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On some dynamic inequalities of Ostrowski, trapezoid, and Grüss type on time scales

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Abstract

In the present manuscript, we prove some new extensions of the dynamic Ostrowski inequality and its companion inequalities on an arbitrary time scale by using two parameters for functions whose second delta derivatives are bounded. In addition to these generalizations, some integral and discrete inequalities are obtained as special cases of main results.

MSC: 26D15; 26E70

Keywords: Montgomery identity; Ostrowski-type inequality; Trapezoid-type inequality; Grüss-type inequality; Time scales

1 Introduction

In 1938, Ostrowski [27] proved the following integral inequality.

Theorem 1.1 *Assume that the function $\digamma : [\theta, \vartheta] \to \mathbb{R}$ is continuous on $[\theta, \vartheta]$ and differentiable on (θ, ϑ), then for all $\lambda \in [\theta, \vartheta]$, we have*

$$\left| \digamma(\lambda) - \frac{1}{\vartheta - \theta} \int_\theta^\vartheta \digamma(\xi)\, d\xi \right| \leq \sup_{\theta < \xi < \vartheta} \left| \digamma'(\xi) \right| (\vartheta - \theta) \left[\frac{(\lambda - \frac{\theta+\vartheta}{2})^2}{(\vartheta - \theta)^2} + \frac{1}{4} \right]. \tag{1.1}$$

Clearly, inequality (1.1) estimates an upper bound for the absolute deviation between the value of $\digamma$ at a point λ in $[\theta, \vartheta]$ and its integral mean over $[\theta, \vartheta]$.

Grüss [18] proved the following inequality to estimate the absolute deviation of the integral mean of the product of two functions from the product of the integral means.

Theorem 1.2 *Let $\digamma$ and ϕ be continuous functions on $[\theta, \vartheta]$ such that*

$$m_1 \leq \digamma(\xi) \leq M_1 \quad \text{and} \quad m_2 \leq \phi(\xi) \leq M_2, \quad \text{for all } \xi \in [\theta, \vartheta].$$

Then, the following inequality

$$\left| \frac{1}{\vartheta - \theta} \int_\theta^\vartheta \digamma(\xi)\phi(\xi)\, d\xi - \frac{1}{(\vartheta - \theta)^2} \int_\theta^\vartheta \digamma(\xi)\, d\xi \int_\theta^\vartheta \phi(\xi)\, d\xi \right|$$

$$\leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{1.2}$$

holds.

Inequality (1.2) is known in the literature as the Grüss inequality.

One of the companion inequalities of the Ostrowski inequality is the following inequality that is known in the literature as the trapezoid inequality [25].

Theorem 1.3 *Assume that $\digamma$ is a twice-differentiable function on $[\theta, \vartheta]$, then*

$$\left| \frac{\digamma(\theta) + \digamma(\vartheta)}{2}(\vartheta - \theta) - \int_\theta^\vartheta \digamma(\xi)\, d\xi \right| \leq \sup_{\theta < \xi < \vartheta} \left| \digamma''(\xi) \right| \frac{(\vartheta - \theta)^3}{12}.$$

In [28], Pachpatte obtained the following trapezoid- and Grüss-type inequalities.

Theorem 1.4 *Assume that $\digamma : [\theta, \vartheta] \to \mathbb{R}$ is continuous and differentiable on (θ, ϑ), whose first derivative $\digamma' : (\theta, \vartheta) \to \mathbb{R}$ is bounded on (θ, ϑ), then*

$$\left| \frac{\digamma^2(\vartheta) - \digamma^2(\theta)}{2} - \frac{\digamma(\vartheta) - \digamma(\theta)}{b - \theta} \int_\theta^\vartheta \digamma(\xi)\, d\xi \right| \leq \frac{M^2(\vartheta - \theta)^2}{3},$$

where $M = \sup_{\theta < \xi < \vartheta} \digamma'(\xi)$.

Theorem 1.5 *Assume that $\digamma, \phi : [\theta, \vartheta] \to \mathbb{R}$ are continuous and differentiable on (θ, ϑ), whose first derivatives $\digamma', \phi' : (\theta, \vartheta) \to \mathbb{R}$ are bounded on (θ, ϑ), then*

$$\begin{aligned} &\left| \frac{1}{\vartheta - \theta} \int_\theta^\vartheta \digamma(\xi)\phi(\xi)\, d\xi - \left(\frac{1}{\vartheta - \theta} \int_\theta^\vartheta \digamma(\xi)\, d\xi \right) \left(\frac{1}{\vartheta - \theta} \int_\theta^\vartheta \phi(\xi)\, d\xi \right) \right| \\ &\quad \leq \frac{1}{2(\vartheta - \theta)^2} \int_\theta^\vartheta \left[M \left| \phi(\xi) \right| + N \left| \digamma(\xi) \right| \right] \left[\frac{(\vartheta - \theta)^2}{4} + \left(\xi - \frac{\theta + \vartheta}{2} \right)^2 \right] d\xi, \end{aligned} \tag{1.3}$$

where $M = \sup_{\theta < \xi < \vartheta} \digamma'(\xi)$ and $N = \sup_{\theta < \xi < \vartheta} \phi'(\xi)$.

Ostrowski's inequality has a significant importance in many fields, particularly in numerical analysis. One of its applications is the estimation of the error in the approximation of integrals.

Many generalizations and refinements of the Ostrowski inequality and its companion inequalities were carried out during the past several decades, we refer the reader to the articles [3, 9, 11–13, 21, 23, 24], the books [25, 26] and the references cited therein.

