\delta)\|\rho_1 - \rho_2\| = \|\rho_1 - \Upsilon\rho\| \le \|\rho_1 - \rho\| = (1-\mu)\|\rho_1 - \rho_2\|$$

and

$$\delta\|\rho_1 - \rho_2\| = \|\rho_2 - \Upsilon\rho\| \le \|\rho_2 - \rho\| = \mu\|\rho_1 - \rho_2\|.$$

Thus, we get $1-\delta \le 1-\mu$, and $\delta \le \mu$ implies $\delta = \mu$. Then, $\Upsilon\rho = \rho$, *i.e.*, $\rho \in \mathrm{Fix}(\Upsilon)$, and hence, $\mathrm{Fix}(\Upsilon)$ is convex. □

Lemma 3.6 *Let $\mathcal{B}$ be a Banach space having Opial's property and $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a weakly K-nonexpansive mapping, where $\mathcal{D}$ is a closed subset of $\mathcal{B}$. If $\{u_n\}$ is a sequence in $\mathcal{D}$ such that $u_n \rightharpoonup x$ for some $x \in \mathcal{D}$ and $\lim_{n\to+\infty}\|u_n - \Upsilon u_n\| = 0$, then $\mathcal{I} - \Upsilon$ is demiclosed at zero, where $\mathcal{I}$ is the identity mapping on $\mathcal{D}$.*

Proof Using (3.3), we have, $\|u_n - \Upsilon x\| \le \|u_n - x\| + \|u_n - \Upsilon u_n\|(1 + K\|x - \Upsilon x\|)$.

Taking lim inf on both sides, we get $\liminf_{n\to+\infty}\|u_n - \Upsilon x\| \le \liminf_{n\to+\infty}\|u_n - x\|$ and by *Opial's property*, we have $\Upsilon x = x$, *i.e.*, $\mathcal{I} - \Upsilon$ is demiclosed at zero. □

4 Stability results

In this section, first we present an example, which does not satisfy (1.11) but satisfies (1.12), and then we prove some stability results of *JF*-iterative process (1.8) for weakly (α, K)-nonexpansive mappings (1.12).

Example 4.1 Let $\mathcal{X} := [2,\infty)$. Define $\Upsilon : \mathcal{X} \to \mathcal{X}$ by $\Upsilon(x) = x^2$. Then, it is easy to check that Υ is a weakly (α, K)-nonexpansive mapping with $\alpha = \frac{1}{2}$ and $K = 1$. But Υ does not satisfy (1.11). For this, we take $x = 2$ and $y = 2^n$ ($n \in \mathbb{N}$). Then, for any $\alpha \in [0,1)$ we have $\psi(2) \ge \frac{1}{2}.[(2^{2n} - 4) - \alpha(2^n - 2)] \to +\infty$ as $n \to +\infty$, which is a contradiction.

Remark 4.2 A weakly (α, K)-nonexpansive mapping does not ensure the existence of fixed point. Example 4.1 shows this.

Theorem 4.3 *Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space and $\Upsilon : \mathcal{X} \to \mathcal{X}$ be a weakly (α, K)-nonexpansive mapping. Suppose that Υ has a fixed point $\rho \in \mathcal{X}$. Let $u_1 \in \mathcal{X}$ and $u_{n+1} = f(\Upsilon, u_n)$ be the JF-iterating process defined by* (1.8). *Then, the JF-iterative process is Υ-stable.*

Proof Here, $u_{n+1} = f(\Upsilon, u_n)$ defined by the iterative scheme (1.8), where $\{\tau_n\}$ and $\{\xi_n\}$ are sequences in $(0,1)$. Let $\{x_n\} \subset \mathcal{X}$ be an arbitrary sequence. Define $\epsilon_n := \|x_{n+1} - f(\Upsilon, x_n)\|$.

