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# Fixed points of single-valued and multi-valued mappings in sb-metric spaces

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## Abstract

In this paper, we have established some fixed point theorems in the context of strong b-metric spaces. For this purpose, Ciric type contraction for single-valued mapping and Nadler's type Banach and Chatterjea contractions for set-valued mappings are applied to obtain fixed point and common fixed points. A simple and different technique has been used to obtain the results. Our results unify, extend and generalize the existence of corresponding present and conventional results existing in the literature of fixed point theory.

**Keywords:** Strong b-metric space; Fixed point; Common fixed points; Hausdorff metric spaces; Single-valued and multivalued mappings

## 1 Introduction

To approximate the solutions of linear and nonlinear differential and integral equations, FP results provide delightful conditions in the study of mathematical analysis [25]. The theory of FP is a strange combination of geometry, topology and analysis, that's why this theory has arisen as an effective and essential tool to study nonlinear phenomena [22]. This theory [13], is an energetic part of both pure and applied mathematics. Normally FP methods have been useful in diverse sectors such as game theory, biology, engineering, nonlinear programming, economics and theory of differential equations [28, 29, 32, 33]. From previous 20–30 years, the theory of FP was a thriving region of analysis for several arithmeticians.

In 1922, Banach [9] introduced a remarkable result in metric FP theory investigated as “Banach contraction principle”. It is the influential research of modern exploration and is extensively perceived as an origin of fixed point theory in metric spaces. It not only ensures the existence but also guarantees the peculiarity of FP. In the same way this theorem provides an impressive illustration of FP in analysis.

Kannan [14], introduced Kannan contractive theorem to find FPs of mappings which are not continuous. Nadler [23] extended Banach's contraction principle and proved an FP theorem for multivalued contraction. Chatterjea [10] contraction was also followed by a number of generalizations. Some other contractions are also studied by Abuloha et al. [1], Alghamdi et al. [3], Patle et al. [26] etc.

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Due to wide applications of the Banach contraction principle [9], the study of the existence and uniqueness of FPs of a mapping and CFPs of two or more mappings has garnered a considerable attention. Many scholars put their efforts in this theory and gave new extensions of Banach and Nadler's theorems in different directions.

Many branches of computer science and mathematics, for example theory of optimization, image processing and fractals [31, 34], rely heavily on the idea of Hausdorff distance. Hausdorff metric is an essential and very important concept, it is the greatest of all the distances from a point in one set to the closest point in the other set. This includes a way for studying "fixed point theory" of set-valued mappings in the spaces which have the structure of generalized metrics.

Bakhtin [8] proposed a domain where a weaker criterion was applied rather than the triangle inequality in 1989, with the purpose of generalizing the Banach contraction principle [9], which was used by Czerwic [12] significantly. These spaces were termed b-metric spaces (b-MS). For a nonempty set  $\Omega$  and  $w \geq 1$  being a real-number, a function  $d : \Omega \times \Omega \rightarrow \mathbb{R}^+$  is called b-metric. If axioms given below are fulfilled for all  $\mu, v, \xi \in \Omega$ :

1.  $d(\mu, v) \geq 0$  and  $d(v, \mu) = 0$  iff  $\mu = v$ ;
2.  $d(\mu, v) = d(v, \mu)$ ;
3.  $d(\mu, \xi) \leq w[d(\mu, v) + d(v, \xi)]$ .

Then  $(\Omega, d)$  is called b-metric space.

A lot of work has been done in the above-mentioned spaces. Fixed point and common fixed point results for single-valued as well as multi-valued mappings have been investigated in b-metric spaces, for example see Afshari et al. [2], Ali et al. [4], Aydi et al. [5, 6], Kanwal et al. [15–17], Karapinar et al. [18–20], Ozyurt [24], Qawaqneh et al. [27] Shoaib et al. [30] and the references therein.

Kirk and Shahzad [21] proposed the concept of strong b-MSs in 2019 by leveraging the disparity in the midst of the classes of b-MSs and MSs.

Strong b-MSs have the benefit over b-MSs in that those open balls are open in the induced topology, and therefore they share a number of features with the traditional metric space. The aim of the present paper is to formulate and prove FP theorems of contractive mappings in sb-MSs. In Theorem 3.1, Ćirić [11] type contraction is applied to find a fixed point of a single-valued map and give extension in sb-MS. Theorem 3.2 and Theorem 3.3 generalize Nadler's fixed-point theorem [23] by using Banach [9] type and Chatterjea [10] type set-valued contractions in the context of complete b-MSs. The paper is organized as follows. Section 2 is devoted to recalling the basic definitions and lemmas that will be crucial throughout the paper. In Sect. 3, the existence and uniqueness theorems for single-valued mappings satisfying certain contractive condition in sb-MS are proved. In addition, two FP theorems for set-valued mappings having Nadler's type contractions are designed and proved.