Stefan Hilger initiated the theory of time scales in his PhD thesis [19] in order to unify discrete and continuous analysis (see [20]). Since then, this theory has received much attention. The basic notion is to establish a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is the so-called time scale $\mathbb{T}$, which is an arbitrary closed subset of the reals $\mathbb{R}$ (see [8]). The three most common examples of calculus on time scales are continuous calculus, discrete calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$. The book due to Bohner and Peterson [7] on the subject of time scales briefs and organizes much of time-scales calculus.

Over the past decade, a great number of dynamic inequalities on time scales have been established by many researchers who were motivated by some applications (see [1, 2]).

In [5], Bohner and Matthews gave the time-scales version of the Montgomery identity as follows:

Theorem 1.6 *Let $\theta, \vartheta, \lambda, \xi \in \mathbb{T}$, with $\theta < \vartheta$ and $F : [\theta, \vartheta]_{\mathbb{T}} \to \mathbb{R}$ be a delta-differentiable function. Then,*

$$F(\lambda) = \frac{1}{\vartheta - \theta} \int_{\theta}^{\vartheta} F^{\sigma}(\xi)\Delta\xi + \frac{1}{\vartheta - \theta} \int_{\theta}^{\vartheta} \Upsilon(\lambda, \xi) F^{\Delta}(\xi)\Delta\xi, \tag{1.4}$$

where

$$\Upsilon(\lambda, \xi) = \begin{cases} \xi - \theta, & \xi \in [\theta, \lambda)_{\mathbb{T}}, \\ \xi - \vartheta, & \xi \in [\lambda, \vartheta]_{\mathbb{T}}. \end{cases}$$

With the help of the above result, Bohner and Matthews [5] extended the Ostrowski inequality (1.1) to time scales. Their result reads as follows:

Theorem 1.7 *Let $\theta, \vartheta, \lambda, \xi \in \mathbb{T}$, $\theta < \vartheta$ and $F : [\theta, \vartheta]_{\mathbb{T}} \to \mathbb{R}$ be a delta-differentiable function. Then, for all $\lambda \in [\theta, \vartheta]_{\mathbb{T}}$, we have*

$$\left| F(\lambda) - \frac{1}{\vartheta - \theta} \int_{\theta}^{\vartheta} F^{\sigma}(\xi)\Delta\xi \right| \le \frac{M}{\vartheta - \theta} \big(h_2(\lambda, \theta) + h_2(\lambda, \vartheta) \big), \tag{1.5}$$

where $h_2(\lambda, \xi) = \int_{\xi}^{\lambda} (s - \xi)\Delta s$ and $M = \sup_{\theta < \xi < \vartheta} |F^{\Delta}(\xi)| < \infty$. Inequality (1.5) is sharp in the sense that the right-hand side can not be replaced by a smaller one.

The same authors, Bohner and Matthews, gave in [4] the time-scales version of the Grüss inequality (1.2) as follows:

Theorem 1.8 *Let $F, \phi \in C_{rd}([\theta, \vartheta]_{\mathbb{T}}, \mathbb{R})$ with*

$$m_1 \le F(\xi) \le M_1 \quad \textit{and} \quad m_2 \le \phi(\xi) \le M_2, \quad \textit{for all } \xi \in [\theta, \vartheta].$$

Then, we have

$$\begin{aligned} &\left| \frac{1}{\vartheta - \theta} \int_{\theta}^{\vartheta} F^{\sigma}(\xi)\phi^{\sigma}(\xi)\Delta\xi - \frac{1}{(\vartheta - \theta)^2} \int_{\theta}^{\vartheta} F^{\sigma}(\xi)\Delta\xi \int_{\theta}^{\vartheta} \phi^{\sigma}(\xi)\Delta\xi \right| \\ &\quad \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2). \end{aligned}$$

Recently, Saker and his research group [17] proved the following weighted Ostrowski-type inequality on an arbitrary time scale.

Theorem 1.9 *Let $a, b \in \mathbb{T}$ with $a < b$, $0 \le k \le 1$, $M = \sup_{a<t<b} |f^{\Delta}(t)| < \infty$, $w : [a, b]_{\mathbb{T}} \to [0, \infty)$ be rd-continuous and positive functions and $f, h : [a, b]_{\mathbb{T}} \to \mathbb{R}$ are Δ-differentiable* *functions such that* $h^{\Delta}(t) = w(t)$ *on* $[a,b]_{\mathbb{T}}$. *Then, for all* $x \in [a,b]_{\mathbb{T}}$, *we have*

$$\begin{aligned}
&\Bigg|\Bigg\{\gamma(1-k)^p f(x) + k^p\Bigg[\bigg(\frac{\int_a^x w(t)\Delta t}{\int_a^b w(t)\Delta t}\bigg)^p f(a) + \bigg(\frac{\int_x^b w(t)\Delta t}{\int_a^b w(t)\Delta t}\bigg)^p f(b)\Bigg]\Bigg\}\bigg(\int_a^b w(t)\Delta t\bigg)^p \\
&\quad - p\bigg[\int_a^x \big[h(t) - \big((1-k)h(a) + kh(x)\big)\big]^{p-1} w(t) f\big(\sigma(t)\big)\Delta t \\
&\quad + \int_x^b \big[h(t) - \big(kh(x) + (1-k)h(b)\big)\big]^{p-1} w(t) f\big(\sigma(t)\big)\Delta t\bigg]\Bigg| \\
&\leq M\int_a^b \big|s^p(x,t)\big|\Delta t,
\end{aligned}$$

where

$$\gamma = \begin{cases} 1, & \text{if } 0 \leq p \leq 1, \\ 2^{1-p}, & \text{if } p < 0 \text{ or } p \geq 1. \end{cases}$$

Some various generalizations and extensions of the dynamic Ostrowski inequality and its companion inequalities can be found in the papers [6, 10, 14–16, 22, 29].

