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An analysis concerning approximate controllability results for second-order Sobolev-type delay differential systems with impulses

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Abstract

This paper is devoted to studying the approximate controllability for second-order impulsive differential inclusions with infinite delay. For proving the main results, we use the results related to the cosine and sine function of operators, Martelli's fixed point theorem, and the results when combined with the properties of differential inclusions. Firstly, we prove the approximate controllability for second-order impulsive differential inclusions with initial conditions. Then, we extend the discussion to the second-order impulsive system with nonlocal conditions. Finally, we provide an example for the illustration of the obtained theoretical results.

MSC: 34G20; 34K09; 35K10; 93B05

Keywords: Sobolev-type differential system; Infinite delay; Impulsive differential inclusions; Approximate controllability; Nonlocal conditions; Cosine function of operators

1 Introduction

It is necessary for the analysis of impulsive differential systems to predict a system's true dynamics. The occurrence of sudden changes in the state of systems gives rise to the notion of impulsive differential systems. Instantaneous forces (disturbances) or changing operational conditions cause these changes in status. Differential systems that handle impulsive changes appear in a variety of applications, including mechanical and biological models that are subjected to shocks, biological systems, population dynamics, and electromagnetic wave radiation. Impulsive differential equations have gotten significant in physical engineering, economics, population dynamics, and social sciences. A critical advancement in the areas of impulsive theory exists, particularly in systems with fixed instants. This is a powerful model for portraying unexpected transform at specific instants in large numbers of the unbroken evolution process and permitting a superior perception of some real circumstances under certain problems in applied science, and one can go through the books [1, 26] and research articles [14, 18–22, 36, 45, 49, 50, 57]. Recently, in [6–13], the authors studied the differential systems with hemivariational inequalities by using various fixed point theorems. In articles [21, 22], the authors discussed the existence of mild solutions for the second-order impulsive differential system by using the sine and cosine functions of operators, classical nonlocal conditions, piecewise continuous functions, and fixed point theorems. In articles [18–20], the authors provided a detailed discussion on the existence of the second-order differential systems by applying evolution operators, sine and cosine functions of operators, theories on nonautonomous systems, and various fixed point theorems.

In recent years, controllability has turned into a fascinating exploration in the fractional dynamical system, and it is also the basic idea in recent mathematical control theory. Control theory plays a vital role in applied mathematics, which engages the construction and inspection of the control framework. For the last few years, in countless dimensional spaces, the controllability of different types of nonlinear has been concentrated in many exploration papers by employing a variety of approaches. A wide rundown of these conveyances may be found in [14, 29–31, 33, 38, 49, 51, 55]. Differential systems of the Sobolev type are also popular in a variety of applications, such as liquid flow across fissured materials, thermodynamics, and shearing in second-order liquids. For more information, refer to articles [3, 14, 16, 17, 24, 28, 34, 48, 49, 52, 54, 56].

In [14], the author proved the controllability of first-order impulsive functional differential systems with infinite delay by using Schauder's fixed point theorem combined with a strongly continuous operator semigroup. In [31], the authors discussed the approximate controllability of second-order evolution differential inclusions by using sine and cosine functions of operators, multivalued maps, and Bohnenblust–Karlin's fixed point theorem. In [38], the authors discussed the approximate controllability of fractional nonlinear differential inclusions by applying the fractional calculus, multivalued maps, and Bohnenblust–Karlin's fixed point theorem. In [39–44], the authors discussed the existence, approximate controllability, and optimal control for first-order, second-order, and fractional-order differential systems by applying semigroup theory, fractional calculus, cosine and sine functions of operators, and various fixed point theorems. In [48, 49, 51–55], the authors discussed the exact and approximate controllability results for first-order, second-order, fractional-order differential systems by applying semigroup theory, resolvent operators, multivalued maps, and various fixed point approaches.

