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On Mann implicit composite subgradient extragradient methods for general systems of variational inequalities with hierarchical variational inequality constraints

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Abstract

In a real Hilbert space, let the VIP, GSVI, HVI, and CFPP denote a variational inequality problem, a general system of variational inequalities, a hierarchical variational inequality, and a common fixed-point problem of a countable family of uniformly Lipschitzian pseudocontractive mappings and an asymptotically nonexpansive mapping, respectively. We design two Mann implicit composite subgradient extragradient algorithms with line-search process for finding a common solution of the CFPP, GSVI, and VIP. The suggested algorithms are based on the Mann implicit iteration method, subgradient extragradient method with line-search process, and viscosity approximation method. Under mild assumptions, we prove the strong convergence of the suggested algorithms to a common solution of the CFPP, GSVI, and VIP, which solves a certain HVI defined on their common solutions set.

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1 Introduction

Let C be a nonempty, closed, and convex subset of a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the induced norm $\|\cdot\|$. Let P_C be the nearest point projection from H onto C. Given a nonlinear operator $T : C \to H$, let $\mathrm{Fix}(T)$ and $\mathbf{R}$ indicate the fixed-points set of T and the set of real numbers, respectively. Let $\to$ and $\rightharpoonup$ represent the strong and weak convergence in H, respectively. An operator $T : C \to C$ is called asymptotically nonexpansive if there exists $\{\theta_l\}_{l=1}^{\infty} \subset [0, +\infty)$ such that $\lim_{l\to\infty} \theta_l = 0$ and

$$\left\|T^l u - T^l v\right\| \le (1 + \theta_l)\|u - v\| \quad \forall l \ge 1, u, v \in C. \tag{1.1}$$

In particular, whenever $\theta_l = 0$ $\forall l \ge 1$, T is called nonexpansive. Given a self-mapping A on H, the classical variational inequality problem (VIP) is finding $u \in C$ such that

$\langle Au, v-u\rangle \geq 0$ $\forall v \in C$. We denote the solutions set of VIP by VI(C,A). To the best of our knowledge, one of the most popular approaches for solving the VIP is the extragradient method put forward by Korpelevich [1] in 1976, i.e., for any initial point $u_0 \in C$, let $\{u_l\}$ be the sequence constructed below

$$\begin{cases} v_l = P_C(u_l - \ell A u_l), \\ u_{l+1} = P_C(u_l - \ell A v_l) \quad \forall l \geq 0, \end{cases} \tag{1.2}$$

where $\ell \in (0, \frac{1}{L})$ and L is Lipschitz constant of A. Whenever VI$(C,A) \neq \emptyset$, the sequence $\{u_l\}$ converges weakly to a point in VI(C,A). At present, the vast literature on Korpelevich's extragradient approach shows that many authors have paid great attention to it and enhanced it in various ways; see, e.g., [2–26] and the references therein.

Suppose that $B_1, B_2 : C \to H$ are two nonlinear operators. Consider the following problem of finding $(u^*, v^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 v^* + u^* - v^*, w - u^*\rangle \geq 0 \quad \forall w \in C, \\ \langle \mu_2 B_2 u^* + v^* - u^*, w - v^*\rangle \geq 0 \quad \forall w \in C, \end{cases} \tag{1.3}$$

with constants $\mu_1, \mu_2 > 0$. Problem (1.3) is called a general system of variational inequalities (GSVI). Note that GSVI (1.3) can be transformed into the fixed-point problem below.

Lemma 1.1 ([6]) *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (1.3) if and only if $x^* \in$ Fix(G), where Fix(G) is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $y^* = P_C(I - \mu_2 B_2)x^*$.*

Suppose that the mappings B_1, B_2 are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $f : C \to C$ be a contraction with coefficient $\delta \in [0,1)$ and $F : C \to H$ be κ-Lipschitzian and η-strongly monotone with constants $\kappa, \eta > 0$ such that $\delta < \zeta := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \in (0,1]$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$. Let $\{S_l\}_{l=1}^{\infty}$ be a countable family of ς-uniformly Lipschitzian pseudocontractive self-mappings on C such that $\Omega := \bigcap_{l=0}^{\infty} \text{Fix}(S_l) \cap \text{Fix}(G) \neq \emptyset$ where $S_0 := S$ and Fix(G) is the fixed-point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Recently, Ceng and Wen [21] proposed the hybrid extragradient-like implicit method for finding an element of Ω, that is, for any initial point $x_1 \in C$, let $\{x_l\}$ be the sequence constructed below

$$\begin{cases} u_l = \beta_l x_l + (1 - \beta_l) S_l u_l, \\ v_l = P_C(u_l - \mu_2 B_2 u_l), \\ y_l = P_C(v_l - \mu_1 B_1 v_l), \\ x_{l+1} = P_C[\alpha_l f(x_l) + (I - \alpha_l \rho F) S^l y_l] \quad \forall l \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_l\}$ and $\{\beta_l\}$ are sequences in $(0,1]$ such that

(i) $\sum_{l=1}^{\infty} |\alpha_{l+1} - \alpha_l| < \infty$ and $\sum_{l=1}^{\infty} \alpha_l < \infty$;

(ii) $\lim_{l\to\infty} \alpha_l = 0$ and $\lim_{l\to\infty} \frac{\theta_l}{\alpha_l} = 0$;

(iii) $\sum_{l=1}^{\infty} |\beta_{l+1} - \beta_l| < \infty$ and $0 < \liminf_{l\to\infty} \beta_l \leq \limsup_{l\to\infty} \beta_l < 1$;

(iv) $\sum_{l=1}^{\infty} \|S^{l+1}y_l - S^l y_l\| < \infty$.

Under appropriate assumptions imposed on $\{S_l\}_{l=1}^{\infty}$, it was proved in [21] that the sequence $\{x_l\}$ converges strongly to an element $x^* \in \Omega$. In 2019, Thong and Hieu [14] proposed the inertial subgradient extragradient method with line-search process for solving the monotone VIP with Lipschitz continuous A and the fixed-point problem (FPP) of a quasinonexpansive mapping S with a demiclosedness property. Assume that $\Omega := \mathrm{Fix}(S) \cap \mathrm{VI}(C,A) \neq \emptyset$. Let the sequences $\{\alpha_l\} \subset [0,1]$ and $\{\beta_l\} \subset (0,1)$ be given.

Algorithm 1.1 ([14]) *Initialization*: Given $\gamma > 0$, $\ell \in (0,1)$, $\mu \in (0,1)$, let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Compute x_{l+1} below:

Step 1. Set $w_l = x_l + \alpha_l(x_l - x_{l-1})$ and calculate $v_l = P_C(w_l - \tau_l A w_l)$, where τ_l is chosen to be the largest $\tau \in \{\gamma, \gamma\ell, \gamma\ell^2, \ldots\}$ satisfying $\tau\|Aw_l - Av_l\| \leq \mu\|w_l - v_l\|$.

Step 2. Calculate $z_l = P_{C_l}(w_l - \tau_l A v_l)$ with $C_l := \{v \in H : \langle w_l - \tau_l A w_l - v_l, v - v_l\rangle \leq 0\}$.

Step 3. Calculate $x_{l+1} = (1 - \beta_l)w_l + \beta_l S z_l$. If $w_l = z_l = x_{l+1}$ then $w_l \in \Omega$.

Again set $l := l + 1$ and go to Step 1.

Under suitable assumptions, it was proven in [14] that $\{x_l\}$ converges weakly to an element of Ω. Very recently, Ceng and Shang [22] introduced the hybrid inertial subgradient extragradient method with line-search process for solving the pseudomonotone VIP with Lipschitz continuous A and the common fixed-point problem (CFPP) of finitely many nonexpansive mappings $\{S_l\}_{l=1}^{N}$ and an asymptotically nonexpansive mapping S in a real Hilbert space H. Assume that $\Omega := \bigcap_{l=0}^{N} \mathrm{Fix}(S_l) \cap \mathrm{VI}(C,A) \neq \emptyset$ with $S_0 := S$. Given a contraction $f : H \to H$ with constant $\delta \in [0,1)$, and an η-strongly monotone and κ-Lipschitzian mapping $F : H \to H$ with $\delta < \zeta := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, 2\eta/\kappa^2)$, let $\{\alpha_l\} \subset [0,1]$ and $\{\beta_l\}, \{\gamma_l\} \subset (0,1)$ with $\beta_l + \gamma_l < 1$ $\forall l \geq 1$. Besides, one writes $S_l := S_{l \bmod N}$ for integer $l \geq 1$ with the mod function taking values in the set $\{1, 2, \ldots, N\}$, i.e., whenever $l = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, one has that $S_l = S_N$ if $q = 0$ and $S_l = S_q$ if $0 < q < N$.

Algorithm 1.2 ([22]) *Initialization*: Given $\gamma > 0$, $\ell \in (0,1)$, $\mu \in (0,1)$, let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{l+1} below:

Step 1. Set $w_l = S_l x_l + \alpha_l(S_l x_l - S_l x_{l-1})$ and calculate $v_l = P_C(w_l - \tau_l A w_l)$, where τ_l is chosen to be the largest $\tau \in \{\gamma, \gamma\ell, \gamma\ell^2, \ldots\}$ satisfying $\tau\|Aw_l - Av_l\| \leq \mu\|w_l - v_l\|$.

Step 2. Calculate $z_l = P_{C_l}(w_l - \tau_l A v_l)$ with $C_l := \{v \in H : \langle w_l - \tau_l A w_l - v_l, v - v_l\rangle \leq 0\}$.

Step 3. Calculate $x_{l+1} = \beta_l f(x_l) + \gamma_l x_l + ((1 - \gamma_l)I - \beta_l \rho F)S^l z_l$.

Again set $l := l + 1$ and go to Step 1.

Under appropriate assumptions, it was proven in [22] that if $S^l z_l - S^{l+1} z_l \to 0$, then $\{x_l\}$ converges strongly to $x^* \in \Omega$ if and only if $x_l - x_{l+1} \to 0$ and $x_l - v_l \to 0$ as $l \to \infty$. In a real Hilbert space H, we always assume that the CFPP and HVI denote a common fixed-point problem of a countable family of uniformly Lipschitzian pseudocontractive mappings $\{S_l\}_{l=1}^{\infty}$ and an asymptotically nonexpansive mapping $S_0 := S$ and a hierarchical variational inequality, respectively. Inspired by the above research works, we design two Mann implicit composite subgradient extragradient algorithms with line-search process for finding a common solution of the CFPP of $\{S_l\}_{l=0}^{\infty}$, the pseudomonotone VIP with Lipschitz continuous A and the GSVI for two inverse-strongly monotone B_1, B_2. The suggested algorithms are based on the viscosity approximation method, subgradient extragradient method with line-search process, and Mann implicit iteration method. Under mild assumptions, we prove the strong convergence of the suggested algorithms to a common solution of the CFPP, GSVI, and VIP, which solves a certain HVI defined on their common solution set. Finally, using the main results, we deal with the CFPP, GSVI, and VIP in an illustrated example.

2 Preliminaries

Let the nonempty set C be convex and closed in a real Hilbert space H. Given a sequence $\{\upsilon_i\} \subset H$, let $\upsilon_i \to \upsilon$ (resp., $\upsilon_i \rightharpoonup \upsilon$) indicate the strong (resp., weak) convergence of $\{\upsilon_i\}$ to υ. An operator $S : C \to H$ is called

(a) L-Lipschitz continuous (or L-Lipschitzian) if $\exists L > 0$ such that $\|Su - S\upsilon\| \le L\|u - \upsilon\|$ $\forall u, \upsilon \in C$;
(b) pseudocontractive if $\langle Su - S\upsilon, u - \upsilon\rangle \le \|u - \upsilon\|^2$ $\forall u, \upsilon \in C$;
(c) pseudomonotone if $\langle Su, \upsilon - u\rangle \ge 0 \Rightarrow \langle S\upsilon, \upsilon - u\rangle \ge 0$ $\forall u, \upsilon \in C$;
(d) α-strongly monotone if $\exists \alpha > 0$ such that $\langle Su - S\upsilon, u - \upsilon\rangle \ge \alpha\|u - \upsilon\|^2$ $\forall u, \upsilon \in C$;
(e) β-inverse-strongly monotone if $\exists \beta > 0$ such that $\langle Su - S\upsilon, u - \upsilon\rangle \ge \beta\|Su - S\upsilon\|^2$ $\forall u, \upsilon \in C$;
(f) sequentially weakly continuous if $\forall \{\upsilon_i\} \subset C$, the following relation holds: $\upsilon_i \rightharpoonup \upsilon \Rightarrow S\upsilon_i \rightharpoonup S\upsilon$.

