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Optimal boundary control problems for a hierarchical age-structured two-species model

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Abstract

This paper discusses the optimal control problems for a nonlinear age-structured two-species model, where elder individuals are more competitive than younger ones, and each species is described by a nonlinear integropartial differential equation with a global feedback boundary condition. First, we establish the existence of a unique nonnegative bounded solution by means of frozen coefficients and the fixed-point theorem. More importantly, we discuss the least deviation-cost problem and the most benefit-cost problem. For the least deviation-cost problem, the existence of an optimal strategy is established by means of Ekeland's variational principle, and the minimum principle is obtained via an adjoint system. Meanwhile, the corresponding results for the most benefit-cost problem are given. In addition, some numerical experiment results are presented to examine the effects of parameters on the optimal policies and indexes.

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Keywords: Hierarchical structure; Optimal control; Least deviation-cost problem; Most benefit-cost problem

1 Introduction

It is well known that populations consist of individuals with many structural differences, which include age, size, gender, and status. Structured population models distinguish individuals from one another according to these structural differences, to determine the birth, growth, and death rates, interaction with each other and with the environment, infectivity, etc. (see [1]). During the past one hundred years, age- and size-structured first-order partial differential equations have provided a main tool for modeling population systems and have been recently employed in economics. To name a few, see [2–12] on the well-posedness, asymptotic behaviors, and optimal control of population models with age structure, while [1, 13–22] discussed the related problems of population systems with size structure.

Ecological studies show that there exist dominance ranks of individuals in many species, such as nonmammalian, primates, rodents, etc., (see [23]). Moreover, [24] pointed out that

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for many species the competition for resources that determine individual fertility and mortality is based on some hierarchy in the population that is related to individuals age or any other physiological structure. They also referred to this competitive phenomenon as "contest competition" (see [25]). Hierarchical structuring in a biotic population is the ranking of individuals one above the other based on their age, body size, height, or any other possible structuring variable that can affect their vital rates (see [26]). Further, Lomnicki [27] showed that the rank of an individual not only influences its access to some factors such as food resource, mates, shelter, and nesting, but also has some influence on the individual's vital rates. Thus, the existence of hierarchy has an impact on the dynamic behaviors of the population. Just as Gurney and Nisbet [26] said, the hierarchy of ranks in a population is one of the significant forces to maintain its ecological stability and species persistence. During the past decade, both discrete-time and continuous hierarchical structured population models have been studied by several researchers (see [28–37]). The hierarchical model is considered as a generalization of the age- and size-structure model. The key is to consider the so-called "internal environment" in the modeling. For example, Ackleh and Deng [29] discussed the following nonlinear hierarchical age-structured population model

$$\begin{cases} \frac{\partial p(a,t)}{\partial t} + \frac{\partial p(a,t)}{\partial a} = -\mu(a,t,Q(p)(a,t))p(a,t), & 0 < a < \infty, t > 0, \\ p(0,t) = \int_0^\infty \beta(a,t,Q(p)(a,t))p(a,t)\,\mathrm{d}a, & t > 0, \\ p(a,0) = p^0(a), & 0 \le a < \infty, \end{cases} \tag{1.1}$$

where the environment $Q(p)(a,t)$ is a function of the density p, which is given by

$$Q(p)(a,t) = \alpha \int_0^a p(r,t)\,\mathrm{d}r + \int_a^\infty p(r,t)\,\mathrm{d}r, \quad 0 \le \alpha < 1. \tag{1.2}$$

Parameter α is used to measure the degree of hierarchy in the population structure, and is called the hierarchical coefficient. More precisely, α is the weight of the lower ranks (i.e., age smaller than a) in the competition for resources. From [38], $\alpha = 0$ (i.e.,"contest competition") implies an absolute hierarchical structure, whereas values of α tending to 1 mean that, when competing for resources, the effect of higher ranks is similar to the effect of lower ranks. Moreover, the limit case $\alpha = 1$ (i.e., "scramble competition") corresponds to the environment given by the total size of population at time t, i.e., without any hierarchical structure.

Most studies on the hierarchical population model mainly discuss the existence and uniqueness of solutions (see [28, 30, 36, 37]), numerical approximation of solutions (see [28, 29, 32, 33, 37]), and the asymptotic behavior of solutions (such as stability and persistence, see [31, 34, 35, 38]). However, studies on optimal control problems of hierarchical species models are rather rare. As far as we know, so far only [39, 40] have investigated optimal control problems in hierarchical species. However, there is no investigation on the optimal boundary control of hierarchical multispecies model. The purpose of this paper is to make some contribution in this direction. In this paper, we will discuss the optimal control problems in a hierarchical system composed of two interacting species with age structure. The control variables describe the inflow rate of the baby individual from an external source (such as artificially stocking fry or artificially planting saplings), which appear in the boundary conditions.

2 Description of the problem

Our study is inspired by that of Ackleh and Deng [29]. To build our model, we assume that there are age structures in two species and they have the same life expectancy (i.e., maximal age) and is $A \in (0, +\infty)$. Moreover, we assume that the mortality and fertility of species depend mainly on the number of elderly individuals in addition to the age of individuals. From [29], this phenomenon can be described as that the vital rates of an individual depend not only on its age, but also on an "internal environment". Let $p_i(a,t)$ be the density of species i with age a at time t $(i = 1,2)$. In a similar way as we developed (1.2), the "internal environment" $E(p_i)$ of species i can be defined as

$$E(p_i)(a,t) = \alpha_i \int_0^a p_i(r,t)\,\mathrm{d}r + \int_a^A p_i(r,t)\,\mathrm{d}r, \quad 0 \le \alpha_i < 1. \tag{2.1}$$

Here, constant α_i is the degree of hierarchy. This means that younger individuals are less competitive than older ones. Moreover, when two species interact, we assume that the fertility of a species is only related to its own "internal environment", and the mortality of one species is not only related to its own "internal environment", but also dependent on the "internal environment" of the another species. Thus, we can assume that the fertility and mortality of species i are $\beta_i(a, E(p_i)(a,t))$ and $\mu_i(a, E(p_1)(a,t), E(p_2)(a,t))$, respectively. Hence, we propose the following hierarchical age-structured two-species model with boundary control $(i = 1,2)$

$$\begin{cases} \frac{\partial p_i(a,t)}{\partial t} + \frac{\partial p_i(a,t)}{\partial a} = -\mu_i(a, E(p_1)(a,t), E(p_2)(a,t))p_i(a,t), & (a,t) \in Q, \\ p_i(0,t) = \int_0^A \beta_i(a, E(p_i)(a,t))p_i(a,t)\,\mathrm{d}a + u_i(t), & t \in (0,T), \\ p_i(a,0) = p_i^0(a), & a \in [0,A), \end{cases} \tag{2.2}$$

where $Q = (0,A) \times (0,T)$ and $T \in (0,+\infty)$ represents the control horizon. $p_i^0(a)$ is the initial age distribution of species i. The control variable $u_i(t)$ stands for the number of baby individuals influx into species i at time t. Moreover, we assume that the control variable $u_i(t)$ belongs to

$$u_i \in \mathcal{U}_i \doteq \big\{ v \in L^\infty(0,T) \mid 0 \le v(t) \le U_i, t \in (0,T) \big\}.$$

The positive constant U_i represents the maximum number of baby individuals influx into species i. Let $\mathcal{U} \doteq \mathcal{U}_1 \times \mathcal{U}_2 = \{(v_1,v_2) \in [L^\infty(0,T)]^2 | 0 \le v_i(t) \le U_i, t \in (0,T), i = 1,2\}$. For any $u = (u_1,u_2) \in \mathcal{U}$, let $(p_1^u(a,t), p_2^u(a,t))$ be the solution of (2.1) and (2.2) with the control variable u.

Compared with known closely related ones, our model has the following features. It is clear that our model includes some existing two species models with age structure. If we take $\mu_i(a, E(p_1)(a,t), E(p_2)(a,t)) = \mu_i(a,t) - \lambda_i(a,t)P_j$ and $\beta_i(a, E(p_i)(a,t)) = \beta_i(t)m_i(a,t)$ $(i,j = 1,2;\ i \neq j)$, then our model can be reduced to the model discussed in [9]. Moreover, if there is only one species and $u_i(t) \equiv 0$, our system can be reduced to the system (1.1) and (1.2). In [29], Ackleh and Deng assumed that the vital signs β and μ in (1.2) are, respectively, decreasing and increasing with respect to "internal environment" Q (i.e., $-\infty < \beta_Q \le 0$ and $0 \le \mu_Q < \infty$). Based on these, the authors proved that the system has a unique solution by establishing a comparison principle and constructing monotone sequences. Moreover, they gave conditions for extinction and persistence of the population.

Undoubtedly, the above works are important and helpful to understand the evolution of hierarchical species. However, it is also an important topic for human beings to use or transform nature (such as renewable-resource development, pest control, etc.) to serve human survival and development on the basis of understanding of nature. The research in this field corresponds to the control problem of the model mathematically. It is widely known that investigations on optimal control problems in hierarchical populations are rather rare. The purpose of this paper is to make some contribution in this direction.

On the one hand, if the system describes the interaction between pests or annoying animals (such as rodents), we always hope that each species can be reduced to an ideal distribution level that does not affect crop growth and will not become extinct. Thus, in this paper, we first consider the following optimization problem

$$\min_{(u_1,u_2)\in\mathcal{U}} J(u_1,u_2) \doteq \sum_{i=1}^{2}\left\{\int_0^A \left[p_i^u(a,T)-\bar{p}_i(a)\right]^2 \,\mathrm{d}a + \sigma_i \int_0^T u_i^2(t)\,\mathrm{d}t\right\}. \tag{2.3}$$

Here, the first integral represents the deviation between final state $p_i^u(a,T)$ and ideal distribution $\bar{p}_i(a)$, while the second one stands for the costs of control with factor σ_i. Thus, an optimal policy for (2.3) is to manipulate the system with any given initial distribution as close as possible to an ideal distribution with least control cost. Hence, control problem (2.3) can be called the least deviation-cost problem.

On the other hand, if species in the system are of renewable resources (such as forest resources or fish resources), optimal harvesting is always one of the most fascinating topics. Thus, in this paper, we also consider the following optimization problem

$$\max_{(u_1,u_2)\in\mathcal{U}} J(u_1,u_2) \doteq \sum_{i=1}^{2}\left\{\int_0^A g_i(a)p_i(a,T)\,\mathrm{d}a - \sigma_i \int_0^T u_i^2(t)\,\mathrm{d}t\right\}. \tag{2.4}$$

Here, function $g_i(a)$ represents the price of individuals with age a in species i. The motivation for (2.4) is as follows: How to input baby individuals during the period $[0,T]$, such that one can obtain the most benefit by exploiting the final distributions of species with the least control cost? Hence, one can say optimization problem (2.4) is the most benefit-cost problem.