This article mainly focuses on the approximate controllability for Sobolev-type impulsive delay differential inclusions of second order with the infinite delay of the form

$$\frac{d^2}{d\varsigma^2}\big(Mz(\varsigma)\big) \in Az(\varsigma) + G(\varsigma, z_\varsigma) + Bu(\varsigma), \quad \varsigma \in V = [0,c], \varsigma \neq \varsigma_j, j = 1,2,\ldots,q, \tag{1.1}$$

$$z(\varsigma) = \alpha(\varsigma) \in \mathcal{P}_\nu, \quad \varsigma \in (-\infty, 0], z'(0) = z_1 \in \mathscr{X}, \tag{1.2}$$

$$\Delta z|_{\varsigma=\varsigma_j} = J_j\big(z\big(\varsigma_j^-\big)\big), \quad j = 1,2,\ldots,q, \tag{1.3}$$

$$\Delta z'|_{\varsigma=\varsigma_j} = \overline{J}_j\big(z\big(\varsigma_j^-\big)\big), \quad j = 1,2,\ldots,q, \tag{1.4}$$

where $z(\cdot)$ takes values in a Banach space $\mathscr{X}$. $u(\cdot)$ is given in $L^2(V,U)$, a Banach space of admissible control functions, $G : V \times \mathcal{P}_\nu \to 2^{\mathcal{P}_\nu}$ is a nonempty, bounded, closed, and convex multivalued map. The histories $z_\varsigma : (-\infty, 0] \to \mathcal{P}_\nu$, $z_\varsigma(\varepsilon) = z(\varsigma + \varepsilon)$, $\varepsilon \leq 0$ are associated with the phase space $\mathcal{P}_\nu$. The linear operator B is bounded from a Banach space U into

$\mathscr{Z}$. $J_j, \overline{J}_j : \mathscr{Z} \to \mathscr{Z}$, $\Delta z|_{\varsigma=\varsigma_j} = z(\varsigma_j^+) - z(\varsigma_j^-)$, $\Delta z'|_{\varsigma=\varsigma_j} = z'(\varsigma_j^+) - z'(\varsigma_j^-)$ for all $j = 1, 2, \ldots, q$. $0 = \varsigma_0 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_j < \varsigma_{k+1} = c$. Here, $z(\varsigma_j^+)$, $z(\varsigma_j^-)$, $z'(\varsigma_j^+)$, and $z'(\varsigma_j^-)$ denote right and left limits of $z(\varsigma)$ at $\varsigma = \varsigma_j$ and $z'(\varsigma)$ at $\varsigma = \varsigma_j$ respectively.

The main contributions of this study are as follows:

- Under the assumption that the associated linear system is approximately controllable, we establish a set of adequate requirements for the approximate controllability of second-order delay differential inclusions of Sobolev type.
- In the recent and vast literature on the exact controllability of abstract control differential issues, the authors [37] pointed out an inaccuracy. However, in our research, we merely define necessary conditions for the approximate controllability results of a second-order differential system to prevent such inaccuracies.
- The cosine function of the operator is considered to be compact, and as a result, the linear control system connected with the cosine function of the operator is only approximately controllable.
- We show that the concept of exact controllability has no analogue in our result. Finally, we show an example of a system that is not exactly controllable but is approximately controllable to some extent.
- To the best of our knowledge, approximate controllability discussion for second-order differential systems with infinite delay by using Martelli's fixed point theorem has not been studied in this connection. This gives the additional motivation for writing this article.

We subdivide this paper into the accompanying sections: Some basic definitions are recalled and preparation outcomes are presented in Sect. 2. Section 3 derives a sort of adequate conditions proving the approximate controllability of system (1.1)–(1.4). The extension of system (1.1)–(1.4) with nonlocal conditions can be found in Sect. 4. At last, in Sect. 5, an example is presented for drawing the theory of our primary outcomes.

2 Preliminaries

This section recalls the necessary things to obtain the primary facts of our discussion. $B_p(z, \mathscr{Z})$ denotes the closed ball with center z and radius $p > 0$ in $\mathscr{Z}$. Now, the sine function signified by $(\mathscr{M}(\varsigma))_{\varsigma\in\mathbb{R}}$ is combined with the cosine function $(\mathscr{N}(\varsigma))_{\varsigma\in\mathbb{R}}$, which is defined by

$$\mathscr{M}(\varsigma)z = \int_0^{\varsigma} \mathscr{N}(\psi)z\,d\psi, \quad z \in \mathscr{Z}, \varsigma \in \mathbb{R}.$$