The aim of the present paper is first to establish a new dynamic Ostrowski-type inequality on time scales for functions whose second delta derivatives are bounded. Then, we prove new generalized dynamic trapezoid- and Grüss-type inequalities on time scales. As special cases of our results, some continuous and discrete inequalities are obtained.

This paper is organized as follows: In Sect. 2, we briefly present the basic definitions and concepts related to the calculus on time scales. In Sect. 3, we state and prove our main results. In Sect. 4, we state the conclusion.

2 Time-scales preliminaries

First, we recall some essentials of time scales, and some symbols that will be used in the present paper. From now on, $\mathbb{R}$ and $\mathbb{Z}$ are the set of real numbers and the set of integers, respectively.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. Throughout the article, we assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ for any $\xi \in \mathbb{T}$ by

$$\sigma(\xi) := \inf\{s \in \mathbb{T} : s > \xi\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ for any $\xi \in \mathbb{T}$ by

$$\rho(\xi) := \sup\{s \in \mathbb{T} : s < \xi\}.$$

In the preceding two definitions, we set $\inf\emptyset = \sup\mathbb{T}$ (i.e., if ξ is the maximum of $\mathbb{T}$, then $\sigma(\xi) = \xi$) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., if ξ is the minimum of $\mathbb{T}$, then $\rho(\xi) = \xi$), where $\emptyset$ denotes the empty set.

A point $\xi \in \mathbb{T}$ with $\inf\mathbb{T} < \xi < \sup\mathbb{T}$ is said to be right-scattered if $\sigma(\xi) > \xi$, right-dense if $\sigma(\xi) = \xi$, left-scattered if $\rho(\xi) < \xi$, and left-dense if $\rho(\xi) = \xi$. Points that are simultaneously right-dense and left-dense are said to be dense points, whereas points that are simultaneously right-scattered and left-scattered are said to be isolated points.

The forward graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined for any $\xi \in \mathbb{T}$ by $\mu(\xi) := \sigma(\xi) - \xi$.

If $\chi : \mathbb{T} \to \mathbb{R}$ is a function, then the function $\chi^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $\chi^\sigma(\xi) = \chi(\sigma(\xi))$, $\forall \xi \in \mathbb{T}$, that is $\chi^\sigma = \chi \circ \sigma$. Similarly, the function $\chi^\rho : \mathbb{T} \to \mathbb{R}$ is defined by $\chi^\rho(\xi) = \chi(\rho(\xi))$, $\forall \xi \in \mathbb{T}$, that is $\chi^\rho = \chi \circ \rho$.

The set $\mathbb{T}^\kappa$ is introduced as follows: If $\mathbb{T}$ has a left-scattered maximum ξ_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{\xi_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

The interval $[\theta, \vartheta]$ in $\mathbb{T}$ is defined by

$$[\theta, \vartheta]_{\mathbb{T}} = \{\xi \in \mathbb{T} : \theta \leq \xi \leq \vartheta\}.$$

We define the open intervals and half-closed intervals similarly.

Assume $\chi : \mathbb{T} \to \mathbb{R}$ is a function and $\xi \in \mathbb{T}^\kappa$. Then, $\chi^\Delta(\xi) \in \mathbb{R}$ is said to be the delta derivative of χ at ξ if for any $\varepsilon > 0$ there exists a neighborhood U of ξ such that, for every $s \in U$, we have

$$\left|\left[\chi\left(\sigma(\xi)\right) - \chi(s)\right] - \chi^\Delta(\xi)\left[\sigma(\xi) - s\right]\right| \leq \varepsilon\left|\sigma(\xi) - s\right|.$$

Moreover, χ is said to be delta differentiable on $\mathbb{T}^\kappa$ if it is delta differentiable at every $\xi \in \mathbb{T}^\kappa$.

Let $\chi, \varphi : \mathbb{T} \to \mathbb{R}$ be delta-differentiable functions at $\xi \in \mathbb{T}^\kappa$. Then, we have the following:

(i) $(\chi + \varphi)^\Delta(\xi) = \chi^\Delta(\xi) + \varphi^\Delta(\xi)$;

(ii) $(\chi\varphi)^\Delta(\xi) = \chi^\Delta(\xi)\varphi(\xi) + \chi(\sigma(\xi))\varphi^\Delta(\xi) = \chi(\xi)\varphi^\Delta(\xi) + \chi^\Delta(\xi)\varphi(\sigma(\xi))$;

(iii) $(\frac{\chi}{\varphi})^\Delta(\xi) = \frac{\chi^\Delta(\xi)\varphi(\xi) - \chi(\xi)\varphi^\Delta(\xi)}{\varphi(\xi)\varphi(\sigma(\xi))}$, $\varphi(\xi)\varphi(\sigma(\xi)) \neq 0$.

The integration by parts on time scales is given by the following formula

$$\int_\theta^\vartheta \chi^\Delta(\xi)\varphi(\xi)\Delta\xi = \chi(\vartheta)\varphi(\vartheta) - \chi(\theta)\varphi(\theta) - \int_\theta^\vartheta \chi^\sigma(\xi)\varphi(\xi)\Delta\xi. \tag{2.1}$$

Assume that $\chi : \mathbb{T} \to \mathbb{T}$ is a function and $n \in \mathbb{N}$. Then, (see [7])

$$\left(\chi^n(\xi)\right)^\Delta = \left\{\sum_{k=0}^{n-1} \chi^k(\xi)\left(\chi^\sigma(\xi)\right)^{n-1-k}\right\}\chi^\Delta(\xi). \tag{2.2}$$

A function $\varphi : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) if φ is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

We say that a function $F : \mathbb{T} \to \mathbb{R}$ is a delta antiderivative of $\chi : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(\xi) = \chi(\xi)$ for all $\xi \in \mathbb{T}^\kappa$. In this case, the definite delta integral of χ is given by

$$\int_\theta^\vartheta \chi(\xi)\Delta\xi = F(\vartheta) - F(\theta) \quad \text{for all } \theta, \vartheta \in \mathbb{T}.$$

If $\varphi \in C_{rd}(\mathbb{T})$ and ξ, $\xi_0 \in \mathbb{T}$, then the definite integral $F(\xi) := \int_{\xi_0}^\xi \varphi(s)\Delta s$ exists, and $F^\Delta(\xi) = \varphi(\xi)$ holds.