It is clear that each monotone mapping is pseudomonotone, but the converse is not true. It is known that $\forall u \in H$, $\exists!$ (nearest point) $P_C u \in C$ such that $\|u - P_C u\| \le \|u - \upsilon\|$ $\forall \upsilon \in C$; P_C is refereed to as a metric (or nearest point) projection of H onto C. Recall that the following conclusions hold (see [27]):

(a) $\langle u - \upsilon, P_C u - P_C \upsilon\rangle \ge \|P_C u - P_C \upsilon\|^2$ $\forall u, \upsilon \in H$;
(b) $w = P_C u \Leftrightarrow \langle u - w, \upsilon - w\rangle \le 0$ $\forall u \in H, \upsilon \in C$;
(c) $\|u - \upsilon\|^2 \ge \|u - P_C u\|^2 + \|\upsilon - P_C u\|^2$ $\forall u \in H, \nu \in C$;
(d) $\|u - \upsilon\|^2 = \|u\|^2 - \|\upsilon\|^2 - 2\langle u - \upsilon, \upsilon\rangle$ $\forall u, \upsilon \in H$;
(e) $\|su + (1 - s)\upsilon\|^2 = s\|u\|^2 + (1 - s)\|\upsilon\|^2 - s(1 - s)\|u - \upsilon\|^2$ $\forall u, \upsilon \in H, s \in [0, 1]$.

The following concept will be used in the convergence analysis of the proposed algorithms.

Definition 2.1 ([21]) Let $\{S_l\}_{l=1}^{\infty}$ be a sequence of continuous pseudocontractive self-mappings on C. Then $\{S_l\}_{l=1}^{\infty}$ is called a countable family of ς-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\varsigma > 0$ such that each S_l is ς-Lipschitz continuous.

The following propositions and lemmas will be needed for demonstrating our main results.

Proposition 2.1 ([28]) *Let C be a nonempty, closed, convex subset of a Banach space X. Suppose that $\{S_l\}_{l=1}^{\infty}$ is a countable family of self-mappings on C such that $\sum_{l=1}^{\infty} \sup\{\|S_l x - S_{l+1}x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{S_l y\}$ converges strongly to some point of C. Moreover, let $\hat{S}$ be a self-mapping on C, defined by $\hat{S}y = \lim_{l\to\infty} S_l y$ for all $y \in C$. Then* $\lim_{l\to\infty} \sup\{\|\hat{S}x - S_l x\| : x \in C\} = 0$.

Proposition 2.2 ([29]) *Let C be a nonempty, closed, convex subset of a Banach space X and $T : C \to C$ be a continuous and strong pseudocontraction mapping. Then, T has a unique fixed point in C.*

The following inequality is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$:

$$\|u+\upsilon\|^2 \le \|u\|^2 + 2\langle \upsilon, u+\upsilon\rangle \quad \forall u,\upsilon \in H.$$

Lemma 2.1 *Let the mapping $B : C \to H$ be β-inverse-strongly monotone. Then, for a given $\lambda \ge 0$,*

$$\left\|(I-\lambda B)u-(I-\lambda B)\upsilon\right\|^2 \le \|u-\upsilon\|^2 - \lambda(2\alpha-\lambda)\|Bu-B\upsilon\|^2.$$

In particular, if $0 \le \lambda \le 2\alpha$, then $I-\lambda B$ is nonexpansive.

Using Lemma 2.1, we immediately derive the following lemma.

Lemma 2.2 *Let the mappings $B_1, B_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : C \to C$ be defined as $G := P_C(I-\mu_1 B_1)P_C(I-\mu_2 B_2)$. If $0 \le \mu_1 \le 2\alpha$ and $0 \le \mu_2 \le 2\beta$, then $G : C \to C$ is nonexpansive.*

Lemma 2.3 ([6, Lemma 2.1]) *Let $A : C \to H$ be pseudomonotone and continuous. Then $u \in C$ is a solution to the VIP $\langle Au, \upsilon - u\rangle \ge 0$ $\forall \upsilon \in C$ if and only if $\langle A\upsilon, \upsilon - u\rangle \ge 0$ $\forall \upsilon \in C$.*

Lemma 2.4 ([30]) *Let $\{a_l\}$ be a sequence of nonnegative numbers satisfying the following conditions: $a_{l+1} \le (1-\lambda_l)a_l + \lambda_l\gamma_l$ $\forall l \ge 1$, where $\{\lambda_l\}$ and $\{\gamma_l\}$ are sequences of real numbers such that (i) $\{\lambda_l\} \subset [0,1]$ and $\sum_{l=1}^{\infty}\lambda_l = \infty$, and (ii) $\limsup_{l\to\infty}\gamma_l \le 0$ or $\sum_{l=1}^{\infty}|\lambda_l\gamma_l| < \infty$. Then $\lim_{l\to\infty} a_l = 0$.*

Lemma 2.5 ([31]) *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty, closed, convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. Then $I-T$ is demiclosed at zero, i.e., if $\{u_k\}$ is a sequence in C such that $u_k \rightharpoonup u \in C$ and $(I-T)u_k \to 0$, then $(I-T)u = 0$, where I is the identity mapping of X.*

The following lemmas are crucial to the convergence analysis of the proposed algorithms.

Lemma 2.6 ([25]) *Let $\{\Gamma_m\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{m_k}\}$ of $\{\Gamma_m\}$ which satisfies $\Gamma_{m_k} < \Gamma_{m_k+1}$ for each integer $k \ge 1$. Define the sequence $\{\tau(m)\}_{m\ge m_0}$ of integers by*

$$\tau(m) = \max\{k \le m : \Gamma_k < \Gamma_{k+1}\},$$

where integer $m_0 \ge 1$ is such that $\{k \le m_0 : \Gamma_k < \Gamma_{k+1}\} \ne \emptyset$. Then the following hold:

(i) $\tau(m_0) \le \tau(m_0+1) \le \cdots$ *and* $\tau(m) \to \infty$;

(ii) $\Gamma_{\tau(m)} \le \Gamma_{\tau(m)+1}$ *and* $\Gamma_m \le \Gamma_{\tau(m)+1}$ $\forall m \ge m_0$.

3 Main results

In this section, let the feasible set C be a nonempty, closed, convex subset of a real Hilbert space H, and assume always that the following conditions hold:

- A is pseudomonotone and L-Lipschitzian self-mapping on H such that $\|Au\| \leq \liminf_{n\to\infty} \|A\upsilon_n\|$ for each $\{\upsilon_n\} \subset C$ with $\upsilon_n \rightharpoonup u$.
- $B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $f : C \to C$ is a δ-contraction with constant $\delta \in [0, 1)$.
- $\{S_n\}_{n=1}^\infty$ is a countable family of ς-uniformly Lipschitzian pseudocontractive self-mappings on C and $S : H \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$.
- $\Omega = \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$ with $S_0 := S$, and $\mathrm{Fix}(G)$ is the fixed point set of mapping $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ for $0 < \mu_1 < 2\alpha$ and $0 < \mu_2 < 2\beta$.
- $\sum_{n=1}^\infty \sup_{x\in D} \|S_n x - S_{n+1}x\| < \infty$ for any bounded subset D of C and $\mathrm{Fix}(\hat{S}) = \bigcap_{n=1}^\infty \mathrm{Fix}(S_n)$ where $\hat{S} : C \to C$ is defined as $\hat{S}x = \lim_{n\to\infty} S_n x$ $\forall x \in C$.
- $\{\sigma_n\} \subset (0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$ such that:
  - (i) $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
  - (ii) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;
  - (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Algorithm 3.1 *Initialization*: Given $\gamma > 0$, $\mu \in (0, 1)$, $\ell \in (0, 1)$, pick an initial $x_1 \in C$ arbitrarily.

Iterative steps: Compute x_{n+1} below:

Step 1. Calculate $u_n = \sigma_n x_n + (1 - \sigma_n)S_n u_n$ and $w_n = Gu_n$, and set $y_n = P_C(w_n - \tau_n Aw_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma\ell, \gamma\ell^2, \dots\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|. \tag{3.1}$$

Step 2. Calculate $z_n = P_{C_n}(w_n - \tau_n Ay_n)$ with $C_n := \{y \in H : \langle w_n - \tau_n Aw_n - y_n, y - y_n\rangle \leq 0\}$.

Step 3. Calculate

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n z_n. \tag{3.2}$$

Again put $n := n + 1$ and return to Step 1.

Lemma 3.1 *The Armijo-like search rule* (3.1) *is well defined, and the following inequality holds*: $\min\{\gamma, \mu\ell/L\} \leq \tau_n \leq \gamma$.

Proof Thanks to $\|Aw_n - AP_C(w_n - \gamma\ell^m Aw_n)\| \leq L\|w_n - P_C(w_n - \gamma\ell^m Aw_n)\|$, we know that (3.1) holds for each $\gamma\ell^m \leq \frac{\mu}{L}$ and so τ_n is well defined. Obviously, $\tau_n \leq \gamma$. In the case of $\tau_n = \gamma$, the conclusion is true. In the case of $\tau_n < \gamma$, from (3.1) one gets $\|Aw_n - AP_C(w_n - \frac{\tau_n}{\ell}Aw_n)\| > \frac{\mu}{(\tau_n/\ell)}\|w_n - P_C(w_n - \frac{\tau_n}{\ell}Aw_n)\|$, which hence leads to $\tau_n > \mu\ell/L$. □

Lemma 3.2 *Let the sequences* $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ *be constructed by Algorithm* 3.1. *Then for each* $p \in \Omega$, *one has*

$$\begin{aligned} \|z_n - p\|^2 \leq{}& \|u_n - p\|^2 - (1-\mu)\big[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\big] \\ & - \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 p\|^2 - \mu_1(2\alpha - \mu_1)\|B_1\upsilon_n - B_1 q\|^2, \end{aligned} \tag{3.3}$$

where $q = P_C(p - \mu_2 B_2 p)$ *and* $\upsilon_n = P_C(u_n - \mu_2 B_2 u_n)$.

Proof Define $T_n x := \beta_n x_n + (1-\beta_n)S_n x$, $x \in C$, for each $n \geq 0$. Then T_n is continuous by the continuity of S_n and

$$\begin{aligned}\langle T_n x - T_n y, x - y\rangle &= (1-\beta_n)\langle S_n x - S_n y, x - y\rangle \\ &\leq (1-\beta_n)\|x-y\|^2 \\ &\leq \bar{\beta}_n\|x-y\|^2,\end{aligned}$$

where $\bar{\beta}_n := 1-\beta_n \in (0,1)$ and this implies that T_n is a strong pseudocontractive mapping. Hence, by Proposition 2.2, there exists a unique element $u_n \in C$ such that for each $n \geq 0$,

$$u_n = \beta_n x_n + (1-\beta_n)S_n u_n.$$

Observe that for each $p \in \Omega \subset C \subset C_n$,

$$\begin{aligned}\|z_n - p\|^2 &= \left\|P_{C_n}(w_n - \tau_n A y_n) - P_{C_n} p\right\|^2 \\ &\leq \langle z_n - p, w_n - \tau_n A y_n - p\rangle \\ &= \frac{1}{2}\left(\|z_n - p\|^2 + \|w_n - p\|^2 - \|z_n - w_n\|^2\right) - \tau_n\langle z_n - p, A y_n\rangle,\end{aligned}$$

which hence yields

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\tau_n\langle z_n - p, A y_n\rangle.$$

Owing to $z_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{y \in H : \langle w_n - \tau_n A w_n - y_n, y - y_n\rangle \leq 0\}$, one gets $\langle w_n - \tau_n A w_n - y_n, z_n - y_n\rangle \leq 0$. Combining (3.1) and the pseudomonotonicity of A guarantees that

$$\begin{aligned}\|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\tau_n\langle A y_n, y_n - p + z_n - y_n\rangle \\ &\leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\tau_n\langle A y_n, z_n - y_n\rangle \\ &= \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \tau_n A y_n - y_n, z_n - y_n\rangle \\ &= \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \tau_n A w_n - y_n, z_n - y_n\rangle \\ &\quad + 2\tau_n\langle A w_n - A y_n, z_n - y_n\rangle \\ &\leq \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\mu\|w_n - y_n\|\|z_n - y_n\| \\ &\leq \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 + \mu\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right) \\ &= \|w_n - p\|^2 - (1-\mu)\left[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\right].\end{aligned} \tag{3.4}$$

Note that $q = P_C(p - \mu_2 B_2 p)$, $\upsilon_n = P_C(u_n - \mu_2 B_2 u_n)$, and $w_n = P_C(\upsilon_n - \mu_1 B_1 \upsilon_n)$. Then $w_n = G u_n$. By Lemma 2.1, one has

$$\|\upsilon_n - q\|^2 \leq \|u_n - p\|^2 - \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 p\|^2$$

and

$$\|w_n - p\|^2 \leq \|\upsilon_n - q\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 \upsilon_n - B_1 q\|^2.$$

Combining the last two inequalities, one gets

$$\|w_n - p\|^2 \le \|u_n - p\|^2 - \mu_2(2\beta - \mu_2)\|B_2u_n - B_2p\|^2 - \mu_1(2\alpha - \mu_1)\|B_1\upsilon_n - B_1q\|^2.$$

This, together with (3.4), implies that inequality (3.3) holds. □

Lemma 3.3 *Suppose that* $\{u_n\}$, $\{x_n\}$ *are bounded sequences constructed by Algorithm* 3.1. *Assume that* $x_n - x_{n+1} \to 0$, $u_n - Gu_n \to 0$, *and* $S^n x_n - S^{n+1}x_n \to 0$, *and suppose there exists a subsequence* $\{x_{n_k}\} \subset \{x_n\}$ *such that* $x_{n_k} \rightharpoonup z \in C$. *Then* $z \in \Omega$.