Similar to the hypothesis of model (2.1) in [2, p.30], here, we make the following assumptions ($i = 1,2$). Denote $R_+ = [0,+\infty)$.

(A1) $0 \le \beta_i(a,x) \le \beta_i^*$ for any $(a,x) \in (0,A) \times R_+$, where β_i^* is a positive constant. This is reasonable because the birth rate of any species is a bounded function. Moreover, for any $M > 0$, there is $L_1(M) > 0$ such that

$$\left|\beta_i(a,x_1) - \beta_i(a,x_2)\right| \le L_1(M)|x_1 - x_2|, \quad \text{for } a \in [0,A), x_1,x_2 \in [0,M].$$

This means that the derivative of the birth rate of the species with respect to its "internal environment" is a bounded function.

(A2) $\mu_i(a,x,y) > 0$ for any $(a,x,y) \in (0,A) \times R_+ \times R_+$. For any $(x,y) \in R_+ \times R_+$, $\mu_i(\cdot,x,y) \in L^1_{\mathrm{loc}}[0,A]$ and $\int_0^A \mu_i(a,x,y)\,\mathrm{d}a = +\infty$. This condition will guarantee that there are no individuals with age beyond A for each species. Moreover, for any $M > 0$, there is $L_2(M) > 0$ such that for $a \in [0, A)$ and $x_1, x_2, y_1, y_2 \in [0, M]$

$$\left|\mu_i(a, x_1, y_1) - \mu_i(a, x_2, y_2)\right| \le L_2(M)\left(|x_1 - x_2| + |y_1 - y_2|\right).$$

(A3) For any $a \in (0, A)$, $0 \le p_i^0(a) \le p_i^*$ with positive constant p_i^*. This ensures that the initial distribution of each species is bounded.

3 Well-posedness of the state system

The purpose of this section is to discuss the well-posedness of (2.1) and (2.2). Here, we assume $T > A$ and omit the proof for the case $T \le A$ (which is analogous, but easier). Let $\mathbf{X} = [L^\infty(0, T; L^1(0, A))]^2 \doteq L^\infty(0, T; L^1(0, A)) \times L^\infty(0, T; L^1(0, A))$. For any $w = (w_1, w_2) \in \mathbf{X}$, define a new norm in $\mathbf{X}$ by

$$\|w\|_* = \operatorname{Ess}\sup_{t \in (0,T)} \left\{ e^{-\lambda t}\left[\int_0^A \left|w_1(a,t)\right| \mathrm{d}a + \int_0^A \left|w_2(a,t)\right| \mathrm{d}a\right]\right\},$$

for some $\lambda > 0$. It is clear that $(\mathbf{X}, \|\cdot\|_*)$ is a Banach space. Let $M_i^* = \max\{(A\beta_i^* p_i^* + U_i)\exp\{T\beta_i^*\}, p_i^*\}$ and $M^* = \max\{M_1^*, M_2^*\}$. Further, define the solution space as follows

$$\mathcal{X} = \left\{(w_1, w_2) \in \mathbf{X} \,\middle|\, \begin{array}{l} w_i(a,t) \ge 0 \text{ a.e. } (a,t) \in Q \text{ and } \int_0^A w_i(a,t)\,\mathrm{d}a \le M^* \\ \text{a.e. } t \in (0,T), i = 1, 2 \end{array}\right\}.$$

Note that $\mathcal{X}$ is a nonempty closed subset in $\mathbf{X}$. Hence, $(\mathcal{X}, \|\cdot\|_*)$ is also a Banach space. Moreover, for any $w = (w_1, w_2) \in \mathbf{X}$, we denote

$$\left\|w(\cdot,t)\right\|_{[L^1(0,A)]^2} = \sum_{i=1}^2 \left\|w_i(\cdot,t)\right\|_{L^1(0,A)} = \sum_{i=1}^2 \int_0^A \left|w_i(a,t)\right| \mathrm{d}a.$$

First, let $q(a,t) = (q_1(a,t), q_2(a,t)) \in \mathbf{X}$ be arbitrary but fixed, so is the function

$$E(q_i)(a,t) = \alpha_i \int_0^a q_i(r,t)\,\mathrm{d}r + \int_a^A q_i(r,t)\,\mathrm{d}r, \quad i = 1, 2. \tag{3.1}$$

Consider the following linear system

$$\begin{cases} \frac{\partial p_i(a,t)}{\partial t} + \frac{\partial p_i(a,t)}{\partial a} = -\mu_i(a, E(q_1)(a,t), E(q_2)(a,t))p_i(a,t), & (a,t) \in Q, \\ p_i(0,t) = \int_0^A \beta_i(a, E(q_i)(a,t))p_i(a,t)\,\mathrm{d}a + u_i(t), & t \in (0,T), \\ p_i(a,0) = p_i^0(a), & a \in [0,A). \end{cases} \tag{3.2}$$

Clearly, system (3.1) and (3.2) has a unique nonnegative solution $(p_1(a,t;q), p_2(a,t;q))$. Using the characteristic lines (see [2]), the solution has the form ($i = 1, 2$):

$$p_i(a,t;q) = \begin{cases} p_i(a-t, 0)\Pi_i(a,t,t;q), & a \ge t; \\ b_i(t-a;q)\Pi_i(a,t,a;q), & a < t, \end{cases} \tag{3.3}$$

where $b_i(t-a;q) \doteq p_i(0, t-a;q)$ and

$$\Pi_i(a,t,s;q) = e^{-\int_0^s \mu_i(a-\tau, E(q_1)(a-\tau,t-\tau), E(q_2)(a-\tau,t-\tau))\,\mathrm{d}\tau}. \tag{3.4}$$

Thus, we can claim that $b_i(t;q)$ $(i=1,2)$ satisfies

$$b_i(t;q)=F_i(t;q)+\int_0^t K_i(t,a;q)b_i(t-a;q)\,\mathrm{d}a,\quad t\in(0,T), \tag{3.5}$$

where

$$K_i(t,a;q)=\begin{cases}\beta_i(a,E(q_i)(a,t))\Pi_i(a,t,a;q), & (a,t)\in Q;\\ 0, & \text{otherwise},\end{cases} \tag{3.6}$$

$$F_i(t;q)=\begin{cases}\int_t^A \beta_i(a,E(q_i)(a,t))p_i^0(a-t)\Pi_i(a,t,t;q)\,\mathrm{d}a+u_i(t), & t\in(0,A);\\ 0, & \text{otherwise}.\end{cases} \tag{3.7}$$

Moreover, **(A1)**–**(A3)** imply that

$$0\le K_i(t,s;q)\le\beta_i^*,\qquad 0\le F_i(t;q)\le A\beta_i^*p_i^*+U_i. \tag{3.8}$$

Lemma 3.1 *For any $q^i=(q_1^i,q_2^i)\in\mathcal{X}$ $(i=1,2)$, there are positive constants M_1 and M_2 such that for $0<t<T$*

$$\begin{aligned}&\left|F_i\left(t;q^1\right)-F_i\left(t;q^2\right)\right|\\ &\quad\le M_1\left[\left\|q^1(\cdot,t)-q^2(\cdot,t)\right\|_{[L^1(0,A)]^2}+\int_0^t\left\|q^1(\cdot,s)-q^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\,\mathrm{d}s\right],\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\left|b_i\left(t;q^1\right)-b_i\left(t;q^2\right)\right|\\ &\quad\le M_2\left[\left\|q^1(\cdot,t)-q^2(\cdot,t)\right\|_{[L^1(0,A)]^2}+\int_0^t\left\|q^1(\cdot,s)-q^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\,\mathrm{d}s\right].\end{aligned} \tag{3.10}$$

Proof It is easy to show that (3.5) has a unique nonnegative solution. Moreover, using (3.5)–(3.8) and Gronwall's inequality, we can derive that $b_i(t;q)\le B_i$ with positive constant $B_i=(A\beta_i^*p_i^*+U_i)\exp\{T\beta_i^*\}$ $(i=1,2)$. For any $q^i=(q_1^i,q_2^i)\in\mathcal{X}$, it follows from (3.1) that $(i,j=1,2)$

$$E\left(q_j^i\right)(a,t)=\alpha_j\int_0^a q_j^i(r,t)\,\mathrm{d}r+\int_a^A q_j^i(r,t)\,\mathrm{d}r\le\int_0^A q_j^i(r,t)\,\mathrm{d}r\le M^*. \tag{3.11}$$

Moreover, we have

$$\begin{aligned}&\left|E\left(q_j^1\right)(a,t)-E\left(q_j^2\right)(a,t)\right|\\ &\quad=\left|\alpha_j\int_0^a\left[q_j^1(r,t)-q_j^2(r,t)\right]\mathrm{d}r+\int_a^A\left[q_j^1(r,t)-q_j^2(r,t)\right]\mathrm{d}r\right|\\ &\quad\le\int_0^A\left|q_j^1(r,t)-q_j^2(r,t)\right|\mathrm{d}r=\left\|q_j^1(\cdot,t)-q_j^2(\cdot,t)\right\|_{L^1(0,A)}.\end{aligned} \tag{3.12}$$

Now, we prove inequality (3.9). From **(A1)–(A3)**, it follows that

$$\begin{aligned}
&\left|\Pi_i\big(a,t,t;q^1\big)-\Pi_i\big(a,t,t;q^2\big)\right| \\
&\quad\le L_2\big(M^*\big)\int_0^t\sum_{j=1}^2\left|E\big(q_j^1\big)(a-\tau,t-\tau)-E\big(q_j^2\big)(a-\tau,t-\tau)\right|\,\mathrm{d}\tau \\
&\quad= L_2\big(M^*\big)\int_0^t\sum_{j=1}^2\left|E\big(q_j^1\big)(a-t+s,s)-E\big(q_j^2\big)(a-t+s,s)\right|\,\mathrm{d}s \\
&\quad\le L_2\big(M^*\big)\int_0^t\left\|q^1(\cdot,s)-q^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\,\mathrm{d}s\quad (i=1,2).
\end{aligned}\tag{3.13}$$

Thus, from (3.7) and (3.11)–(3.13), we have

$$\begin{aligned}
\left|F_1\big(t;q^1\big)-F_1\big(t;q^2\big)\right| &\le L_1\big(M^*\big)p_1^*\int_0^A\left|E\big(q_1^1\big)(a,t)-E\big(q_1^2\big)(a,t)\right|\,\mathrm{d}a \\
&\quad+p_1^*\beta_1^*\int_0^A\left|\Pi_1\big(a,t,t;q^1\big)-\Pi_1\big(a,t,t;q^2\big)\right|\,\mathrm{d}a \\
&\le M_{1,1}D(t),
\end{aligned}$$

where $D(t)=\|q^1(\cdot,t)-q^2(\cdot,t)\|_{[L^1(0,A)]^2}+\int_0^t\|q^1(\cdot,s)-q^2(\cdot,s)\|_{[L^1(0,A)]^2}\,\mathrm{d}s$ and positive constant $M_{1,1}=p_1^*A\max\{L_1(M^*),\beta_1^*L_2(M^*)\}$. Similarly, we also have

$$\left|F_2\big(t;q^1\big)-F_2\big(t;q^2\big)\right|\le M_{1,2}D(t),$$

where $M_{1,2}=p_2^*A\max\{L_1(M^*),\beta_2^*L_2(M^*)\}$. Thus, the inequality (3.9) is true with $M_1=\max\{M_{1,1},M_{1,2}\}$.