Now we define the constants P_1, P_2 such that $\|\mathscr{N}(\varsigma)\| \leq P_1$ and $\|\mathscr{M}(\varsigma)\| \leq P_2$ for each $\varsigma \in V$. $[D(A)]$ signifies the domain of A equipped along with the norm $\|z\|_A = \|z\| + \|Az\|$, $z \in D(A)$. Furthermore, E means the space composed by $z \in \mathscr{Z}$ for which $\mathscr{N}(\cdot)z$ is a class C^1. Kisyński [25] demonstrated that space E provided with

$$\|z\|_E = \|z\| + \sup_{0\leq\varsigma\leq 1} \big\|A\mathscr{M}(\varsigma)z\big\|, \quad z \in E,$$

is a Banach space. A group of linear operators

$$G(\varsigma) = \begin{bmatrix} \mathscr{N}(\varsigma) & \mathscr{M}(\varsigma) \\ A\mathscr{M}(\varsigma) & \mathscr{N}(\varsigma) \end{bmatrix}$$

is strongly continuous on $E \times \mathscr{X}$ which is generated by $\mathcal{A} = \left[\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right]$ and defined on $D(A) \times E$. Accordingly, the linear operator $A\mathscr{M}(\varsigma) : E \to \mathscr{X}$ is bounded and $A\mathscr{M}(\varsigma)z \to 0$, $\varsigma \to 0$, for all $z \in E$. Moreover, if $z : [0,\infty) \to \mathscr{X}$ is locally integrable, then $z(\varsigma) = \int_0^\varsigma \mathscr{M}(\varsigma - \psi)z(\psi)\,d\psi$ establishes an E-valued continuous function. This is an outcome of the way that

$$\int_0^\varsigma G(\varsigma - \psi)\begin{bmatrix} 0 \\ z(\psi) \end{bmatrix} d\psi = \begin{bmatrix} \int_0^\varsigma \mathscr{M}(\varsigma - \psi)z(\psi)\,d\psi \\ \int_0^\varsigma \mathscr{N}(\varsigma - \psi)z(\psi)\,d\psi \end{bmatrix}$$

defines a function which is $(E \times \mathscr{X})$-valued continuous.

Consider the abstract Cauchy problem of a second-order differential system

$$\begin{cases} z''(\varsigma) = Az(\varsigma) + G(\varsigma), & 0 \le \varsigma \le c, \\ z(0) = z_0, & z'(0) = z_1, \end{cases} \tag{2.1}$$

where $G : [0,c] \to \mathscr{X}$ is an integrable function, which can be examined in [46, 47]. Now $z(\cdot)$ presented by

$$z(\varsigma) = \mathscr{N}(\varsigma)z_0 + \mathscr{M}(\varsigma)z_1 + \int_0^\varsigma \mathscr{M}(\varsigma - \psi)G(\psi)\,d\psi, \quad 0 \le \varsigma \le c, \tag{2.2}$$

which is known as the mild solution of system (2.1). When $z_0 \in E$, $z(\cdot)$ is continuously differentiable and

$$z'(\varsigma) = A\mathscr{M}(\varsigma)z_0 + \mathscr{N}(\varsigma)z_1 + \int_0^\varsigma \mathscr{N}(\varsigma - \psi)F(\psi)\,d\psi. \tag{2.3}$$

We now show that $A : D(A) \subset \mathscr{X} \to \mathscr{X}$ and $M : D(A) \subset \mathscr{X} \to \mathscr{X}$ satisfy the following conditions discussed in [28]:

($\mathbf{E_1}$) The linear operators A and M are closed.

($\mathbf{E_2}$) $D(M) \subset D(A)$ and M is bijective.

($\mathbf{E_3}$) $M^{-1} : \mathscr{X} \to D(M)$ is continuous.

Additionally, because ($\mathbf{E_1}$) and ($\mathbf{E_2}$) M^{-1} are closed, by ($\mathbf{E_3}$) and applying closed graph theorem, we get the boundedness of $AM^{-1} : \mathscr{X} \to \mathscr{X}$. Define $\|M^{-1}\| = \widetilde{P}_m$ and $\|M\| = \widehat{P}_m$.