Proof From Algorithm 3.1, we obtain that for each $p \in \Omega$,

$$\begin{aligned}\|u_n - p\|^2 &= \sigma_n\langle x_n - p, u_n - p\rangle + (1 - \sigma_n)\langle S_n u_n - p, u_n - p\rangle \\ &\le \sigma_n\langle x_n - p, u_n - p\rangle + (1 - \sigma_n)\|u_n - p\|^2,\end{aligned}$$

which hence yields

$$\begin{aligned}\|u_n - p\|^2 &\le \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}\big[\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2\big].\end{aligned}$$

This immediately implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.5}$$

So it follows from (3.3) and the last inequality that

$$\begin{aligned}\|z_n - p\|^2 &\le \|u_n - p\|^2 - (1 - \mu)\big[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\big] \\ &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 - (1 - \mu)\big[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\big],\end{aligned}$$

which, together with Algorithm 3.1, leads to

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad= \big\|\alpha_n\big(f(x_n) - p\big) + \beta_n(x_n - p) + \gamma_n\big(S^n z_n - p\big)\big\|^2 \\ &\quad\le \alpha_n\big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\big\|S^n z_n - p\big\|^2 - \beta_n\gamma_n\big\|x_n - S^n z_n\big\|^2 \\ &\quad\le \alpha_n\big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(1 + \theta_n)^2\|z_n - p\|^2 - \beta_n\gamma_n\big\|x_n - S^n z_n\big\|^2 \\ &\quad\le \alpha_n\big\|f(x_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(1 + \theta_n)^2\big\{\|x_n - p\|^2 - \|x_n - u_n\|^2 \\ &\qquad - (1 - \mu)\big[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\big]\big\} - \beta_n\gamma_n\big\|x_n - S^n z_n\big\|^2 \\ &\quad\le \alpha_n\big\|f(x_n) - p\big\|^2 + \|x_n - p\|^2 + \theta_n(2 + \theta_n)\|x_n - p\|^2 - \gamma_n(1 + \theta_n)^2\big\{\|x_n - u_n\|^2 \\ &\qquad + (1 - \mu)\big[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\big]\big\} - \beta_n\gamma_n\big\|x_n - S^n z_n\big\|^2.\end{aligned}$$

This immediately ensures that

$$\begin{aligned}&\gamma_n(1+\theta_n)^2\big\{\|x_n-u_n\|^2+(1-\mu)\big[\|y_n-z_n\|^2+\|y_n-w_n\|^2\big]\big\}+\beta_n\gamma_n\big\|x_n-S^nz_n\big\|^2\\&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\big\|f(x_n)-p\big\|^2+\theta_n(2+\theta_n)\|x_n-p\|^2\\&\quad\le\|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big)+\alpha_n\big\|f(x_n)-p\big\|^2+\theta_n(2+\theta_n)\|x_n-p\|^2.\end{aligned}$$

Note that $\lim_{n\to\infty}\alpha_n=0$ and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$. Thus we know that $\liminf_{n\to\infty}\gamma_n=\liminf_{n\to\infty}(1-\alpha_n-\beta_n)=1-\limsup_{n\to\infty}\beta_n>0$. Since $\theta_n\to0$, $x_n-x_{n+1}\to 0$ and $\mu\in(0,1)$, by the boundedness of $\{x_n\}$, we get

$$\lim_{n\to\infty}\|x_n-u_n\|=\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\|y_n-w_n\|=\lim_{n\to\infty}\big\|x_n-S^nz_n\big\|=0. \tag{3.6}$$

So it follows that $\|w_n-x_n\|\le\|Gu_n-u_n\|+\|u_n-x_n\|\to0$ $(n\to\infty)$,

$$\begin{aligned}\|z_n-x_n\|&\le\|z_n-w_n\|+\|w_n-x_n\|\\&\le\|z_n-y_n\|+\|y_n-w_n\|+\|w_n-x_n\|\to0\quad(n\to\infty),\end{aligned}$$

and $\|x_n-y_n\|\le\|x_n-z_n\|+\|z_n-y_n\|\to0$ $(n\to\infty)$.

We show that $\lim_{n\to\infty}\|x_n-Sx_n\|=0$. In fact, using the asymptotical nonexpansivity of S, one obtains that

$$\begin{aligned}\|x_n-Sx_n\|&\le\big\|x_n-S^nz_n\big\|+\big\|S^nz_n-S^nx_n\big\|+\big\|S^nx_n-S^{n+1}x_n\big\|\\&\quad+\big\|S^{n+1}x_n-S^{n+1}z_n\big\|+\big\|S^{n+1}z_n-Sx_n\big\|\\&\le\big\|x_n-S^nz_n\big\|+(1+\theta_n)\|z_n-x_n\|+\big\|S^nx_n-S^{n+1}x_n\big\|\\&\quad+(1+\theta_{n+1})\|x_n-z_n\|+(1+\theta_1)\big\|S^nz_n-x_n\big\|\\&=(2+\theta_1)\big\|x_n-S^nz_n\big\|+(2+\theta_n+\theta_{n+1})\|z_n-x_n\|+\big\|S^nx_n-S^{n+1}x_n\big\|.\end{aligned}$$

Since $x_n-S^nz_n\to0$, $x_n-z_n\to0$ and $S^nx_n-S^{n+1}x_n\to0$, we obtain

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{3.7}$$

We show that $\lim_{n\to\infty}\|x_n-\bar{S}x_n\|=0$ where $\bar{S}:=(2I-\hat{S})^{-1}$. In fact, noticing $u_n=\sigma_nx_n+(1-\sigma_n)S_nu_n$ and $x_n-u_n\to0$, we get

$$(1-\sigma_n)\|S_nu_n-u_n\|=\sigma_n\|x_n-u_n\|\le\|x_n-u_n\|\to0\quad(n\to\infty),$$

which, together with $0<\liminf_{n\to\infty}(1-\sigma_n)$, yields

$$\lim_{n\to\infty}\|S_nu_n-u_n\|=0.$$

Since $\{S_n\}_{n=1}^\infty$ is ς-uniformly Lipschitzian on C, we deduce from $x_n-u_n\to0$ and $S_nu_n-u_n\to0$ that

$$\begin{aligned}\|S_nx_n-x_n\|&\le\|S_nx_n-S_nu_n\|+\|S_nu_n-u_n\|+\|u_n-x_n\|\\&\le(\varsigma+1)\|u_n-x_n\|+\|S_nu_n-u_n\|\to0\quad(n\to\infty).\end{aligned}$$

It is clear that $\hat{S}: C \to C$ is pseudocontractive and ς-Lipschitzian where $\hat{S}x = \lim_{n\to\infty} S_n x$ $\forall x \in C$. We claim that $\lim_{n\to\infty} \|\hat{S}x_n - x_n\| = 0$. Using the boundedness of $\{x_n\}$ and putting $D = \overline{\mathrm{conv}}\{x_n : n \geq 1\}$ (the closed convex hull of the set $\{x_n : n \geq 1\}$), by the hypothesis, we get $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n+1}x\| < \infty$. So, by Proposition 2.1, we have $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - \hat{S}x\| = 0$, which immediately arrives at

$$\lim_{n\to\infty} \|S_n x_n - \hat{S}x_n\| = 0.$$

Consequently,

$$\|x_n - \hat{S}x_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - \hat{S}x_n\| \to 0 \quad (n \to \infty).$$

Now, let us show that if we define $\bar{S} := (2I - \hat{S})^{-1}$, then $\bar{S}: C \to C$ is nonexpansive, $\mathrm{Fix}(\bar{S}) = \mathrm{Fix}(\hat{S}) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$, and $\lim_{n\to\infty} \|x_n - \bar{S}x_n\| = 0$. As a matter of fact, it is known that $\bar{S}$ is nonexpansive and $\mathrm{Fix}(\bar{S}) = \mathrm{Fix}(\hat{S}) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$ as a consequence of [32, Theorem 6]. From $x_n - \hat{S}x_n \to 0$, it follows that

$$\begin{aligned} \|x_n - \bar{S}x_n\| &= \left\|\bar{S}\bar{S}^{-1}x_n - \bar{S}x_n\right\| \\ &\leq \left\|\bar{S}^{-1}x_n - x_n\right\| = \left\|(2I - \hat{S})x_n - x_n\right\| = \|x_n - \hat{S}x_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.8}$$

Next, let us show $z \in \mathrm{VI}(C,A)$. Indeed, noticing $w_n - x_n \to 0$ and $x_{n_k} \rightharpoonup z$, we have $w_{n_k} \rightharpoonup z$. We consider two cases below.

If $Az = 0$, then it is clear that $z \in \mathrm{VI}(C,A)$ because $\langle Az, x - z\rangle \geq 0$ $\forall x \in C$.

Assume that $Az \neq 0$. Since $w_{n_k} \rightharpoonup z$ as $k \to \infty$, utilizing the assumption on A, instead of the sequentially weak continuity of A, we get $0 < \|Az\| \leq \liminf_{k\to\infty} \|Aw_{n_k}\|$. So, we could suppose that $\|Aw_{n_k}\| \neq 0$ $\forall k \geq 1$. Moreover, from $y_n = P_C(w_n - \tau_n Aw_n)$, we have $\langle w_n - \tau_n Aw_n - y_n, x - y_n\rangle \leq 0$ $\forall x \in C$, and hence

$$\frac{1}{\tau_n}\langle w_n - y_n, x - y_n\rangle + \langle Aw_n, y_n - w_n\rangle \leq \langle Aw_n, x - w_n\rangle \quad \forall x \in C. \tag{3.9}$$

According to the Lipschitz continuity of A, one knows that $\{Aw_n\}$ is bounded. Note that $\{y_n\}$ is bounded as well. Using Lemma 3.1, from (3.9) we get $\liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0$ $\forall x \in C$.

To show that $z \in \mathrm{VI}(C,A)$, we now choose a sequence $\{\varepsilon_k\} \subset (0,1)$ satisfying $\varepsilon_k \downarrow 0$ as $k \to \infty$. For each $k \geq 1$, we denote by m_k the smallest positive integer such that

$$\langle Aw_{n_j}, x - w_{n_j}\rangle + \varepsilon_k \geq 0 \quad \forall j \geq m_k. \tag{3.10}$$

Since $\{\varepsilon_k\}$ is decreasing, it can be readily seen that $\{m_k\}$ is increasing. Noticing that $Aw_{m_k} \neq 0$ $\forall k \geq 1$ (due to $\{Aw_{m_k}\} \subset \{Aw_{n_k}\}$), we set $\varrho_{m_k} = \frac{Aw_{m_k}}{\|Aw_{m_k}\|^2}$, we get $\langle Aw_{m_k}, \varrho_{m_k}\rangle = 1$ $\forall k \geq 1$. So, from (3.10) we get $\langle Aw_{m_k}, x + \varepsilon_k\varrho_{m_k} - w_{m_k}\rangle \geq 0$ $\forall k \geq 1$. Again from the pseudomonotonicity of A, we have $\langle A(x + \varepsilon_k\varrho_{m_k}), x + \varepsilon_k\varrho_{m_k} - w_{m_k}\rangle \geq 0$ $\forall k \geq 1$. This immediately leads to

$$\langle Ax, x - w_{m_k}\rangle \geq \left\langle Ax - A(x + \varepsilon_k\varrho_{m_k}), x + \varepsilon_k\varrho_{m_k} - w_{m_k}\right\rangle - \varepsilon_k\langle Ax, \varrho_{m_k}\rangle \quad \forall k \geq 1. \tag{3.11}$$

We claim that $\lim_{k\to\infty}\varepsilon_k\varrho_{m_k}=0$. Note that $\{w_{m_k}\}\subset\{w_{n_k}\}$ and $\varepsilon_k\downarrow 0$ as $k\to\infty$. So it follows that $0\le\limsup_{k\to\infty}\|\varepsilon_k\varrho_{m_k}\|=\limsup_{k\to\infty}\frac{\varepsilon_k}{\|Aw_{m_k}\|}\le\frac{\limsup_{k\to\infty}\varepsilon_k}{\liminf_{k\to\infty}\|Aw_{n_k}\|}=0$. Hence we get $\varepsilon_k\varrho_{m_k}\to 0$ as $k\to\infty$. Thus, letting $k\to\infty$, we deduce that the right-hand side of (3.11) tends to zero by the Lipschitz continuity of A, the boundedness of $\{w_{m_k}\}$, $\{\varrho_{m_k}\}$ and the limit $\lim_{k\to\infty}\varepsilon_k\varrho_{m_k}=0$. Therefore, we get $\langle Ax,x-z\rangle=\liminf_{k\to\infty}\langle Ax,x-w_{m_k}\rangle\ge 0$ $\forall x\in C$. By Lemma 2.3, we have $z\in\mathrm{VI}(C,A)$.