Next, we prove inequality (3.10). From **(A1)–(A3)** and (3.5), it follows that

$$\begin{aligned}
&\left|b_1\big(t;q^1\big)-b_1\big(t;q^2\big)\right| \\
&\quad\le\left|F_1\big(t;q^1\big)-F_1\big(t;q^2\big)\right|+\int_0^t K_1\big(t,t-s;q^2\big)\left|b_1\big(s;q^1\big)-b_1\big(s;q^2\big)\right|\,\mathrm{d}s \\
&\qquad+\int_0^t\left|K_1\big(t,t-s;q^1\big)-K_1\big(t,t-s;q^2\big)\right|b_1\big(s;q^1\big)\,\mathrm{d}s \\
&\quad\le M_1D(t)+L_1\big(M^*\big)B_1T\left\|q_1^1(\cdot,t)-q_1^2(\cdot,t)\right\|_{L^1(0,A)} \\
&\qquad+\beta_1^*\int_0^t\left|b_1\big(s;q^1\big)-b_1\big(s;q^2\big)\right|\,\mathrm{d}s \\
&\qquad+L_2\big(M^*\big)B_1T\beta_1^*\int_0^t\sum_{j=1}^2\left\|q_j^1(\cdot,s)-q_j^2(\cdot,s)\right\|_{L^1(0,A)}\,\mathrm{d}s \\
&\quad\le M_3D(t)+\beta_1^*\int_0^t\left|b_1\big(s;q^1\big)-b_1\big(s;q^2\big)\right|\,\mathrm{d}s,
\end{aligned}$$

where $M_3 = M_1 + B_1 T \max\{L_1(M), \beta_1^* L_2(M)\}$. Then, using Gronwall's inequality, we can obtain

$$\left|b_1\left(t;q^1\right) - b_1\left(t;q^2\right)\right| \le M_{2,1} D(t),$$

where $M_{2,1} = M_3[1 + \beta_1^* \exp\{\beta_1^* T\}(1+T)]$. Similarly, we also have

$$\left|b_2\left(t;q^1\right) - b_2\left(t;q^2\right)\right| \le M_{2,2} D(t),$$

where $M_{2,2} = [M_1 + B_2 T \max\{L_1(M), \beta_2^* L_2(M)\}][1 + \beta_2^* \exp\{\beta_2^* T\}(1+T)]$. Hence, (3.10) is true with $M_2 = \max\{M_{2,1}, M_{2,2}\}$. The proof is complete. □

Now, for any $q = (q_1, q_2) \in \mathcal{X}$, we define the mapping $\mathcal{A} : \mathcal{X} \to \mathbf{X}$ by

$$\mathcal{A}q = \mathcal{A}(q_1, q_2) = \left(\mathcal{A}_1(q_1, q_2), \mathcal{A}_2(q_1, q_2)\right),$$

where $\mathcal{A}_i(q_1, q_2)$ $(i = 1, 2)$ is defined by the right-hand sides of (3.3). Moreover, for any $q \in \mathcal{X}$, it is easy to show that $\mathcal{A}q \in \mathcal{X}$. Thus, if $\mathcal{A}$ is a contractive mapping on Banach space $(\mathcal{X}, \|\cdot\|_*)$, then system (2.1) and (2.2) has a unique solution.

Theorem 3.2 *Let* **(A1)–(A3)** *hold. Then, system* (2.1) *and* (2.2) *has a unique solution, which is nonnegative and bounded.*

Proof For any $q^i = (q_1^i, q_2^i) \in \mathcal{X}$ $(i = 1, 2)$, it follows from Lemma 3.1 that

$$\begin{aligned}
&\int_0^A \left|\mathcal{A}_1\left(q_1^1, q_2^1\right) - \mathcal{A}_1\left(q_1^2, q_2^2\right)\right|(a,t)\,\mathrm{d}a \\
&\quad \le \int_0^t \left|b_1\left(s;q^1\right) - b_1\left(s;q^2\right)\right|\mathrm{d}s + B_1 \int_0^t \left|\Pi_1\left(a,t,a;q^1\right) - \Pi_1\left(a,t,a;q^2\right)\right|\mathrm{d}a \\
&\qquad + p_1^* \int_t^A \left|\Pi_1\left(a,t,t;q^1\right) - \Pi_1\left(a,t,t;q^2\right)\right|\mathrm{d}a \\
&\quad \le M_2 \int_0^t \left[\left\|q^1(\cdot,s) - q^2(\cdot,s)\right\|_{[L^1(0,A)]^2} + \int_0^s \left\|q^1(\cdot,\tau) - q^2(\cdot,\tau)\right\|_{[L^1(0,A)]^2}\mathrm{d}\tau\right]\mathrm{d}s \\
&\qquad + B_1 L_2\left(M^*\right) \int_0^t \int_0^a \sum_{j=1}^2 \left|E\left(q_j^1\right)(a-\tau, t-\tau) - E\left(q_j^2\right)(a-\tau, t-\tau)\right|\mathrm{d}\tau\,\mathrm{d}a \\
&\qquad + p_1^* L_2\left(M^*\right) \int_t^A \int_0^t \sum_{j=1}^2 \left|E\left(q_j^1\right)(a-\tau, t-\tau) - E\left(q_j^2\right)(a-\tau, t-\tau)\right|\mathrm{d}\tau\,\mathrm{d}a \\
&\quad \le H_1 \int_0^t \left\|q^1(\cdot,s) - q^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\mathrm{d}s,
\end{aligned}$$

where $H_1 = M_2(1+T) + B_1 L_2(M^*) T + p_1^* L_2(M^*) A$. Similarly, there is a constant $H_2 = M_2(1+T) + B_2 L_2(M^*) T + p_2^* L_2(M^*) A$ such that

$$\int_0^A \left|\mathcal{A}_2\left(q_1^1, q_2^1\right) - \mathcal{A}_2\left(q_1^2, q_2^2\right)\right|(a,t)\,\mathrm{d}a \le H_2 \int_0^t \left\|q^1(\cdot,s) - q^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\mathrm{d}s.$$

Thus,

$$\big\|(\mathcal{A}q^1)(\cdot,t)-(\mathcal{A}q^2)(\cdot,t)\big\|_{[L^1(0,A)]^2}\le M_4\int_0^t\big\|q^1(\cdot,s)-q^2(\cdot,s)\big\|_{[L^1(0,A)]^2}\,\mathrm{d}s, \tag{3.14}$$

where $M_4=2M_2(1+T)+L_2(M)T(B_1+B_2)+L_2(M)A(p_1^*+p_2^*)$. Hence, for any $q^i=(q_1^i,q_2^i)\in\mathcal{X}$ $(i=1,2)$, it follows from (3.14) that

$$\begin{aligned}\big\|\mathcal{A}q^1-\mathcal{A}q^2\big\|_* &= \operatorname{Ess}\sup_{t\in(0,T)}\big\{e^{-\lambda t}\big\|(\mathcal{A}q^1)(\cdot,t)-(\mathcal{A}q^2)(\cdot,t)\big\|_{[L^1(0,A)]^2}\big\}\\ &\le M_4\operatorname{Ess}\sup_{t\in(0,T)}\left\{e^{-\lambda t}\int_0^t\big\|q^1(\cdot,s)-q^2(\cdot,s)\big\|_{[L^1(0,A)]^2}\,\mathrm{d}s\right\}\\ &\le\frac{M_4}{\lambda}\big\|q^1-q^2\big\|_*.\end{aligned}$$

Choose λ such that $\lambda>M_4$. Then, $\mathcal{A}$ is a contraction on Banach space $(\mathcal{X},\|\cdot\|_*)$. By the Banach fixed-point theorem, $\mathcal{A}$ has a unique fixed point, which is the solution of system (2.1) and (2.2). Moreover, from the definition of solution space $\mathcal{X}$, it follows that the solution is nonnegative and bounded. □

Theorem 3.3 *Let* **(A1)–(A3)** *hold. Then, the solution* $p^u=(p_1^u,p_2^u)$ *of system* (2.1) *and* (2.2) *is continuous with respect to the control variable* $u=(u_1,u_2)\in\mathcal{U}$*, that is, for any* $u^i=(u_1^i,u_2^i)\in\mathcal{U}$ $(i=1,2)$*, there is a constant* $\bar{M}>0$ *such that*

$$\sum_{j=1}^2\big\|p_j^1(\cdot,t)-p_j^2(\cdot,t)\big\|_{L^1(0,A)}\le\bar{M}\int_0^t\sum_{j=1}^2\big|u_j^1(s)-u_j^2(s)\big|\,\mathrm{d}s,$$

where $p^i=(p_1^i,p_2^i)$ *is the solution of system* (2.1) *and* (2.2) *corresponding to* u^i*.*

Proof For any $u=(u_1,u_2)\in\mathcal{U}$, the solution of system (2.1) and (2.2) has the form

$$p_i(a,t)=\begin{cases}p_i^0(a-t)\Pi_i(a,t,t), & a\ge t;\\ b_i(t-a)\Pi_i(a,t,a), & a<t,\end{cases} \tag{3.15}$$

where $b_i(t-a)\doteq p_i(0,t-a)$ and

$$\Pi_i(a,t,s)=e^{-\int_0^s\mu_i(a-\tau,E(p_1)(a-\tau,t-\tau),E(p_2)(a-\tau,t-\tau))\,\mathrm{d}\tau}. \tag{3.16}$$

Similarly, $b_i(t)$ $(i=1,2)$ satisfies

$$b_i(t)=F_i(t)+\int_0^t K_i(t,a)b_i(t-a)\,\mathrm{d}a,\quad t\in(0,T). \tag{3.17}$$

Here,

$$K_i(t,a)=\begin{cases}\beta_i(a,E(p_i)(a,t))\Pi_i(a,t,a), & (a,t)\in Q;\\ 0, & \text{otherwise},\end{cases} \tag{3.18}$$

$$F_i(t) = \begin{cases} \int_t^A \beta_i(a, E(p_i)(a,t)) p_i^0(a-t) \Pi_i(a,t,t)\, \mathrm{d}a + u_i(t), & t \in (0,A); \\ 0, & \text{otherwise.} \end{cases} \tag{3.19}$$