By referring to [14, 57], we define a phase space as follows:

Consider the function $g : (-\infty, 0] \to (0, +\infty)$ which is continuous along $j = \int_{-\infty}^0 \nu(\varsigma)\,d\varsigma < +\infty$. For any $c > 0$,

$$\mathcal{P} = \bigl\{\alpha : [-c, 0] \to \mathscr{X} \text{ such that } \alpha(\varsigma) \text{ is bounded and measurable}\bigr\},$$

along

$$\|\alpha\|_{[-c,0]} = \sup_{\psi \in [-c,0]} \bigl\|\alpha(\psi)\bigr\|, \quad \text{for all } \alpha \in \mathcal{P}.$$

Now, we define

$$\mathcal{P}_\nu = \Biggl\{\alpha : (-\infty, 0] \to \mathscr{X} \text{ such that for any } b > 0, \alpha|_{[-b,0]} \in \mathcal{P} \text{ and}$$

$$\int_{-\infty}^{0} \nu(\psi)\|\alpha\|_{[\psi,0]}\,d\psi < +\infty\biggr\}.$$

Provided that $\mathcal{P}_\nu$ is endowed along

$$\|\alpha\|_{\mathcal{P}_\nu} = \int_{-\infty}^{0} \nu(\psi)\|\alpha\|_{[\psi,0]}\,d\psi \quad \text{for all } \alpha \in \mathcal{P}_\nu,$$

then it is clear that $(\mathcal{P}_\nu, \|\cdot\|_{\mathcal{P}_\nu})$ is a Banach space.

Presently we discuss

$$\mathcal{P}'_\nu = \bigl\{z : (-\infty, b] \to \mathscr{Z} \text{ such that } z|_V \in \mathcal{C}(V, \mathscr{Z}), z_0 = \alpha \in \mathcal{P}_\nu\bigr\}.$$

Set $\|\cdot\|'_g$ to be a seminorm in $\mathcal{P}'_\nu$ defined by

$$\|z\|'_\nu = \|\alpha\|_{\mathcal{P}_\nu} + \sup\bigl\{\bigl\|z(\psi)\bigr\| : \psi \in [0, c]\bigr\}, \quad z \in \mathcal{P}'_\nu.$$

In view of [15, 23], we present some fundamental ideas and facts related to multimaps.

Definition 2.1 ([15, 23]) The multimap $\mathcal{K}$ is said to be upper semicontinuous on $\mathscr{Z}$ provided that, for every $z_0 \in \mathscr{Z}$, $\mathcal{K}(z_0)$ is a nonempty closed subset of $\mathscr{Z}$ and provided that, for each open set H of $\mathscr{Z}$ including $\mathcal{K}(z_0)$, there exists an open neighborhood V of z_0 such that $\mathcal{K}(V) \subseteq H$.

Definition 2.2 ([15, 23]) The multimap $\mathcal{K}$ is said to be completely continuous provided that $\mathcal{K}(H)$ is relatively compact for every bounded subset H of $\mathscr{Z}$. Provided that $\mathcal{K}$ is completely continuous with nonempty values, at another time $\mathcal{K}$ is upper semicontinuous, if and only if $\mathcal{K}$ has a closed graph, that is, $z_n \to z_*$, $v_n \to v_*$, $v_n \in \mathcal{K}z_n$ imply $z_* \in \mathcal{K}z_*$. The multimap $\mathcal{K}$ has a fixed point provided that there is $z \in \mathscr{Z}$ such that $z \in \mathcal{K}(z)$.

Definition 2.3 ([15]) A multivalued function $\mathcal{K}$ mapping from V into $BCC(\mathscr{Z})$ is called measurable provided that, for all $z \in \mathscr{Z}$, the function χ mapping from V into $\mathcal{V}_{cl}$ defined by

$$\chi(\varsigma) = d\bigl(z, \mathcal{K}(\varsigma)\bigr) = \inf\bigl\{\|z - y\|^1 : y \in \mathcal{K}(\varsigma)\bigr\} \in L^1(V, \mathbb{R}).$$

An upper semicontinuous map $\mathcal{K} : \mathscr{Z} \to \mathscr{Z}$ is said to be condensing if, for any bounded subset $Q \subseteq \mathscr{Z}$ with $\jmath(Q) \neq 0$, we have

$$\jmath\bigl(\mathcal{K}(Q)\bigr) < \jmath(Q).$$

In the above, $\jmath$ denotes the Kuratowski measure of noncompactness. For additional details, one can refer to [2].

We point out that the simplest example of a condensing map is a completely continuous multivalued map.

We need to provide the following appropriate operators and basic assumption on the operators:

$$\aleph_0^c = \int_0^c M^{-1}\mathscr{M}(c-\psi)BB^*M^{-1}\mathscr{M}^*(c-\psi)\,d\psi : \mathscr{Z} \to \mathscr{Z},$$
$$R(\delta,\aleph_0^c) = (\delta I + \aleph_0^c)^{-1} : \mathscr{Z} \to \mathscr{Z},$$

where B^*, $\mathscr{M}^*(c)$ denote the adjoint of B and $\mathscr{M}(c)$ respectively, and it is easy to conclude that the linear operator $\aleph_0^c$ is bounded.