First suppose that $\lim_{n\to+\infty}\epsilon_n = 0$. Then,

$$
\begin{aligned}
&\|x_{n+1} - \rho\| \\
&\quad \le \big\|x_{n+1} - f(\Upsilon, x_n)\big\| + \big\|f(\Upsilon, x_n) - \rho\big\| \\
&\quad = \epsilon_n + \big\|\Upsilon\big[(1-\tau_n)\Upsilon\Upsilon\big\{(1-\xi_n)x_n + \xi_n\Upsilon x_n\big\} + \tau_n\Upsilon\Upsilon\Upsilon\big\{(1-\xi_n)x_n + \xi_n\Upsilon x_n\big\}\big] - \rho\big\| \\
&\quad \le \epsilon_n + \alpha\big\|(1-\tau_n)\Upsilon\Upsilon\big\{(1-\xi_n)x_n + \xi_n\Upsilon x_n\big\} + \tau_n\Upsilon\Upsilon\Upsilon\big\{(1-\xi_n)x_n + \xi_n\Upsilon x_n\big\} - \rho\big\| \\
&\quad \le \epsilon_n + \alpha\big\|(1-\tau_n)(y_n - \rho) + \tau_n(\Upsilon y_n - \rho)\big\|, \quad \text{where } y_n = \Upsilon\Upsilon\big\{(1-\xi_n)x_n + \xi_n\Upsilon x_n\big\} \\
&\quad \le \epsilon_n + \alpha[(1-\tau_n)\|y_n - \rho\| + \tau_n\|\Upsilon y_n - \rho\| \\
&\quad \le \epsilon_n + \alpha(1-\tau_n+\alpha\tau_n)\|y_n - \rho\| \\
&\quad = \epsilon_n + \alpha(1-\tau_n+\alpha\tau_n)\big\|\Upsilon\Upsilon\big\{(1-\xi_n)x_n + \xi_n\Upsilon x_n\big\} - \rho\big\| \\
&\quad \le \epsilon_n + \alpha^2(1-\tau_n+\alpha\tau_n)\big\|\Upsilon\big\{(1-\xi_n)x_n + \xi_n\Upsilon x_n\big\} - \rho\big\| \\
&\quad \le \epsilon_n + \alpha^3(1-\tau_n+\alpha\tau_n)\big\|(1-\xi_n)x_n + \xi_n\Upsilon x_n - \rho\big\| \\
&\quad \le \epsilon_n + \alpha^3(1-\tau_n+\alpha\tau_n)\big[(1-\xi_n)\|x_n - \rho\| + \xi_n\|\Upsilon x_n - \rho\|\big] \\
&\quad \le \epsilon_n + \alpha^3(1-\tau_n+\alpha\tau_n).(1-\xi_n+\alpha\xi_n)\|x_n - \rho\|.
\end{aligned}
$$

Since $\alpha \in (0,1)$, so $0 \le 1-\tau_n+\alpha\tau_n = 1-\tau_n(1-\alpha) < 1$, and $0 \le 1-\xi_n+\alpha\xi_n = 1-\xi_n(1-\alpha) < 1$. Therefore, by Lemma 2.3, we have $\lim_{n\to+\infty}\|x_n - \rho\| = 0$, *i.e.*, $\lim_{n\to+\infty}x_n = \rho$. Consequently, *JF*-iterative process is Υ-stable. □

5 Convergence results

In this section, we present some convergence results for weakly K-nonexpansive mappings using *JF* iterative algorithm (1.8). For this purpose, the following Lemmas are crucial.

Lemma 5.1 *Let $\mathcal{D}$ be a non-empty, closed and convex subset of a uniformly convex Banach space $\mathcal{B}$ and $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a weakly K-nonexpansive mapping with* $\mathrm{Fix}(\Upsilon) \neq \emptyset$. *Let $\{u_n\}$ be the iterative sequence defined by* (1.8). *Then,* $\lim_{n\to+\infty} \|u_n - \rho\|$ *exists for all* $\rho \in \mathrm{Fix}(\Upsilon)$.