Note that $p^i = (p_1^i, p_2^i)$ is a solution of system (2.1) and (2.2) corresponding to $u^i \in \mathcal{U}$ ($i = 1,2$). It follows from (3.17) that

$$\begin{aligned}
&\left|b_1^1(t) - b_1^2(t)\right| \\
&\leq p_1^* L_1(M^*) \int_t^A \left|E(p_1^1)(a,t) - E(p_1^2)(a,t)\right| \mathrm{d}a + \left|u_1^1(t) - u_1^2(t)\right| \\
&\quad + p_1^* \beta_1^* \int_t^A \left|\Pi_1^1(a,t,t) - \Pi_1^2(a,t,t)\right| \mathrm{d}a + \beta_1^* \int_0^t \left|b_1^1(s) - b_1^2(s)\right| \mathrm{d}s \\
&\quad + B_1 \int_0^t \left|\beta_1(a, E(p_1^1)(a,s)) \Pi_1^1(a,s,a) - \beta_1(a, E(p_1^2)(a,s)) \Pi_1^2(a,s,a)\right| \mathrm{d}s \\
&\leq (p_1^* A + B_1 T) L_1(M^*) \left\|p_1^1(\cdot,t) - p_1^2(\cdot,t)\right\|_{L^1(0,A)} + \left|u_1^1(t) - u_1^2(t)\right| \\
&\quad + \beta_1^* \int_0^t \left|b_1^1(s) - b_1^2(s)\right| \mathrm{d}s \\
&\quad + p_1^* \beta_1^* L_2(M^*) \int_t^A \int_0^t \sum_{j=1}^2 \left\|p_j^1(\cdot,t-\tau) - p_j^2(\cdot,t-\tau)\right\|_{L^1(0,A)} \mathrm{d}\tau\, \mathrm{d}a \\
&\quad + B_1 \beta_1^* L_2(M^*) \int_0^t \int_0^a \sum_{j=1}^2 \left\|p_j^1(\cdot,s-\tau) - p_j^2(\cdot,s-\tau)\right\|_{L^1(0,A)} \mathrm{d}\tau\, \mathrm{d}s \\
&\leq M_5 \left\|p^1(\cdot,t) - p^2(\cdot,t)\right\|_{[L^1(0,A)]^2} + \left|u_1^1(t) - u_1^2(t)\right| + \beta_1^* \int_0^t \left|b_1^1(s) - b_1^2(s)\right| \mathrm{d}s,
\end{aligned} \tag{3.20}$$

where $M_5 = (p_1^* A + B_1 T) L_1(M^*) + (p_1^* + B_1) A T \beta_1^* L_2(M^*)$. Using Gronwall's inequality, we can derive that

$$\begin{aligned}
\left|b_1^1(t) - b_1^2(t)\right| &\leq \left|u_1^1(t) - u_1^2(t)\right| + M_5 \left\|p^1(\cdot,t) - p^2(\cdot,t)\right\|_{[L^1(0,A)]^2} \\
&\quad + \exp\{\beta_1^* T\} \int_0^t \Big[\left|u_1^1(s) - u_1^2(s)\right| \\
&\quad + M_5 \left\|p^1(\cdot,s) - p^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\Big] \mathrm{d}s.
\end{aligned} \tag{3.21}$$

Further, for $a \geq t$, from (3.12), (3.15), and (3.16), it follows that

$$\left|p_1^1(a,t) - p_1^2(a,t)\right| \leq p_1^* L_2(M^*) \int_0^t \left\|p^1(\cdot,s) - p^2(\cdot,s)\right\|_{[L^1(0,A)]^2} \mathrm{d}s. \tag{3.22}$$

For $a < t$, from (3.12), (3.15), and (3.16), it follows that

$$\begin{aligned}
\left|p_1^1(a,t) - p_1^2(a,t)\right| &\leq B_1 L_2(M^*) \int_0^t \left\|p^1(\cdot,s) - p^2(\cdot,s)\right\|_{[L^1(0,A)]^2} \mathrm{d}s \\
&\quad + \left|b_1^1(t-a) - b_1^2(t-a)\right|.
\end{aligned} \tag{3.23}$$

Thus, from (3.21)–(3.23), it follows that

$$\begin{aligned}
&\left\|p_1^1(\cdot,t)-p_1^2(\cdot,t)\right\|_{L^1(0,A)}\\
&\quad=\int_0^t\left|p^1(a,t)-p^2(a,t)\right|\mathrm{d}a+\int_t^A\left|p^1(a,t)-p^2(a,t)\right|\mathrm{d}a\\
&\quad\le\int_0^t\left|b_1^1(s)-b_1^2(s)\right|\mathrm{d}s+H_3AL_2\left(M^*\right)\int_0^t\left\|p^1(\cdot,s)-p^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\mathrm{d}s\\
&\quad\le\left(1+\exp\left\{\beta_1^*T\right\}T\right)\int_0^t\left|u_1^1(s)-u_1^2(s)\right|\mathrm{d}s\\
&\qquad+M_6\int_0^t\left\|p^1(\cdot,s)-p^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\mathrm{d}s,
\end{aligned}$$

where $H_3=\max\{B_1,p_1^*\}$ and $M_6=M_5(1+\exp\{\beta_1^*T\})+H_3AL_2(M^*)$. Similarly, there exists a positive constant M_7 such that

$$\begin{aligned}
\left\|p_2^1(\cdot,t)-p_2^2(\cdot,t)\right\|_{L^1(0,A)}\le{}&\left(1+\exp\left\{\beta_2^*T\right\}T\right)\int_0^t\left|u_2^1(s)-u_2^2(s)\right|\mathrm{d}s\\
&+M_7\int_0^t\left\|p^1(\cdot,s)-p^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\mathrm{d}s.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sum_{j=1}^2\left\|p_j^1(\cdot,t)-p_j^2(\cdot,t)\right\|_{L^1(0,A)}\le{}&\left(1+\exp\left\{\beta^*T\right\}T\right)\int_0^t\sum_{j=1}^2\left|u_j^1(s)-u_j^2(s)\right|\mathrm{d}s\\
&+M_8\int_0^t\left\|p^1(\cdot,s)-p^2(\cdot,s)\right\|_{[L^1(0,A)]^2}\mathrm{d}s,
\end{aligned}$$

where $\beta^*=\max\{\beta_1^*,\beta_2^*\}$ and $M_8=\max\{M_6,M_7\}$. Then, Gronwall's inequality implies that

$$\sum_{j=1}^2\left\|p_j^1(\cdot,t)-p_j^2(\cdot,t)\right\|_{L^1(0,A)}\le H_4\int_0^t\sum_{j=1}^2\left|u_j^1(s)-u_j^2(s)\right|\mathrm{d}s,$$

where $H_4=(1+\exp\{\beta^*T\}T)(1+\exp\{M_8T\}T)$. The proof is complete. □

Remark 3.4 In this section, we first prove that the model has a unique nonnegative bounded solution by constructing a suitable solution space and equivalent norm, and using the freezing-coefficient method and fixed-point theory. The boundedness of the solution ensures that the species described in the system will not grow indefinitely. Then, using the basic inequality theory, we obtain the continuous dependence of solutions on the control variable, which will play a key role in the following discussions.

4 Least deviation-cost problem

In this section, we will discuss the optimization problem (2.3). Here and after, we denote by $\mathcal{T}_{\mathcal{U}_i}(u_i)$ and $\mathcal{N}_{\mathcal{U}_i}(u_i)$ the tangent cone and the normal cone of $\mathcal{U}_i$ at u_i, respectively. Moreover, from [41], we have the following result.

Lemma 4.1 *For any $v_i \in \mathcal{T}_{\mathcal{U}_i}(u_i)$, suppose that $\vartheta_i(t) \in L^1(0,T)$ satisfies*

$$\sum_{i=1}^{2} \int_0^T \left[\vartheta_i(t)v_i(t) + \rho_i\left|v_i(t)\right|\right] \mathrm{d}t \geq 0.$$

Then, there exists $\theta_i \in L^\infty(0,T)$ such that $\|\theta_i\|_{L^\infty(0,T)} \leq 1$ and $\rho_i\theta_i - \vartheta_i \in \mathcal{N}_{\mathcal{U}_i}(u_i)$.

4.1 Minimum principle

In this subsection, we will establish first-order necessary conditions of optimality in the form of an Euler–Lagrange system.

Theorem 4.2 *Let $u^* = (u_1^*, u_2^*)$ and $p^* = (p_1^*, p_2^*)$. If (u^*, p^*) is an optimal pair for the optimization problem (2.3), then optimal strategies u_i^* must be structured as*

$$u_i^*(t) = \mathcal{F}_i\left(\frac{q_i(0,\cdot)}{\sigma_i}\right)(t) \doteq \begin{cases} 0, & \text{if } q_i(0,t) < 0, \\ \frac{q_i(0,t)}{\sigma_i}, & \text{if } 0 \leq \frac{q_i(0,t)}{\sigma_i} < U_i, \\ U_i, & \text{if } \frac{q_i(0,t)}{\sigma_i} \geq U_i, \end{cases} \tag{4.1}$$

where $q = (q_1, q_2)$ is the solution of the adjoint system

$$\begin{cases} \frac{\partial q_i}{\partial t} + \frac{\partial q_i}{\partial a} = \alpha_i \int_a^A [\sum_{j=2}^3 \mu_{ij}(r, E(p_1^*)(r,t), E(p_2^*)(r,t))q_i(r,t)p_i^*(r,t) \\ \qquad - \beta_{i2}(r, E(p_i^*)(r,t))q_i(0,t)p_i^*(r,t)]\,\mathrm{d}r \\ \qquad + \int_0^a [\sum_{j=2}^3 \mu_{ij}(r, E(p_1^*)(r,t), E(p_2^*)(r,t))q_i(r,t)p_i^*(r,t) \\ \qquad - \beta_{i2}(r, E(p_i^*)(r,t))q_i(0,t)p_i^*(r,t)]\,\mathrm{d}r \\ \qquad + [\mu_i(a, E(p_1^*)(a,t), E(p_2^*)(a,t))q_i(a,t) - \beta_i(a, E(p_i^*)(a,t))q_i(0,t)], \\ q_i(a,T) = \bar{p}_i(a) - p_i^*(a,T), \qquad q_i(A,t) = 0, \\ E(p_i^*)(a,t) = \alpha_i \int_0^a p_i^*(r,t)\,\mathrm{d}r + \int_a^A p_i^*(r,t)\,\mathrm{d}r, \quad (a,t) \in Q, i = 1,2, \end{cases} \tag{4.2}$$

in which μ_{i2} and β_{i2} are partial derivatives of μ_i and β_i with respect to the second variable, respectively; μ_{i3} is a partial derivative of μ_i with respect to the third variable.