Next we show that $z\in\Omega$. In fact, from $x_n-u_n\to 0$ and $x_{n_k}\rightharpoonup z$, we get $u_{n_k}\rightharpoonup z$. Note that the condition $u_n-Gu_n\to 0$ guarantees $u_{n_k}-Gu_{n_k}\to 0$. From Lemma 2.5, it follows that $I-G$ is demiclosed at zero. Hence we get $(I-G)z=0$, i.e., $z\in\mathrm{Fix}(G)$. In the meantime, let us show that $z\in\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)$. Again from Lemma 2.5, we know that $I-S$ and $I-\bar{S}$ are demiclosed at zero. Noticing $x_{n_k}-Sx_{n_k}\to 0$ (due to (3.7)) and $x_{n_k}-\bar{S}x_{n_k}\to 0$ (due to (3.8)), we deduce from $x_{n_k}\rightharpoonup z$ that $z\in\mathrm{Fix}(S)$ and $z\in\mathrm{Fix}(\bar{S})=\bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i)$. Consequently, $z\in\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)=\Omega$ with $S_0:=S$. This completes the proof. □

Theorem 3.1 *Let $\{x_n\}$ be the sequence constructed in Algorithm 3.1. Then $x_n\to x^*\in\Omega$, provided $S^nx_n-S^{n+1}x_n\to 0$, where $x^*\in\Omega$ is the unique solution to the HVI, $\langle(I-f)x^*,p-x^*\rangle\ge 0$ $\forall p\in\Omega$.*

Proof First of all, since $0<\liminf_{n\to\infty}\sigma_n\le\limsup_{n\to\infty}\sigma_n<1$ and $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}=0$, we may assume, without loss of generality, that $\{\sigma_n\}\subset[a,b]\subset(0,1)$ and $\theta_n\le\frac{\alpha_n(1-\delta)}{2}$ $\forall n\ge 1$. We claim that $P_\Omega\circ f:C\to C$ is a contraction. In fact, it is clear that $P_\Omega\circ f$ is a contraction. Banach's contraction mapping principle guarantees that $P_\Omega\circ f$ has a unique fixed point, say $x^*\in C$, i.e., $x^*=P_\Omega f(x^*)$. Thus, there exists a unique solution $x^*\in\Omega=\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ of the HVI

$$\bigl\langle(I-f)x^*,p-x^*\bigr\rangle\ge 0\quad\forall p\in\Omega.\tag{3.12}$$

Next we divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, take an arbitrary $p\in\Omega=\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$. Then $Sp=p$, $S_np=p$ $\forall n\ge 1$, $Gp=p$ and (3.3) holds, i.e.,

$$\begin{aligned}\|z_n-p\|^2&\le\|u_n-p\|^2-(1-\mu)\bigl[\|y_n-z_n\|^2+\|y_n-w_n\|^2\bigr]\\&\quad-\mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2-\mu_1(2\alpha-\mu_1)\|B_1\upsilon_n-B_1q\|^2,\end{aligned}\tag{3.13}$$

where $q=P_C(p-\mu_2B_2p)$ and $\upsilon_n=P_C(u_n-\mu_2B_2u_n)$. Again from (3.4) and (3.5), we deduce that

$$\|z_n-p\|\le\|w_n-p\|=\|Gu_n-p\|\le\|u_n-p\|\le\|x_n-p\|\quad\forall n\ge 1.\tag{3.14}$$

Thus, using (3.14) and $\alpha_n+\beta_n+\gamma_n=1$ $\forall n\ge 1$, from the asymptotical nonexpansivity of S, we obtain

$$\begin{aligned}\|x_{n+1}-p\|&\le\alpha_n\bigl\|f(x_n)-p\bigr\|+\beta_n\|x_n-p\|+\gamma_n\bigl\|S^nz_n-p\bigr\|\\&\le\alpha_n\bigl(\bigl\|f(x_n)-f(p)\bigr\|+\bigl\|f(p)-p\bigr\|\bigr)+\beta_n\|x_n-p\|+\gamma_n(1+\theta_n)\|z_n-p\|\\&\le\alpha_n\delta\|x_n-p\|+\alpha_n\bigl\|f(p)-p\bigr\|+\beta_n\|x_n-p\|+(\gamma_n+\theta_n)\|x_n-p\|\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\delta\|x_n-p\| + \alpha_n\big\|f(p)-p\big\| + (1-\alpha_n)\|x_n-p\| + \frac{\alpha_n(1-\delta)}{2}\|x_n-p\| \\
&= \left[1-\frac{\alpha_n(1-\delta)}{2}\right]\|x_n-p\| + \alpha_n\big\|f(p)-p\big\| \\
&= \left[1-\frac{\alpha_n(1-\delta)}{2}\right]\|x_n-p\| + \frac{\alpha_n(1-\delta)}{2}\frac{2\|f(p)-p\|}{1-\delta} \\
&\le \max\left\{\|x_n-p\|, \frac{2\|f(p)-p\|}{1-\delta}\right\}.
\end{aligned}
$$

By induction, we obtain $\|x_n-p\| \le \max\{\|x_1-p\|, \frac{2\|f(p)-p\|}{1-\delta}\}$ $\forall n\ge 1$. Therefore, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Ay_n\}$, $\{S_nu_n\}$, $\{S^nz_n\}$.

Step 2. We show that

$$
\begin{aligned}
&\gamma_n\big\{\|x_n-u_n\|^2 + (1-\mu)\big[\|y_n-z_n\|^2 + \|y_n-w_n\|^2\big] + \mu_2(2\beta-\mu_2) \\
&\quad\times \|B_2u_n-B_2p\|^2 + \mu_1(2\alpha-\mu_1)\|B_1\upsilon_n-B_1q\|^2\big\} \\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \theta_n(2+\theta_n)M_0 + 2\alpha_nM_0
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
&\gamma_n\big[\|u_n-\upsilon_n+q-p\|^2 + \|\upsilon_n-w_n+p-q\|^2\big] \\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\mu_2\|B_2p-B_2u_n\|\|\upsilon_n-q\| \\
&\quad + 2\mu_1\|B_1q-B_1\upsilon_n\|\|w_n-p\| + \theta_n(2+\theta_n)M_0 + 2\alpha_nM_0,
\end{aligned}
\tag{3.16}
$$

for some $M_0>0$. In fact, using (3.5), (3.13), (3.14), and the convexity of the function $\phi(s) = s^2$ $\forall s\in\mathbf{R}$, we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&= \big\|\alpha_n\big(f(x_n)-f(p)\big) + \beta_n(x_n-p) + \gamma_n\big(S^nz_n-p\big) + \alpha_n\big(f(p)-p\big)\big\|^2 \\
&\le \big\|\alpha_n\big(f(x_n)-f(p)\big) + \beta_n(x_n-p) + \gamma_n\big(S^nz_n-p\big)\big\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\le \alpha_n\big\|f(x_n)-f(p)\big\|^2 + \beta_n\|x_n-p\|^2 + \gamma_n\big\|S^nz_n-p\big\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 + \gamma_n(1+\theta_n)^2\|z_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 + \big[\gamma_n+\theta_n(2+\theta_n)\big]\|z_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 + \gamma_n\big\{\|u_n-p\|^2 - (1-\mu)\big[\|y_n-z_n\|^2 + \|y_n-w_n\|^2\big] \\
&\quad - \mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2 - \mu_1(2\alpha-\mu_1)\|B_1\upsilon_n-B_1q\|^2\big\} \\
&\quad + \theta_n(2+\theta_n)\|x_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \qquad (3.17) \\
&\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 + \gamma_n\big\{\|x_n-p\|^2 - \|x_n-u_n\|^2 - (1-\mu)\big[\|y_n-z_n\|^2 \\
&\quad + \|y_n-w_n\|^2\big] - \mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2 - \mu_1(2\alpha-\mu_1)\|B_1\upsilon_n-B_1q\|^2\big\} \\
&\quad + \theta_n(2+\theta_n)\|x_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&= \big[1-\alpha_n(1-\delta)\big]\|x_n-p\|^2 - \gamma_n\big\{\|x_n-u_n\|^2 + (1-\mu)\big[\|y_n-z_n\|^2 + \|y_n-w_n\|^2\big] \\
&\quad + \mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2 + \mu_1(2\alpha-\mu_1)\|B_1\upsilon_n-B_1q\|^2\big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \theta_n(2+\theta_n)\|x_n - p\|^2 + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \|x_n - p\|^2 - \gamma_n\bigl\{\|x_n - u_n\|^2 + (1-\mu)\bigl[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\bigr] \\
&+ \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 p\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 \upsilon_n - B_1 q\|^2\bigr\} \\
&+ \theta_n(2+\theta_n)M_0 + 2\alpha_n M_0,
\end{aligned}
$$

where $\sup_{n\ge 1}\{\|x_n - p\|^2 + \|f(p) - p\|\|x_n - p\|\} \le M_0$ for some $M_0 > 0$. This ensures that (3.15) holds.

On the other hand, by the firm nonexpansivity of P_C we obtain that

$$
\begin{aligned}
\|w_n - p\|^2 &\le \langle \upsilon_n - q, w_n - p\rangle + \mu_1\langle B_1 q - B_1\upsilon_n, w_n - p\rangle \\
&\le \frac{1}{2}\bigl[\|\upsilon_n - q\|^2 + \|w_n - p\|^2 - \|\upsilon_n - w_n + p - q\|^2\bigr] \\
&\quad + \mu_1\|B_1 q - B_1\upsilon_n\|\|w_n - p\|,
\end{aligned}
$$

which hence gives

$$
\|w_n - p\|^2 \le \|\upsilon_n - q\|^2 - \|\upsilon_n - w_n + p - q\|^2 + 2\mu_1\|B_1 q - B_1\upsilon_n\|\|w_n - p\|. \tag{3.18}
$$

In a similar way, we have

$$
\|\upsilon_n - q\|^2 \le \|u_n - p\|^2 - \|u_n - \upsilon_n + q - p\|^2 + 2\mu_2\|B_2 p - B_2 u_n\|\|\upsilon_n - q\|. \tag{3.19}
$$

Substituting (3.19) for (3.18), from (3.14) we deduce that

$$
\begin{aligned}
\|w_n - p\|^2 &\le \|x_n - p\|^2 - \|u_n - \upsilon_n + q - p\|^2 - \|\upsilon_n - w_n + p - q\|^2 \\
&\quad + 2\mu_2\|B_2 p - B_2 u_n\|\|\upsilon_n - q\| + 2\mu_1\|B_1 q - B_1\upsilon_n\|\|w_n - p\|,
\end{aligned}
$$

which, together with (3.14) and (3.17), leads to

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \bigl[\gamma_n + \theta_n(2+\theta_n)\bigr]\|z_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|w_n - p\|^2 + \theta_n(2+\theta_n)\|x_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + \gamma_n\bigl\{\|x_n - p\|^2 - \|u_n - \upsilon_n + q - p\|^2 - \|\upsilon_n - w_n + p - q\|^2 \\
&\quad + 2\mu_2\|B_2 p - B_2 u_n\|\|\upsilon_n - q\| + 2\mu_1\|B_1 q - B_1\upsilon_n\|\|w_n - p\|\bigr\} \\
&\quad + \theta_n(2+\theta_n)\|x_n - p\|^2 + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \bigl[1 - \alpha_n(1-\delta)\bigr]\|x_n - p\|^2 - \gamma_n\bigl[\|u_n - \upsilon_n + q - p\|^2 + \|\upsilon_n - w_n + p - q\|^2\bigr] \\
&\quad + 2\mu_2\|B_2 p - B_2 u_n\|\|\upsilon_n - q\| + 2\mu_1\|B_1 q - B_1\upsilon_n\|\|w_n - p\| \\
&\quad + \theta_n(2+\theta_n)\|x_n - p\|^2 + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
&\leq \|x_n - p\|^2 - \gamma_n\big[\|u_n - \upsilon_n + q - p\|^2 + \|\upsilon_n - w_n + p - q\|^2\big] \\
&\quad + 2\mu_2\|B_2p - B_2u_n\|\|\upsilon_n - q\| \\
&\quad + 2\mu_1\|B_1q - B_1\upsilon_n\|\|w_n - p\| + \theta_n(2+\theta_n)M_0 + 2\alpha_nM_0.
\end{aligned}
$$

This ensures that (3.16) holds.