Proof Let $\rho \in \mathrm{Fix}(\Upsilon)$. Since Υ is a weakly K-nonexpansive mapping, so for every sequence $\{x_n\} \subset \mathcal{D}$, we can get $\|\Upsilon x_n - \rho\| \leq \|x_n - \rho\|$. Then using the iteration (1.8), we have

$$\begin{aligned}
\|w_n - \rho\| &= \big\|\Upsilon\big((1-\xi_n)u_n + \xi_n \Upsilon u_n\big) - \rho\big\| \\
&\leq \big\|(1-\xi_n)u_n + \xi_n \Upsilon u_n - \rho\big\| \\
&\leq (1-\xi_n)\|u_n - \rho\| + \xi_n\|\Upsilon u_n - \rho\| \\
&\leq (1-\xi_n)\|u_n - \rho\| + \xi_n\|u_n - \rho\| \\
&= \|u_n - \rho\|. \qquad (5.1)
\end{aligned}$$

Now, using (5.1), we have

$$\begin{aligned}
\|v_n - \rho\| &= \|\Upsilon w_n - \rho\| \\
&\leq \|w_n - \rho\| \qquad (5.2) \\
&\leq \|u_n - \rho\|. \qquad (5.3)
\end{aligned}$$

Finally, using (5.3), we have

$$\begin{aligned}
\|u_{n+1} - \rho\| &= \big\|\Upsilon\big((1-\tau_n)v_n + \tau_n \Upsilon v_n\big) - \rho\big\| \\
&\leq \big\|(1-\tau_n)v_n + \tau_n \Upsilon v_n - \rho\big\| \\
&\leq (1-\tau_n)\|v_n - \rho\| + \tau_n\|\Upsilon v_n - \rho\| \\
&\leq (1-\tau_n)\|v_n - \rho\| + \tau_n\|v_n - \rho\| \\
&= \|v_n - \rho\| \qquad (5.4) \\
&\leq \|u_n - \rho\|. \qquad (5.5)
\end{aligned}$$

Thus, we get $\{\|u_n - \rho\|\}_n$ is a non-increasing sequence of reals, which is bounded below by zero. Hence, $\lim_{n\to+\infty} \|u_n - \rho\|$ exists for all $\rho \in \mathrm{Fix}(\Upsilon)$. □

Lemma 5.2 *Let $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a weakly K-nonexpansive mapping defined on a non-empty closed convex subset $\mathcal{D}$ of a uniformly convex Banach space $\mathcal{B}$. Let $\{u_n\}$ be the iterative sequence defined by* (1.8). *Then,* $\mathrm{Fix}(\Upsilon) \neq \emptyset$ *if and only if $\{u_n\}$ is bounded and* $\lim_{n\to+\infty} \|u_n - \Upsilon u_n\| = 0$.

Proof First suppose that $\mathrm{Fix}(\Upsilon) \neq \emptyset$ and let $\rho \in \mathrm{Fix}(\Upsilon)$. Then, from Lemma 5.1, we have $\lim_{n\to+\infty} \|u_n - \rho\|$ exists and consequently $\{u_n\}$ becomes bounded.

Let $\lim_{n\to+\infty}\|u_n-\rho\|=\theta$. Then, from (5.1) and (5.3), we have $\limsup_{n\to+\infty}\|w_n-\rho\|\le\theta$ and $\limsup_{n\to+\infty}\|v_n-\rho\|\le\theta$.

Since Υ is weakly K-nonexpansive mapping, we have $\|\Upsilon u_n-\rho\|=\|\Upsilon u_n-\Upsilon\rho\|\le\|u_n-\rho\|$ and therefore $\limsup_{n\to+\infty}\|\Upsilon u_n-\rho\|\le\theta$.

Now, taking lim inf on both sides of (5.4), we have

$$\theta=\liminf_{n\to+\infty}\|u_{n+1}-\rho\|\le\liminf_{n\to+\infty}\|v_n-\rho\|\le\limsup_{n\to+\infty}\|v_n-\rho\|\le\theta,$$

which yields $\lim_{n\to+\infty}\|v_n-\rho\|=\theta$.

Again by taking lim inf on both sides in (5.2), we have

$$\theta=\liminf_{n\to+\infty}\|v_n-\rho\|\le\liminf_{n\to+\infty}\|w_n-\rho\|\le\limsup_{n\to+\infty}\|w_n-\rho\|\le\theta,$$

implying that $\lim_{n\to+\infty}\|w_n-\rho\|=\theta$.