Proof The existence of a unique bounded solution to (4.2) can be treated in the same manner as that for (2.1) and (2.2). For any $v_i \in \mathcal{T}_{\mathcal{U}_i}(u_i^*)$ (see [41]), we have $u_i^\varepsilon \doteq u_i^* + \varepsilon v_i \in \mathcal{U}_i$ $(i = 1,2)$ for sufficiently small $\varepsilon > 0$. Let $p^\varepsilon = (p_1^\varepsilon, p_2^\varepsilon)$ be a solution of (2.1) and (2.2) corresponding to $u = u^\varepsilon$. Then, the optimality of u^* implies $J(u^*) \leq J(u^\varepsilon)$. Hence,

$$\int_0^A \frac{[p_i^\varepsilon(a,T) - p_i^*(a,T)][p_i^\varepsilon(a,T) + p_i^*(a,T) - 2\bar{p}_i(a)]}{\varepsilon}\,\mathrm{d}a + \int_0^T \frac{[2u_i^*(t) + \varepsilon v_i(t)]\varepsilon v_i}{\varepsilon}\,\mathrm{d}t \geq 0. \tag{4.3}$$

Using the Theorem 3.3 and passing to the limit $\varepsilon \to 0^+$, we can derive that

$$\sum_{i=1}^{2}\left\{\int_0^A z_i(a,T)\left[p_i^*(a,T) - \bar{p}_i(a)\right]\mathrm{d}a + \sigma_i \int_0^T u_i^*(t)v_i(t)\,\mathrm{d}t\right\} \geq 0, \tag{4.4}$$

in which $z_i(a,t) = \lim_{\varepsilon\to 0^+} \frac{p_i^\varepsilon(a,t)-p_i^*(a,t)}{\varepsilon}$ $(i = 1,2)$. Via a similar discussion to that in [2], $z_1(a,t)$ and $z_2(a,t)$ do make sense. Further, it follows from (2.1) and (2.2) that (z_1,z_2) satisfies

$$\begin{cases} \frac{\partial z_i}{\partial t} + \frac{\partial z_i}{\partial a} = -\sum_{j=2}^{3} \mu_{ij}(a, E(p_1^*)(a,t), E(p_2^*)(a,t))E(z_j)(a,t)p_i^*(a,t) \\ \qquad\qquad - \mu_i(a, E(p_1^*)(a,t), E(p_2^*)(a,t))z_i(a,t), \\ z_i(0,t) = \nu_i(t) + \int_0^A [\beta_{i2}(a, E(p_i^*)(a,t))p_i^*(a,t)E(z_i)(a,t) \\ \qquad\qquad + \beta_i(a, E(p_i^*)(a,t))z_i(a,t)]\,\mathrm{d}a, \\ z_i(a,0) = 0, \\ E(z_i)(a,t) = \alpha_i \int_0^a z_i(r,t)\,\mathrm{d}r + \int_a^A z_i(r,t)\,\mathrm{d}r, \quad (a,t) \in Q, i = 1,2. \end{cases} \tag{4.5}$$

Multiplying the first equation in (4.5) by $q_i(a,t)$ and integrating on Q, we have

$$\begin{aligned} &\int_0^A z_i(a,T)\big[p_i^*(a,T) - \bar{p}_i(a)\big]\,\mathrm{d}a \\ &\quad = -\int_Q \big[\beta_{i2}\big(a, E\big(p_i^*\big)(a,t)\big)E(z_i)(a,t)p_i^* + \beta_i\big(a, E\big(p_i^*\big)(a,t)\big)z_i\big]q_i(0,t)\,\mathrm{d}a\,\mathrm{d}t \\ &\qquad + \int_Q \Bigg\{\sum_{j=2}^{3} \mu_{ij}\big(a, E\big(p_1^*\big)(a,t), E\big(p_2^*\big)(a,t)\big)E(z_j)(a,t)p_i^*(a,t) \\ &\qquad + \mu_i\big(a, E\big(p_1^*\big)(a,t), E\big(p_2^*\big)(a,t)\big)z_i\Bigg\}q_i(a,t)\,\mathrm{d}a\,\mathrm{d}t - \int_0^T \nu_i(t)q_i(0,t)\,\mathrm{d}t \\ &\qquad - \int_Q z_i\left(\frac{\partial q_i}{\partial t} + \frac{\partial q_i}{\partial a}\right)\mathrm{d}a\,\mathrm{d}t. \end{aligned}$$

Hence, we can obtain

$$\begin{aligned} &\sum_{i=1}^{2} \int_0^A z_i(a,T)\big[p_i^*(a,T) - \bar{p}_i(a)\big]\,\mathrm{d}a \\ &\quad = \sum_{i=1}^{2} \int_Q z_i(a,t)\alpha_i \int_a^A \Bigg[\sum_{j=2}^{3} \mu_{ij}\big(r, E\big(p_1^*\big)(r,t), E\big(p_2^*\big)(r,t)\big)q_i(r,t)p_i^*(r,t) \\ &\qquad - \beta_{i2}\big(r, E\big(p_i^*\big)(r,t)\big)q_i(0,t)p_i^*(r,t)\Bigg]\,\mathrm{d}r\,\mathrm{d}a\,\mathrm{d}t \\ &\qquad + \sum_{i=1}^{2} \int_Q z_i(a,t) \int_0^a \Bigg[\sum_{j=2}^{3} \mu_{ij}\big(r, E\big(p_1^*\big)(r,t), E\big(p_2^*\big)(r,t)\big)q_i(r,t)p_i^*(r,t) \\ &\qquad - \beta_{i2}\big(r, E\big(p_i^*\big)(r,t)\big)q_i(0,t)p_i^*(r,t)\Bigg]\,\mathrm{d}r\,\mathrm{d}a\,\mathrm{d}t \\ &\qquad + \sum_{i=1}^{2} \int_Q z_i\big[\mu_i\big(a, E\big(p_1^*\big), E\big(p_2^*\big)(a,t)\big)q_i - \beta_i\big(a, E\big(p_i^*\big)(a,t)\big)q_i(0,t)\big]\,\mathrm{d}a\,\mathrm{d}t \\ &\qquad - \sum_{i=1}^{2} \int_Q z_i\left(\frac{\partial q_i}{\partial t} + \frac{\partial q_i}{\partial a}\right)\mathrm{d}a\,\mathrm{d}t - \sum_{i=1}^{2} \int_0^T \nu_i(t)q_i(0,t)\,\mathrm{d}t. \end{aligned}$$

This, together with (4.2), yields

$$\sum_{i=1}^{2}\int_0^A z_i(a,T)\big[p_i^*(a,T)-\bar{p}_i(a)\big]\,\mathrm{d}a = -\sum_{i=1}^{2}\int_0^T v_i(t)q_i(0,t)\,\mathrm{d}t. \tag{4.6}$$

It follows from (4.4) and (4.6) that $\sum_{i=1}^{2}\int_0^T[q_i(0,t)-\sigma_i u_i^*(t)]v_i(t)\,\mathrm{d}t \le 0$ holds for all $v \in \mathcal{T}_{\mathcal{U}}(u^*)$. Thus, $q_i(0,\cdot)-\sigma_i u_i^* \in \mathcal{N}_{\mathcal{U}_i}(u_i^*)$ $(i=1,2)$, which implies the conclusion of this theorem. The proof is complete. □

4.2 Existence of a unique optimal policy

In this subsection, we study the existence and uniqueness of the optimal control. Define the embedding mapping $\varphi : [L^1(0,T)]^2 \to (-\infty,+\infty]$ by

$$\tilde{J}(u_1,u_2) = \begin{cases} \sum_{i=1}^{2}\{\int_0^A[p_i^u(a,T)-\bar{p}_i(a)]^2\,\mathrm{d}a+\sigma_i\int_0^T u_i^2(t)\,\mathrm{d}t\}, & (u_1,u_2)\in\mathcal{U},\\ +\infty, & (u_1,u_2)\notin\mathcal{U}. \end{cases}$$

Lemma 4.3 *The mapping $\tilde{J}$ is lower semicontinuous.*

Proof Let $\{(u_1^n,u_2^n)\}$ be a sequence in $[L^1(0,T)]^2$ such that $(u_1^n,u_2^n)\to(u_1,u_2)$ as $n\to\infty$. Without loss of generality, we assume that $(u_1^n,u_2^n)\in\mathcal{U}\doteq\mathcal{U}_1\times\mathcal{U}_2$ for any $n\ge 1$. Moreover, let (p_1^n,p_2^n) and (p_1,p_2) be solutions of (2.1) and (2.2) corresponding to (u_1^n,u_2^n) and (u_1,u_2), respectively. From Theorem 3.3, it follows that

$$p_1^n(\cdot,t)\to p_1(\cdot,t)\quad\text{and}\quad p_2^n(\cdot,t)\to p_2(\cdot,t)\quad\text{for any } t\in(0,T)$$

as $n\to\infty$. Riesz's theorem implies that there is a subsequence, denoted still by $\{(u_1^n,u_2^n)\}$, such that, for any $(a,t)\in Q$ and $t\in(0,T)$,

$$\big(u_1^n(t)\big)^2\to\big(u_1(t)\big)^2\quad\text{and}\quad\big(u_2^n(t)\big)^2\to\big(u_2(t)\big)^2, \tag{4.7}$$

$$p_1^n(a,t)\to p_1(a,t)\quad\text{and}\quad p_2^n(a,t)\to p_2(a,t), \tag{4.8}$$

as $n\to\infty$. From (4.7), using Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^T\big(u_i^n(t)\big)^2\,\mathrm{d}t=\int_0^T\big(u_i(t)\big)^2\,\mathrm{d}t\quad(i=1,2).$$

On the other hand, it follows from (4.8) and Theorem 3.3 that

$$\begin{aligned}&\left|\int_0^A\big[p_i^n(a,T)-\bar{p}_i(a)\big]^2\,\mathrm{d}a-\int_0^A\big[p_i(a,T)-\bar{p}_i(a)\big]^2\,\mathrm{d}a\right|\\ &\quad\le\int_0^A\big|p_i^n(a,T)-p_i(a,T)\big|\cdot\big|p_i^n(a,T)+p_i(a,T)-2\bar{p}_i(a)\big|\,\mathrm{d}a\\ &\quad\to 0\quad\text{as } n\to+\infty.\end{aligned}$$

Thus, $\lim_{n\to\infty}\int_0^A[p_i^n(a,T)-\bar{p}_i(a)]^2\,\mathrm{d}a=\int_0^A[p_i(a,T)-\bar{p}_i(a)]^2\,\mathrm{d}a$ $(i=1,2)$. It follows from Fatou's lemma that

$$\lim_{n\to+\infty}\inf_{(u_1^n,u_2^n)\in\mathcal{U}}\tilde{J}(u_1^n,u_2^n)\geq\tilde{J}(u_1,u_2),$$

which shows that $\tilde{J}(u_1,u_2)$ is lower semicontinuous. □

In a similar manner to that in Theorem 3.3, we can obtain the following result.