## 2 Preliminaries

**Definition 2.1** A FP of a self-mapping  $G : \Omega \rightarrow \Omega$  on a nonempty set  $\Omega$  is an element  $a \in \Omega$  which is mapped onto itself, i.e.  $a$  is called a FP of  $G$  if  $G(a) = a$ .

*Example* The mapping  $F : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $F(a) = \sin a$  has 0 as a fixed point.

**Definition 2.2** A pair of self-mappings have a common FP  $F, G : \Omega \rightarrow \Omega$  is a point  $a \in \Omega$  for which

$$F(a) = G(a) = a.$$

**Definition 2.3** Let  $(\Omega, d)$  be a MS and  $CB(\Omega)$  denote the family of all non-empty bounded and closed subsets of  $\Omega$ . Consider a map  $H : CB(\Omega) \times CB(\Omega) \rightarrow R$  for  $U, V \in CB(\Omega)$ , define

$$H(U, V) = \max \left\{ \sup_{u \in U} d(u, V), \sup_{v \in V} d(v, U) \right\},$$

where  $d(u, V) = \inf\{d(u, v) : v \in V\}$  is the distance of a point  $u$  to the set  $V$ . This  $H$  is a metric on  $CB(\Omega)$ , called Hausdorff metric induced by the metric  $d$ .

**Definition 2.4** ([21]) Let  $\Omega$  be an arbitrary nonempty set and  $s \geq 1$  be a given real number. Strong b-metric on  $\Omega$  is a function  $d : \Omega \times \Omega \rightarrow R$  satisfying the following axioms for all  $\eta_1, \eta_2, \eta_3 \in \Omega$ :

$$(sbM1) d(\eta_1, \eta_2) \geq 0;$$

$$(sbM2) d(\eta_1, \eta_2) = 0 \Leftrightarrow \eta_1 = \eta_2;$$

$$(sbM3) d(\eta_1, \eta_2) = d(\eta_2, \eta_1);$$

$$(sbM4) d(\eta_1, \eta_2) \leq d(\eta_1, \eta_3) + sd(\eta_3, \eta_2).$$

The triplet  $(\Omega, d, s)$  is known as strong b- MS.

**Definition 2.5** Let  $(\Omega, d, s)$  be an sb-MS. Suppose that  $\{a_n\}$  is a sequence in  $\Omega$  and  $a \in \Omega$ , then

- i.  $\{a_n\}$  will converge to  $a$  if  $\lim_{n \rightarrow \infty} d(a_n, a) = 0$ .
- ii.  $\{a_n\}$  in  $\Omega$  is known as Cauchy if for every  $\varepsilon > 0$  there exists a natural number  $N = N(\varepsilon)$  such that  $d(a_n, a_m) < \varepsilon$  for every  $m, n > N$ .
- iii.  $\Omega$  is called complete if every Cauchy sequence in  $\Omega$  is convergent in  $\Omega$ .

**Definition 2.6** ([9]) Let  $\Omega = (\Omega, d)$  be an MS. A mapping  $G : \Omega \rightarrow \Omega$  is known as a Banach contraction on  $G$  if there is a positive real number  $0 < \alpha < 1$  such that  $\forall a, b \in \Omega$ ,

$$d(Ga, Gb) \leq \alpha d(a, b).$$

**Definition 2.7** ([10]) Let  $(\Omega, d)$  be an MS and  $G : \Omega \rightarrow \Omega$  be a mapping if there exists  $\alpha \in (0, \frac{1}{2})$  such that, for all  $a_1, a_2 \in \Omega$ , we have

$$d(Ga_1, Ga_2) \leq \alpha \{d(a_1, Ga_2) + d(a_2, Ga_1)\}.$$

Then  $G$  is known as Chatterjee contraction.

**Definition 2.8** ([7]) Consider a multivalued mapping  $G: \Omega \rightarrow CB(\Omega)$  on a nonempty set  $\Omega$ ,  $CB(\Omega)$  be the family of all nonempty closed and bounded subsets of  $\Omega$ . A point  $y \in \Omega$  is called FP of  $G$  if  $y \in Gy$ .

