Yang et al. *Journal of Inequalities and Applications* (2022) 2022:77
https://doi.org/10.1186/s13660-022-02816-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Pareto Z-eigenvalue inclusion theorems for tensor eigenvalue complementarity problems

Ping Yang[1], Yiju Wang[1*], Gang Wang[2] and Qiuling Hou[1]

*Correspondence:
wyiju163@163.com
[1]School of Management Science, Qufu Normal University, 276826, Rizhao, Shandong, P.R. China
Full list of author information is available at the end of the article

Abstract

This paper presents some sharp Pareto Z-eigenvalue inclusion intervals and discusses the relationships among different Pareto Z-eigenvalue inclusion intervals for tensor eigenvalue complementarity problems. As an application, we propose a sufficient condition for identifying the strict copositivity of tensors. Some examples are provided to illustrate the obtained results.

MSC: 15A18; 15A69; 90C33

Keywords: Tensor eigenvalue complementarity problems; Pareto Z-eigenvalues; Pareto Z-eigenvalue inclusion intervals; Copositive tensors

1 Introduction

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an mth-order n-dimensional real tensor, x be a real n-vector and $N = \{1, 2, \ldots, n\}$. Denote by $\mathcal{A}x^{m-1}$ the vector in $\mathbb{R}^n$ with entries

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Consider the tensor eigenvalue complementarity problems of finding $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n_+ \backslash \{0\}$ such that

$$0 \le x \perp \left(\lambda x - \mathcal{A}x^{m-1}\right) \ge 0 \quad \text{and} \quad x^\top x = 1,$$

where $a \perp b$ means that vectors a and b are perpendicular to each other. For the problem, its solution $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n_+ \backslash \{0\}$ is called a Pareto Z-eigenpair of tensor $\mathcal{A}$.

The Pareto Z-eigenpair of a tensor was introduced by Song [1], which is a natural generalization of that of a matrix [2–5]. It is worth noting that Pareto Z-eigenvalues of $\mathcal{A}$ are closely related to Z (Z^+)-eigenvalues of $\mathcal{A}$ introduced by Lim [6] and Qi [7, 8], respectively.

Definition 1 For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, if there exist $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \backslash \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x, \quad x^\top x = 1,$$

then (λ, x) is called a Z-eigenpair of tensor $\mathcal{A}$. Further, Z-eigenvalue λ of $\mathcal{A}$ is said to be a Z^+-eigenvalue, if its eigenvector $x \in \mathbb{R}^n_+ \backslash \{0\}$.

Obviously, Z^+-eigenvalues of $\mathcal{A}$ are Pareto Z-eigenvalues. However, the converse may not hold as pointed by Zeng [9]. Therefore, the tensor Pareto Z-eigenvalue received much attentions of researchers [9–12]. For instance, Zeng [9] proposed a semidefinite relaxation algorithm to obtain Pareto Z-eigenvalues of tensor eigenvalue complementarity problems. Since it is not easy to find all Pareto Z-eigenvalues in practice [1, 9, 13], it is significant to make some characterizations to the distribution of Pareto Z-eigenvalues. Inspired by the results obtained in [14–18], we establish some Pareto Z-eigenvalues inclusion intervals, give comparisons among these Pareto Z-eigenvalue inclusion intervals, and propose a sufficient condition to identify the strict copositivity of real tensors in this paper.

The remainder of this paper is organized as follows. In Sect. 2, we recall some preliminary results and establish Pareto Z-eigenvalue inclusion intervals. Further, we give comparisons among these Pareto Z-eigenvalue inclusion intervals. In Sect. 3, we propose a sufficient condition to identify the strict copositivity of tensors.

To end this section, we give some notations needed. The set of all real numbers is denoted by $\mathbb{R}$, and the n-dimensional real Euclidean space is denoted by $\mathbb{R}^n$. For any $a \in \mathbb{R}$, we denote $[a]_+ := \max\{0, a\}$ and $[a]_- := \max\{0, -a\}$. For any $x \in \mathbb{R}^n$, $x^{\otimes m}$ denotes a tensor whose entries are defined by $(x^{\otimes m})_{i_1 i_2 \cdots i_m} = x_{i_1} x_{i_2} \cdots x_{i_m}$ for all $i_1, i_2, \ldots, i_m \in N$. For any $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $x \in \mathbb{R}^n$, we define

$$\mathcal{A}x^m := x^\top \mathcal{A}x^{m-1} = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

$$\|\mathcal{A}\|_F := \left(\sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m}^2 \right)^{\frac{1}{2}},$$

$$[\mathcal{A}]_+ := \left([a_{i_1 i_2 \cdots i_m}]_+\right) \in \mathbb{R}^{[m,n]}, \qquad [\mathcal{A}]_- := \left([a_{i_1 i_2 \cdots i_m}]_-\right) \in \mathbb{R}^{[m,n]}.$$

For any $i, j \in N$, set

$$R_i(\mathcal{A})_+ := \sum_{i_2, \ldots, i_m = 1}^{n} [a_{i i_2 \cdots i_m}]_+, \qquad R_i(\mathcal{A})_- := \sum_{i_2, \ldots, i_m = 1}^{n} [a_{i i_2 \cdots i_m}]_-,$$

$$R_i^j(\mathcal{A})_+ := R_i(\mathcal{A})_+ - [a_{ij \cdots j}]_+, \qquad R_i^j(\mathcal{A})_- := R_i(\mathcal{A})_- - [a_{ij \cdots j}]_-,$$

$$P_i^j(\mathcal{A})_+ := \sum_{\substack{i_2, \ldots, i_m \in N \\ j \notin \{i_2, \ldots, i_m\}}} [a_{i i_2 \cdots i_m}]_+, \qquad P_i^j(\mathcal{A})_- := \sum_{\substack{i_2, \ldots, i_m \in N \\ j \notin \{i_2, \ldots, i_m\}}} [a_{i i_2 \cdots i_m}]_-.$$

2 Pareto Z-eigenvalues inclusion intervals

First, we recall some results of strictly copositive tensors [19, 20], and then establish Pareto Z-eigenvalue inclusion theorems of tensor $\mathcal{A}$. Some comparisons among different Pareto Z-eigenvalue inclusion intervals are also made in this section.