Lemma 4.4 *For any $u=(u_1,u_2)$, $v=(v_1,v_2)\in\mathcal{U}$, let (p_1^u,p_2^u) and (p_1^v,p_2^v) be solutions of (2.1) and (2.2) corresponding to u and v, respectively. For (4.2), there is a constant $C\doteq C(U,T)>0$, which depends on $U=\max\{U_1,U_2\}$ and T, such that*

$$\sum_{i=1}^{2}\bigl[\|q_i^u(\cdot,t)-q_i^v(\cdot,t)\|_{L^\infty(0,A)}+|q_i^u(0,t)-q_i^v(0,t)|\bigr]\leq C\|u-v\|_{[L^\infty(0,T)]^2},$$

holds for all $t\in(0,T)$. Here, (q_1^u,q_2^u) and (q_1^v,q_2^v) are solutions to (4.2) with (p_1^*,p_2^*) replaced by (p_1^u,p_2^u) and (p_1^v,p_2^v), respectively.

Theorem 4.5 *If $\max\{\sigma_1^{-1},\sigma_2^{-1}\}C(U,T)<1$, then optimization problem (2.3) has a unique solution.*

Proof From Lemma 4.3 and Ekeland's variational principle, for each $\varepsilon>0$, there exists $(u_1^\varepsilon,u_2^\varepsilon)\in\mathcal{U}$ such that

$$\tilde{J}(u_1^\varepsilon,u_2^\varepsilon)\leq\inf_{(u_1,u_2)\in\mathcal{U}}\tilde{J}(u_1,u_2)+\varepsilon, \tag{4.9}$$

$$\tilde{J}(u_1^\varepsilon,u_2^\varepsilon)\leq\inf_{(u_1,u_2)\in\mathcal{U}}\bigl\{\tilde{J}(u_1,u_2)+\sqrt{\varepsilon}\|u_1-u_1^\varepsilon\|_{L^1(0,T)}+\sqrt{\varepsilon}\|u_2-u_2^\varepsilon\|_{L^1(0,T)}\bigr\}. \tag{4.10}$$

Thus, functional $\tilde{J}_\varepsilon(u_1,u_2)=\tilde{J}(u_1,u_2)+\sqrt{\varepsilon}\|u_1-u_1^\varepsilon\|_{L^1(0,T)}+\sqrt{\varepsilon}\|u_2-u_2^\varepsilon\|_{L^1(0,T)}$ reaches its infimum at $(u_1^\varepsilon,u_2^\varepsilon)$. Then, in the same manner as that in Theorem 4.2, for any $v_i\in\mathcal{T}_{\mathcal{U}_i}(u_i^\varepsilon)$ $(i=1,2)$, we have

$$\sum_{i=1}^{2}\left\{\int_0^T\bigl[\sigma_iu_i^\varepsilon(t)-q_i^{u^\varepsilon}(0,t)\bigr]v_i(t)\,\mathrm{d}t+\sqrt{\varepsilon}\int_0^T|v_i(t)|\,\mathrm{d}t\right\}\geq 0.$$

By Lemma 4.1, there is $\theta_i\in L^\infty(0,T)$, $\|\theta_i\|_{L^\infty(0,T)}\leq 1$, such that $\sqrt{\varepsilon}\theta_i+q_i^{u^\varepsilon}(0,\cdot)-\sigma_iu_i^\varepsilon\in\mathcal{N}_{\mathcal{U}_i}(u_i^\varepsilon)$. Hence,

$$u_i^\varepsilon(t)=\mathcal{F}_i\left[\frac{q_i^{u^\varepsilon}(0,t)}{\sigma_i}+\frac{\sqrt{\varepsilon}\theta_i(t)}{\sigma_i}\right]. \tag{4.11}$$

First, we show the uniqueness via the fixed-point principle. For any $u=(u_1,u_2)\in\mathcal{U}$, let (p_1^u,p_2^u) be a solution of (2.1) and (2.2) corresponding to u. Define the mapping $\mathcal{C}:\mathcal{U}_1\times\mathcal{U}_2\to L^\infty(0,T)\times L^\infty(0,T)$ by

$$\mathcal{C}(u_1,u_2)=\left(\mathcal{F}_1\left[\frac{q_1^u(0,t)}{\sigma_1}\right],\mathcal{F}_2\left[\frac{q_2^u(0,t)}{\sigma_2}\right]\right), \tag{4.12}$$

where (q_1^u, q_2^u) is the solution to system (4.2) with (p_1^*, p_2^*) replaced by (p_1^u, p_2^u). It is easy to show that $(\mathcal{U}, \|\cdot\|_{[L^\infty(0,T)]^2})$ is a Banach space. It is easy to show that $\mathcal{C}$ maps $\mathcal{U}$ to itself. In addition, for any $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathcal{U}$, it follows from Lemma 4.1 that

$$\begin{aligned}
\|\mathcal{C}(u_1,u_2) - \mathcal{C}(v_1,v_2)\|_{[L^\infty(0,T)]^2} &= \sum_{i=1}^{2} \left\| \mathcal{F}_i\left[\frac{q_i^u(0,t)}{\sigma_i}\right] - \mathcal{F}_i\left[\frac{q_i^v(0,t)}{\sigma_i}\right] \right\|_{L^\infty(0,T)} \\
&\le \sum_{i=1}^{2} \left\| \frac{q_i^u(0,t)}{\sigma_i} - \frac{q_i^v(0,t)}{\sigma_i} \right\|_{L^\infty(0,T)} \\
&\le \max\{\sigma_1^{-1}, \sigma_2^{-1}\} C(U,T) \|u - v\|_{[L^\infty(0,T)]^2}.
\end{aligned}$$

This, together with $\max\{\sigma_1^{-1}, \sigma_2^{-1}\} C(U,T) < 1$, yields that $\mathcal{C}$ is a contraction on Banach space $(\mathcal{U}, \|\cdot\|_{[L^\infty(0,T)]^2})$. Then, $\mathcal{C}$ has a unique fixed point $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in \mathcal{U}$. Moreover, from Theorem 4.2, we know that any optimal controller, if it exists, must be the fixed point of $\mathcal{C}$. Thus, the uniqueness is proved.

Next, we prove that $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ is the optimal control for optimization problem (2.3). That is to show $\tilde{J}(\tilde{u}_1, \tilde{u}_2) = \inf\{\tilde{J}(u_1, u_2) : \tilde{u}_1, \tilde{u}_2 \in \mathcal{U}\}$. Note that $\|\theta_i\|_{L^\infty(0,T)} \le 1$. Thus, from (4.11) and (4.12), it follows that

$$\begin{aligned}
&\|\mathcal{C}(u_1^\varepsilon, u_2^\varepsilon) - (u_1^\varepsilon, u_2^\varepsilon)\|_{[L^\infty(0,T)]^2} \\
&\quad \le \sum_{i=1}^{2} \left\| \frac{q_i^{u^\varepsilon}(0,t)}{\sigma_i} + \frac{\sqrt{\varepsilon}\theta_i(t)}{\sigma_i} - \frac{q_i^{u^\varepsilon}(0,t)}{\sigma_i} \right\|_{L^\infty(0,T)} \\
&\quad \le \sum_{i=1}^{2} \sigma_i^{-1} \sqrt{\varepsilon}.
\end{aligned}$$

Note that $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in \mathcal{U}$ is the unique fixed point for the mapping $\mathcal{P}$. Thus,

$$\begin{aligned}
&\|u^\varepsilon - \tilde{u}\|_{[L^\infty(0,T)]^2} \\
&\quad \le \max\{\sigma_1^{-1}, \sigma_2^{-1}\} C(U,T) \|u^\varepsilon - \tilde{u}\|_{[L^\infty(0,T)]^2} + \sqrt{\varepsilon}(\sigma_1^{-1} + \sigma_2^{-2}),
\end{aligned}$$

which implies $\|u^\varepsilon - \tilde{u}\|_{[L^\infty(0,T)]^2} \le [1 - \max\{\sigma_1^{-1}, \sigma_2^{-1}\} C(U,T)] \sqrt{\varepsilon}(\sigma_1^{-1} + \sigma_2^{-2})$. Hence, we have $u^\varepsilon \to \tilde{u}$ as $\varepsilon \to 0^+$. From Lemma 4.3 and (4.9), one has $\tilde{J}(\tilde{u}) = \inf\{\tilde{J}(u) : u \in \mathcal{U}\}$. Thus, $\tilde{u} \in \mathcal{U}$ is the optimal policy. □

Remark 4.6 In this section, using Ekeland's variational principle and fixed-point reasoning, we show the existence and uniqueness of optimal policies for the least deviation-cost problem, which gives us a solid theoretical ground for a practical application. As for the structure of the optimal policy, we have presented a feedback strategy, in Theorem 4.2, via the normal cone technique and adjoint system. Due to the high nonlinearities in our problem, one cannot expect an explicit optimal controller. However, the feedback strategy would be helpful for a numerical computation of the optimal policy (see Sect. 6).

5 Most benefit-cost problem

Here, we give the corresponding results for optimization problem (2.4). A similar discussion as that in Theorem 4.5, we can show that the most benefit-cost problem (2.4) has a unique solution. Further, similar to the proof of Theorem 4.2, we have the following result.

Theorem 5.1 (Maximum principle) *Let $u^* = (u_1^*, u_2^*)$ and $p^* = (p_1^*, p_2^*)$. If (u^*, p^*) is an optimal pair for the optimization problem* (2.4), *then*

$$u_i^*(t) = \mathcal{F}_i\left(\frac{q_i(0,\cdot)}{2\sigma_i}\right)(t) \doteq \begin{cases} 0, & \text{if } q_i(0,t) < 0, \\ \frac{q_i(0,t)}{2\sigma_i}, & \text{if } 0 \le \frac{q_i(0,t)}{2\sigma_i} < U_i, \\ U_i, & \text{if } \frac{q_i(0,t)}{2\sigma_i} \ge U_i, \end{cases} \tag{5.1}$$

where $q = (q_1, q_2)$ is the solution of the following adjoint system

$$\begin{cases} \frac{\partial q_i}{\partial t} + \frac{\partial q_i}{\partial a} = \alpha_i \int_a^A [\sum_{j=2}^3 \mu_{ij}(r, E(p_1^*)(r,t), E(p_2^*)(r,t)) q_i(r,t) p_i^*(r,t) \\ \qquad - \beta_{i2}(r, E(p_i^*)(r,t)) q_i(0,t) p_i^*(r,t)]\, \mathrm{d}r \\ \qquad + \int_0^a [\sum_{j=2}^3 \mu_{ij}(r, E(p_1^*)(r,t), E(p_2^*)(r,t)) q_i(r,t) p_i^*(r,t) \\ \qquad - \beta_{i2}(r, E(p_i^*)(r,t)) q_i(0,t) p_i^*(r,t)]\, \mathrm{d}r \\ \qquad + [\mu_i(a, E(p_1^*)(a,t), E(p_2^*)(a,t)) q_i(a,t) - \beta_i(a, E(p_i^*)(a,t)) q_i(0,t)], \\ q_i(a,T) = g_i(a), \qquad q_i(A,t) = 0, \\ E(p_i^*)(a,t) = \alpha_i \int_0^a p_i^*(r,t)\,\mathrm{d}r + \int_a^A p_i^*(r,t)\,\mathrm{d}r, \quad (a,t) \in Q, i = 1,2, \end{cases} \tag{5.2}$$

in which μ_{i2} and β_{i2} are partial derivatives of μ_i and β_i with respect to the second variable, respectively; *μ_{i3} is a partial derivative of μ_i with respect to the third variable*.