Therefore,

$$\begin{aligned}\theta&=\lim_{n\to+\infty}\|w_n-\rho\|\\&=\lim_{n\to+\infty}\big\|\Upsilon\big((1-\xi_n)u_n+\xi_n\Upsilon u_n\big)-\rho\big\|\\&\le\lim_{n\to+\infty}\big\|(1-\xi_n)u_n+\xi_n\Upsilon u_n-\rho\big\|\\&=\lim_{n\to+\infty}\big\|(1-\xi_n)(u_n-\rho)+\xi_n(\Upsilon u_n-\rho)\big\|\\&\le\lim_{n\to+\infty}\big[(1-\xi_n)\|u_n-\rho\|+\xi_n\|\Upsilon u_n-\rho\|\big]\\&\le\lim_{n\to+\infty}\|u_n-\rho\|=\theta,\end{aligned}$$

which implies that $\lim_{n\to+\infty}\|(1-\xi_n)(u_n-\rho)+\xi_n(\Upsilon u_n-\rho)\|=\theta$. Consequently, using Lemma 2.2, we can conclude that $\lim_{n\to+\infty}\|u_n-\Upsilon u_n\|=0$.

Conversely, suppose that $\{u_n\}$ be bounded and $\lim_{n\to+\infty}\|u_n-\Upsilon u_n\|=0$. Since $\mathcal{B}$ is a uniformly convex Banach space, and $\mathcal{D}$ is a non-empty closed and convex subset of $\mathcal{B}$, $A(\mathcal{D},\{u_n\})$ is a singleton set, say $\{\rho\}$.

Now, we claim that ρ is a fixed point of Υ. Using (3.3), we have

$$\begin{aligned}r\big(\Upsilon\rho,\{u_n\}\big)&=\limsup_{n\to+\infty}\|u_n-\Upsilon\rho\|\\&\le\limsup_{n\to+\infty}\big[\|u_n-\rho\|+\|u_n-\Upsilon u_n\|\big(1+K\|\rho-\Upsilon\rho\|\big)\big]\\&=\limsup_{n\to+\infty}\|u_n-\rho\|=r\big(\rho,\{u_n\}\big)=r\big(\mathcal{D},\{u_n\}\big).\end{aligned}$$

Therefore, $\Upsilon\rho\in A(\mathcal{D},\{u_n\})$ and consequently, $\Upsilon\rho=\rho$, *i.e.*, ρ is a fixed point of Υ, and we are done. □

Now, we are ready to prove a weak convergence result and a strong convergence result for a weakly K-nonexpansive mapping using the iterative scheme given by (1.8).

Theorem 5.3 *Let $\mathcal{B}$ be a uniformly convex Banach space having the Opial property and $\mathcal{D}(\neq \emptyset)$ be a closed and convex subset of $\mathcal{B}$. Let $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a weakly K-nonexpansive mapping and let $\{u_n\}$ be an iterated sequence defined by (1.8). If $\operatorname{Fix}(\Upsilon) \neq \emptyset$, then $\{u_n\}$ converges weakly to a fixed point of Υ.*

Proof Suppose that $\operatorname{Fix}(\Upsilon) \neq \emptyset$. Then, from Lemma 5.2, we have $\lim_{n\to+\infty}\|u_n - \Upsilon u_n\| = 0$. Since $\mathcal{B}$ is uniformly convex, it is reflexive, and hence there exists a sub-sequence $\{u_{n_i}\}_i$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup \rho$ for some $\rho \in \mathcal{D}$. Then, by Lemma 3.6, $\mathcal{I} - \Upsilon$ is demiclosed at zero, where $\mathcal{I}$ is the identity mapping on $\mathcal{D}$, *i.e.*, $\rho \in \operatorname{Fix}(\Upsilon)$.

By the contrary, suppose that $u_n \not\rightharpoonup \rho$. Then, there exists a subsequence $\{u_{n_j}\}_j$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup \rho'$ for some $\rho' (\neq \rho) \in \mathcal{D}$. Then, by Lemma 3.6, $\rho' \in \operatorname{Fix}(\Upsilon)$.