Let $\theta, \vartheta, \gamma \in \mathbb{T}$, $c \in \mathbb{R}$, and χ, φ be right-dense continuous functions on $[\theta, \vartheta]_{\mathbb{T}}$. Then,

(i) $\int_\theta^\vartheta [\chi(\xi) + \varphi(\xi)]\Delta\xi = \int_\theta^\vartheta \chi(\xi)\Delta\xi + \int_\theta^\vartheta \varphi(\xi)\Delta\xi$;

(ii) $\int_\theta^\vartheta c\chi(\xi)\Delta\xi = c\int_\theta^\vartheta \chi(\xi)\Delta\xi$;
(iii) $\int_\theta^\vartheta \chi(\xi)\Delta\xi = \int_\theta^\gamma \chi(\xi)\Delta\xi + \int_\gamma^\vartheta \chi(\xi)\Delta\xi$;
(iv) $\int_\theta^\vartheta \chi(\xi)\Delta\xi = -\int_\vartheta^\theta \chi(\xi)\Delta\xi$;
(v) $\int_\theta^\theta \chi(\xi)\Delta\xi = 0$;
(vi) if $\chi(\xi) \geq \varphi(\xi)$ on $[\theta, b]_{\mathbb{T}}$, then $\int_\theta^\vartheta \chi(\xi)\Delta\xi \geq \int_\theta^\vartheta \varphi(\xi)\Delta\xi$.

We use the following crucial relations between calculus on time scales $\mathbb{T}$ and differential calculus on $\mathbb{R}$ and difference calculus on $\mathbb{Z}$. Note that:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned} &\sigma(\xi) = \xi, \qquad \mu(\xi) = 0, \qquad \chi^\Delta(\xi) = \chi'(\xi), \\ &\int_\theta^\vartheta \chi(\xi)\Delta\xi = \int_\theta^\vartheta \chi(\xi)\,d\xi. \end{aligned} \tag{2.3}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &\sigma(\xi) = \xi + 1, \qquad \mu(\xi) = 1, \qquad \chi^\Delta(\xi) = \chi(\xi+1) - \chi(\xi), \\ &\int_\theta^\vartheta \chi(\xi)\Delta\xi = \sum_{\xi=\theta}^{\vartheta-1} \chi(\xi). \end{aligned} \tag{2.4}$$

3 Main results

3.1 An Ostrowski-type inequality on time scales

Theorem 3.1 *Let $\mathbb{T}$ be a time scale with $\theta, \vartheta, \lambda, \xi \in \mathbb{T}$ and $\theta < \vartheta$. Further, assume that $\digamma : [\theta, \vartheta]_{\mathbb{T}} \to \mathbb{T}$ is a twice delta-differentiable function. Then, for all $\lambda \in [\theta, \vartheta]_{\mathbb{T}}$ and $\eta, \gamma \in \mathbb{R}$, we have*

$$\begin{aligned} &\Bigg| \digamma(\lambda) - \frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda \digamma^\sigma(\xi)\Delta\xi + \frac{\gamma}{\vartheta-\lambda}\int_\lambda^\vartheta \digamma^\sigma(\xi)\Delta\xi\right] \\ &\quad - \frac{1}{\eta+\gamma}\bigg[\int_\theta^\vartheta \int_\theta^\xi \frac{\eta}{\xi-\theta}\Upsilon(\lambda,\xi)\digamma^\Delta\big(\sigma(s)\big)\Delta s\Delta\xi \\ &\quad + \int_\theta^\vartheta \int_\xi^\vartheta \frac{\gamma}{\vartheta-\lambda}\Upsilon(\lambda,\xi)\digamma^\Delta\big(\sigma(s)\big)\Delta s\Delta\xi\bigg]\Bigg| \\ &\leq \frac{K}{(\eta+\gamma)^2}\left(\frac{\eta}{\lambda-\theta}h_2(\lambda,\theta) + \frac{\gamma}{\vartheta-\lambda}h_2(\lambda,\vartheta)\right)^2, \end{aligned} \tag{3.1}$$

where

$$\Upsilon(\lambda,\xi) = \begin{cases} \frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \theta \leq \xi < \lambda, \\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \lambda \leq \xi \leq \vartheta \end{cases}$$

and

$$K = \sup_{\theta<\xi<\vartheta} \left|\digamma^{\Delta\Delta}(\xi)\right| < \infty.$$