6 Numerical simulations

In this section, we provide some examples to verify the effectiveness of the obtained results and find other dynamic properties of the system. From Theorem 4.2, optimal strategy u_i^* ($i = 1,2$) is a fixed point of mapping $\mathcal{F}_i$. Clearly, the fixed point can be approximated by an iteration scheme.

First, we discuss the effects of ideal distribution $\bar{p}(a) = (\bar{p}_1(a), \bar{p}_2(a))$ on optimal policy $u^*(t) = (u_1^*(t), u_2^*(t))$ and then on optimal functional $J(u^*)$ for optimization problem (2.3).

Example 1 (The effects of $\bar{p}$ on $J(u^*)$). Let $\alpha_1 = 0.4$, $\alpha_2 = 0.5$, $A = 10$, $T = 20$, $\sigma_1 = 3$, $\sigma_2 = 4$, and the error bound $\varepsilon = 0.005$. The mortality and fertility are, respectively,

$$\mu_1(a, x_1, x_2) = \begin{cases} 0.01\cos^2(4a) + 0.008x_1 + 0.007x_2 + \frac{2-a}{50}, & 0 \le a < 2; \\ 0.01\cos^2(4a) + 0.008x_1 + 0.007x_2, & 2 \le a < 8; \\ 0.01\cos^2(4a) + 0.008x_1 + 0.007x_2 + \frac{a-8}{10-a}, & 8 \le a < 10; \end{cases}$$

$$\mu_2(a, x_1, x_2) = \begin{cases} 0.02\cos^2(4a) + 0.005x_1 + 0.006x_2 + \frac{2-a}{50}, & 0 \le a < 2; \\ 0.02\cos^2(4a) + 0.005x_1 + 0.006x_2, & 2 \le a < 8; \\ 0.02\cos^2(4a) + 0.005x_1 + 0.006x_2 + \frac{a-8}{10-a}, & 8 \le a < 10; \end{cases}$$

and

$$\beta_1(a,x_1) = \begin{cases} 0.4\cos^2(a-2), & 1 \le a < 2; \\ 0.4, & 2 \le a < 9; \\ 0, & \text{otherwise}; \end{cases}$$

$$\beta_2(a,x_2) = \begin{cases} 0.8(a-1)\cos^2(a), & 1 \le a < 2; \\ 0.8\cos^2(a), & 2 \le a < 9; \\ 0, & \text{otherwise}. \end{cases}$$

The initial age distributions of species $p_1(a,t)$ and $p_2(a,t)$ are, respectively, $p_1^0(a) = 8(10 - a)(3 + \cos(2a))$ and $p_2^0(a) = 10(10 - a)\cos^2(a)$. Also, the initial controls are

$$u_1^0(t) = u_2^0(t) = \begin{cases} 0, & t < 4; \\ 1, & t \ge 4. \end{cases}$$

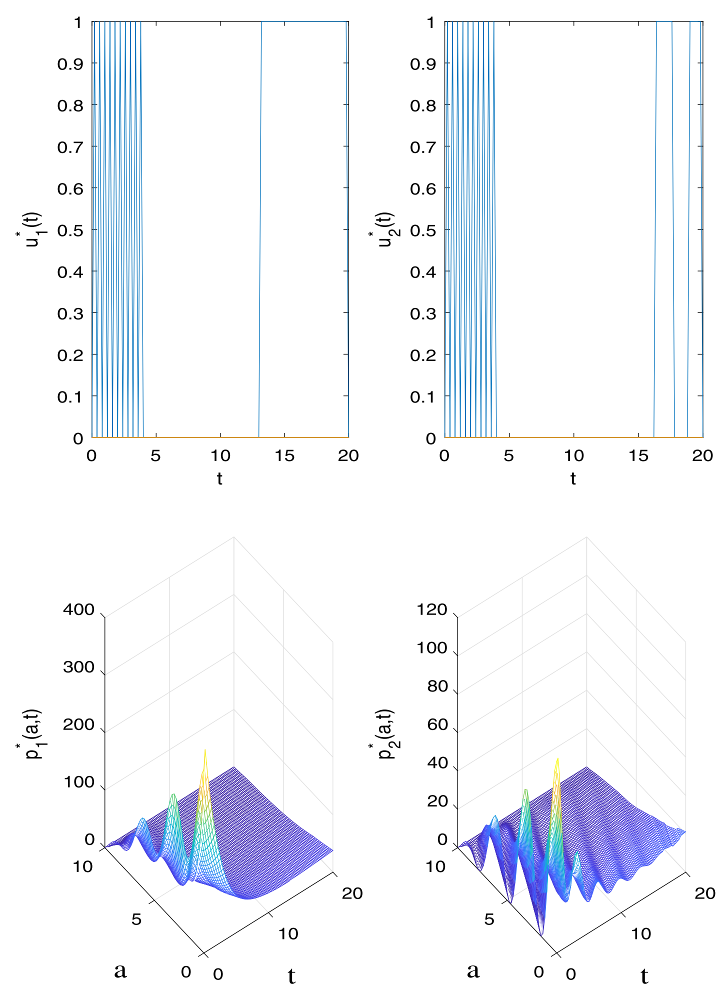

Figure 1 Optimal controls u_1^*, u_2^* and optimal densities p_1^*, p_2^* with $s = 2$

Table 1 The effects of s on $J(u^*)$

s	2	3	4	5	6	7	8	9	10
J	7.7	18.0	32.7	51.7	75.1	102.8	134.9	171.3	212.1
s	11	12	13	14	15	16	17	18	19
J	257.2	306.6	360.4	418.6	481.1	547.9	619.1	694.7	774.6
s	20	21	22	23	24	25	26	27	28
J	858.8	947.4	1040.3	1137.6	1239.2	1345.2	1455.5	1570.2	1689.2
s	29	30	31	32	33	34	35	36	37
J	1812.5	1940.2	2072.3	2208.7	2349.4	2494.5	2644.0	2797.8	2955.9
s	38	39	40	41	42	43	44	45	46
J	3118.4	3285.2	3456.4	3631.9	3811.8	3996.0	4184.6	4377.5	4574.8
s	47	48	49	50	51	52	53	54	55
J	4776.4	4982.3	5192.7	5407.3	5626.3	5849.7	6077.3	6309.4	6545.8
s	56	57	58	59	60	61	62	63	64
J	6786.5	7031.6	7281.0	7534.8	7792.9	8055.4	8322.2	8593.4	8868.9
s	65	66	67	68	69	70	71	72	73
J	9148.8	9433.0	9721.5	10,014.0	10,312.0	10,613.0	10,919.0	11,230.0	11,544.0
s	74	75	76	77	78	79	80	81	82
J	11,863.0	12,186.0	12,514.0	12,846.0	13,183.0	13,523.0	13,868.0	14,218.0	14,572.0
s	83	84	85	86	87	88	89	90	91
J	14,930.0	15,292.0	15,659.0	16,030.0	16,406.0	16,785.0	17,170.0	17,558.0	17,951.0
s	92	93	94	95	96	97	98	99	100
J	18,348.0	18,750.0	19,156.0	19,566.0	19,981.0	20,400.0	20,823.0	21,251.0	21,683.0
s	101	102	103	104	105	106			
J	22,119.0	22,560.0	23,005.0	23,454.0	23,908.0	24,366.0			

Moreover, the ideal distributions we hope to reach are (with the coefficient s)

$$\bar{p}_1(a) = 8s(10-a)\bigl(3+\cos(2a)\bigr) \quad \text{and} \quad \bar{p}_2(a) = 10s(10-a)\cos^2(a).$$

Calculating in MATLAB with the above parameters, we obtain the numerical results for the optimal policy u^* and optimal state p^* corresponding to $s = 2$ (see Fig. 1). Moreover, the indexes $J(u^*) = j \times 10^5$ corresponding to $s = 2, 3, 4, \dots, 106$ are listed in Table 1.

Figure 1 shows the optimal strategy $u^*(t)$ and the optimal population density $p^*(a,t)$ with $s = 2$ within the ideal distribution $\bar{p}(a)$. Moreover, as can be seen from Table 1, optimal objective functional $J(u^*)$ is increasing with respect to the coefficient s. This means that as the ideal distribution increases, the more baby individuals (i.e., the greater the control cost) should be input to make the final distribution of the species sufficiently close to the ideal distribution.

Now, we investigate the effects of price function $g(a) = (g_1(a), g_2(a))$ on optimal policy $u^*(t) = (u_1^*(t), u_2^*(t))$ and then on optimal functional $J(u^*)$ for optimization problem (2.4) in the numerical approach.