Definition 2 Tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is said to be:

(i) strictly copositive if $\mathcal{A}x^m > 0$ for any $x \in \mathbb{R}^n_+ \backslash \{0\}$;

(ii) symmetric if $a_{i_1 i_2 \cdots i_m} = a_{i_{\pi(1)} \cdots i_{\pi(m)}}, \forall \pi \in \Gamma_m$, where Γ_m is the permutation group of m indices.

Lemma 1 ([1, Corollary 3.5]) *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be symmetric. Then $\mathcal{A}$ always has Pareto Z-eigenvalues; $\mathcal{A}$ is strictly copositive if and only if all of its Pareto Z-eigenvalues are positive.*

Lemma 2 ([20, Proposition 2.1]) *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. If $\mathcal{A}$ is strictly copositive, then $a_{i \cdots i} > 0, \forall i \in N$.*

Based on the above lemmas, we have the following conclusion.

Theorem 1 *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. Denote the set of Pareto Z-eigenvalues by $\sigma(\mathcal{A})$ and assume $\sigma(\mathcal{A}) \neq \emptyset$. Then,*

$$\sigma(\mathcal{A}) \subseteq \Psi(\mathcal{A}) := \left\{ \lambda \in \mathbb{R} : \max\left\{ -\bar{a} \cdot n^{\frac{m}{2}}, -\left\| [\mathcal{A}]_- \right\|_F \right\} \leq \lambda \leq \min\left\{ \bar{a} \cdot n^{\frac{m}{2}}, \left\| [\mathcal{A}]_+ \right\|_F \right\} \right\}, \quad (1)$$

where $\bar{a} = \max_{i_1,\ldots,i_m \in N} |a_{i_1 i_2 \cdots i_m}|$.

Proof Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. Then

$$\begin{aligned} \lambda \sum_{i=1}^n x_i^2 = \mathcal{A}x^m \leq [\mathcal{A}]_+ x^m &\leq \left\| [\mathcal{A}]_+ \right\|_F \left\| x^{\otimes m} \right\|_F \\ &= \left\| [\mathcal{A}]_+ \right\|_F \left(\sum_{i_1, i_2, \ldots, i_m = 1}^n x_{i_1}^2 x_{i_2}^2 \cdots x_{i_m}^2 \right)^{\frac{1}{2}} \\ &= \left\| [\mathcal{A}]_+ \right\|_F \left(\sum_{i=1}^n x_i^2 \right)^{\frac{m}{2}} = \left\| [\mathcal{A}]_+ \right\|_F \end{aligned} \quad (2)$$

and

$$\begin{aligned} -\lambda \sum_{i=1}^n x_i^2 = -\mathcal{A}x^m \leq [\mathcal{A}]_- x^m &\leq \left\| [\mathcal{A}]_- \right\|_F \left\| x^{\otimes m} \right\|_F \\ &= \left\| [\mathcal{A}]_- \right\|_F \left(\sum_{i_1, i_2, \ldots, i_m = 1}^n x_{i_1}^2 x_{i_2}^2 \cdots x_{i_m}^2 \right)^{\frac{1}{2}} \\ &= \left\| [\mathcal{A}]_- \right\|_F \left(\sum_{i=1}^n x_i^2 \right)^{\frac{m}{2}} = \left\| [\mathcal{A}]_- \right\|_F. \end{aligned} \quad (3)$$

Combining (2) with (3) yields

$$-\left\|[\mathcal{A}]_-\right\|_F \leq \lambda \leq \left\|[\mathcal{A}]_+\right\|_F. \tag{4}$$

Meanwhile, from the definition of Pareto Z-eigenpair, we obtain

$$\begin{aligned}|\lambda| &= \frac{|\mathcal{A}x^m|}{\sum_{i=1}^n x_i^2} \leq \frac{\bar{a}(\sum_{i=1}^n x_i)^m}{\sum_{i=1}^n x_i^2} \leq \bar{a}\left(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \cdot \sqrt{1+1+\cdots+1}\right)^m \\ &= \bar{a}n^{\frac{m}{2}},\end{aligned} \tag{5}$$

where the second inequality holds via Cauchy–Schwartz inequality. The desired result follows by combining (4) and (5). □

In the following, we will use some important elements of tensor to describe Pareto Z-eigenvalues inclusion intervals.