Proof Using integration by parts formula on time scales (2.1), we have

$$\int_\theta^\lambda \frac{\eta}{\eta+\gamma}\left(\frac{\xi-\theta}{\lambda-\theta}\right)\digamma^\Delta(\xi)\Delta\xi = \frac{\eta}{\eta+\gamma}\digamma(\lambda) - \frac{\eta}{(\eta+\gamma)(\lambda-\theta)}\int_\theta^\lambda \digamma^\sigma(\xi)\Delta\xi, \tag{3.2}$$

and

$$\int_{\lambda}^{\vartheta} \frac{-\gamma}{\eta+\gamma}\left(\frac{\vartheta-\xi}{\vartheta-\lambda}\right)F^{\Delta}(\xi)\Delta\xi = \frac{\gamma}{\eta+\gamma}F(\lambda) - \frac{\gamma}{(\eta+\gamma)(\vartheta-\lambda)}\int_{\lambda}^{\vartheta} F^{\sigma}(\xi)\Delta\xi. \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$\begin{aligned}
&\int_{\theta}^{\vartheta} \Upsilon(\lambda,\xi)F^{\Delta}(\xi)\Delta\xi \\
&\quad = F(\lambda) - \frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda} F^{\sigma}(\xi)\Delta\xi + \frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta} F^{\sigma}(\xi)\Delta\xi\right].
\end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned}
&\int_{\theta}^{\vartheta} \Upsilon(\xi,s)F^{\Delta\Delta}(s)\Delta s \\
&\quad = F^{\Delta}(\xi) - \frac{1}{\eta+\gamma}\left[\frac{\eta}{\xi-\theta}\int_{\theta}^{\xi} F^{\Delta}\big(\sigma(s)\big)\Delta s + \frac{\gamma}{\vartheta-\xi}\int_{\xi}^{\vartheta} F^{\Delta}\big(\sigma(s)\big)\Delta s\right].
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) leads to

$$\begin{aligned}
&\int_{\theta}^{\vartheta}\int_{\theta}^{\vartheta} \Upsilon(\lambda,\xi)\Upsilon(\xi,s)F^{\Delta\Delta}(s)\Delta s\Delta\xi + \frac{1}{\eta+\gamma}\left[\int_{\theta}^{\vartheta}\int_{\theta}^{\xi} \frac{\eta}{\xi-\theta}\Upsilon(\lambda,\xi)F^{\Delta}\big(\sigma(s)\big)\Delta s\Delta\xi\right. \\
&\qquad \left. + \int_{\theta}^{\vartheta}\int_{\xi}^{\vartheta} \frac{\gamma}{\vartheta-\xi}\Upsilon(\lambda,\xi)F^{\Delta}\big(\sigma(s)\big)\Delta s\Delta\xi\right] \\
&\quad = F(\lambda) - \frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda} F^{\sigma}(\xi)\Delta\xi + \frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta} F^{\sigma}(\xi)\Delta\xi\right].
\end{aligned} \tag{3.6}$$

Inequality (3.1) follows directly from (3.6), the properties of modulus and the definition of $h_2(\cdot,\cdot)$. This completes the proof. □

Corollary 3.2 *If we take* $\mathbb{T}=\mathbb{R}$ *in Theorem* 3.1, *then, with the help of relation* (2.3), *inequality* (3.1) *becomes*

$$\begin{aligned}
&\left|F(\lambda) - \frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda} F(\xi)\,d\xi + \frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta} F(\xi)\,d\xi\right]\right. \\
&\qquad \left. - \frac{1}{\eta+\gamma}\left[\int_{\theta}^{\vartheta}\int_{\theta}^{\xi} \frac{\eta}{\xi-\theta}\Upsilon(\lambda,\xi)F'(s)\,ds\,d\xi + \int_{\theta}^{\vartheta}\int_{\xi}^{\vartheta} \frac{\gamma}{\vartheta-\lambda}\Upsilon(\lambda,\xi)F'(s)\,ds\,d\xi\right]\right| \\
&\quad \leq \frac{K}{(\eta+\gamma)^2}\left(\frac{\eta(\lambda-\theta)+\gamma(\vartheta-\lambda)}{2}\right)^2,
\end{aligned}$$

where

$$\Upsilon(\lambda,\xi) = \begin{cases} \frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \theta \leq \xi < \lambda, \\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \lambda \leq \xi \leq \vartheta \end{cases}$$

and

$$K=\sup_{\theta<\xi<\vartheta}\left|F''(\xi)\right|<\infty.$$

Corollary 3.3 *If we take $\mathbb{T}=\mathbb{Z}$ in Theorem 3.1, then, with the help of relation (2.4), inequality (3.1) becomes*

$$\begin{aligned}&\Bigg|F(\lambda)-\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\sum_{\xi=\theta}^{\lambda-1}F(\xi+1)+\frac{\gamma}{\vartheta-\lambda}\sum_{\xi=\lambda}^{\vartheta-1}F(\xi+1)\right]\\&\quad-\frac{1}{\eta+\gamma}\left[\sum_{\xi=\theta}^{\vartheta-1}\sum_{s=\theta}^{\xi-1}\frac{\eta}{\xi-\theta}\Upsilon(\lambda,\xi)\Delta F(s+1)\right.\\&\quad\left.+\sum_{\xi=\theta}^{\vartheta-1}\sum_{s=\xi}^{\vartheta-1}\frac{\gamma}{\vartheta-\lambda}\Upsilon(\lambda,\xi)\Delta F(s+1)\right]\Bigg|\\&\leq\frac{K}{(\eta+\gamma)^2}\left(\frac{\eta}{\lambda-\theta}\sum_{\tau=\lambda}^{\theta-1}(\tau-\theta)+\frac{\gamma}{\vartheta-\lambda}\sum_{\tau=\lambda}^{\vartheta-1}(\tau-\vartheta)\right)^2,\end{aligned}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \xi=\theta,\dots,\lambda-1,\\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \xi=\lambda,\dots,\vartheta\end{cases}$$

and

$$K=\max_{\theta<\xi<\vartheta}\left|\Delta^2F(\xi)\right|<\infty.$$

3.2 A trapezoid-type inequality on time scales

Theorem 3.4 *Under the same assumptions as in Theorem 3.1, we have*

$$\begin{aligned}&\Bigg|F^2(\vartheta)-F^2(\theta)-\frac{1}{\eta+\gamma}\int_\theta^\vartheta\left[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\left[F^\sigma(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right.\\&\quad\left.+\frac{\gamma}{\vartheta-\lambda}\int_\lambda^\vartheta\left[F^\sigma(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right]\Delta\lambda\Bigg|\\&\leq M(M+P)\int_\theta^\vartheta\int_\theta^\vartheta\left|\Upsilon(\lambda,\xi)\right|\Delta\xi\,\Delta\lambda,\end{aligned}\tag{3.7}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \theta\leq\xi<\lambda,\\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \lambda\leq\xi\leq\vartheta\end{cases}$$

and

$$M=\sup_{\theta<\xi<\vartheta}\left|F^\Delta(\xi)\right|\quad\textit{and}\quad P=\sup_{\theta<\xi<\vartheta}\left|\left(F^\sigma\right)^\Delta(\xi)\right|.$$