Again from Lemma 5.1, we conclude that $\lim_{n\to+\infty}\|u_n - \rho\|$ exists, $\forall \rho \in \operatorname{Fix}(\Upsilon)$. Therefore,

$$\begin{aligned}
\lim_{n\to+\infty}\|u_n - \rho\| &= \lim_{i\to+\infty}\|u_{n_i} - \rho\| \\
&< \lim_{i\to+\infty}\|u_{n_i} - \rho'\|, \quad \text{using Opial's property} \\
&= \lim_{n\to+\infty}\|u_n - \rho'\| \\
&= \lim_{j\to+\infty}\|u_{n_i} - \rho'\| \\
&< \lim_{j\to+\infty}\|u_{n_j} - \rho\|, \quad \text{using Opial's property} \\
&= \lim_{n\to+\infty}\|u_n - \rho\|
\end{aligned}$$

arrives at a contradiction and consequently, $u_n \rightharpoonup \rho$, which completes the proof. □

Theorem 5.4 *Let $\mathcal{B}$ be a uniformly convex Banach space and $\mathcal{D}$ be a closed and convex subset of $\mathcal{B}$. Let $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a weakly K-nonexpansive mapping with $\operatorname{Fix}(\Upsilon) \neq \emptyset$ and let $\{u_n\}$ be the iterated sequence defined by (1.8). Then, $\{u_n\}$ converges strongly to a fixed point of Υ if one of the followings hold:*

(i) $\liminf_{n\to+\infty} d(u_n, \operatorname{Fix}(\Upsilon)) = 0$,

(ii) *Υ satisfies Condition (I).*

Proof (i) Assume that $\liminf_{n\to+\infty} d(u_n, \operatorname{Fix}(\Upsilon)) = 0$. Since $\operatorname{Fix}(\Upsilon) \neq \emptyset$, let us choose $\rho \in \operatorname{Fix}(\Upsilon)$. From (5.5), we have $\|u_{n+1} - \rho\| \leq \|u_n - \rho\|$, which implies that $d(u_{n+1}, \operatorname{Fix}(\Upsilon)) \leq d(u_n, \operatorname{Fix}(\Upsilon))$. Thus, $\{d(u_n, \operatorname{Fix}(\Upsilon))\}_n$ is a non-increasing sequence, which is bounded below by zero. Therefore, $\lim_{n\to+\infty} d(u_n, \operatorname{Fix}(\Upsilon))$ exists and by our assumption, $\lim_{n\to+\infty} d(u_n, \operatorname{Fix}(\Upsilon)) = 0$. Then, there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}$ and a sequence $\{\rho_k\}$ of $\operatorname{Fix}(\Upsilon)$ such that

$$\|u_{n_k} - \rho_k\| < \frac{1}{2^k}, \quad \text{for all } k \in \mathbb{N}.$$

Again, we have $\|u_{n_{k+1}} - \rho_k\| \leq \|u_{n_k} - \rho_k\| < \frac{1}{2^k}$.

Therefore, $\|\rho_{k+1} - \rho_k\| \leq \|\rho_{k+1} - u_{n_{k+1}}\| + \|u_{n_{k+1}} - \rho_k\| \leq \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}$. Now, can be easily proved that $\{\rho_k\}$ is a Cauchy sequence in $\operatorname{Fix}(\Upsilon)$ and since $\operatorname{Fix}(\Upsilon)$ is closed, $\rho_k \to \rho'$ for some $\rho' \in \operatorname{Fix}(\Upsilon)$.

Table 1 Convergence behavior of various iterative process

Name of the iterations	Number of iterations needed
Mann	30
Ishikawa	24
Agarwal	13
M	8
*M**	8
Thakur-new	8
JF	5

Again, $\|u_{n_k} - \rho'\| \leq \|u_{n_k} - \rho_k\| + \|\rho_k - \rho'\| \to 0$ as $k \to \infty$. So, $u_{n_k} \to \rho'$. Since $\lim_{n\to+\infty}\|u_n - \rho'\|$ exists, by Lemma 5.1 $u_n \to \rho'$.