**Lemma 2.1** ([7]) Let  $(\Omega, d)$  be a  $b$ -MS,  $CB(\Omega)$  be the family of all nonempty closed and bounded subsets of  $\Omega$ . Then, for  $U, V \in CB(\Omega)$ ,

- (1)  $d(a, U) \leq H(U, V), a \in U$ ;
- (2) For  $\varepsilon > 0$  and  $a \in U, \exists b \in V$  such that

$$d(a, b) \leq H(U, V) + \varepsilon.$$

### 3 Main result

**Theorem 3.1** Consider a complete  $sb$ -MS  $(\Omega, d, s)$  with  $s \geq 1$ . Let  $G: \Omega \rightarrow \Omega$  be a single-valued mapping such that

$$\begin{aligned} d(Ga, Gb) &\leq \varpi_1 d(a, b) + \varpi_2 d(a, Ga) + \varpi_3 d(b, Gb) + \varpi_4 [d(b, Ga) + d(a, Gb)], \\ \text{where } \varpi_1 + (1+s)\varpi_2 + \varpi_3 + (1+s)\varpi_4 &< 1 \end{aligned} \quad (3.1)$$

$\forall a, b \in \Omega$ . Then there exists  $a^* \in \Omega$  such that  $a_n \rightarrow a^*$  and  $a^*$  is the unique FP.

*Proof* Let  $a_0 \in \Omega$  and  $\{a_n\}$  be a sequence in  $\Omega$  defined as

$$a_n = Ga_{n-1} = G^n a_0, \quad n = 1, 2, 3, \dots \quad (3.2)$$

Now

$$\begin{aligned} d(a_n, a_{n+1}) &= d(Ga_{n-1}, Ga_n), \\ d(a_n, a_{n+1}) &\leq \varpi_1 d(a_{n-1}, a_n) + \varpi_2 d(a_{n-1}, a_n) + \varpi_3 d(a_n, a_{n+1}) \\ &\quad + \varpi_4 [d(a_n, a_n) + d(a_{n-1}, a_{n+1})]. \end{aligned}$$

Using the triangular inequality of  $(sbM4)$ , we get

$$\begin{aligned} &\leq \varpi_1 d(a_{n-1}, a_n) + \varpi_2 d(a_{n-1}, a_n) + \varpi_3 d(a_n, a_{n+1}) + \varpi_4 [d(a_{n-1}, a_n) + sd(a_n, a_{n+1})] \\ &\leq \varpi_1 d(a_{n-1}, a_n) + \varpi_2 d(a_{n-1}, a_n) + \varpi_3 d(a_n, a_{n+1}) + \varpi_4 d(a_{n-1}, a_n) + s\varpi_4 d(a_n, a_{n+1}), \\ (1 - \varpi_3 - s\varpi_4)d(a_n, a_{n+1}) &\leq (\varpi_1 + \varpi_2 + \varpi_4)d(a_{n-1}, a_n), \\ \Rightarrow d(a_n, a_{n+1}) &\leq \left( \frac{\varpi_1 + \varpi_2 + \varpi_4}{1 - s\varpi_3 - \varpi_3 - s\varpi_4} \right) d(a_{n-1}, a_n) \\ &\leq kd(a_{n-1}, a_n), \end{aligned}$$

where  $k = \frac{\varpi_1 + \varpi_2 + \varpi_4}{1 - s\varpi_3 - \varpi_3 - s\varpi_4}$ .

$$\begin{aligned} d(a_n, a_{n+1}) &\leq kd(a_{n-1}, a_n) \\ &\leq k^2 d(a_{n-2}, a_{n-1}). \end{aligned}$$

Continuing this process, we get

$$\leq k^n d(a_o, a_1). \quad (3.3)$$

Now we will show that  $\{a_n\}$  is a Cauchy sequence in  $\Omega$ .

Consider  $m, n \in \mathbb{N}$  with  $m > n$ :

$$d(a_n, a_m) \leq d(a_n, a_{n+1}) + sd(a_{n+1}, a_{n+2}) + s^2 d(a_{n+2}, a_{n+3}) + \cdots.$$

Using (3.3) we can write

$$\begin{aligned} &\leq k^n d(a_o, a_1) + sk^{n+1} d(a_o, a_1) + s^2 k^{n+2} d(a_o, a_1) + \cdots + s^{m-1} k^{n+m-1} d(a_o, a_1) \\ &\leq k^n d(a_o, a_1) [1 + sk + (sk)^2 + \cdots + (sk)^{m-1}] \\ &\leq k^n d(a_o, a_1) \left[ \frac{1 - (sk)^m}{1 - sk} \right]. \end{aligned}$$

When  $m, n \rightarrow \infty$ ,  $d(a_n, a_m) \rightarrow 0$ .

Hence  $\{a_n\}$  is a Cauchy sequence in  $\Omega$ . Since  $\Omega$  is complete,  $\{a_n\}$  converges to an element of  $\Omega$ , say  $a^*$ ,  $a^* \in \Omega$ .