Theorem 2 *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\sigma(\mathcal{A}) \neq \emptyset$. Then,*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \bigcup_{i\in N} \Omega_i(\mathcal{A}) := \left\{\lambda \in \mathbb{R} : |\lambda| \leq \max\left\{R_i(\mathcal{A})_+, R_i(\mathcal{A})_-\right\}\right\}.$$

Proof Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. Then

$$\lambda x_i^2 = \sum_{i_2,\ldots,i_m=1}^n a_{ii_2\cdots i_m} x_i x_{i_2} \cdots x_{i_m}. \tag{6}$$

Denote $x_p = \max_{i\in N}\{x_i\}$. Then, $0 < x_p \leq 1$ as $x^\top x = 1$. Recalling the pth equation of (6), we get

$$\lambda x_p^2 = \sum_{i_2,\ldots,i_m=1}^n a_{pi_2\cdots i_m} x_p x_{i_2} \cdots x_{i_m}.$$

Taking the absolute value of the equation above, one has

$$\begin{aligned}|\lambda| x_p^2 &= \left|\sum_{i_2,\ldots,i_m=1}^n [a_{pi_2\cdots i_m}]_+ x_p x_{i_2} \cdots x_{i_m} - \sum_{i_2,\ldots,i_m=1}^n [a_{pi_2\cdots i_m}]_- x_p x_{i_2} \cdots x_{i_m}\right| \\ &\leq \max\left\{\sum_{i_2,\ldots,i_m=1}^n [a_{pi_2\cdots i_m}]_+ x_p x_{i_2} \cdots x_{i_m}, \sum_{i_2,\ldots,i_m=1}^n [a_{pi_2\cdots i_m}]_- x_p x_{i_2} \cdots x_{i_m}\right\} \\ &\leq \max\left\{\sum_{i_2,\ldots,i_m=1}^n [a_{pi_2\cdots i_m}]_+, \sum_{i_2,\ldots,i_m=1}^n [a_{pi_2\cdots i_m}]_-\right\} x_p^2.\end{aligned} \tag{7}$$

Dividing both sides by x_p^2, one has

$$|\lambda| \leq \max\left\{R_p(\mathcal{A})_+, R_p(\mathcal{A})_-\right\},$$

which implies $\lambda \in \Omega_p(\mathcal{A})$, and hence $\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A})$. □

Theorem 3 *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\sigma(\mathcal{A}) \neq \emptyset$. Then,*

$$\sigma(\mathcal{A}) \subseteq \Phi(\mathcal{A}) = \bigcup_{i\in N} \bigcap_{j\in N, i\neq j} \Phi_{i,j}(\mathcal{A}),$$

where

$$\begin{aligned}\Phi_{i,j}(\mathcal{A}) &:= \left\{\lambda \in \mathbb{R} : \left|\lambda - R_i^j(\mathcal{A})_+\right||\lambda| \leq [a_{ij\cdots j}]_+ \max\left\{R_j(\mathcal{A})_+, R_j(\mathcal{A})_-\right\}\right\} \\ &\quad \cup \left\{\lambda \in \mathbb{R} : \left|\lambda - R_i^j(\mathcal{A})_-\right||\lambda| \leq [a_{ij\cdots j}]_- \max\left\{R_j(\mathcal{A})_+, R_j(\mathcal{A})_-\right\}\right\}.\end{aligned}$$

Proof Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. Setting $0 < x_p = \max_{i\in N}\{x_i\}$ and referring to the pth equation of (6), for any $q \in N, q \neq p$, we obtain

$$\begin{aligned}|\lambda| x_p^2 &= \left|\sum_{i_2,\ldots,i_m=1}^{n} a_{pi_2\cdots i_m} x_p x_{i_2}\cdots x_{i_m}\right| \\ &= \left|\sum_{i_2,\ldots,i_m=1}^{n} [a_{pi_2\cdots i_m}]_+ x_p x_{i_2}\cdots x_{i_m} - \sum_{i_2,\ldots,i_m=1}^{n} [a_{pi_2\cdots i_m}]_- x_p x_{i_2}\cdots x_{i_m}\right| \\ &\leq \max\left\{\sum_{i_2,\ldots,i_m=1}^{n} [a_{pi_2\cdots i_m}]_+ x_p x_{i_2}\cdots x_{i_m}, \sum_{i_2,\ldots,i_m=1}^{n} [a_{pi_2\cdots i_m}]_- x_p x_{i_2}\cdots x_{i_m}\right\} \\ &\leq \max\Bigg\{[a_{pq\cdots q}]_+ x_p x_q^{m-1} + \sum_{\delta_{qi_2\ldots i_m}=0} [a_{pi_2\cdots i_m}]_+ x_p x_{i_2}\cdots x_{i_m}, \\ &\qquad [a_{pq\cdots q}]_- x_p x_q^{m-1} + \sum_{\delta_{qi_2\ldots i_m}=0} [a_{pi_2\cdots i_m}]_- x_p x_{i_2}\cdots x_{i_m}\Bigg\} \\ &\leq \max\Bigg\{[a_{pq\cdots q}]_+ x_p x_q + \sum_{\delta_{qi_2\ldots i_m}=0} [a_{pi_2\cdots i_m}]_+ x_p^2, \\ &\qquad [a_{pq\cdots q}]_- x_p x_q + \sum_{\delta_{qi_2\ldots i_m}=0} [a_{pi_2\cdots i_m}]_- x_p^2\Bigg\} \\ &= \max\left\{R_p^q(\mathcal{A})_+ x_p^2 + [a_{pq\cdots q}]_+ x_p x_q, R_p^q(\mathcal{A})_- x_p^2 + [a_{pq\cdots q}]_- x_p x_q\right\},\end{aligned}$$

which implies

$$|\lambda| x_p^2 \leq \max\left\{R_p^q(\mathcal{A})_+ x_p^2 + [a_{pq\cdots q}]_+ x_p x_q, R_p^q(\mathcal{A})_- x_p^2 + [a_{pq\cdots q}]_- x_p x_q\right\}. \tag{8}$$