Proof From (3.4) we have

$$F(\lambda)=\int_{\theta}^{\vartheta}\Upsilon(\lambda,\xi)F^{\Delta}(\xi)\Delta\xi+\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}F^{\sigma}(\xi)\Delta\xi+\frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta}F^{\sigma}(\xi)\Delta\xi\right] \tag{3.8}$$

and similarly

$$F^{\sigma}(\lambda)=\int_{\theta}^{\vartheta}\Upsilon(\lambda,\xi)\left(F^{\sigma}\right)^{\Delta}(\xi)\Delta\xi+\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}F^{\sigma^2}(\xi)\Delta\xi+\frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta}F^{\sigma^2}(\xi)\Delta\xi\right]. \tag{3.9}$$

Now, adding (3.8) and (3.9) produces

$$\begin{aligned}F(\lambda)+F^{\sigma}(\lambda)={}&\int_{\theta}^{\vartheta}\Upsilon(\lambda,\xi)\left[F^{\Delta}(\xi)+\left(F^{\sigma}\right)^{\Delta}(\xi)\right]\Delta\xi\\&+\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right.\\&\left.+\frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right].\end{aligned}$$

Multiplying the last identity by $F^{\Delta}(\lambda)$, using (2.3) and integrating the resulting identity with respect to λ from θ to ϑ yields

$$\begin{aligned}F^2(\vartheta)-F^2(\theta)={}&\int_{\theta}^{\vartheta}\int_{\theta}^{\vartheta}F^{\Delta}(\lambda)\Upsilon(\lambda,\xi)\left[F^{\Delta}(\xi)+\left(F^{\sigma}\right)^{\Delta}(\xi)\right]\Delta\xi\,\Delta\lambda\\&+\frac{1}{\eta+\gamma}\int_{\theta}^{\vartheta}F^{\Delta}(\lambda)\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right.\\&\left.+\frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right]\Delta\lambda.\end{aligned}$$

Equivalently,

$$\begin{aligned}&F^2(\vartheta)-F^2(\theta)-\frac{1}{\eta+\gamma}\int_{\theta}^{\vartheta}F^{\Delta}(\lambda)\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right.\\&\quad\left.+\frac{\gamma}{b-\lambda}\int_{\lambda}^{\vartheta}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right]\Delta\lambda\\&=\int_{\theta}^{\vartheta}\int_{\theta}^{\vartheta}F^{\Delta}(\lambda)\Upsilon(\lambda,\xi)\left[F^{\Delta}(\xi)+\left(F^{\sigma}\right)^{\Delta}(\xi)\right]\Delta\xi\,\Delta\lambda.\end{aligned}$$

Taking the absolute value on both sides, we obtain

$$\begin{aligned}&\left|F^2(\vartheta)-F^2(\theta)-\frac{1}{\eta+\gamma}\int_{\theta}^{\vartheta}F^{\Delta}(\lambda)\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right.\right.\\&\quad\left.\left.+\frac{\gamma}{\vartheta-\lambda}\int_{\lambda}^{\vartheta}\left[F^{\sigma}(\xi)+F^{\sigma^2}(\xi)\right]\Delta\xi\right]\Delta\lambda\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_\theta^\vartheta \int_\theta^\vartheta F^\Delta(\lambda)\Upsilon(\lambda,\xi)\big[F^\Delta(\xi) + \big(F^\sigma\big)^\Delta(\xi)\big]\Delta\xi\,\Delta\lambda \right| \\
&\le \int_\theta^\vartheta \int_\theta^\vartheta \big|F^\Delta(\lambda)\big|\big|\Upsilon(\lambda,\xi)\big|\big[\big|F^\Delta(\xi)\big| + \big|\big(F^\sigma\big)^\Delta(\xi)\big|\big]\Delta\xi\,\Delta\lambda \\
&\le M(M+P)\int_\theta^\vartheta \int_\theta^\vartheta \big|\Upsilon(\lambda,\xi)\big|\Delta\xi\,\Delta\lambda.
\end{aligned}
$$

This shows the validity of (3.7). □

Corollary 3.5 *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 3.4, then, with the help of relation (2.3), inequality (3.7) becomes*

$$
\begin{aligned}
&\left| \frac{F^2(\vartheta) - F^2(\theta)}{2} - \frac{1}{\eta+\gamma}\int_\theta^\vartheta F'(\lambda)\left[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda F(\xi)\,d\xi + \frac{\gamma}{\vartheta-\lambda}\int_\lambda^\vartheta F(\xi)\,d\xi\right]d\lambda \right| \\
&\le M^2 \int_\theta^\vartheta \int_\theta^\vartheta \big|\Upsilon(\lambda,\xi)\big|\,dt\,dx,
\end{aligned}
$$

where

$$
\Upsilon(\lambda,\xi) = \begin{cases} \frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \theta \le \xi < \lambda, \\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \lambda \le \xi \le \vartheta \end{cases}
$$