Now,

$$\begin{aligned} d(a^*, Ga^*) &\leq d(a^*, a_{n+1}) + sd(a_{n+1}, Ga^*) \\ &\leq d(a^*, a_{n+1}) + sd(Ga_n, Ga^*) \\ &\leq d(a^*, a_{n+1}) + s\varpi_1 d(a_n, a^*) + s\varpi_2 d(a_n, Ga_n) \\ &\quad + s\varpi_3 d(a^*, Ga^*) + s\varpi_4 d(a^*, Ga_n) + s\varpi_4 d(a_n, Ga^*) \\ &\leq d(a^*, a_{n+1}) + s\varpi_1 d(a_n, a^*) + s\varpi_2 d(a_n, Ga_n) + s\varpi_3 d(a^*, Ga^*) \\ &\quad + s\varpi_4 d(a^*, a_{n+1}) + s\varpi_4 d(a_n, a^*) + s^2 \varpi_4 d(a^*, Ga^*) \\ \Rightarrow (1 - s\varpi_3 - s^2 \varpi_4) d(a^*, Ga^*) &\leq (1 + s\varpi_4) d(a^*, a_{n+1}) + (s\varpi_1 + s\varpi_4) d(a_n, a^*) + s\varpi_2 d(a_n, a_{n+1}) \\ \Rightarrow d(a^*, Ga^*) &\leq \left[ \frac{1 + s\varpi_4}{1 - s\varpi_3 - s^2 \varpi_4} \right] d(a^*, a_{n+1}) + \left[ \frac{s\varpi_1 + s\varpi_4}{1 - s\varpi_3 - s^2 \varpi_4} \right] d(a_n, a^*) \\ &\quad + \left[ \frac{s\varpi_2}{1 - s\varpi_3 - s^2 \varpi_4} \right] d(a_n, a_{n+1}) \\ \Rightarrow d(a^*, Ga^*) &\leq 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

$\Rightarrow a^* = Ga^*$ . Hence  $a^*$  is an FP of  $G$ .

*Uniqueness.* Assume that  $a^\circ$  is another FP of  $G$ . Then we have

$$Ga^\circ = a^\circ.$$

Consider

$$\begin{aligned}
 d(a^*, a^\circ) &= d(Ga^*, Ga^\circ) \\
 &\leq \varpi_1 d(a^*, a^\circ) + \varpi_2 d(a^*, Ga^*) + \varpi_3 d(a^\circ, Ga^\circ) + \varpi_4 [d(a^\circ, Ga^*) + d(a^*, Ga^\circ)] \\
 &\leq \varpi_1 d(a^*, a^\circ) + \varpi_2 d(a^*, a^*) + \varpi_3 d(a^\circ, a^\circ) + \varpi_4 [d(a^\circ, a^*) + d(a^*, a^\circ)] \\
 &\leq \varpi_1 d(a^*, a^\circ) + \varpi_4 d(a^\circ, a^*) + \varpi_4 d(a^*, a^\circ), \\
 d(a^\circ, a^*) &\leq (\varpi_1 + 2\varpi_4) d(a^\circ, a^*).
 \end{aligned}$$

It is a contradiction, hence  $a^\circ = a^*$ .  $\square$

**Corollary 1** Let  $(\Omega, d)$  be a complete MS and  $G: \Omega \rightarrow \Omega$  be a mapping such that

$$\begin{aligned}
 d(Ga, Gb) &\leq \varpi_1 d(a, b) + \varpi_2 d(a, Ga) + \varpi_3 d(b, Gb) + \varpi_4 [d(b, Ga) + d(a, Gb)], \\
 &\text{where } \varpi_1 + 2\varpi_2 + \varpi_3 + 2\varpi_4 < 1
 \end{aligned}$$

for all  $a, b \in \Omega$ . Then there exists  $a^* \in \Omega$  such that  $a_n \rightarrow a^*$  and  $a^*$  is the unique FP of  $G$ .

**Theorem 3.2** Let  $(\Omega, d, s)$  be a complete sb-MS with constant  $s \geq 1$ . Let  $F, G: \Omega \rightarrow \Omega$  be two maps for which  $\eta_1, \eta_2 \in [0, \frac{1}{3})$  such that

$$d(Fx, Gy) \leq \eta_1 d(x, y) + \eta_2 [d(x, Fx) + d(y, Gx)].$$

Then there exists a CFP of  $F$  and  $G$ .