(ii) From Lemma 5.2, we have $\lim_{n\to+\infty}\|u_n - \Upsilon u_n\| = 0$. Again, from *Condition (I)*, we have,

$$0 \leq \lim_{n\to+\infty} \varphi\big(d\big(u_n, \operatorname{Fix}(\Upsilon)\big)\big) \leq \lim_{n\to+\infty}\|u_n - \Upsilon u_n\| = 0,$$

which implies $\lim_{n\to+\infty}\varphi(d(u_n, \operatorname{Fix}(\Upsilon))) = 0$ and hence $\lim_{n\to+\infty} d(u_n, \operatorname{Fix}(\Upsilon)) = 0$, which reduces to (i) and completes the proof. □

Now, we compare the behavior of convergence of some known iterative scheme for the weakly K-nonexpansive mappings by choosing the parameter sequences $\{\tau_n\}$ and $\{\xi_n\}$ in $(0,1)$.

Example 5.5 Let $\mathcal{B} = \mathbb{R}$ be equipped with the usual norm and $\mathcal{D} = [1, +\infty)$. Define a map $\Upsilon : \mathcal{D} \to \mathcal{D}$ by

$$\Upsilon x = \begin{cases} \frac{x+2}{3}, & \text{if } x \in [1,3], \\ \frac{x}{x+1}, & \text{if } x \in (3, +\infty). \end{cases} \tag{5.6}$$

Then, it can be easily checked that Υ is a weakly K-nonexpansive mapping for $K = 2$.

It is clear that $x = 1$ is the unique fixed point of Υ. Now, to approximate this fixed point, consider $\tau_n = \frac{5n}{7n+4}$ and $\xi_n = \frac{2n}{3n+1}$ and let the initial guess be $u_1 = 3$. Using these sequences of scalars and the weakly K-nonexpansive mapping defined in (5.6), in Table 1, we compare the convergence behavior of the Mann-iteration, Ishikawa-iteration, Agarwal-iteration, Thakur-new iteration, M-iteration, M^*-iteration, JF-iteration, and we stop the process when the result is correct up to 7-decimal places (*i.e.*, we stop the process when the result comes 1.0000000).

6 Application to nonlinear fractional differential equation

During the last three decades, fractional differential calculus has became an interesting and fruitful area of research in science and engineering. It has several applications in the field of signal processing, fluid flow, diffusive transport, electrical networks, electronics, robotics, telecommunication, etc.; for more details, one can refer to ([3–5, 7, 8, 17, 25], and [27]). Sometimes, it is observed that a particular nonlinear fractional differential equation may have no analytic solution. In this case, we need to find out an approximate solution. In this section, we will estimate an approximate solution of a nonlinear fractional differential equation using the iterative algorithm (1.8).

Type-I:

Consider the fractional differential equation:

$$D^{\gamma} y(x) + f\big(x, y(x)\big) = 0, \quad 0 \le x \le 1 \text{ and } 1 < \gamma < 2 \tag{6.1}$$

with the boundary conditions $y(0) = 0$ and $y(1) = 1$, where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $D^{\gamma} (= \frac{d^{\gamma}}{dx^{\gamma}})$ denotes the fractional derivative of order γ.

Let $\mathcal{B} = C[0,1]$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the sup-norm. The Green function [12] corresponding to the equation (6.1) is defined by

$$G(x,t) = \begin{cases} \frac{1}{\Gamma(\gamma)}[x(1-t)^{\gamma-1} - (x-t)^{\gamma-1}] & \text{for } 0 \le t \le x, \\ \frac{1}{\Gamma(\gamma)} x(1-t)^{\gamma-1} & \text{for } x \le t \le 1. \end{cases}$$

Now, we approximate the solution of the fractional differential equation (6.1) using the iterative scheme (1.8).

Theorem 6.1 *Let $\mathcal{B} = C[0,1]$ be a Banach space equipped with the sup-norm and $\{u_n\}$ be a sequence defined by JF-iterative scheme for the function $\Upsilon : \mathcal{B} \to \mathcal{B}$ defined by*

$$\Upsilon y(x) = \int_0^1 G(x,t) f\big(t, y(t)\big)\, dt, \quad \text{for all } x \in [0,1] \text{ and } y \in \mathcal{B}.$$

Moreover, assume that f is a Lipschitz function with respect to the second variable, i.e., $|f(x,y_1) - f(x,y_2)| \le |y_1 - y_2|$, for all $x \in [0,1]$ and $y_1, y_2 \in \mathcal{B}$. Then JF-iterative sequence converges to a solution of the problem (6.1).