and

$$
M = \sup_{\theta<\xi<\vartheta}\big|F'(\xi)\big|.
$$

Corollary 3.6 *If we take $\mathbb{T} = \mathbb{Z}$ in Theorem 3.4, then, with the help of relation (2.4), inequality (3.7) becomes*

$$
\begin{aligned}
&\left| F^2(\vartheta) - F^2(\theta) - \frac{1}{\eta+\gamma}\sum_{\lambda=\theta}^{\vartheta-1}\Delta F(\lambda)\left[\frac{\eta}{\lambda-\theta}\sum_{\xi=\theta}^{\lambda-1}\big[F(\xi+1)+F(\xi+2)\big]\right.\right. \\
&\qquad \left.\left. + \frac{\gamma}{\vartheta-\lambda}\sum_{\xi=\lambda}^{\vartheta-1}\big[F(\xi+1)+F(\xi+2)\big]\right]\right| \\
&\le M(M+N)\sum_{\lambda=\theta}^{\vartheta-1}\sum_{\xi=\theta}^{\vartheta-1}\big|\Upsilon(\lambda,\xi)\big|,
\end{aligned}
$$

where

$$
\Upsilon(\lambda,\xi) = \begin{cases} \frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \xi = \theta,\dots,\lambda-1, \\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \xi = \lambda,\dots,\vartheta \end{cases}
$$

and

$$
M = \max_{\theta<\xi<\vartheta}\big|\Delta F(\xi)\big| \quad \textit{and} \quad P = \max_{\theta<\xi<\vartheta}\big|\Delta F(\xi+1)\big|.
$$

3.3 A Grüss-type inequality on time scales

Theorem 3.7 *Let $\mathbb{T}$ be a time scale with $\theta, \vartheta, \lambda, \xi \in \mathbb{T}$ and $\theta < \vartheta$. Moreover, assume that $F, \phi : [\theta, \vartheta]_{\mathbb{T}} \to \mathbb{R}$ are delta-differentiable functions. Then, for all $\lambda \in [\theta, \vartheta]_{\mathbb{T}}$ and $\eta, \gamma \in \mathbb{R}$, we have*

$$\begin{aligned} &\Bigg|2\int_\theta^\vartheta F(\lambda)\phi(\lambda)\Delta\lambda - \frac{1}{\eta+\gamma}\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\vartheta\int_\theta^\lambda \big(F^\sigma(\xi)\phi(\lambda)+\phi^\sigma(\xi)F(\lambda)\big)\Delta\xi\,\Delta\lambda \\ &\quad + \frac{\gamma}{\vartheta-\lambda}\int_\theta^\vartheta\int_\lambda^\vartheta \big(F^\sigma(\xi)\phi(\lambda)+\phi^\sigma(\xi)F(\lambda)\big)\Delta\xi\,\Delta\lambda\Bigg]\Bigg| \\ &\leq \int_\theta^\vartheta\int_\theta^\vartheta \big|\Upsilon(\lambda,\xi)\big|\big[M\big|\phi(\lambda)\big| + N\big|F(\lambda)\big|\big]\Delta\xi\,\Delta\lambda, \end{aligned} \tag{3.10}$$

where

$$\Upsilon(\lambda,\xi) = \begin{cases} \frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \theta \leq \xi < \lambda, \\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \lambda \leq \xi \leq \vartheta \end{cases}$$

and

$$M = \sup_{\theta<\xi<\vartheta}\big|F^\Delta(\xi)\big| < \infty \quad \textit{and} \quad N = \sup_{\theta<\xi<\vartheta}\big|\phi^\Delta(\xi)\big| < \infty.$$

Proof From (3.4) we have

$$\begin{aligned} F(\lambda) &= \int_\theta^\vartheta \Upsilon(\lambda,\xi)F^\Delta(\xi)\Delta\xi \\ &\quad + \frac{1}{\eta+\gamma}\bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda F^\sigma(\xi)\Delta\xi + \frac{\gamma}{\vartheta-\lambda}\int_\lambda^\vartheta F^\sigma(\xi)\Delta\xi\bigg] \end{aligned} \tag{3.11}$$

and similarly

$$\begin{aligned} \phi(\lambda) &= \int_\theta^\vartheta \Upsilon(\lambda,\xi)\phi^\Delta(\xi)\Delta\xi \\ &\quad + \frac{1}{\eta+\gamma}\bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda \phi^\sigma(\xi)\Delta\xi + \frac{\gamma}{\vartheta-\lambda}\int_\lambda^\vartheta \phi^\sigma(\xi)\Delta\xi\bigg]. \end{aligned} \tag{3.12}$$

Multiplying (3.11) by $\phi(\lambda)$ and (3.12) by $F(\lambda)$, adding them and integrating the resulting identity with respect to λ from θ to ϑ yields:

$$\begin{aligned} 2\int_\theta^\vartheta F(\lambda)\phi(\lambda)\Delta\lambda &= \int_\theta^\vartheta\int_\theta^\vartheta \Upsilon(\lambda,\xi)\big[F^\Delta(\xi)\phi(\lambda)+\phi^\Delta(\xi)F(\lambda)\big]\Delta\xi\,\Delta\lambda \\ &\quad + \frac{1}{\eta+\gamma}\bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\vartheta\int_\theta^\lambda \big(F^\sigma(\xi)\phi(\lambda)+\phi^\sigma(\xi)F(\lambda)\big)\Delta\xi\,\Delta\lambda \\ &\quad + \frac{\gamma}{\vartheta-\lambda}\int_\theta^\vartheta\int_\lambda^\vartheta \big(F^\sigma(\xi)\phi(\lambda)+\phi^\sigma(\xi)F(\lambda)\big)\Delta\xi\,\Delta\lambda\bigg]. \end{aligned}$$