Recalling the qth equation of (6), one has

$$\begin{aligned}|\lambda| x_q^2 &= \left|\sum_{i_2,\ldots,i_m=1}^{n} a_{qi_2\cdots i_m} x_q x_{i_2}\cdots x_{i_m}\right| \\ &\leq \max\left\{\sum_{i_2,\ldots,i_m=1}^{n} [a_{qi_2\cdots i_m}]_+ x_q x_{i_2}\cdots x_{i_m}, \sum_{i_2,\ldots,i_m=1}^{n} [a_{qi_2\cdots i_m}]_- x_q x_{i_2}\cdots x_{i_m}\right\} \\ &\leq \max\left\{\sum_{i_2,\ldots,i_m=1}^{n} [a_{qi_2\cdots i_m}]_+ x_q x_p, \sum_{i_2,\ldots,i_m=1}^{n} [a_{qi_2\cdots i_m}]_- x_q x_p\right\}\end{aligned}$$

$$= \max\{R_q(\mathcal{A})_+ x_p x_q, R_q(\mathcal{A})_- x_p x_q\},$$

which shows

$$|\lambda| x_q^2 \le \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\} x_p x_q. \tag{9}$$

We now break up the argument into two cases for (8).

Case I. $|\lambda| x_p^2 \le R_p^q(\mathcal{A})_+ x_p^2 + [a_{pq\cdots q}]_+ x_p x_q$. In this case, if $x_q > 0$, multiplying (8) with (9) and dividing $x_p^2 x_q^2$ yields

$$(|\lambda| - R_p^q(\mathcal{A})_+)|\lambda| \le [a_{pq\cdots q}]_+ \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\},$$

which implies $\lambda \in \Phi_{p,q}(\mathcal{A})$.

Otherwise, $x_q = 0$. From (8), it holds that

$$(|\lambda| - R_p^q(\mathcal{A})_+)|\lambda| \le 0 \le [a_{pq\cdots q}]_+ \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\},$$

which shows that $\lambda \in \Phi_{p,q}(\mathcal{A})$.

Case II. $|\lambda| x_p^2 \le R_p^q(\mathcal{A})_- x_p^2 + [a_{pq\cdots q}]_- x_p x_q$. Following similar arguments as in the proof of Case I, we obtain $\lambda \in \Phi_{p,q}(\mathcal{A})$.

Combining Cases I and II, we obtain the desired results. □

Compared with Theorem 2, the result of Theorem 3 requires relatively many calculations but has accurate results. Detailed investigation is given in Corollary 1.

Corollary 1 *For a tensor* $\mathcal{A} \in \mathbb{R}^{[m,n]}$, *it holds that*

$$\Phi(\mathcal{A}) \subseteq \Omega(\mathcal{A}),$$

where $\Phi(\mathcal{A})$ *and* $\Omega(\mathcal{A})$ *are defined in Theorems 2 and 3.*

Proof For any $\lambda \in \Phi(\mathcal{A})$, there exist $p, q \in N$ with $p \neq q$ such that

$$(|\lambda| - R_p^q(\mathcal{A})_+)|\lambda| \le [a_{pq\cdots q}]_+ \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\},$$

or

$$(|\lambda| - R_p^q(\mathcal{A})_-)|\lambda| \le [a_{pq\cdots q}]_- \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\}.$$

We now break up the argument into two cases.

Case I. $(|\lambda| - R_p^q(\mathcal{A})_+)|\lambda| \le [a_{pq\cdots q}]_+ \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\}$.

If $[a_{pq\cdots q}]_+ \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\} = 0$, it holds that

$$|\lambda| \le R_p^q(\mathcal{A})_+ \le R_p(\mathcal{A})_+ \le \max\{R_p(\mathcal{A})_+, R_p(\mathcal{A})_-\},$$

or

$$|\lambda| = 0 \le \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\},$$

which indicates that

$$\lambda \in \Omega_p(\mathcal{A}) \subseteq \Omega(\mathcal{A}) \quad \text{or} \quad \lambda \in \Omega_q(\mathcal{A}) \subseteq \Omega(\mathcal{A}). \tag{10}$$

Otherwise, $[a_{pq\cdots q}]_+ \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\} > 0$. Then,

$$\frac{|\lambda| - R_p^q(\mathcal{A})_+}{[a_{pq\cdots q}]_+} \cdot \frac{|\lambda|}{\max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\}} \le 1,$$

which implies

$$\frac{|\lambda| - R_p^q(\mathcal{A})_+}{[a_{pq\cdots q}]_+} \le 1 \quad \text{or} \quad \frac{|\lambda|}{\max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\}} \le 1.$$

Consequently, (10) holds.

Case II. $(|\lambda| - R_p^q(\mathcal{A})_-)|\lambda| \le [a_{pq\cdots q}]_- \max\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\}$. Following similar arguments as in the proof of Case I, we can prove that $\lambda \in \Omega(\mathcal{A})$.

Combining Case I with Case II, we conclude that $\Phi(\mathcal{A}) \subseteq \Omega(\mathcal{A})$. □

To get accurate results, we divide precisely the index set of $\mathcal{A}$ and establish Theorem 4.

Theorem 4 *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\sigma(\mathcal{A}) \neq \emptyset$. Then,*

$$\sigma(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) = \bigcup_{i\in N} \bigcap_{j\in N, j\neq i} \mathcal{N}_{i,j}(\mathcal{A}),$$

where $\mathcal{N}_{i,j}(\mathcal{A}) = \{\lambda \in \mathbb{R} : (|\lambda| - \max\{P_i^j(\mathcal{A})_+, P_i^j(\mathcal{A})_-\})|\lambda| \le \max\{R_i(\mathcal{A})_+ - P_i^j(\mathcal{A})_+, R_i(\mathcal{A})_- - P_i^j(\mathcal{A})_-\} \cdot \max\{R_j(\mathcal{A})_+, R_j(\mathcal{A})_-\}\}$.