Proof We know that the solution of the fractional differential equation (6.1) in terms of Green's function is

$$y(x) = \int_0^1 G(x,t) f\big(t, y(t)\big)\, dt.$$

Then for all $y_1, y_2 \in \mathcal{B}$ and $x \in [0,1]$, we have

$$\begin{aligned} \big|\Upsilon y_1(x) - \Upsilon y_2(x)\big| &= \left| \int_0^1 G(x,t) f\big(t, y_1(t)\big)\, dt - \int_0^1 G(x,t) f\big(t, y_2(t)\big)\, dt \right| \\ &= \left| \int_0^1 G(x,t) \big\{ f\big(t, y_1(t)\big) - f\big(t, y_2(t)\big) \big\}\, dt \right| \\ &\le \int_0^1 G(x,t) \big| f\big(t, y_1(t)\big) - f\big(t, y_2(t)\big) \big|\, dt \\ &\le \int_0^1 G(x,t) \big| y_1(t) - y_2(t) \big|\, dt \end{aligned}$$

Table 2 Approximate solution of Example 6.2

Sl. no.	x	u_2	u_4	u_7	u_9	u_{10}
1	0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
2	0.1	0.04573922	0.04573922	0.04573922	0.04573922	0.04573922
3	0.2	0.09039085	0.09039085	0.09039085	0.09039085	0.09039085
4	0.3	0.13196066	0.13196066	0.13196066	0.13196066	0.13196066
5	0.4	0.16763088	0.16763088	0.16763088	0.16763088	0.16763088
6	0.5	0.19379645	0.19379645	0.19379645	0.19379645	0.19379645
7	0.6	0.20606552	0.20606552	0.20606552	0.20606552	0.20606552
8	0.7	0.19924752	0.19924752	0.19924752	0.19924752	0.19924752
9	0.8	0.16733569	0.16733569	0.16733569	0.16733569	0.16733569
10	0.9	0.10348689	0.10348689	0.10348689	0.10348689	0.10348689
11	1.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

$$\le \|y_1 - y_2\|.\left(\sup_{x\in[0,1]} \int_0^1 G(x,t)\,dt\right)$$
$$\le \|y_1 - y_2\|.$$

Thus, we get $\|\Upsilon y_1 - \Upsilon y_2\| \le \|y_1 - y_2\|$, $\forall y_1, y_2 \in \mathcal{B}$. Then, Υ is a non-expansive mapping and so is a weakly K-nonexpansive mapping, and hence the JF-iterative scheme converges to the solution of (6.1). □

Now, we present a numerical example, corresponding to the above theorem.

Example 6.2 Consider the following fractional differential equation:

$$D^\gamma y(x) + x^2(x+2) = 0 \quad 0 \le x \le 1, \gamma \in (1,2) \tag{6.2}$$

with the boundary conditions $y(0) = 0$ and $y(1) = 1$.

Consider the mapping $\Upsilon : C[0,1] \to C[0,1]$ defined by

$$\Upsilon y(x) := \frac{1}{\Gamma(\gamma)} \int_{t=0}^{x} \left[x(1-t)^{\gamma-1} - (x-t)^{\gamma-1}\right] t^2(t+2)\,dt + \frac{x}{\Gamma(\gamma)} \int_{t=x}^{1} (1-t)^{\gamma-1} t^2 (t+2)\,dt. \tag{6.3}$$

Take $\gamma = \frac{3}{2}$, initial guess $u_1(x) = x^2(1-x)^2$ and $x \in [0,1]$. Choose the sequences $\xi_n = 0.87$ and $\tau_n = 0.69$ for all $n \in \mathbb{N}$. Then, the JF-iterative scheme (1.8) converges to the solution of (6.2) shown in Table 2.

Type-II:

Now, we consider the nonlinear fractional differential equation

$$D^\gamma y(x) + D^\delta y(x) + \phi\big(x, y(x)\big) = 0, \quad 0 \le x \le 1, \text{and } 0 < \delta < \gamma < 1, \tag{6.4}$$

with the boundary conditions $y(0) = 1$ and $y(1) = 1$, and $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The Green function associated with (6.4) is given by $G(t) = t^{\gamma-1} E_{\gamma-\delta,\gamma}(-t^{\gamma-\delta})$, where $E_{p,q}(z) := \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(mp+q)}$, $p, q > 0$, denotes the two parameter Mittag-Leffler function (see [12]).