Example 2 (The effects of g_i on $J(u^*)$) Take $\alpha_1 = 0.3$, $\alpha_2 = 0.4$, $A = 10$, $T = 20$, $\sigma_1 = 3$, $\sigma_2 = 2$, and the error bound $\varepsilon = 0.005$. The mortality and fertility of species $p_1(a,t)$ and

$p_2(a,t)$ are

$$\mu_1(a,x_1,x_2) = \begin{cases} 0.02(1-\sin^2(4a)) + 0.02x_1 + 0.01x_2 + \frac{2-a}{50}, & 0 \le a < 2; \\ 0.02(1-\sin^2(4a)) + 0.02x_1 + 0.01x_2, & 2 \le a < 8; \\ 0.02(1-\sin^2(4a)) + 0.02x_1 + 0.01x_2 + \frac{a-8}{10-a}, & 8 \le a < 10; \end{cases}$$

$$\mu_2(a,x_1,x_2) = \begin{cases} 0.05(1-\sin^2(4a)) + 0.01x_1 + 0.02x_2 + \frac{2-a}{50}, & 0 \le a < 2; \\ 0.05(1-\sin^2(4a)) + 0.01x_1 + 0.02x_2, & 2 \le a < 8; \\ 0.05(1-\sin^2(4a)) + 0.01x_1 + 0.02x_2 + \frac{a-8}{10-a}, & 8 \le a < 10; \end{cases}$$

$$\beta_1(a,x_1) = \begin{cases} 0.473(\cos(0.4a)+1)(a-1), & 1 \le a < 2; \\ 0.473(\cos(0.4a)+1), & 2 \le a < 7; \\ 0.473(\cos(0.4a)+1)(8-a), & 7 \le a < 8; \\ 0, & \text{otherwise}; \end{cases}$$

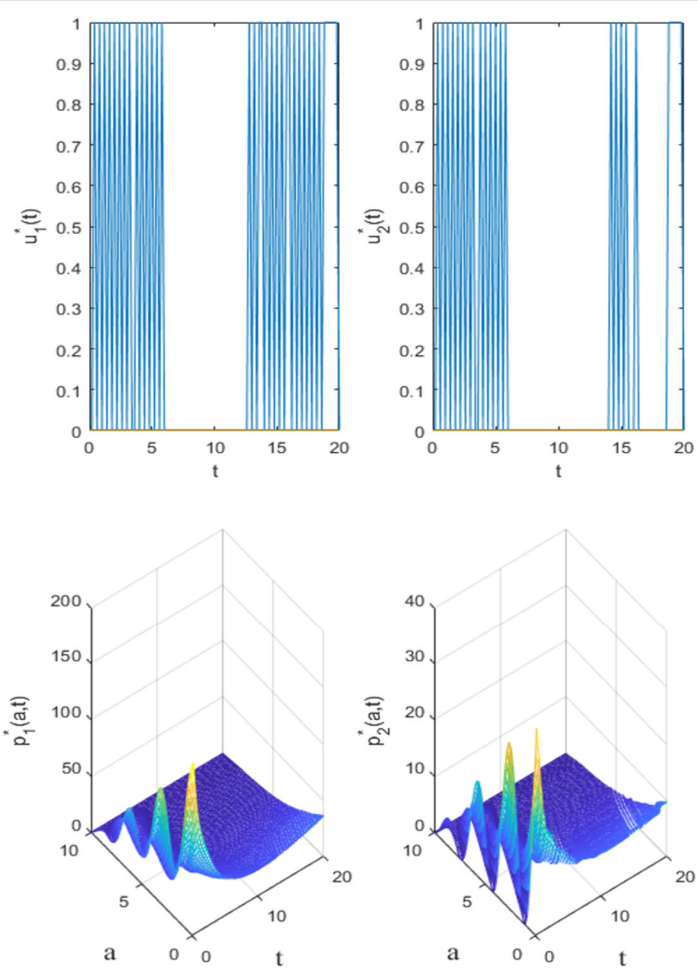

Figure 2 Optimal controls u_1^*, u_2^* and optimal densities p_1^*, p_2^* with $s = 2$

$$\beta_2(a,x_2)=\begin{cases}0.998\sin^2(2a)(a-1), & 1\le a<2;\\ 0.998\sin^2(2a), & 2\le a<7;\\ 0.998\sin^2(2a)(8-a), & 7\le a<8;\\ 0, & \text{otherwise.}\end{cases}$$

The initial age distributions are $p_1^0(a)=5(10-a)(\cos(2a)+2)$ and $p_2^0(a)=4(10-a)\cos^2(a+\frac{\pi}{6})$. Also, the initial controls are

$$u_1^0(t)=u_2^0(t)=\begin{cases}0, & t<5;\\ 1, & t\ge 5.\end{cases}$$

Moreover, the price functions are (with the coefficient s)

$$g_1(a)=\begin{cases}0.5s, & 0\le a<2;\\ a(\sin(a)+4)+0.5s, & 2\le a<8;\\ 2+0.5s, & 8\le a<10;\end{cases}$$

$$g_2(a)=\begin{cases}0.5s, & 0\le a<2;\\ a(\cos(a)+4)+0.5s, & 2\le a<8;\\ 2+0.5s, & 8\le a<10.\end{cases}$$

The optimal strategies $u^*(t)$ and optimal densities $p^*(a,t)$ corresponding to $s=0$ are illustrated in Fig. 2. Table 2 shows the benefits corresponding to $s=0,1,2,\dots,103$.

Table 2 Effects of g_i on $J(u^*)$

s	0	1	2	3	4	5	6	7	8	9
J	1059.2	1062.1	1065.0	1068.8	1071.6	1074.5	1077.4	1080.2	1083.1	1086.0
s	10	11	12	13	14	15	16	17	18	19
J	1088.9	1091.7	1094.6	1097.5	1100.3	1103.8	1106.6	1109.5	1112.4	1115.3
s	20	21	22	23	24	25	26	27	28	29
J	1118.1	1121.0	1123.9	1126.8	1129.8	1132.7	1135.6	1138.4	1141.3	1144.2
s	30	31	32	33	34	35	36	37	38	39
J	1147.1	1150.3	1153.2	1156.1	1159.0	1161.9	1164.7	1167.6	1170.5	1173.4
s	40	41	42	43	44	45	46	47	48	49
J	1176.2	1179.1	1182.0	1184.9	1187.8	1190.6	1193.5	1196.4	1199.3	1202.1
s	50	51	52	53	54	55	56	57	58	59
J	1205.0	1207.9	1210.8	1213.7	1216.5	1219.4	1222.3	1225.2	1228.1	1230.9
s	60	61	62	63	64	65	66	67	68	69
J	1233.8	1236.7	1239.6	1242.4	1245.3	1248.2	1252.2	1254.0	1256.8	1259.7
s	70	71	72	73	74	75	76	77	78	79
J	1262.6	1265.5	1268.4	1271.2	1274.1	1277.0	1279.9	1283.0	1285.9	1288.8
s	80	81	82	83	84	85	86	87	88	89
J	1291.7	1294.5	1297.4	1300.3	1303.2	1306.1	1308.9	1311.8	1314.7	1317.6
s	90	91	92	93	94	95	96	97	98	99
J	1320.5	1323.4	1326.2	1329.1	1332.0	1334.5	1337.4	1340.3	1343.2	1346.1
s	100	101	102	103						
J	1349.0	1351.8	1354.7	1357.6						

Figure 2 presents the optimal policy $u^*(t)$ and the optimal population density $p^*(a,t)$ with $s = 2$ in the price function $g(a) = (g_1(a), g_2(a))$ for the most benefit-cost problem (2.4). Further, from Table 2, we can see that optimal objective functional $J(u^*)$ is increasing with respect to s. Thus, one can say that the total net economic benefit yielded from harvesting the species in the final state will increase as the price of the individual rises.

Finally, we will discuss the effects of the costs factor σ_i on the optimal functional $J(u^*)$ for optimization problem (2.4).

Example 3 (The effects of σ_i on $J(u^*)$). Take $\sigma_1 = \sigma_2 = k$, $k = 1, 2, 3, \ldots, 104$, and the price functions are

$$g_1(a) = \begin{cases} 0, & 0 \le a < 2; \\ a(\sin(a) + 4), & 2 \le a < 8; \\ 2, & 8 \le a < 10; \end{cases}$$

$$g_2(a) = \begin{cases} 0, & 0 \le a < 2; \\ a(\cos(a) + 4), & 2 \le a < 8; \\ 2, & 8 \le a < 10. \end{cases}$$

The other parameters are the same as those in Example 2. Table 3 demonstrates the changes of optimal benefits $J(u^*)$ with respect to the control cost factor σ_i. From the data in Table 3 alone, the objective optimal benefit $J(u^*)$ decreases as the cost factor σ_i increases. This means that as the cost factor increases, the total net economic benefit yielded from harvesting the species in the final state will decrease.

Table 3 $J(u^*)$ corresponding to σ_i

k	1	2	3	4	5	6	7	8	9	10
J	1079.4	1066.3	1055.0	1044.4	1035.8	1025.3	1018.6	1014.3	1010.5	1001.3
k	11	12	13	14	15	16	17	18	19	20
J	993.5	990.2	983.9	977.9	972.4	964.6	956.8	949.0	944.8	937.2
k	21	22	23	24	25	26	27	28	29	30
J	929.6	922.0	914.4	906.8	899.2	891.6	884.0	876.4	868.8	861.2
k	31	32	33	34	35	36	37	38	39	40
J	853.6	846.0	844.4	843.6	836.4	829.2	822.0	814.8	807.6	800.4
k	41	42	43	44	45	46	47	48	49	50
J	793.2	786.0	779.4	772.2	765.0	757.8	750.6	743.4	736.2	729.0
k	51	52	53	54	55	56	57	58	59	60
J	731.6	724.6	717.6	710.6	703.6	696.6	689.6	682.6	675.6	668.6
k	61	62	63	64	65	66	67	68	69	70
J	661.6	654.6	647.6	640.6	646.3	639.5	632.7	625.9	619.1	612.3
k	71	72	73	74	75	76	77	78	79	80
J	605.5	598.7	591.9	585.1	578.3	571.5	564.7	557.9	551.1	544.3
k	81	82	83	84	85	86	87	88	89	90
J	537.5	530.7	524.3	517.5	510.7	503.9	497.1	490.3	483.5	476.7
k	91	92	93	94	95	96	97	98	99	100
J	469.9	463.1	456.3	449.5	442.7	435.9	429.1	422.3	415.5	408.7
k	101	102	103	104						
J	401.9	415.2	408.6	402.0						

7 Conclusion and discussion

This paper is concerned with the optimal boundary control problems for a hierarchical age-structured two-species model. In the previous sections, we have established the well-posedness of the state system provided the control variable (the influx rate of baby for each species) is put in. Meanwhile, the continuous dependence of density of each species on the control parameters is established. More importantly, the least deviation-cost problem and the most benefit-cost problem are discussed. Here, the least deviation-cost problem means that the system with any initial distribution is manipulated to be as close to an ideal distribution as possible at the least cost. The most benefit-cost problem refers to maximizing the benefit achieved by developing the final population distribution with the least cost of control (i.e., the cost of inputting baby individuals). For the least deviation-cost problem, Theorem 4.2 establishes the minimum principle by using an adjoint system, and Theorem 4.5 shows the existence of a unique optimal strategy by means of Ekeland's variational principle. Meanwhile, the corresponding results, the existence of a unique optimal control policy and the maximum principle (see Theorem 5.1), for the most benefit-cost problem are also given.

From Theorems 4.2 and 5.1, the optimal strategies can be regarded as fixed points of contraction mappings about the adjoint variables. Thus, we can approximate the optimal policies by the standard iterative paradigm. Moreover, some examples and numerical results are presented to verify the effectiveness of the obtained results and find other dynamic properties of the system. Although the phenomena observed in Tables 1, 2, and 3 have certain biological significance, they are intuitive in the general sense and do not have universal significance. However, the theoretical analysis of these phenomena is very difficult or even impossible to deal with.

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Abbreviations
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Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RL analyzed the well-posedness of the system and discussed the optimal control problem. NZ carried out the numerical analysis. ZH reviewed and edited the manuscript, and was a major contributor in writing the manuscript. All the authors read and approved the final manuscript.

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