Step 3. We show that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \big[1 - \alpha_n(1-\delta)\big]\|x_n - p\|^2 \\
&\quad + \alpha_n(1-\delta)\left\{\frac{2\langle (f-I)p, x_{n+1} - p\rangle}{1-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{(2+\theta_n)M_0}{1-\delta}\right\}.
\end{aligned}
$$

In fact, from (3.14) and (3.17), we have

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad\leq \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \big[\gamma_n + \theta_n(2+\theta_n)\big]\|z_n - p\|^2 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\quad\leq \alpha_n\delta\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \theta_n(2+\theta_n)M_0 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\quad= \big[1 - \alpha_n(1-\delta)\big]\|x_n - p\|^2 + \theta_n(2+\theta_n)M_0 + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\quad= \big[1 - \alpha_n(1-\delta)\big]\|x_n - p\|^2 \\
&\qquad + \alpha_n(1-\delta)\left\{\frac{2\langle (f-I)p, x_{n+1} - p\rangle}{1-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{(2+\theta_n)M_0}{1-\delta}\right\}.
\end{aligned}
\tag{3.21}
$$

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (3.12). In fact, putting $p = x^*$, we deduce from (3.21) that

$$
\begin{aligned}
\big\|x_{n+1} - x^*\big\|^2 &\leq \big[1 - \alpha_n(1-\delta)\big]\big\|x_n - x^*\big\|^2 + \alpha_n(1-\delta)\bigg[\frac{2\langle (f-I)x^*, x_{n+1} - x^*\rangle}{1-\delta} \\
&\quad + \frac{\theta_n}{\alpha_n}\cdot\frac{(2+\theta_n)M_0}{1-\delta}\bigg].
\end{aligned}
\tag{3.22}
$$

Putting $\Gamma_n = \|x_n - x^*\|^2$, we show the convergence of $\{\Gamma_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $n_0 \geq 1$ such that $\{\Gamma_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\Gamma_n = \hbar < +\infty$ and $\lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$. Putting $p = x^*$ and $q = y^*$, from (3.15) and (3.16) we obtain

$$
\begin{aligned}
&\gamma_n\big\{\|x_n - u_n\|^2 + (1-\mu)\big[\|y_n - z_n\|^2 + \|y_n - w_n\|^2\big] + \mu_2(2\beta - \mu_2) \\
&\qquad\times \big\|B_2u_n - B_2x^*\big\|^2 + \mu_1(2\alpha - \mu_1)\big\|B_1\upsilon_n - B_1y^*\big\|^2\big\} \\
&\quad\leq \big\|x_n - x^*\big\|^2 - \big\|x_{n+1} - x^*\big\|^2 + \theta_n(2+\theta_n)M_0 + 2\alpha_nM_0 \\
&\quad= \Gamma_n - \Gamma_{n+1} + \theta_n(2+\theta_n)M_0 + 2\alpha_nM_0
\end{aligned}
\tag{3.23}
$$

and

$$\begin{aligned}
&\gamma_n\bigl[\bigl\|u_n-\upsilon_n+y^*-x^*\bigr\|^2+\bigl\|\upsilon_n-w_n+x^*-y^*\bigr\|^2\bigr] \\
&\quad\le \bigl\|x_n-x^*\bigr\|^2-\bigl\|x_{n+1}-x^*\bigr\|^2+2\mu_2\bigl\|B_2x^*-B_2u_n\bigr\|\bigl\|\upsilon_n-y^*\bigr\| \\
&\qquad+2\mu_1\bigl\|B_1y^*-B_1\upsilon_n\bigr\|\bigl\|w_n-x^*\bigr\|+\theta_n(2+\theta_n)M_0+2\alpha_nM_0 \\
&\quad=\Gamma_n-\Gamma_{n+1}+2\mu_2\bigl\|B_2x^*-B_2u_n\bigr\|\bigl\|\upsilon_n-y^*\bigr\| \\
&\qquad+2\mu_1\bigl\|B_1y^*-B_1\upsilon_n\bigr\|\bigl\|w_n-x^*\bigr\|+\theta_n(2+\theta_n)M_0+2\alpha_nM_0.
\end{aligned} \tag{3.24}$$

Noticing $0<\liminf_{n\to\infty}(1-\alpha_n-\beta_n)=\liminf_{n\to\infty}\gamma_n$, $\alpha_n\to0$, $\theta_n\to0$ and $\Gamma_n-\Gamma_{n+1}\to0$, one has from (3.23) that

$$\lim_{n\to\infty}\|x_n-u_n\|=\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\|y_n-w_n\|=0, \tag{3.25}$$

and

$$\lim_{n\to\infty}\bigl\|B_2u_n-B_2x^*\bigr\|=\lim_{n\to\infty}\bigl\|B_1\upsilon_n-B_1y^*\bigr\|=0. \tag{3.26}$$

Since $0<\liminf_{n\to\infty}\gamma_n$, $\alpha_n\to0$, $\theta_n\to0$ and $\Gamma_n-\Gamma_{n+1}\to0$, from (3.24), (3.26), and the boundedness of $\{\upsilon_n\}$, $\{w_n\}$, we deduce that

$$\lim_{n\to\infty}\bigl\|u_n-\upsilon_n+y^*-x^*\bigr\|=\lim_{n\to\infty}\bigl\|\upsilon_n-w_n+x^*-y^*\bigr\|=0. \tag{3.27}$$

Therefore,

$$\begin{aligned}
\|u_n-Gu_n\|&=\|u_n-w_n\| \\
&\le\bigl\|u_n-\upsilon_n+y^*-x^*\bigr\|+\bigl\|\upsilon_n-w_n+x^*-y^*\bigr\| \\
&\to0\quad(n\to\infty).
\end{aligned} \tag{3.28}$$

Furthermore, using (3.14), gives

$$\begin{aligned}
&\bigl\|x_{n+1}-x^*\bigr\|^2 \\
&\quad\le\bigl\|\alpha_n\bigl(f(x_n)-x^*\bigr)+\beta_n\bigl(x_n-x^*\bigr)+\gamma_n\bigl(S^nz_n-x^*\bigr)\bigr\|^2 \\
&\quad\le\alpha_n\bigl\|f(x_n)-x^*\bigr\|^2+\beta_n\bigl\|x_n-x^*\bigr\|+\gamma_n\bigl\|S^nz_n-x^*\bigr\|^2-\beta_n\gamma_n\bigl\|x_n-S^nz_n\bigr\|^2 \\
&\quad\le\alpha_n\bigl\|f(x_n)-x^*\bigr\|^2+\beta_n\bigl\|x_n-x^*\bigr\|+\gamma_n(1+\theta_n)^2\bigl\|z_n-x^*\bigr\|^2-\beta_n\gamma_n\bigl\|x_n-S^nz_n\bigr\|^2 \\
&\quad\le\alpha_n\bigl\|f(x_n)-x^*\bigr\|^2+(1-\alpha_n)\bigl\|x_n-x^*\bigr\|^2+\theta_n(2+\theta_n)\bigl\|x_n-x^*\bigr\|^2-\beta_n\gamma_n\bigl\|x_n-S^nz_n\bigr\|^2 \\
&\quad\le\alpha_n\bigl\|f(x_n)-x^*\bigr\|^2+(1-\alpha_n)\bigl\|x_n-x^*\bigr\|^2+\theta_n(2+\theta_n)\bigl\|x_n-x^*\bigr\|^2-\beta_n\gamma_n\bigl\|x_n-S^nz_n\bigr\|^2 \\
&\quad\le\bigl\|x_n-x^*\bigr\|^2+\alpha_nM_1+\theta_n(2+\theta_n)M_1-\beta_n\gamma_n\bigl\|x_n-S^nz_n\bigr\|^2,
\end{aligned}$$

where $\sup_{n\ge1}\{\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2\}\le M_1$ for some $M_1>0$. This immediately implies

$$\begin{aligned}
\beta_n\gamma_n\bigl\|x_n-S^nz_n\bigr\|^2&\le\bigl\|x_n-x^*\bigr\|^2-\bigl\|x_{n+1}-x^*\bigr\|^2+\alpha_nM_1+\theta_n(2+\theta_n)M_1 \\
&=\Gamma_n-\Gamma_{n+1}+\alpha_nM_1+\theta_n(2+\theta_n)M_1.
\end{aligned} \tag{3.29}$$

Since $0 < \liminf_{n\to\infty} \beta_n$, $0 < \liminf_{n\to\infty} \gamma_n$, $\alpha_n \to 0$, $\theta_n \to 0$, and $\Gamma_n - \Gamma_{n+1} \to 0$, we infer from (3.29) that

$$\lim_{n\to\infty} \|x_n - S^n z_n\| = 0,$$

which, together with the boundedness of $\{x_n\}$, implies that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n(f(x_n) - x_n) + \gamma_n(S^n z_n - x_n)\| \\ &\le \alpha_n \|f(x_n) - x_n\| + \gamma_n \|S^n z_n - x_n\| \\ &\le \alpha_n \|f(x_n) - x_n\| + \|S^n z_n - x_n\| \to 0 \quad (n\to\infty). \end{aligned} \tag{3.30}$$

From the boundedness of $\{x_n\}$, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (f-I)x^*, x_n - x^*\rangle = \lim_{k\to\infty} \langle (f-I)x^*, x_{n_k} - x^*\rangle. \tag{3.31}$$

Since H is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_k} \rightharpoonup \widetilde{x}$. Thus, from (3.31) one gets

$$\begin{aligned} \limsup_{n\to\infty} \langle (f-I)x^*, x_n - x^*\rangle &= \lim_{k\to\infty} \langle (f-I)x^*, x_{n_k} - x^*\rangle \\ &= \langle (f-I)x^*, \widetilde{x} - x^*\rangle. \end{aligned} \tag{3.32}$$

Since $S^n x_n - S^{n+1} x_n \to 0$ (due to the assumption), $u_n - Gu_n \to 0$ (due to (3.28)), $x_n - x_{n+1} \to 0$ (due to (3.30)), and $x_{n_k} \rightharpoonup \widetilde{x}$ for $\{x_{n_k}\} \subset \{x_n\}$, by Lemma 3.3, we obtain that $\widetilde{x} \in \Omega$. Hence from (3.12) and (3.32), one gets

$$\limsup_{n\to\infty} \langle (f-I)x^*, x_n - x^*\rangle = \langle (f-I)x^*, \widetilde{x} - x^*\rangle \le 0, \tag{3.33}$$

which, together with (3.30), leads to

$$\begin{aligned} &\limsup_{n\to\infty} \langle (f-I)x^*, x_{n+1} - x^*\rangle \\ &\quad = \limsup_{n\to\infty} \big[\langle (f-I)x^*, x_{n+1} - x_n\rangle + \langle (f-I)x^*, x_n - x^*\rangle\big] \\ &\quad \le \limsup_{n\to\infty} \big[\|(f-I)x^*\| \|x_{n+1} - x_n\| + \langle (f-I)x^*, x_n - x^*\rangle\big] \le 0. \end{aligned} \tag{3.34}$$

Note that $\{\alpha_n(1-\delta)\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n(1-\delta) = \infty$, and

$$\limsup_{n\to\infty} \left[\frac{2\langle (f-I)x^*, x_{n+1} - x^*\rangle}{1-\delta} + \frac{\theta_n}{\alpha_n} \cdot \frac{(2+\theta_n)M_0}{1-\delta}\right] \le 0.$$

Consequently, applying Lemma 2.4 to (3.22), one has $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$.