Proof Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. Setting $0 < x_p = \max_{i\in N}\{x_i\}$ and referring to the pth equation of (6), for any $q \in N, q \neq p$, one has

$$\lambda x_p^2 = \sum_{\substack{i_2,\dots,i_m\in N\\ q\in\{i_2,\dots,i_m\}}} a_{pi_2\dots i_m} x_p x_{i_2}\cdots x_{i_m} + \sum_{\substack{i_2,\dots,i_m\in N\\ q\notin\{i_2,\dots,i_m\}}} a_{pi_2\dots i_m} x_p x_{i_2}\cdots x_{i_m}.$$

Taking the absolute value of the equation above, we obtain

$$\begin{aligned}
|\lambda| x_p^2 \le{}& \left| \sum_{\substack{i_2,\dots,i_m\in N\\ q\in\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_+ x_p x_{i_2}\cdots x_{i_m} - \sum_{\substack{i_2,\dots,i_m\in N\\ q\in\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_- x_p x_{i_2}\cdots x_{i_m} \right| \\
&+ \left| \sum_{\substack{i_2,\dots,i_m\in N\\ q\notin\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_+ x_p x_{i_2}\cdots x_{i_m} - \sum_{\substack{i_2,\dots,i_m\in N\\ q\notin\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_- x_p x_{i_2}\cdots x_{i_m} \right| \\
\le{}& \max\left\{ \sum_{\substack{i_2,\dots,i_m\in N\\ q\in\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_+ x_p x_{i_2}\cdots x_{i_m}, \sum_{\substack{i_2,\dots,i_m\in N\\ q\in\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_- x_p x_{i_2}\cdots x_{i_m} \right\} \\
&+ \max\left\{ \sum_{\substack{i_2,\dots,i_m\in N\\ q\notin\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_+ x_p x_{i_2}\cdots x_{i_m}, \sum_{\substack{i_2,\dots,i_m\in N\\ q\notin\{i_2,\dots,i_m\}}} [a_{pi_2\dots i_m}]_- x_p x_{i_2}\cdots x_{i_m} \right\}
\end{aligned}$$

$$\begin{aligned}&\leq x_p x_q \max\Bigg\{\sum_{\substack{i_2,\ldots,i_m\in N\\ q\in\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_+,\sum_{\substack{i_2,\ldots,i_m\in N\\ q\in\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_-\Bigg\}\\&\quad + x_p^2\max\Bigg\{\sum_{\substack{i_2,\ldots,i_m\in N\\ q\notin\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_+,\sum_{\substack{i_2,\ldots,i_m\in N\\ q\notin\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_-\Bigg\},\end{aligned}$$

where the third inequality holds from $0 < x_p^{m-1} \leq x_p \leq 1$ and $0 \leq x_q < 1$. Further,

$$\begin{aligned}&\Bigg[|\lambda| - \max\Bigg\{\sum_{\substack{i_2,\ldots,i_m\in N\\ q\notin\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_+,\sum_{\substack{i_2,\ldots,i_m\in N\\ q\notin\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_-\Bigg\}\Bigg]x_p^2\\&\quad\leq x_p x_q \max\Bigg\{\sum_{\substack{i_2,\ldots,i_m\in N\\ q\in\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_+,\sum_{\substack{i_2,\ldots,i_m\in N\\ q\in\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_-\Bigg\}.\end{aligned}\tag{11}$$

In view of the qth equation of (6), we deduce

$$\begin{aligned}|\lambda|x_q^2 &= \Bigg|\sum_{i_2,\ldots,i_m\in N} a_{qi_2\cdots i_m}x_q x_{i_2}\cdots x_{i_m}\Bigg|\\&= \Bigg|\sum_{i_2,\ldots,i_m\in N}[a_{qi_2\cdots i_m}]_+ x_q x_{i_2}\cdots x_{i_m} - \sum_{i_2,\ldots,i_m\in N}[a_{qi_2\cdots i_m}]_- x_q x_{i_2}\cdots x_{i_m}\Bigg|\\&\leq \max\Bigg\{\sum_{i_2,\ldots,i_m\in N}[a_{qi_2\cdots i_m}]_+ x_q x_{i_2}\cdots x_{i_m},\sum_{i_2,\ldots,i_m\in N}[a_{qi_2\cdots i_m}]_- x_q x_{i_2}\cdots x_{i_m}\Bigg\}\\&\leq x_p x_q\max\Bigg\{\sum_{i_2,\ldots,i_m\in N}[a_{qi_2\cdots i_m}]_+,\sum_{i_2,\ldots,i_m\in N}[a_{qi_2\cdots i_m}]_-\Bigg\}.\end{aligned}\tag{12}$$

We now break up the argument into two cases.

Case I: $x_q > 0$. Multiplying (11) with (12) and dividing $x_p^2 x_q^2$, we obtain

$$\begin{aligned}\big(|\lambda| - \max\big\{P_p^q(\mathcal{A})_+, P_p^q(\mathcal{A})_-\big\}\big)|\lambda| \leq{}& \max\big\{R_p(\mathcal{A})_+ - P_p^q(\mathcal{A})_+, R_p(\mathcal{A})_- - P_p^q(\mathcal{A})_-\big\}\\&\times\max\big\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\big\},\end{aligned}$$

which implies $\lambda \in \mathcal{N}_{p,q}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A})$.