Theorem 6.3 *Consider the Banach space $\mathcal{B} := C[0,1]$ equipped with the sup-norm and $\{u_n\}$, which is a sequence defined by JF-iterative scheme for the function $\Upsilon : \mathcal{B} \to \mathcal{B}$ defined by*

$$\Upsilon y(x) = \int_0^x G(x-s)\phi\bigl(s, y(s)\bigr)\, ds \quad \forall x \in [0,1] \text{ and } y \in \mathcal{B}.$$

Moreover, assume that ϕ satisfies the following condition:

$$\bigl|\phi\bigl(x, y_1(x)\bigr) - \phi\bigl(x, y_2(x)\bigr)\bigr| \le \gamma \bigl|y_1(x) - y_2(x)\bigr| \quad \text{for all } x \in [0,1] \text{ and } y_1, y_2 \in \mathcal{B}.$$

Then, the JF-iterative sequence converges to a solution of the fractional differential equation (6.4).

Proof For all $y_1, y_2 \in \mathcal{B}$ and $x \in [0,1]$, we have

$$\begin{aligned}
\bigl|\Upsilon y_1(x) - \Upsilon y_2(x)\bigr| &= \left|\int_0^x G(x-s)\phi\bigl(s, y_1(s)\bigr)\, ds - \int_0^x G(x-s)\phi\bigl(s, y_2(s)\bigr)\, ds\right| \\
&= \left|\int_0^x G(x-s)\bigl(\phi\bigl(s, y_1(s)\bigr) - \phi\bigl(s, y_2(s)\bigr)\bigr)\right| ds \\
&\le \int_0^x \bigl|G(x-s)\bigr|\bigl|\phi\bigl(s, y_1(s)\bigr) - \phi\bigl(s, y_2(s)\bigr)\bigr|\, ds \\
&\le \int_0^x \bigl|G(x-s)\bigr|.\gamma\bigl|y_1(s) - y_2(s)\bigr|\, ds \\
&\le \gamma \|y_1 - y_2\|.\left(\sup_{x\in[0,1]} \int_0^x \bigl|G(x-s)\bigr|\, ds\right).
\end{aligned}$$

Using the properties of the Mittag-Leffler function, it can be seen [12, 13] that $G(t) = t^{\gamma-1}E_{\gamma-\delta,\gamma}(-t^{\gamma-\delta}) \le t^{\gamma-1}$ for all $t \in [0,1]$. Then, $\sup_{x\in[0,1]} \int_0^x |G(x-s)|\, ds \le \sup_{x\in[0,1]} \frac{x^\gamma}{\gamma} = \frac{1}{\gamma}$.

Therefore, we get $\|\Upsilon y_1 - \Upsilon y_2\| \le \|y_1 - y_2\|$ for all $y_1, y_2 \in \mathcal{B}$. Thus, Υ is a weakly K-nonexpansive mapping for $K = 0$, and hence the *JF*-iterative scheme converges to a fixed point of Υ. Again, it is well known that the exact solution of (6.4) is given by $y(x) = \int_0^x G(x-s)\phi(s, y(s))\, ds$. Consequently, the *JF*-iterative scheme converges to the solution of the equation (6.4). □

7 Conclusion

The main purpose of this paper is to introduce a new type of non-expansive mappings, which is different from any other previously defined non-expensive type mappings. We have used the latest *JF*-iterative algorithm to approximate fixed points for our new non-expansive mappings, and we have established some weak and strong convergence theorems. Also, here, we have introduced the concept of (α, K)-nonexpansive mappings and have proved a stability result for the *JF*-iterative process, which is more general than other previous stability results. Furthermore, we have presented a numerical example for our mappings and have compared the convergence behavior of various iterative processes with respect to it. We have also shown that α-nonexpansive mappings and generalized α-nonexpansive mappings are independent of each other. Moreover, an application of our fixed point theorems is given to the nonlinear fractional differential equations.

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