Case 2. Suppose that $\exists \{\Gamma_{n_k}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_k} < \Gamma_{n_k+1}$ $\forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau : \mathcal{N} \to \mathcal{N}$ by

$$\tau(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

By Lemma 2.6, we get

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \quad \text{and} \quad \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Putting $p = x^*$ and $q = y^*$, from (3.15) and (3.16), we obtain

$$\begin{aligned} &\gamma_{\tau(n)}\Big\{\|x_{\tau(n)} - u_{\tau(n)}\|^2 + (1-\mu)\big[\|y_{\tau(n)} - z_{\tau(n)}\|^2 + \|y_{\tau(n)} - w_{\tau(n)}\|^2\big] + \mu_2(2\beta - \mu_2) \\ &\quad \times \big\|B_2 u_{\tau(n)} - B_2 x^*\big\|^2 + \mu_1(2\alpha - \mu_1)\big\|B_1 \upsilon_{\tau(n)} - B_1 y^*\big\|^2\Big\} \\ &\quad \leq \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + \theta_{\tau(n)}(2 + \theta_{\tau(n)})M_0 + 2\alpha_{\tau(n)}M_0 \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} &\gamma_{\tau(n)}\Big[\big\|u_{\tau(n)} - \upsilon_{\tau(n)} + y^* - x^*\big\|^2 + \big\|\upsilon_{\tau(n)} - w_{\tau(n)} + x^* - y^*\big\|^2\Big] \\ &\quad \leq \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + 2\mu_2\big\|B_2 x^* - B_2 u_{\tau(n)}\big\|\big\|\upsilon_{\tau(n)} - y^*\big\| \\ &\qquad + 2\mu_1\big\|B_1 y^* - B_1 \upsilon_{\tau(n)}\big\|\big\|w_{\tau(n)} - x^*\big\| + \theta_{\tau(n)}(2 + \theta_{\tau(n)})M_0 + 2\alpha_{\tau(n)}M_0. \end{aligned} \tag{3.36}$$

So it follows from (3.35) that

$$\lim_{n\to\infty}\|x_{\tau(n)} - u_{\tau(n)}\| = \lim_{n\to\infty}\|y_{\tau(n)} - z_{\tau(n)}\| = \lim_{n\to\infty}\|y_{\tau(n)} - w_{\tau(n)}\| = 0, \tag{3.37}$$

and

$$\lim_{n\to\infty}\big\|B_2 u_{\tau(n)} - B_2 x^*\big\| = \lim_{n\to\infty}\big\|B_1 \upsilon_{\tau(n)} - B_1 y^*\big\| = 0. \tag{3.38}$$

Further, from (3.36), (3.38), and the boundedness of $\{\upsilon_{\tau(n)}\}$, $\{w_{\tau(n)}\}$, we deduce that

$$\lim_{n\to\infty}\big\|u_{\tau(n)} - \upsilon_{\tau(n)} + y^* - x^*\big\| = \lim_{n\to\infty}\big\|\upsilon_{\tau(n)} - w_{\tau(n)} + x^* - y^*\big\| = 0.$$

Therefore,

$$\begin{aligned} \|u_{\tau(n)} - Gu_{\tau(n)}\| &= \|u_{\tau(n)} - w_{\tau(n)}\| \\ &\leq \big\|u_{\tau(n)} - \upsilon_{\tau(n)} + y^* - x^*\big\| + \big\|\upsilon_{\tau(n)} - w_{\tau(n)} + x^* - y^*\big\| \\ &\to 0 \quad (n \to \infty). \end{aligned} \tag{3.39}$$

Utilizing the same inferences as in the proof of Case 1, we deduce that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \tag{3.40}$$

and

$$\limsup_{n\to\infty}\big\langle(f - I)x^*, x_{\tau(n)+1} - x^*\big\rangle \leq 0. \tag{3.41}$$

On the other hand, from (3.22) we obtain

$$\begin{aligned}\alpha_{\tau(n)}(1-\delta)\Gamma_{\tau(n)} &\le \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + \alpha_{\tau(n)}(1-\delta)\biggl[\frac{2\langle (f-I)x^*, x_{\tau(n)+1}-x^*\rangle}{1-\delta} \\ &\quad + \frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\cdot\frac{(2+\theta_{\tau(n)})M_0}{1-\delta}\biggr] \\ &\le \alpha_{\tau(n)}(1-\delta)\biggl[\frac{2\langle (f-I)x^*, x_{\tau(n)+1}-x^*\rangle}{1-\delta} + \frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\cdot\frac{(2+\theta_{\tau(n)})M_0}{1-\delta}\biggr],\end{aligned}$$

which hence yields

$$\limsup_{n\to\infty}\Gamma_{\tau(n)} \le \limsup_{n\to\infty}\biggl[\frac{2\langle (f-I)x^*, x_{\tau(n)+1}-x^*\rangle}{1-\delta} + \frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\cdot\frac{(2+\theta_{\tau(n)})M_0}{1-\delta}\biggr] \le 0.$$

Thus, $\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2 = 0$. Also, note that

$$\begin{aligned}&\|x_{\tau(n)+1}-x^*\|^2 - \|x_{\tau(n)}-x^*\|^2 \\ &\quad = 2\langle x_{\tau(n)+1}-x_{\tau(n)}, x_{\tau(n)}-x^*\rangle + \|x_{\tau(n)+1}-x_{\tau(n)}\|^2 \\ &\quad \le 2\|x_{\tau(n)+1}-x_{\tau(n)}\|\|x_{\tau(n)}-x^*\| + \|x_{\tau(n)+1}-x_{\tau(n)}\|^2.\end{aligned} \tag{3.42}$$

Owing to $\Gamma_n \le \Gamma_{\tau(n)+1}$, we get

$$\begin{aligned}\|x_n-x^*\|^2 &\le \|x_{\tau(n)+1}-x^*\|^2 \\ &\le \|x_{\tau(n)}-x^*\|^2 + 2\|x_{\tau(n)+1}-x_{\tau(n)}\|\|x_{\tau(n)}-x^*\| + \|x_{\tau(n)+1}-x_{\tau(n)}\|^2 \\ &\to 0 \quad (n\to\infty).\end{aligned}$$

That is, $x_n \to x^*$ as $n\to\infty$. This completes the proof. □

Theorem 3.2 *Let $S: H \to C$ be nonexpansive and the sequence $\{x_n\}$ be constructed by the modified version of Algorithm 3.1, that is, for any initial $x_1 \in C$,*

$$\begin{cases} u_n = \sigma_n x_n + (1-\sigma_n)S_n u_n, \\ w_n = Gu_n, \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n \quad \forall n \ge 1, \end{cases} \tag{3.43}$$

where for each $n \ge 1$, C_n and τ_n are chosen as in Algorithm 3.1. Then $x_n \to x^* \in \Omega$, where $x^* \in \Omega$ is the unique solution to the HVI, $\langle (I-f)x^*, p-x^*\rangle \ge 0$ $\forall p \in \Omega$.

Proof We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, using the same arguments as in Step 1 of the proof of Theorem 3.1, we obtain the desired assertion.

Step 2. We show that

$$\begin{aligned}&\gamma_n\{\|x_n-u_n\|^2+(1-\mu)\big[\|y_n-z_n\|^2+\|y_n-w_n\|^2\big]+\mu_2(2\beta-\mu_2)\\&\quad\times\|B_2u_n-B_2p\|^2+\mu_1(2\alpha-\mu_1)\|B_1\upsilon_n-B_1q\|^2\}\\&\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_nM_0\end{aligned}$$

and

$$\begin{aligned}&\gamma_n\big[\|u_n-\upsilon_n+q-p\|^2+\|\upsilon_n-w_n+p-q\|^2\big]\\&\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_2\|B_2p-B_2u_n\|\|\upsilon_n-q\|\\&\quad+2\mu_1\|B_1q-B_1\upsilon_n\|\|w_n-p\|+2\alpha_nM_0,\end{aligned}$$

where $\sup_{n\geq1}\{\|x_n-p\|^2+\|f(p)-p\|\|x_n-p\|\}\leq M_0$ for some $M_0>0$. In fact, using the same arguments as in Step 2 of the proof of Theorem 3.1, we obtain the desired assertion.

Step 3. We show that

$$\|x_{n+1}-p\|^2\leq\big[1-\alpha_n(1-\delta)\big]\|x_n-p\|^2+\alpha_n(1-\delta)\frac{2\langle(f-I)p,x_{n+1}-p\rangle}{1-\delta}.$$

In fact, using the same arguments as in Step 3 of the proof of Theorem 3.1, we obtain the desired assertion.

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^*\in\Omega$ to the HVI (3.12), with $S_0=S$ a nonexpansive mapping. In fact, putting $p=x^*$, we deduce from Step 3 that

$$\|x_{n+1}-x^*\|^2\leq\big[1-\alpha_n(1-\delta)\big]\|x_n-x^*\|^2+\alpha_n(1-\delta)\frac{2\langle(f-I)x^*,x_{n+1}-x^*\rangle}{1-\delta}.\qquad(3.44)$$

Putting $\Gamma_n=\|x_n-x^*\|^2$, we show the convergence of $\{\Gamma_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $n_0\geq1$ such that $\{\Gamma_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\Gamma_n=\hbar<+\infty$ and $\lim_{n\to\infty}(\Gamma_n-\Gamma_{n+1})=0$. Putting $p=x^*$ and $q=y^*$, from Step 2 we obtain

$$\begin{aligned}&\gamma_n\{\|x_n-u_n\|^2+(1-\mu)\big[\|y_n-z_n\|^2+\|y_n-w_n\|^2\big]+\mu_2(2\beta-\mu_2)\\&\quad\times\|B_2u_n-B_2x^*\|^2+\mu_1(2\alpha-\mu_1)\|B_1\upsilon_n-B_1y^*\|^2\}\\&\leq\Gamma_n-\Gamma_{n+1}+2\alpha_nM_0\end{aligned}$$

and

$$\begin{aligned}&\gamma_n\big[\|u_n-\upsilon_n+y^*-x^*\|^2+\|\upsilon_n-w_n+x^*-y^*\|^2\big]\\&\leq\Gamma_n-\Gamma_{n+1}+2\mu_2\|B_2x^*-B_2u_n\|\|\upsilon_n-y^*\|\\&\quad+2\mu_1\|B_1y^*-B_1\upsilon_n\|\|w_n-x^*\|+2\alpha_nM_0.\end{aligned}$$

By the same inferences as in Case 1 of the proof of Theorem 3.1, we deduce that

$$\lim_{n\to\infty}\|u_n-Gu_n\|=0,\qquad(3.45)$$

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0 \quad \text{and} \quad \limsup_{n\to\infty}\langle (f-I)x^*, x_{n+1} - x^*\rangle \le 0. \tag{3.46}$$

Consequently, applying Lemma 2.4 to (3.44), we obtain $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$.

Case 2. Suppose that $\exists\{\Gamma_{n_k}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_k} < \Gamma_{n_k+1}$ $\forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau : \mathcal{N} \to \mathcal{N}$ by

$$\tau(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

By Lemma 2.6, we get

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \quad \text{and} \quad \Gamma_n \le \Gamma_{\tau(n)+1}.$$

The conclusion follows using the same arguments as in Case 2 of the proof of Theorem 3.1. □

Next, we introduce another composite subgradient extragradient algorithm.

Algorithm 3.2 *Initialization*: Given $\gamma > 0$, $\mu \in (0,1)$, $\ell \in (0,1)$, pick an initial $x_1 \in C$ arbitrarily.

Iterative steps: Compute x_{n+1} below:

Step 1. Calculate $u_n = \sigma_n x_n + (1-\sigma_n)S_n u_n$ and $w_n = Gu_n$, and set $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma\ell, \gamma\ell^2, \dots\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \le \mu\|w_n - y_n\|. \tag{3.47}$$

Step 2. Calculate $z_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{y \in H : \langle w_n - \tau_n A w_n - y_n, y - y_n\rangle \le 0\}$.

Step 3. Calculate

$$x_{n+1} = \alpha_n f(x_n) + \beta_n u_n + \gamma_n S^n z_n. \tag{3.48}$$

Again put $n := n + 1$ and return to Step 1.

It is worth pointing out that inequality (3.5) and Lemmas 3.1–3.3 are still valid for Algorithm 3.2.

Theorem 3.3 *Let $\{x_n\}$ be the sequence constructed in Algorithm 3.2. Then $x_n \to x^* \in \Omega$, provided $S^n x_n - S^{n+1}x_n \to 0$, where $x^* \in \Omega$ is the unique solution to the HVI, $\langle (I-f)x^*, p - x^*\rangle \ge 0$ $\forall p \in \Omega$.*

Proof Using the same arguments as in the proof of Theorem 3.1, we deduce that there exists the unique solution $x^* \in \Omega = \bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C,A)$ to the HVI (3.12). We divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, using the same arguments as in Step 1 of the proof of Theorem 3.1, we obtain that inequalities (3.13) and (3.14) hold. Thus, from

(3.14) it follows that

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|f(x_n)-p\| + \beta_n\|u_n-p\| + \gamma_n\|S^n z_n - p\| \\ &\le \alpha_n\big(\|f(x_n)-f(p)\| + \|f(p)-p\|\big) + \beta_n\|u_n-p\| + \gamma_n(1+\theta_n)\|z_n-p\| \\ &\le \alpha_n\big(\delta\|x_n-p\| + \|f(p)-p\|\big) + \beta_n\|x_n-p\| + (\gamma_n+\theta_n)\|x_n-p\| \\ &\le \left[1-\frac{\alpha_n(1-\delta)}{2}\right]\|x_n-p\| + \alpha_n\|f(p)-p\| \\ &= \left[1-\frac{\alpha_n(1-\delta)}{2}\right]\|x_n-p\| + \frac{\alpha_n(1-\delta)}{2}\frac{2\|f(p)-p\|}{1-\delta} \\ &\le \max\left\{\|x_n-p\|, \frac{2\|f(p)-p\|}{1-\delta}\right\}.\end{aligned}$$

By induction, we obtain $\|x_n-p\| \le \max\{\|x_1-p\|, \frac{2\|f(p)-p\|}{1-\delta}\}$ $\forall n \ge 1$. Therefore, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Ay_n\}$, $\{S_n u_n\}$, $\{S^n z_n\}$.