To prove the approximate controllability of system (1.1)–(1.4), we provide the following hypothesis:

$\mathbf{H_0}$ $\delta R(\delta,\aleph_0^c) \to 0$ as $\delta \to 0^+$ in the strong operator topology.

In terms of [30], $\mathbf{H_0}$ is satisfied if and only if the linear system

$$\frac{d^2}{d\varsigma^2}\big(Mz(\varsigma)\big) = Az(\varsigma) + (Bu)(\varsigma), \quad \varsigma \in [0,c], \tag{2.4}$$

$$z(0) = z_0, \qquad z'(0) = z_1, \tag{2.5}$$

is approximately controllable on $[0,c]$.

Lemma 2.4 ([27, Lasota and Opial]) *Assume that V is a compact real interval, the nonempty set $BCC(\mathscr{Z})$ is a bounded, closed, and convex subset of $\mathscr{Z}$, and the multimap G satisfying $G : V \times \mathscr{Z} \to BCC(\mathscr{Z})$ is measurable to ς for each fixed $z \in \mathscr{Z}$, upper semicontinuous to z for each $\varsigma \in V, z \in \mathcal{C}$ the set*

$$T_{G,z} = \{g \in L^1(V,\mathscr{Z}) : g(\varsigma) \in G(\varsigma, z(\varsigma)), \varsigma \in V\}$$

is nonempty. Assume that the linear operator $\mathscr{G}$ is continuous from $L^1(V,\mathscr{Z})$ to $\mathcal{C}$, at another time

$$\mathscr{G} \circ T_G : \mathcal{C} \to BCC(\mathcal{C}), \qquad z \to (\mathscr{G} \circ T_G)(z) = \mathscr{G}(T_{G,z})$$

is closed in $\mathcal{C} \times \mathcal{C}$.

Theorem 2.5 ([32]) *Assume that $\mathscr{Z}$ is a Banach space and $\Omega : \mathscr{Z} \to BCC(\mathscr{Z})$ is an upper semicontinuous and condensing function. If*

$$\mathcal{R} = \{z \in \mathscr{Z} : \lambda z \in \Omega z \text{ for some } \varphi > 1\}$$

is bounded, then Ω has a fixed point.

3 Approximate controllability

By applying Martelli's fixed point theorem, we discuss the primary results in this section. We present the mild solution of system (1.1)–(1.4) as follows.

Definition 3.1 A function $z:(-\infty,c]\to\mathscr{X}$ is said to be a mild solution of system (1.1)–(1.4) if $z_0=\alpha\in\mathcal{P}_\nu, z'(0)=z_1\in\mathscr{X}$ on $(-\infty,0]$, $\Delta z|_{\varsigma=\varsigma_j}=J_j(z(\varsigma_j^-)), \Delta z'|_{\varsigma=\varsigma_j}=\overline{J_j}(z(\varsigma_j^-)), j=1,2,\dots,q$; $z(\cdot)$ to J_j $(j=0,1,\dots,q)$ is continuous and

$$\begin{aligned}z(\varsigma)&=M^{-1}\mathscr{N}(\varsigma)M\alpha(0)+M^{-1}\mathscr{M}(\varsigma)Mz_1+\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi\\&\quad+\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)Bu(\psi)\,d\psi+\sum_{0<\varsigma_j<c}M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j(z_{\varsigma_j})\\&\quad+\sum_{0<\varsigma_j<c}M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J_j}(z_{\varsigma_j}),\quad\varsigma\in V,\end{aligned}$$

is satisfied.

To discuss the controllability performance and achieve the goal, we introduce the necessary hypotheses as follows:

H_1 The operator $\mathscr{M}(\varsigma)$, $\varsigma>0$ is compact.

H_2 The function $G:V\times\mathcal{P}_\nu\to BCC(\mathscr{X})$ is L^1-Caratheodory and satisfies the following conditions:

For each $\varsigma\in V$, $G(\varsigma,\cdot)$ is upper semicontinuous; for each $z\in\mathcal{P}_\nu$, $G(\cdot,z)$ is measurable and $z\in\mathcal{P}_\nu$,

$$T_{G,z}=\{g\in L^1(V,\mathscr{X}):g(\varsigma)\in G(\varsigma,z_\varsigma)\text{ for almost everywhere }\varsigma\in V\},$$

is nonempty.