*Proof* Let  $a_o \in \Omega$ . Consider the sequence  $\{a_n\}$  so that  $a_{2n+2} = Ga_{2n+1}$ ,  $a_{2n+1} = Fa_{2n}$ . Then

$$\begin{aligned}
 d(a_{2n+2}, a_{2n+1}) &= d(Ga_{2n+1}, Fa_{2n}) \\
 &\leq \eta_1 d(a_{2n+1}, a_{2n}) + \eta_2 [d(a_{2n+1}, Ga_{2n+1}) + d(a_{2n}, Fa_{2n})] \\
 &\leq \eta_1 d(a_{2n+1}, a_{2n}) + \eta_2 d(a_{2n+1}, a_{2n+2}) + \eta_2 d(a_{2n}, a_{2n+1}), \\
 (1 - \eta_2) d(a_{2n+1}, a_{2n+2}) &\leq (\eta_1 + \eta_2) \eta_2 d(a_{2n}, a_{2n+1}), \\
 d(a_{2n+1}, a_{2n+1}) &\leq \left[ \frac{\eta_1 + \eta_2}{1 - \eta_2} \right] d(a_{2n+1}, a_{2n}) \leq kd(a_{2n+1}, a_{2n}),
 \end{aligned}$$

where  $k = \left[ \frac{\eta_1 + \eta_2}{1 - \eta_2} \right]$ . As  $\eta_1, \eta_2 \in [0, \frac{1}{3}]$ , So  $\eta_1 + 2\eta_2 < 1$

$$\Rightarrow \eta_1 + \eta_2 < 1 - \eta_2.$$

This implies that  $\frac{\eta_1 + \eta_2}{1 - \eta_2} < 1$ , i.e.  $k < 1$ .

So,

$$\begin{aligned}
 d(a_{2n+2}, a_{2n+1}) &\leq kd(a_{2n+1}, a_{2n}) \\
 &\leq k^2 d(a_{2n}, a_{2n-1}).
 \end{aligned}$$

Continuing this process, we obtain

$$d(a_{2n+2}, a_{2n+1}) \leq k^n d(a_0, a_1).$$

In general,

$$d(a_n, a_{n+1}) \leq k^n d(a_0, a_1).$$

Now, let  $m, n \in \mathbb{N}$  with  $m > n$

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, a_{n+1}) + s d(a_{n+1}, a_{n+2}) + s^2 d(a_{n+2}, a_{n+3}) + \cdots + s^{m-1} d(a_{m-1}, a_m) \\ &\leq k^n d(a_1, a_0) + s k^{n+2} d(a_1, a_0) + s^2 k^{n+2} d(a_1, a_0) + \cdots + s^{m-1} k^{n+m-1} d(a_1, a_0) \\ &\leq k^n d(a_1, a_0) [1 + sk + (sk)^2 + \cdots + (sk)^{m-1}], \\ d(a_n, a_m) &\leq k^n d(a_1, a_0) \left[ \frac{1 - (sk)^m}{1 - sk} \right]. \end{aligned}$$

When  $m, n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} d(a_n, a_m) = 0$ .

Hence  $\{a_n\}_{n=1}^\infty$  is a Cauchy sequence in  $\Omega$ . Since  $\Omega$  is complete,  $\{a_n\}$  converges to  $b \in \Omega$ .

Now,

$$\begin{aligned} d(b, Gb) &\leq d(b, Fa_{2n}) + s d(Fa_{2n}, Gb) \\ &\leq d(b, a_{2n+1}) + s [\eta_1 d(a_{2n}, b) + \eta_2 d(b, Gb) + \eta_2 d(a_{2n}, Fa_{2n})], \\ d(b, Gb) &\leq \left( \frac{1}{1 - s\eta_2} \right) d(b, a_{2n+1}) + \left( \frac{s\eta_1}{1 - s\eta_2} \right) d(a_{2n}, b) + \left( \frac{s\eta_2}{1 - s\eta_2} \right) d(a_{2n}, a_{2n+1}). \end{aligned}$$

When  $n \rightarrow \infty$ ,  $d(b, Gb) \leq 0$

$$\Rightarrow b = Gb.$$

Now

$$\begin{aligned} d(b, Fb) &\leq d(b, Ga_{2n+1}) + s d(Ga_{2n}, Fb), \\ (1 - s\eta_2) d(b, Fb) &\leq d(b, a_{2n+2}) + s\eta_1 d(a_{2n+1}, b) + s\eta_2 d(a_{2n+1}, a_{2n+2}), \\ d(b, Fb) &\leq \frac{1}{1 - s\eta_2} [d(b, a_{2n+2}) + s\eta_1 d(a_{2n+1}, b) + s\eta_2 d(a_{2n+1}, a_{2n+2})]. \end{aligned}$$