Step 2. We show that

$$\begin{aligned}&\gamma_n\big\{\|x_n-u_n\|^2 + (1-\mu)\big[\|y_n-z_n\|^2 + \|y_n-w_n\|^2\big] + \mu_2(2\beta-\mu_2) \\ &\quad\times\|B_2u_n - B_2p\|^2 + \mu_1(2\alpha-\mu_1)\|B_1\upsilon_n - B_1q\|^2\big\} \\ &\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \theta_n(2+\theta_n)M_0 + 2\alpha_n M_0\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}&\gamma_n\big[\|u_n-\upsilon_n+q-p\|^2 + \|\upsilon_n - w_n + p - q\|^2\big] \\ &\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\mu_2\|B_2p - B_2u_n\|\|\upsilon_n - q\| \\ &\quad + 2\mu_1\|B_1q - B_1\upsilon_n\|\|w_n-p\| + \theta_n(2+\theta_n)M_0 + 2\alpha_n M_0,\end{aligned} \tag{3.50}$$

for some $M_0 > 0$. In fact, using (3.5), (3.13), (3.14), and the convexity of the function $\phi(s) = s^2$ $\forall s \in \mathbf{R}$, we get

$$\begin{aligned}&\|x_{n+1}-p\|^2 \\ &\le \alpha_n\|f(x_n)-f(p)\|^2 + \beta_n\|u_n-p\|^2 + \gamma_n\|S^n z_n - p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\ &\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|u_n-p\|^2 + \big[\gamma_n + \theta_n(2+\theta_n)\big]\|z_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\ &\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 + \gamma_n\big\{\|x_n-p\|^2 - \|x_n-u_n\|^2 - (1-\mu)\big[\|y_n-z_n\|^2 \\ &\quad + \|y_n-w_n\|^2\big] - \mu_2(2\beta-\mu_2)\|B_2u_n - B_2p\|^2 - \mu_1(2\alpha-\mu_1)\|B_1\upsilon_n - B_1q\|^2\big\} \\ &\quad + \theta_n(2+\theta_n)\|x_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\ &\le \|x_n-p\|^2 - \gamma_n\big\{\|x_n-u_n\|^2 + (1-\mu)\big[\|y_n-z_n\|^2 + \|y_n-w_n\|^2\big] \\ &\quad + \mu_2(2\beta-\mu_2)\|B_2u_n - B_2p\|^2 + \mu_1(2\alpha-\mu_1)\|B_1\upsilon_n - B_1q\|^2\big\} \\ &\quad + \theta_n(2+\theta_n)M_0 + 2\alpha_n M_0\end{aligned} \tag{3.51}$$

where $\sup_{n\ge1}\{\|x_n-p\|^2 + \|f(p)-p\|\|x_n-p\|\} \le M_0$ for some $M_0 > 0$. This ensures that (3.49) holds. Further, using similar arguments to those of (3.16), we obtain that (3.50) holds.

Step 3. We show that

$$\|x_{n+1}-p\|^2 \le \big[1-\alpha_n(1-\delta)\big]\|x_n-p\|^2 + \alpha_n(1-\delta)\left\{\frac{2\langle (f-I)p, x_{n+1}-p\rangle}{1-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{(2+\theta_n)M_0}{1-\delta}\right\}.$$

In fact, from (3.14) and (3.51), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|u_n-p\|^2 + \big[\gamma_n+\theta_n(2+\theta_n)\big]\|z_n-p\|^2 + 2\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\ &\le \alpha_n\delta\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 + \gamma_n\|x_n-p\|^2 + \theta_n(2+\theta_n)M_0 \\ &\quad + 2\alpha_n\big\langle f(p)-p, x_{n+1}-p\big\rangle \\ &= \big[1-\alpha_n(1-\delta)\big]\|x_n-p\|^2 + \alpha_n(1-\delta)\left\{\frac{2\langle (f-I)p, x_{n+1}-p\rangle}{1-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{(2+\theta_n)M_0}{1-\delta}\right\}.\end{aligned}$$

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (3.12). In fact, putting $p = x^*$, we deduce from Step 3 that

$$\big\|x_{n+1}-x^*\big\|^2 \le \big[1-\alpha_n(1-\delta)\big]\big\|x_n-x^*\big\|^2 + \alpha_n(1-\delta)\left\{\frac{2\langle (f-I)x^*, x_{n+1}-x^*\rangle}{1-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{(2+\theta_n)M_0}{1-\delta}\right\}. \tag{3.52}$$

Putting $\Gamma_n = \|x_n - x^*\|^2$, we show the convergence of $\{\Gamma_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $n_0 \ge 1$ such that $\{\Gamma_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\Gamma_n = \hbar < +\infty$ and $\lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$. Putting $p = x^*$ and $q = y^*$, from (3.49) and (3.50), we obtain that

$$\begin{aligned}&\gamma_n\big\{\|x_n-u_n\|^2 + (1-\mu)\big[\|y_n-z_n\|^2 + \|y_n-w_n\|^2\big] + \mu_2(2\beta-\mu_2) \\ &\quad\times \big\|B_2u_n - B_2x^*\big\|^2 + \mu_1(2\alpha-\mu_1)\big\|B_1v_n - B_1y^*\big\|^2\big\} \\ &\le \Gamma_n - \Gamma_{n+1} + \theta_n(2+\theta_n)M_0 + 2\alpha_nM_0\end{aligned}$$

and

$$\begin{aligned}&\gamma_n\big[\big\|u_n-v_n+y^*-x^*\big\|^2 + \big\|v_n-w_n+x^*-y^*\big\|^2\big] \\ &\le \Gamma_n - \Gamma_{n+1} + 2\mu_2\big\|B_2x^* - B_2u_n\big\|\big\|v_n-y^*\big\| \\ &\quad + 2\mu_1\big\|B_1y^* - B_1v_n\big\|\big\|w_n - x^*\big\| + \theta_n(2+\theta_n)M_0 + 2\alpha_nM_0.\end{aligned}$$

By the same inferences as in Case 1 of the proof of Theorem 3.1, we deduce that $u_n - Gu_n \to 0$, $x_n - x_{n+1} \to 0$ and

$$\limsup_{n\to\infty}\big\langle (f-I)x^*, x_{n+1}-x^*\big\rangle \le 0.$$

Consequently, applying Lemma 2.4 to (3.52), we obtain $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$.

Case 2. Suppose that $\exists\{\Gamma_{n_k}\}\subset\{\Gamma_n\}$ such that $\Gamma_{n_k}<\Gamma_{n_k+1}$ $\forall k\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau:\mathcal{N}\to\mathcal{N}$ by

$$\tau(n):=\max\{k\le n:\Gamma_k<\Gamma_{k+1}\}.$$

By Lemma 2.6, we get

$$\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1}\quad\text{and}\quad\Gamma_n\le\Gamma_{\tau(n)+1}.$$

In the remainder of the proof, using the same arguments as in Case 2 of Step 4 in the proof of Theorem 3.1, we obtain the desired conclusion. □

Theorem 3.4 *Let $S:H\to C$ be nonexpansive and the sequence $\{x_n\}$ be constructed by the modified version of Algorithm 3.1, that is, for any initial $x_1\in C$,*

$$\begin{cases} u_n=\sigma_n x_n+(1-\sigma_n)S_n u_n,\\ w_n=Gu_n,\\ y_n=P_C(w_n-\tau_n Aw_n),\\ z_n=P_{C_n}(w_n-\tau_n Ay_n),\\ x_{n+1}=\alpha_n f(x_n)+\beta_n u_n+\gamma_n Sz_n\quad\forall n\ge 1,\end{cases}\tag{3.53}$$

where for each $n\ge1$, C_n and τ_n are chosen as in Algorithm 3.2. Then $x_n\to x^*\in\Omega$, where $x^*\in\Omega$ is the unique solution to the HVI, $\langle(I-f)x^*,p-x^*\rangle\ge0$ $\forall p\in\Omega$.

Proof We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, using the same arguments as in Step 1 of the proof of Theorem 3.3, we obtain the desired assertion.

Step 2. We show that

$$\begin{aligned}&\gamma_n\bigl\{\|x_n-u_n\|^2+(1-\mu)\bigl[\|y_n-z_n\|^2+\|y_n-w_n\|^2\bigr]+\mu_2(2\beta-\mu_2)\\&\qquad\times\|B_2u_n-B_2p\|^2+\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2\bigr\}\\&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_nM_0\end{aligned}$$

and

$$\begin{aligned}&\gamma_n\bigl[\|u_n-v_n+q-p\|^2+\|v_n-w_n+p-q\|^2\bigr]\\&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|\\&\qquad+2\mu_1\|B_1q-B_1v_n\|\|w_n-p\|+2\alpha_nM_0,\end{aligned}$$

where $\sup_{n\ge1}\{\|x_n-p\|^2+\|f(p)-p\|\|x_n-p\|\}\le M_0$ for some $M_0>0$. In fact, using the same arguments as in Step 2 of the proof of Theorem 3.3, we obtain the desired assertion.

Step 3. We show that

$$\|x_{n+1}-p\|^2\le\bigl[1-\alpha_n(1-\delta)\bigr]\|x_n-p\|^2+\alpha_n(1-\delta)\frac{2\langle(f-I)p,x_{n+1}-p\rangle}{1-\delta}.$$

In fact, using the same arguments as in Step 3 of the proof of Theorem 3.3, we obtain the desired assertion.

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ to the HVI (3.12), with $S_0 = S$ a nonexpansive mapping. In fact, putting $p = x^*$, we deduce from Step 3 that

$$\|x_{n+1} - x^*\|^2 \le [1 - \alpha_n(1-\delta)]\|x_n - x^*\|^2 + \alpha_n(1-\delta)\frac{2\langle (f-I)x^*, x_{n+1} - x^*\rangle}{1-\delta}. \tag{3.54}$$

Putting $\Gamma_n = \|x_n - x^*\|^2$, we show the convergence of $\{\Gamma_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $n_0 \ge 1$ such that $\{\Gamma_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty} \Gamma_n = \hbar < +\infty$ and $\lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$. Putting $p = x^*$ and $q = y^*$, from Step 2 we obtain

$$\begin{aligned} &\gamma_n\{\|x_n - u_n\|^2 + (1-\mu)[\|y_n - z_n\|^2 + \|y_n - w_n\|^2] + \mu_2(2\beta - \mu_2) \\ &\quad \times \|B_2 u_n - B_2 x^*\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2\} \\ &\le \Gamma_n - \Gamma_{n+1} + 2\alpha_n M_0 \end{aligned}$$

and

$$\begin{aligned} &\gamma_n[\|u_n - v_n + y^* - x^*\|^2 + \|v_n - w_n + x^* - y^*\|^2] \\ &\le \Gamma_n - \Gamma_{n+1} + 2\mu_2\|B_2 x^* - B_2 u_n\|\|v_n - y^*\| \\ &\quad + 2\mu_1\|B_1 y^* - B_1 v_n\|\|w_n - x^*\| + 2\alpha_n M_0. \end{aligned}$$

By the same arguments as in Case 1 of the proof of Theorem 3.3, we deduce that $u_n - Gu_n \to 0$, $x_n - x_{n+1} \to 0$ and

$$\limsup_{n\to\infty}\langle (f-I)x^*, x_{n+1} - x^*\rangle \le 0.$$

Consequently, applying Lemma 2.4 to (3.54), we obtain $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$.

Case 2. Suppose that $\exists\{\Gamma_{n_k}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_k} < \Gamma_{n_k+1}$ $\forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\tau : \mathcal{N} \to \mathcal{N}$ by

$$\tau(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

By Lemma 2.6, we get

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \quad \text{and} \quad \Gamma_n \le \Gamma_{\tau(n)+1}.$$

The conclusion follows using the same arguments as in Case 2 of the proof of Theorem 3.3. □

Remark 3.1 Compared with the corresponding results in Ceng and Wen [21], Ceng and Shang [22], and Thong and Hieu [14], our results improve and extend them in the following aspects:

(i) The problem of finding an element of $\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)$ in [21] is extended to develop our problem of finding an element of $\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ where $\{S_i\}_{i=1}^{\infty}$ is a countable family of ς-uniformly Lipschitzian pseudocontractive mappings and $S_0 = S$ is asymptotically nonexpansive. The hybrid extragradient-like implicit method for finding an element of $\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)$ in [21] is extended to develop our Mann implicit composite subgradient extragradient method with line-search process for finding an element of $\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$, which is based on the Mann implicit iteration method, subgradient extragradient method with line-search process, and viscosity approximation method.