H_3 There exists $\rho:V\to[0,\infty)$ such that

$$\|G(\varsigma,z_\varsigma)\|=\sup\{\|g\|:g(\varsigma)\in G(\varsigma,z_\varsigma)\}\le\rho(\varsigma)\Theta(\|z\|_{\mathcal{P}_\nu}),\quad\varsigma\in V.$$

In the above, the continuous increasing function Θ maps from $[0,\infty)$ into itself.

H_4 The functions $J_j\in C(\mathscr{X},\mathscr{X})$, and there exists $P_m:[0,+\infty)\to(0,+\infty)$ to be continuous nondecreasing such that

$$|J_j(z)|\le P_m(|z|),\quad j=1,2,\dots,q,z\in\mathscr{X},$$

and

$$\liminf_{p\to\infty}\frac{P_m(p)}{p}=\tau_j<\infty,\quad j=1,2,\dots,q.$$

H_5 The function $\overline{J_j}\in C(\mathscr{X},\mathscr{X})$, and there exists $\overline{P_m}:[0,+\infty)\to(0,+\infty)$ to be continuous nondecreasing such that

$$|\overline{J_j}(z)|\le\overline{P_m}(|z|),\quad j=1,2,\dots,q,z\in\mathscr{X}$$

and

$$\liminf_{p\to\infty}\frac{\overline{P_m}(p)}{p}=\overline{\tau_j}<\infty,\quad j=1,2,\dots,q.$$

$\mathbf{H_6}$ The following inequalities hold:

$$
\begin{aligned}
&\|B\| = P_B, \qquad \Lambda = \widetilde{P}_m\widehat{P}_mP_1\big\|\alpha(0)\big\| + l\widetilde{P}_m\widehat{P}_mP_2\|z_1\|,\\
&\Lambda_1 = \widetilde{P}_mP_2, \qquad \eta = \frac{1}{\delta}\widetilde{P}_m^2P_2^2P_B^2c,\\
&\Lambda_2 = \eta\left[\|z_c\| + \Lambda + \widetilde{P}_mP_1\sum_{j=1}^{q}P_m\big(l^{-1}p'\big) + \widetilde{P}_mP_2\sum_{j=1}^{q}\overline{P_m}\big(l^{-1}p'\big)\right]\\
&\qquad + \widetilde{P}_mP_1\sum_{j=1}^{q}P_m\big(l^{-1}p'\big) + \widetilde{P}_mP_2\sum_{j=1}^{q}\overline{P_m}\big(l^{-1}p'\big), \qquad \varpi = l\Lambda_1(1+\eta).
\end{aligned}
$$

We demonstrate that system (1.1)–(1.4) is approximately controllable if for all $\delta > 0$ there exists $z(\cdot)$ which is continuous such that

$$
\begin{aligned}
z(\varsigma) &= M^{-1}\mathscr{N}(\varsigma)M\alpha(0) + M^{-1}\mathscr{M}(\varsigma)Mz_1 + \int_0^{\varsigma}M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi\\
&\quad + \int_0^{\varsigma}M^{-1}\mathscr{M}(\varsigma-\psi)Bu_\delta(\psi,z)\,d\psi + \sum_{0<\varsigma_j<c}M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j(z_{\varsigma_j})\\
&\quad + \sum_{0<\varsigma_j<c}M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j(z_{\varsigma_j}), \quad g\in T_{G,z},
\end{aligned}
\tag{3.1}
$$

$$
u_\delta(\varsigma,z) = B^*M^{-1}\mathscr{M}(c-\varsigma)R\big(\delta,\aleph_0^c\big)\sigma\big(z(\cdot)\big),
\tag{3.2}
$$

where

$$
\begin{aligned}
\sigma\big(z(\cdot)\big) &= z_c - M^{-1}\mathscr{N}(\varsigma)M\alpha(0) - M^{-1}\mathscr{M}(\varsigma)Mz_1 - \int_0^{\varsigma}M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi\\
&\quad - \sum_{0<\varsigma_j<c}M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j(z_{\varsigma_j}) - \sum_{0<\varsigma_j<c}M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j(z_{\varsigma_j}).
\end{aligned}
$$

Remark 3.2 ([27, 35])

(a) Provided that $\dim\mathscr{X} < \infty$, then for all $z\in\mathscr{X}$, $T_{G,z} = \emptyset$.