When  $n \rightarrow \infty$ ,  $d(b, Fb) \rightarrow 0$ ,  $\Rightarrow b = Fb$ . Thus  $Gb = Fb = b$ . Hence  $b$  is a common fixed point of  $G$  and  $F$ .  $\square$

**Corollary** Let  $(\Omega, d)$  be a complete MS. Let  $F, G : \Omega \rightarrow \Omega$  be two maps for which  $\eta_1, \eta_2 \in [0, \frac{1}{3})$  such that

$$d(Fa, Gb) \leq \eta_1 d(a, b) + \eta_2 [d(a, Fa) + d(b, Ga)], \quad \forall a, b \in \Omega.$$

Then there exists a CFP of  $F$  and  $G$ .

**Theorem 3.3** Let  $(\Omega, d, s)$  be a complete sb-MS and  $F : \Omega \rightarrow CB(\Omega)$  be a set-valued mapping with contraction

$$H(Fa, Fb) \leq \alpha [d(a, Fb) + d(b, Fa)] \quad (3.4)$$

for all  $a, b \in \Omega$  and  $\alpha \in (0, \frac{1}{2s})$  ( $s \geq 1$ ). Then there exists an FP of  $F$  (i.e.  $\exists u \in \Omega$  such that  $u \in Fu$ ), where  $CB(\Omega)$  is the set of all closed and bounded subsets of  $\Omega$ .

*Proof* Consider a sequence  $\{a_n : n \in \mathbb{N}\}$  such that  $a_{n+1} \in Fa_n$ . Then, by Lemma 2.1, for  $a_1 \in Fa_0$ , there exists  $a_2 \in Fa_1$  such that we have

$$\begin{aligned} d(a_1, a_2) &\leq H(Fa_0, Fa_1) + \alpha \\ &\leq \alpha [d(a_0, Fa_1) + d(a_1, Fa_0)] + \alpha \\ &\leq \alpha [d(a_0, a_2) + d(a_1, a_1)] + \alpha, \\ d(a_1, a_2) &\leq \alpha d(a_0, a_2) + \alpha. \end{aligned}$$

Using (sbM4),

$$\begin{aligned} d(a_1, a_2) &\leq \alpha d(a_0, a_1) + \alpha s d(a_1, a_2) + \alpha, \\ (1 - \alpha s) d(a_1, a_2) &\leq \alpha d(a_0, a_1) + \alpha, \\ d(a_1, a_2) &\leq \frac{\alpha}{(1 - \alpha s)} d(a_0, a_1) + \frac{\alpha}{(1 - \alpha s)}, \\ d(a_1, a_2) &\leq \beta d(a_0, a_1) + \beta, \end{aligned} \quad (3.5)$$

where  $\beta = \frac{\alpha}{1 - \alpha s}$  as  $\alpha \in (0, \frac{1}{2s})$ , then  $\beta \in (0, \frac{1}{s})$ .

Now, again by Lemma 2.1,

$$\begin{aligned} d(a_2, a_3) &\leq H(Fa_2, Fa_1) + \alpha \beta \\ &\leq \alpha [d(a_1, Fa_2) + d(a_2, Fa_1)] + \alpha \beta \quad (\text{by given contraction}) \\ &\leq \alpha [d(a_1, a_3) + d(a_2, a_2)] + \alpha \beta \\ &\leq \alpha d(a_1, a_3) + \alpha \beta. \end{aligned}$$

Using (sbM4), we have

$$\begin{aligned} d(a_2, a_3) &\leq \alpha [d(a_1, a_2) + s d(a_2, a_3)] + \alpha \beta \\ &\leq \alpha d(a_1, a_2) + \alpha s d(a_2, a_3) + \alpha \beta, \\ (1 - \alpha s) d(a_2, a_3) &\leq \alpha d(a_1, a_2) + \alpha \beta \\ \Rightarrow d(a_2, a_3) &\leq \frac{\alpha}{(1 - \alpha s)} d(a_1, a_2) + \frac{\alpha \beta}{(1 - \alpha s)} \\ &\leq \beta d(a_1, a_2) + \beta^2. \end{aligned}$$



Using (3.5), we can write

$$\begin{aligned} d(a_2, a_3) &\leq \beta[\beta d(a_0, a_1) + \beta] + \beta^2 \\ &= \beta^2 d(a_0, a_1) + 2\beta^2 \\ \Rightarrow d(a_2, a_3) &\leq \beta^2 d(a_0, a_1) + 2\beta^2. \end{aligned}$$