Case II: $x_q = 0$. It follows from (11) that

$$|\lambda| \leq \max\Bigg\{\sum_{\substack{i_2,\ldots,i_m\in N\\ q\notin\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_+,\sum_{\substack{i_2,\ldots,i_m\in N\\ q\notin\{i_2,\ldots,i_m\}}}[a_{pi_2\ldots i_m}]_-\Bigg\},$$

that is,

$$\begin{aligned}\big[|\lambda| - \max\big\{P_p^q(\mathcal{A})_+, P_p^q(\mathcal{A})_-\big\}\big]|\lambda| \leq{}& \max\big\{R_p(\mathcal{A})_+ - P_p^q(\mathcal{A})_+, R_p(\mathcal{A})_- - P_p^q(\mathcal{A})_-\big\}\\&\times\max\big\{R_q(\mathcal{A})_+, R_q(\mathcal{A})_-\big\},\end{aligned}$$

which implies $\lambda \in \mathcal{N}_{p,q}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A})$. □

In what follows, we now test the efficiency of the obtained results.

Example 1 Consider a 3rd order 3-dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by

$$a_{ijk} = \begin{cases} a_{111} = 1; \quad a_{112} = -1; \quad a_{131} = 1; \quad a_{133} = 1; \\ a_{211} = -1; \quad a_{222} = 2; \quad a_{232} = 1; \\ a_{311} = 1; \quad a_{322} = 3; \quad a_{323} = 1; \\ a_{ijk} = 0, \quad \text{otherwise.} \end{cases}$$

By calculating, we have

$$\begin{aligned}
&\|[\mathcal{A}]_-\|_F = \sqrt{2}, \quad \|[\mathcal{A}]_+\|_F = \sqrt{19}, \quad \bar{a} \cdot n^{\frac{m}{2}} = 9\sqrt{3}, \\
&R_1(\mathcal{A})_+ = 3, \quad R_1(\mathcal{A})_- = 1, \quad R_2(\mathcal{A})_+ = 3, \quad R_2(\mathcal{A})_- = 1, \\
&R_3(\mathcal{A})_+ = 5, \quad R_3(\mathcal{A})_- = 0, \\
&R_1^2(\mathcal{A})_+ = 3, \quad R_1^2(\mathcal{A})_- = 1, \quad R_1^3(\mathcal{A})_+ = 2, \quad R_1^3(\mathcal{A})_- = 1, \\
&R_2^1(\mathcal{A})_+ = 3, \quad R_2^1(\mathcal{A})_- = 0, \quad R_2^3(\mathcal{A})_+ = 3, \quad R_2^3(\mathcal{A})_- = 1, \\
&R_3^1(\mathcal{A})_+ = 4, \quad R_3^1(\mathcal{A})_- = 0, \quad R_3^2(\mathcal{A})_+ = 2, \quad R_3^2(\mathcal{A})_- = 0, \\
&P_1^2(\mathcal{A})_+ = 3, \quad P_1^2(\mathcal{A})_- = 0, \quad P_1^3(\mathcal{A})_+ = 1, \quad P_1^3(\mathcal{A})_- = 1, \\
&P_2^1(\mathcal{A})_+ = 3, \quad P_2^1(\mathcal{A})_- = 0, \quad P_2^3(\mathcal{A})_+ = 2, \quad P_2^3(\mathcal{A})_- = 1, \\
&P_3^1(\mathcal{A})_+ = 4, \quad P_3^1(\mathcal{A})_- = 0, \quad P_3^2(\mathcal{A})_+ = 1, \quad P_3^2(\mathcal{A})_- = 0.
\end{aligned}$$

According to Theorem 1, we obtain

$$\Psi(\mathcal{A}) = \{\lambda \in \mathbb{R} : -\sqrt{2} \le \lambda \le \sqrt{19}\}.$$

Referring to Theorem 2, we deduce

$$\Omega(\mathcal{A}) = \bigcup_{i \in N} \Omega_i(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \le 5\}.$$

Recalling Theorem 3, one has

$$\Phi(\mathcal{A}) = \bigcup_{i \in N} \bigcap_{j \in N, i \ne j} \Phi_{i,j}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \le 1 + \sqrt{10}\},$$

where

$\Phi_{1,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 3\}$	$\Phi_{1,3}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 1 + \sqrt{6}\}$
$\Phi_{2,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 3\}$	$\Phi_{2,3}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 3\}$
$\Phi_{3,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 2 + \sqrt{7}\}$	$\Phi_{3,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 1 + \sqrt{10}\}$.

It follows from Theorem 4 that

$$\mathcal{N}(\mathcal{A}) = \bigcup_{i \in N} \bigcap_{j \in N, i \ne j} \mathcal{N}_{i,j}(\mathcal{A}) = \left\{\lambda \in \mathbb{R} : |\lambda| \le \frac{1 + \sqrt{41}}{2}\right\},$$

where

$\mathcal{N}_{1,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le \frac{3+\sqrt{21}}{2}\}$	$\mathcal{N}_{1,3}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le \frac{1+\sqrt{41}}{2}\}$
$\mathcal{N}_{2,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le \frac{3+\sqrt{21}}{2}\}$	$\mathcal{N}_{2,3}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 1 + \sqrt{6}$
$\mathcal{N}_{3,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 2 + \sqrt{7}\}$	$\mathcal{N}_{3,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \lvert\lambda\rvert \le 3\}$.

3 Judging strict copositivity of tensors

In this section, we mainly propose a sufficient condition for judging strict copositivity of $\mathcal{A}$.