By using modulus properties, we obtain

$$\begin{aligned}
&\bigg|2\int_{\theta}^{\vartheta} F(\lambda)\phi(\lambda)\Delta\lambda - \frac{1}{\eta+\gamma}\bigg[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\vartheta}\int_{\theta}^{\lambda}\big(F^{\sigma}(\xi)\phi(\lambda)+\phi^{\sigma}(\xi)F(\lambda)\big)\Delta\xi\,\Delta\lambda \\
&\quad + \frac{\gamma}{\vartheta-\lambda}\int_{\theta}^{\vartheta}\int_{\lambda}^{\vartheta}\big(F^{\sigma}(\xi)\phi(\lambda)+\phi^{\sigma}(\xi)F(\lambda)\big)\Delta\xi\,\Delta\lambda\bigg]\bigg| \\
&= \bigg|\int_{\theta}^{\vartheta}\int_{\theta}^{\vartheta}\Upsilon(\lambda,\xi)\big[F^{\Delta}(\xi)\phi(\lambda)+\phi^{\Delta}(\xi)F(\lambda)\big]\Delta\xi\,\Delta\lambda\bigg| \\
&\le \int_{\theta}^{\vartheta}\int_{\theta}^{\vartheta}\big|\Upsilon(\lambda,\xi)\big|\big[\big|F^{\Delta}(\xi)\big|\big|\phi(\lambda)\big|+\big|\phi^{\Delta}(\xi)\big|\big|F(\lambda)\big|\big]\Delta\xi\,\Delta\lambda \\
&\le \int_{\theta}^{\vartheta}\int_{\theta}^{\vartheta}\big|\Upsilon(\lambda,\xi)\big|\big[M\big|\phi(\lambda)\big|+N\big|F(\lambda)\big|\big]\Delta\xi\,\Delta\lambda.
\end{aligned}$$

This concludes the proof. □

Corollary 3.8 *If we take $\mathbb{T}=\mathbb{R}$ in Theorem 3.7, then, with the help of relation (2.3), inequality (3.10) becomes*

$$\begin{aligned}
&\bigg|2\int_{\theta}^{\vartheta} F(\lambda)\phi(\lambda)\,d\lambda - \frac{1}{\eta+\gamma}\bigg[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\vartheta}\int_{\theta}^{\lambda}\big(F(\xi)\phi(\lambda)+\phi(\xi)F(\lambda)\big)\,d\xi\,d\lambda \\
&\quad + \frac{\gamma}{\vartheta-\lambda}\int_{\theta}^{\vartheta}\int_{\lambda}^{\vartheta}\big(F(\xi)\phi(\lambda)+\phi(\xi)F(\lambda)\big)\,d\xi\,d\lambda\bigg]\bigg| \\
&\le \int_{\theta}^{\vartheta}\int_{\theta}^{\vartheta}\big|\Upsilon(\lambda,\xi)\big|\big[M\big|\phi(\lambda)\big|+N\big|F(\lambda)\big|\big]\,d\xi\,d\lambda,
\end{aligned}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \theta\le\xi<\lambda,\\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \lambda\le\xi\le\vartheta\end{cases}$$

and

$$M=\sup_{\theta<\xi<\vartheta}\big|F'(\xi)\big|<\infty \quad \textit{and} \quad N=\sup_{\theta<\xi<\vartheta}\big|\phi'(\xi)\big|<\infty.$$

Corollary 3.9 *If we take $\mathbb{T}=\mathbb{Z}$ in Theorem 3.7, then, with the help of relation (2.4), inequality (3.10) becomes*

$$\begin{aligned}
&\bigg|2\sum_{\lambda=\theta}^{\vartheta-1} F(\lambda)\phi(\lambda) - \frac{1}{\eta+\gamma}\bigg[\frac{\eta}{\lambda-\theta}\sum_{\lambda=\theta}^{\vartheta-1}\sum_{\xi=\theta}^{\lambda-1}\big(F(\xi+1)\phi(\lambda)+\phi(\xi+1)F(\lambda)\big) \\
&\quad + \frac{\gamma}{\vartheta-\lambda}\sum_{\lambda=\theta}^{\vartheta-1}\sum_{\xi=\lambda}^{\vartheta-1}\big(F(\xi+1)\phi(\lambda)+\phi(\xi+1)F(\lambda)\big)\bigg]\bigg| \\
&\le \sum_{\lambda=\theta}^{\vartheta-1}\sum_{\xi=\theta}^{\vartheta-1}\big|\Upsilon(\lambda,\xi)\big|\big[M\big|\phi(\lambda)\big|+N\big|F(\lambda)\big|\big],
\end{aligned}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\frac{\eta}{\eta+\gamma}(\frac{\xi-\theta}{\lambda-\theta}), & \xi=\theta,\dots,\lambda-1,\\ \frac{-\gamma}{\eta+\gamma}(\frac{\vartheta-\xi}{\vartheta-\lambda}), & \xi=\lambda,\dots,\vartheta\end{cases}$$

and

$$M=\max_{\theta<\xi<\vartheta}\left|\Delta\digamma(\xi)\right|<\infty \quad \textit{and} \quad N=\max_{\theta<\xi<\vartheta}\left|\Delta\phi(\xi)\right|<\infty.$$

4 Conclusion

In this manuscript we discussed some new investigations of the dynamic Ostrowski inequality and its companion inequalities on an arbitrary time scale by using two parameters. These inequalities have certain conditions that have not been studied before. For example, in Theorem 3.1, we are dealing with a function $\digamma$ whose second derivative is bounded, while all the existing literature deals with functions whose first derivatives are bounded. In addition, in order to obtain some new inequalities as special cases, we extended our inequalities to discrete and continuous calculus.

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