In general,

$$d(a_n, a_{n+1}) \leq \beta^n d(a_0, a_1) + n\beta^n. \quad (3.6)$$

Now we show that  $\{a_n\}_{n=1}^\infty$  is a Cauchy sequence in  $\Omega$ . For this, let  $m, n \in \mathbb{N}$  with  $m > n$

$$d(a_n, a_m) \leq d(a_n, a_{n+1}) + s d(a_{n+1}, a_{n+2}) + s^2 d(a_{n+2}, a_{n+3}) + \cdots + s^{m-n-1} d(a_{m-1}, a_m).$$

Using (3.6), we get

$$\begin{aligned} &\leq \beta^n d(a_0, a_1) + n\beta^n + s\beta^{n+1} d(a_0, a_1) + s(n+1)\beta^{n+1} + s^2\beta^{n+2} d(a_0, a_1) \\ &\quad + s^3(n+2)\beta^{n+3} + \cdots + s^{m-n-1}\beta^{m-1} d(a_0, a_1) + s^{m-n-1}(m-1)\beta^{m-1} \\ &\leq \beta^n d(a_0, a_1) \left[ \frac{1 - (s\beta)^{m-n-1}}{1 - s\beta} \right] + \sum_{i=n}^{m-1} i s^{i-n} \beta^i. \end{aligned}$$

When we take  $m, n \rightarrow \infty$ ,  $\Rightarrow d(a_n, a_m) = 0$ .

Accordingly,  $\{a_n\}$  is a Cauchy grouping in  $\Omega$ . Since  $\Omega$  is finished, so there exists  $u \in \Omega$  to such an extent that  $a_n \rightarrow u$ .

Now,

$$\begin{aligned} d(u, Fu) &\leq d(u, a_n) + s d(a_n, Fu) \\ &\leq d(u, a_n) + s H(Fa_{n-1}, Fu) \\ &\leq d(u, a_n) + s\alpha [d(a_{n-1}, Fu) + d(u, Fa_{n-1})]. \end{aligned}$$

As  $n \rightarrow \infty$ ,  $d(u, Fu) \leq d(u, u) + s[\alpha d(u, Fu) + d(u, Fu)]$

$$\Rightarrow d(u, Fu) \leq 0.$$

The only possibility is  $d(u, Fu) = 0 \Rightarrow u \in Fu$ .

Thus,  $u$  is the FP of  $F$ . □

**Corollary** Let  $(\Omega, d)$  be a complete MS and  $F : \Omega \rightarrow CB(\Omega)$  be a set-valued mapping with contraction

$$H(Fa, Fb) \leq \alpha [d(a, Fb) + d(b, Fa)] \quad \text{for all } a, b \in \Omega \text{ and } \alpha \in \left(0, \frac{1}{2}\right).$$

Then there exists an FP of  $F$  (i.e., there is  $u \in \Omega$  such that  $u \in Fu$ ).

**Theorem 3.4** Consider a complete strong  $b$ -MS  $(\Omega, d)$  and  $G : \Omega \rightarrow CB(\Omega)$  be a multivalued map defined as

$$H(Ga, Gb) \leq \alpha d(a, b), \quad \forall a, b \in \Omega \text{ and } \alpha \in [0, 1), s \geq 1.$$

Then there exists  $b \in \Omega$  such that  $b \in Gb$ .

*Proof* Let  $a_0 \in \Omega$ ,  $Ga_0 \neq \emptyset$  be closed and bounded subsets of  $\Omega$ . Furthermore, let  $a_1 \in Ga_0$ ,  $Ga_1 \neq \emptyset$  be closed and bounded subsets of  $\Omega$ . By Lemma 2.1, there exists  $a_2 \in Ga_1$  such that

$$d(a_1, a_2) \leq H(Ga_0, Ga_1) + \alpha. \quad (3.7)$$

Now,  $Ga_2 \neq \emptyset$  closed and bounded subsets of  $\Omega$ , there exists  $a_3 \in Ga_2$  such that

$$d(a_2, a_3) \leq H(Ga_1, Ga_2) + \alpha^2. \quad (3.8)$$

By a given contraction condition,

$$\begin{aligned} d(a_2, a_3) &\leq \alpha d(a_1, a_2) + \alpha^2, \\ d(a_3, a_4) &\leq H(Ga_2, Ga_3) + \alpha^3 \\ &\leq \alpha d(a_2, a_3) + \alpha^3. \end{aligned}$$