Theorem 5 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be symmetric with $a_{i\cdots i} > 0$ for $i \in N$. Then $\mathcal{A}$ is strictly copositive provided that*

$$a_{i\cdots i}\left(\frac{1}{\sqrt{n}}\right)^{m-2} - R_i(\mathcal{A})_- > 0. \tag{13}$$

Proof Suppose that (λ, x) is a Pareto Z-eigenpair of $\mathcal{A}$. Setting $0 < x_p = \max_{i\in N}\{x_i\}$ and referring to the pth equation of (6), we obtain

$$\begin{aligned}\lambda x_p^2 &= \sum_{i_2,\ldots,i_m=1}^{n} a_{pi_2\cdots i_m} x_p x_{i_2}\cdots x_{i_m} \\ &= a_{p\cdots p} x_p^m + \sum_{\delta_{pi_2\cdots i_m}=0} [a_{pi_2\cdots i_m}]_+ x_p x_{i_2}\cdots x_{i_m} - \sum_{\delta_{pi_2\cdots i_m}=0} [a_{pi_2\cdots i_m}]_- x_p x_{i_2}\cdots x_{i_m}.\end{aligned}$$

Further,

$$\begin{aligned}\lambda x_p^2 &\ge a_{p\cdots p} x_p^m - \sum_{\delta_{pi_2\cdots i_m}=0} [a_{pi_2\cdots i_m}]_- x_p x_{i_2}\cdots x_{i_m} \\ &\ge a_{p\cdots p} x_p^m - \sum_{\delta_{pi_2\cdots i_m}=0} [a_{pi_2\cdots i_m}]_- x_p^2.\end{aligned}$$

Dividing both sides by x_p^2, we have

$$\lambda \ge a_{p\cdots p} x_p^{m-2} - \sum_{\delta_{pi_2\cdots i_m}=0} [a_{pi_2\cdots i_m}]_- = a_{p\cdots p} x_p^{m-2} - R_p(\mathcal{A})_-. \tag{14}$$

Since $x_p = \max_{i\in N}\{x_i\}$ and $x^\top x = 1$, we deduce $x_p \ge \frac{1}{\sqrt{n}}$. It follows from $a_{i\cdots i} > 0$ and (14) that

$$\lambda \ge a_{p\cdots p}\left(\frac{1}{\sqrt{n}}\right)^{m-2} - \sum_{\delta_{pi_2\cdots i_m}=0} [a_{pi_2\cdots i_m}]_- = a_{p\cdots p}\left(\frac{1}{\sqrt{n}}\right)^{m-2} - R_p(\mathcal{A})_-. \tag{15}$$

Combining (13) with (15), we have $\lambda > 0$ and $\mathcal{A}$ is strictly copositive. □

From the conclusion, identifying the strict copositivity of tensor $\mathcal{A}$ requires that it is symmetric. For general tensors, symmetry is a relatively strict condition. To tackle this problem, we may symmetrize the tensors $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ as follows:

$$\widetilde{a}_{i_1 i_2 \cdots i_m} = \begin{cases} a_{i_1 i_2 \cdots i_m} & \text{if } i_1 = i_2 = \cdots = i_m, \\ \frac{1}{m!} \sum_{i_2 \cdots i_m \in \Gamma_m} a_{i_1 i_2 \cdots i_m} & \text{otherwise,} \end{cases}$$

where $\widetilde{\mathcal{A}} = (\widetilde{a}_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is the symmetrization tensor under permutation group Γ_m.

The following example shows that the result given in Theorem 5 can verify the strict copositivity of tensors.

Example 2 Consider a 3rd order 2-dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by

$$a_{ijk} = \begin{cases} a_{111} = 4; & a_{112} = -1; & a_{121} = -1; & a_{122} = 0; \\ a_{222} = 2; & a_{211} = -1; & a_{212} = 0; & a_{221} = 0. \end{cases}$$

It is easy to see that $\mathcal{A}$ is symmetric with

$$R_1(\mathcal{A})_- = 2, \qquad R_2(\mathcal{A})_- = 1.$$

According to Theorem 5, we have

$$a_{111}\left(\frac{1}{\sqrt{2}}\right)^{3-2} - R_1(\mathcal{A})_- = 2(\sqrt{2} - 1) > 0,$$
$$a_{222}\left(\frac{1}{\sqrt{2}}\right)^{3-2} - R_2(\mathcal{A})_- = \sqrt{2} - 1 > 0,$$

which means that $\mathcal{A}$ is strictly copositive.

When $\mathcal{A}$ is asymmetric, we still identify the strict copositivity by Theorem 5.

Example 3 Consider a 3rd order 2-dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by

$$a_{ijk} = \begin{cases} a_{111} = 4; & a_{112} = -1; & a_{121} = -2; & a_{122} = 0; \\ a_{222} = 2; & a_{211} = -1; & a_{212} = 0; & a_{221} = 0. \end{cases}$$

Since $a_{112} = -1, a_{121} = -2$, and $a_{211} = -1$, we know that $\mathcal{A}$ is asymmetric. Therefore, we cannot directly use Theorem 5 to judge whether $\mathcal{A}$ is strictly copositive. Symmetrizing $\mathcal{A}$, we obtain $\widetilde{\mathcal{A}}$ with

$$\widetilde{a}_{ijk} = \begin{cases} \widetilde{a}_{111} = 4; & \widetilde{a}_{112} = -\frac{4}{3}; & \widetilde{a}_{121} = -\frac{4}{3}; & \widetilde{a}_{122} = 0; \\ \widetilde{a}_{222} = 2; & \widetilde{a}_{211} = -\frac{4}{3}; & \widetilde{a}_{212} = 0; & \widetilde{a}_{221} = 0. \end{cases}$$

It is easy to see that $\widetilde{\mathcal{A}}$ is symmetric with

$$R_1(\widetilde{\mathcal{A}})_- = \frac{8}{3}, \qquad R_2(\widetilde{\mathcal{A}})_- = \frac{4}{3}.$$

According to Theorem 5, we have

$$a_{111}\left(\frac{1}{\sqrt{2}}\right)^{3-2} - R_1(\widetilde{\mathcal{A}})_- = \frac{6\sqrt{2}-8}{3} > 0,$$
$$a_{222}\left(\frac{1}{\sqrt{2}}\right)^{3-2} - R_2(\widetilde{\mathcal{A}})_- = \frac{3\sqrt{2}-4}{3} > 0,$$

which implies that $\widetilde{\mathcal{A}}$ is strictly copositive. Taking into account that $\mathcal{A}x^3 = \widetilde{\mathcal{A}}x^3 > 0$, we deduce that $\mathcal{A}$ is strictly copositive.