(ii) The problem of finding an element of $\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ with quasinonexpansive mapping S in [14] is extended to develop our problem of finding an element of $\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ where $\{S_i\}_{i=1}^{\infty}$ is a countable family of ς-uniformly Lipschitzian pseudocontractive mappings and $S_0 = S$ is asymptotically nonexpansive. The inertial subgradient extragradient method with linear-search process for finding an element of $\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ in [14] is extended to develop our Mann implicit composite subgradient extragradient method with line-search process for finding an element of $\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$, which is based on the Mann implicit iteration method, subgradient extragradient method with line-search process, and viscosity approximation method.

(iii) The problem of finding an element of $\Omega = \bigcap_{i=0}^{N}\mathrm{Fix}(S_i)\cap\mathrm{VI}(C,A)$ with finitely many nonexpansive mappings $\{S_i\}_{i=1}^{N}$ is extended to develop our problem of finding an element of $\Omega = \bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ with a countable family of ς-uniformly Lipschitzian pseudocontractive mappings $\{S_i\}_{i=1}^{\infty}$. The hybrid inertial subgradient extragradient method with line-search process in [22] is extended to develop our Mann implicit composite subgradient extragradient method with line-search process, e.g., the original inertial approach $w_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1})$ is replaced by Mann implicit composite iteration method $u_n = \sigma_n x_n + (1-\sigma_n)Su_n$ and $w_n = Gu_n$. In addition, it was shown in [22] that, under condition $S^n z_n - S^{n+1} z_n \to 0$, the conclusion holds:

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \|x_n - y_n\| + \|x_n - x_{n+1}\| \to 0 \quad \text{with } x^* = P_\Omega(I - \rho F + f)x^*.$$

In this paper, using Lemma 2.6, we show that, under condition $S^n x_n - S^{n+1} x_n \to 0$, the following conclusion holds:

$$x_n \to x^* \in \Omega \quad \text{with } x^* = P_\Omega f\bigl(x^*\bigr).$$

4 Applications

In this section, applying our main results, we deal with the GSVI, VIP, and CFPP in an illustrated example. Put $\mu_1 = \mu_2 = \frac{1}{3}$, $\gamma = 1$, $\mu = \ell = \frac{1}{2}$, $\sigma_n = \frac{2}{3}$, $\alpha_n = \frac{1}{3(n+1)}$, $\beta_n = \frac{n}{3(n+1)}$, and $\gamma_n = \frac{2}{3}$.

We first provide an example of two inverse-strongly monotone mappings $B_1, B_2 : C \to H$, Lipschitz continuous and pseudomonotone mapping A, asymptotically nonexpansive mapping S, and countably many ς-uniformly Lipschitzian pseudocontractive mappings $\{S_i\}_{i=1}^{\infty}$ with $\Omega = \bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A) \neq \emptyset$ with $S_0 := S$. Let $C = [-3, 3]$ and $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The initial point x_1 is randomly chosen in C. Take $f(x) = \frac{1}{2}x$ $\forall x \in C$ with $\delta = \frac{1}{2}$, and put $B_1 x = B_2 x := Bx = x - \frac{1}{2}\sin x$ $\forall x \in C$. Let $A : H \to H$ and $S, S_i : C \to C$ be defined as $Au := \frac{1}{1+|\sin u|} - \frac{1}{1+|u|}$, $Su := \frac{5}{6}\sin u$, and $S_i u = Tu = \sin u$ $\forall u \in H$, $i \geq 1$. We now claim that B is $\frac{2}{9}$-inverse-strongly monotone. In fact, since B is $\frac{1}{2}$-strongly monotone and $\frac{3}{2}$-Lipschitz continuous, we know that B is $\frac{2}{9}$-inverse-strongly monotone with $\alpha = \beta = \frac{2}{9}$. Let us show that A is pseudomonotone and Lipschitz continuous. In fact, for all $u, v \in H$, we have

$$\begin{aligned}\|Au - Av\| &\leq \left|\frac{\|v\| - \|u\|}{(1 + \|u\|)(1 + \|v\|)}\right| + \left|\frac{\|\sin v\| - \|\sin u\|}{(1 + \|\sin u\|)(1 + \|\sin v\|)}\right| \\ &\leq \frac{\|v - u\|}{(1 + \|u\|)(1 + \|v\|)} + \frac{\|\sin v - \sin u\|}{(1 + \|\sin u\|)(1 + \|\sin v\|)} \\ &\leq \|u - v\| + \|\sin u - \sin v\| \leq 2\|u - v\|.\end{aligned}$$

This implies that A is Lipschitz continuous with $L = 2$. Next, we show that A is pseudomonotone. For each $u, v \in H$, it is easy to see that

$$\begin{aligned}&\langle Au, v - u\rangle = \left(\frac{1}{1 + |\sin u|} - \frac{1}{1 + |u|}\right)(v - u) \geq 0 \\ &\quad\Rightarrow \quad \langle Av, v - u\rangle = \left(\frac{1}{1 + |\sin v|} - \frac{1}{1 + |v|}\right)(v - u) \geq 0.\end{aligned}$$

Besides, it is easy to verify that S is asymptotically nonexpansive with $\theta_n = (\frac{5}{6})^n$ $\forall n \geq 1$, such that $\|S^{n+1}x_n - S^n x_n\| \to 0$ as $n \to \infty$. Indeed, we observe that

$$\left\|S^n u - S^n v\right\| \leq \frac{5}{6}\left\|S^{n-1}u - S^{n-1}v\right\| \leq \cdots \leq \left(\frac{5}{6}\right)^n \|u - v\| \leq (1 + \theta_n)\|u - v\|$$

and

$$\begin{aligned}\left\|S^{n+1}x_n - S^n x_n\right\| &\leq \left(\frac{5}{6}\right)^{n-1}\left\|S^2 x_n - Sx_n\right\| = \left(\frac{5}{6}\right)^{n-1}\left\|\frac{5}{6}\sin(Sx_n) - \frac{5}{6}\sin x_n\right\| \\ &\leq 2\left(\frac{5}{6}\right)^n \to 0.\end{aligned}$$

It is clear that $\mathrm{Fix}(S) = \{0\}$ and

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = \lim_{n\to\infty}\frac{(5/6)^n}{1/3(n+1)} = 0.$$

In addition, it is clear that $S_i = T$ is nonexpansive and $\mathrm{Fix}(T) = \{0\}$. Therefore, $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. In this case, noticing $S_n = T$ and $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2) = [P_C(I - \frac{1}{3}B)]^2$, we rewrite Algorithm 3.1 as follows:

$$\begin{cases} u_n = \frac{2}{3}x_n + \frac{1}{3}Tu_n, \\ w_n = [P_C(I - \frac{1}{3}B)]^2 u_n, \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}x_n + \frac{2}{3}S^n z_n \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.1. Then, by Theorem 3.1, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$.

In particular, since $Su := \frac{5}{6} \sin u$ is also nonexpansive, we consider the modified version of Algorithm 3.1, that is,

$$\begin{cases} u_n = \frac{2}{3}x_n + \frac{1}{3}Tu_n, \\ w_n = [P_C(I - \frac{1}{3}B)]^2 u_n, \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + \frac{n}{3(n+1)}x_n + \frac{2}{3}Sz_n \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where for each $n \geq 1$, C_n and τ_n are chosen as above. Then, by Theorem 3.2, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$.

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Competing interests
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Authors' contributions
Conceptualization and Formal analysis are done by L-CC, YS and J-CY. Funding acquisition, Project administration and Supervision are done by J-CY. Investigation and Methodology are done by L-CC, YS and J-CY. All authors have read and agreed to the published version of the manuscript.

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References

1. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Èkon. Mat. Metody **12**, 747–756 (1976)
2. Yao, Y., Liou, Y.C., Kang, S.M.: Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. Comput. Math. Appl. **59**, 3472–3480 (2010)
3. Jolaoso, L.O., Shehu, Y., Yao, J.C.: Inertial extragradient type method for mixed variational inequalities without monotonicity. Math. Comput. Simul. **192**, 353–369 (2022)
4. Ceng, L.C., Petrusel, A., Yao, J.C., Yao, Y.: Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. Fixed Point Theory **19**(2), 487–501 (2018)
5. Ceng, L.C., Petrusel, A., Qin, X., Yao, J.C.: Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints. Optimization **70**, 1337–1358 (2021)
6. Ceng, L.C., Wang, C.Y., Yao, J.C.: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math. Methods Oper. Res. **67**, 375–390 (2008)

7. Yao, Y., Shahzad, N., Yao, J.C.: Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems. Carpath. J. Math. **37**, 541–550 (2021)
8. He, L., Cui, Y.L., Ceng, L.C., Zhao, T.-Y., Wang, D.-Q., Hu, H.-Y.: Strong convergence for monotone bilevel equilibria with constraints of variational inequalities and fixed points using subgradient extragradient implicit rule. J. Inequal. Appl. **2021**, Paper No. 146, 37 pp. (2021)
9. Zhao, T.Y., Wang, D.Q., Ceng, L.C., et al.: Quasi-inertial Tseng's extragradient algorithms for pseudomonotone variational inequalities and fixed point problems of quasi-nonexpansive operators. Numer. Funct. Anal. Optim. **42**, 69–90 (2020)
10. Denisov, S.V., Semenov, V.V., Chabak, L.M.: Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. Cybern. Syst. Anal. **51**, 757–765 (2015)
11. Cai, G., Shehu, Y., Iyiola, O.S.: Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules. Numer. Algorithms **77**, 535–558 (2018)
12. Yang, J., Liu, H., Liu, Z.: Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization **67**, 2247–2258 (2018)
13. Vuong, P.T.: On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. J. Optim. Theory Appl. **176**, 399–409 (2018)
14. Thong, D.V., Hieu, D.V.: Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. Numer. Algorithms **80**, 1283–1307 (2019)
15. Shehu, Y., Iyiola, O.S.: Strong convergence result for monotone variational inequalities. Numer. Algorithms **76**, 259–282 (2017)
16. Thong, D.V., Dong, Q.L., Liu, L.L., Triet, N.A., Lan, N.P.: Two new inertial subgradient extragradient methods with variable step sizes for solving pseudomonotone variational inequality problems in Hilbert spaces. J. Comput. Appl. Math. **410**, Paper No. 114260 (2022)
17. Vuong, P.T., Shehu, Y.: Convergence of an extragradient-type method for variational inequality with applications to optimal control problems. Numer. Algorithms **81**, 269–291 (2019)
18. Shehu, Y., Dong, Q.L., Jiang, D.: Single projection method for pseudo-monotone variational inequality in Hilbert spaces. Optimization **68**, 385–409 (2019)
19. Thong, D.V., Hieu, D.V.: Modified subgradient extragradient method for variational inequality problems. Numer. Algorithms **79**, 597–610 (2018)
20. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
21. Ceng, L.C., Wen, C.F.: Systems of variational inequalities with hierarchical variational inequality constraints for asymptotically nonexpansive and pseudocontractive mappings. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**, 2431–2447 (2019)
22. Ceng, L.C., Shang, M.J.: Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings. Optimization **70**, 715–740 (2021)
23. Ceng, L.C., Petrusel, A., Qin, X., Yao, J.C.: Pseudomonotone variational inequalities and fixed points. Fixed Point Theory **22**, 543–558 (2021)
24. Reich, S., Thong, D.V., Dong, Q.L., Li, X.H., Dung, V.T.: New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings. Numer. Algorithms **87**, 527–549 (2021)
25. Maingé, P.E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899–912 (2008)
26. Shehu, Y.: Fixed point solutions of generalized equilibrium problems for nonexpansive mappings. J. Comput. Appl. Math. **234**, 892–898 (2010)
27. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker, New York (1984)
28. Reich, S.: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274–276 (1979)
29. Deimling, K.: Zeros of accretive operators. Manuscr. Math. **13**, 365–374 (1974)
30. Xu, H.K., Kim, T.H.: Convergence of hybrid steepest-descent methods for variational inequalities. J. Optim. Theory Appl. **119**, 185–201 (2003)
31. Ceng, L.C., Xu, H.K., Yao, J.C.: The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces. Nonlinear Anal. **69**, 1402–1412 (2008)
32. Martin, Jr. R.H.M.: Differential equations on closed subsets of a Banach space. Trans. Am. Math. Soc. **179**, 399–414 (1973)