Using (3.8), we have

$$\begin{aligned} d(a_3, a_4) &\leq \alpha [\alpha d(a_1, a_2) + \alpha^2] + \alpha^3 \\ &\leq \alpha^2 d(a_1, a_2) + 2\alpha^3 \\ &\leq \alpha^2 [H(Ga_0, Ga_1) + \alpha] + 2\alpha^3 \\ &\leq \alpha^2 [\alpha d(a_0, a_1) + \alpha] + 2\alpha^3 \\ &\leq \alpha^3 d(a_0, a_1) + \alpha^3 + 2\alpha^3 \\ &\leq \alpha^3 d(a_0, a_1) + 3\alpha^3. \end{aligned}$$

In general,

$$d(a_n, a_{n+1}) \leq \alpha^n d(a_0, a_1) + n\alpha^n.$$

For convenience, we set

$d(a_n, a_{n+1}) = d_n$ , so the above result can be written as

$$d_n \leq \alpha^n d_0 + n\alpha^n. \quad (3.9)$$

For  $m, n \in \mathbb{N}$ ,  $m \geq n$ , we have

$$d(a_n, a_m) \leq d(a_n, a_{n+1}) + s d(a_{n+1}, a_{n+2}) + s^2 d(a_{n+2}, a_{n+3}) + \cdots + s^{m-n-1} d(a_{m-1}, a_m).$$

Using (3.9), we get

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, a_{n+1}) + sd(a_{n+1}, a_{n+2}) + s^2 d(a_{n+2}, a_{n+3}) \\ &\quad + \cdots + s^{m-n-1} \alpha^{m-1} d(a_{m-1}, a_m) + s^{m-n-1} (m-1) \alpha^{m-1}. \end{aligned}$$

Using (3.9), we get

$$\begin{aligned} &\leq \alpha^n d_0 + s \alpha^{n+1} d_0 + s^2 \alpha^{n+2} d_0 + \cdots + s^{m-n-1} \alpha^{m-1} d_0 \\ &\quad + n \alpha^n + s(n+1) \alpha^{n+1} + s^2(n+2) \alpha^{n+2} + \cdots + s^{m-n-1} (m-1) \alpha^{m-1} \\ &\leq \alpha^n d_0 (1 + \alpha s + (\alpha s)^2 + (\alpha s)^3 + \cdots + s^{m-n-1} \alpha^{m-n-1}) + \sum_{i=n}^{m-1} i s^{i-n} \alpha^i, \\ d(a_n, a_m) &\leq \alpha^n d_0 \left[ \frac{1 + (s\alpha)^{m-n-1}}{1 - s\alpha} \right] + \sum_{i=n}^{m-1} i s^{i-n} \alpha^i. \end{aligned}$$

In the limiting case when  $m, n \rightarrow \infty$ ,

$$d(a_n, a_m) = 0$$

$\Rightarrow \{a_n\}$  is a Cauchy sequence in  $\Omega$ , the completeness of  $\Omega$  implies that there exists  $b \in \Omega$  such that

$$x_n \rightarrow b.$$

Now we will prove that  $b$  is a fixed point of  $G$ .

$$d(b, Gb) \leq d(b, a_n) + sd(a_n, Gb).$$

By Lemma 2.1,

$$\begin{aligned} &\leq d(b, a_n) + sH(Ga_{n-1}, Gb) \\ &\leq d(b, a_n) + s\alpha d(a_{n-1}, b). \end{aligned}$$

In the limiting case when  $n \rightarrow \infty$ ,  $d(b, Gb) \leq 0$ .

This implies that  $b \in Gb$ . Hence  $b$  is an FP of  $G$ .  $\square$

**Corollary** Let  $(\Omega, d)$  be a complete metric space and  $G : \Omega \rightarrow CB(\Omega)$  be a multivalued map such that

$$H(Ga, Gb) \leq \alpha d(a, b), \quad \forall a, b \in \Omega \text{ and } \alpha \in [0, 1).$$

Then there exists  $y \in \Omega$  such that  $y \in Gy$ .

## 4 Conclusion

Fixed point techniques are extremely helpful and appealing tools. Functional inclusions, optimization theory, fractal graphics, discrete dynamics for set-valued operators and other fields of nonlinear functional analysis could all benefit from this theory. In sb-MS, we have generalized and proven FP and CFP theorems for single-valued mappings satisfying Ciric type contractions. Furthermore, in these spaces, two FP theorems for multi-valued mappings with Nadler's type contractions have been established and demonstrated. These generalizations could be useful in future research and applications.

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## Abbreviations

FP, fixed point; CFP, common fixed point; MS, metric space.

## Availability of data and materials

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## Declarations

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

The main idea of this paper was proposed by AT and SK. SP prepared the manuscript initially and performed all the steps of the proofs in this research. RS has supervised the entire work and guided. AT and SK also carried out a theoretical analysis of the proposed idea. Each author equally contributed towards writing and finalizing the article. All authors read and approved the final manuscript.

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