4 Conclusion

In this paper, we proposed sharp Pareto Z-eigenvalue inclusion intervals and established comparisons among different Pareto Z-eigenvalue inclusion intervals for tensor eigenvalue complementarity problems. Meanwhile, we gave a sufficient condition to check strict copositivity of real tensors. Further studies can be considered to develop some algorithms by Pareto Z-eigenvalue inclusion intervals for tensor eigenvalue complementarity problems, as done in [5] for solving the matrix eigenvalue complementarity problems.

Acknowledgements
The authors would like to thank anonymous referees and editors for their helpful comments and suggestions, which greatly improved the quality of this paper.

Funding
This research is supported by the Natural Science Foundation of Shandong Province (ZR2020MA025, ZR2019PA016) and the National Natural Science Foundation of China (12071250, 11901343).

Availability of data and materials
Data sharing not applicable to this article as no data were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
PY: original draft writing, review writing, and editing. YJW: conceptualization, supervision, and funding acquisition. GW: computation and review writing. QLH: computation and review writing. All authors read and approved the final manuscript.

Author details
[1]School of Management Science, Qufu Normal University, 276826, Rizhao, Shandong, P.R. China. [2]Institute of Operations Research, Qufu Normal University, 276826, Rizhao, Shandong, P.R. China.

Publisher's Note

Received: 28 March 2022 Accepted: 24 May 2022 Published online: 09 June 2022

References

1. Song, Y.S., Qi, L.Q.: Eigenvalue analysis of constrained minimization problem for homogeneous polynomial. J. Glob. Optim. **64**, 563–575 (2016)
2. Seeger, A.: Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions. Linear Algebra Appl. **292**, 1–14 (1999)
3. Adly, S., Rammal, H.: A new method for solving Pareto eigenvalue complementarity problems. Comput. Optim. Appl. **55**, 703–731 (2013)
4. Judice, J., Sherali, H., Ribeiro, I.: The eigenvalue complementarity problem. Comput. Optim. Appl. **37**, 139–156 (2007)
5. Fernandes, L., Judice, J., Sherali, H., Fukushima, M.: On the computation of all eigenvalues for the eigenvalue complementarity problem. J. Glob. Optim. **59**, 307–326 (2014)

6. Lim, L.H.: Singular values and eigenvalues of tensors: a variational approach. In: CAMSAP'05: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, Puerto Vallarta, pp. 129–132 (2005)
7. Qi, L.Q.: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. **40**(6), 1302–1324 (2005)
8. Qi, L.Q., Luo, Z.Y.: Tensor Analysis: Spectral Properties and Special Tensors. SIAM, Philadelphia (2017)
9. Zeng, M.L.: Tensor *Z*-eigenvalue complementarity problems. Comput. Optim. Appl. **78**, 559–573 (2021)
10. Fan, J.Y., Nie, J.W., Zhou, A.W.: Tensor eigenvalue complementarity problems. Math. Program. **170**(2), 507–539 (2018)
11. Ling, C., He, H.J., Qi, L.Q.: On the cone eigenvalue complementarity problem for higher-order tensors. Comput. Optim. Appl. **63**, 143–168 (2016)
12. Song, Y.S., Qi, L.Q.: Spectral properties of positively homogeneous operators induced by higher order tensors. SIAM J. Matrix Anal. Appl. **34**, 1581–1595 (2013)
13. Wang, K.N., Cao, J.D., Pei, H.M.: Robust extreme learning machine in the presence of outliers by iterative reweighted algorithm. Appl. Math. Comput. **377**, 125–186 (2020)
14. Sang, C.L.: A new Brauer-type *Z*-eigenvalue inclusion set for tensors. Numer. Algorithms **32**, 781–794 (2019)
15. Wang, G., Zhou, G.L., Caccetta, L.: *Z*-eigenvalue inclusion theorems for tensors. Discrete Contin. Dyn. Syst., Ser. B **22**(1), 87–198 (2017)
16. Wang, Y.N., Wang, G.: Two *S*-type *Z*-eigenvalue inclusion sets for tensors. J. Inequal. Appl. **2017**, Article ID 152 (2017)
17. Wang, G., Wang, Y.N., Zhang, Y.: Brauer-type upper bounds for *Z*-spectral radius of weakly symmetric nonnegative tensors. J. Math. Inequal. **13**(4), 1105–1116 (2019)
18. Zhao, J.X.: A new *Z*-eigenvalue localization set for tensors. J. Inequal. Appl. **2017**, Article ID 85 (2017)
19. Che, M.L., Qi, L.Q., Wei, Y.M.: Positive definite tensors to nonlinear complementarity problems. J. Optim. Theory Appl. **168**, 475–487 (2016)
20. Song, Y.S., Qi, L.Q.: Tensor complementarity problem and semi-positive tensors. J. Optim. Theory Appl. **169**, 1069–1078 (2016)