(b) $T_{G,z}$ is nonempty if and only if $\psi(\varsigma)$ maps from V into $\mathbb{R}$ defined by

$$
\psi(\varsigma) = \inf\big\{\|g\| : g\in\mathbb{G}(\varsigma,z_\varsigma)\big\}\in L^1(V,\mathbb{R}).
$$

Lemma 3.3 (See [14]) *Assume $z\in\mathcal{P}_\nu'$, then for $\varsigma\in V, z_\varsigma\in\mathcal{P}_\nu$. Moreover,*

$$
j\big|z(\varsigma)\big| \le \|z_\varsigma\|_{\mathcal{P}_\nu} \le \|\alpha\|_{\mathcal{P}_\nu} + j\sup_{\psi\in[0,\varsigma]}\big|z(\psi)\big|,
$$

where $j = \int_{-\infty}^{0}\nu(\varsigma)\,d\varsigma < +\infty$.

For any $\varrho > 0$, we define $\bigwedge^{\varrho} : \mathcal{P}'_{\nu} \to 2^{\mathcal{P}'_{\nu}}$ by $\bigwedge^{\varrho} z$ the set of $z \in \mathcal{P}'_{\nu}$ such that

$$z(\varsigma) = \begin{cases} \alpha(\varsigma), \quad \varsigma \in (-\infty, 0], \\ M^{-1}\mathscr{N}(\varsigma)M\alpha(0) + M^{-1}\mathscr{M}(\varsigma)Mz_1 + \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma - \psi)g(\psi)\,d\psi \\ \quad + \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma - \psi)Bu_{\delta}(\psi, z)\,d\psi + \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(\varsigma - \varsigma_j)J_j(z_{\varsigma_j}) \\ \quad + \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(\varsigma - \varsigma_j)\overline{J}_j(z_{\varsigma_j}), \quad \varsigma \in V, \end{cases}$$

where $g \in T_{G,z}$. To demonstrate $\bigwedge^{\varrho}$ has a fixed point, we conclude that it is the solution of system (1.1)–(1.4). Obviously, $z_c = z(c) \in (\bigwedge^{\varrho} z)(c)$, which means that $u_{\varrho}(z, \varsigma)$ drives (1.1)–(1.4) from z_0 to z_c in finite time c.

For $\alpha \in \mathcal{P}_{\nu}$, we now define $\widehat{\alpha}$ by

$$\widehat{\alpha}(\varsigma) = \begin{cases} \alpha(\varsigma), \quad \varsigma \in (-\infty, 0], \\ M^{-1}\mathscr{N}(\varsigma)M\alpha(0) + M^{-1}\mathscr{M}(\varsigma)Mz_1, \quad \varsigma \in V, \end{cases}$$

then $\widehat{\alpha} \in \mathcal{P}'_{\nu}$. Assume $z(\varsigma) = x(\varsigma) + \widehat{\alpha}(\varsigma)$, $-\infty < \varsigma \leq c$. We come to an end that x satisfies $x_0 = 0$ and

$$\begin{aligned} x(\varsigma) = {} & \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma - \psi)g(\psi)\,d\psi + \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma - \zeta)BB^{*}M^{-1}\mathscr{M}^{*}(c - \varsigma)R(\delta, \aleph_0^c) \\ & \times \Bigg[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c - \zeta)g(\zeta)\,d\zeta \\ & - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c - \varsigma_j)J_j\big(x(\varsigma_j^-) + \widehat{\alpha}(\varsigma_j^-)\big) \\ & - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c - \varsigma_j)\overline{J}_j\big(x(\varsigma_j^-) + \widehat{\alpha}(\varsigma_j^-)\big) \Bigg](\psi)\,d\psi \\ & + \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma - \varsigma_j)J_j\big(x(\varsigma_j^-) + \widehat{\alpha}(\varsigma_j^-)\big) \\ & + \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma - \varsigma_j)\overline{J}_j\big(x(\varsigma_j^-) + \widehat{\alpha}(\varsigma_j^-)\big), \quad \varsigma \in V. \end{aligned}$$

If x satisfies the following:

$$\begin{aligned} z(\varsigma) = {} & M^{-1}\mathscr{N}(\varsigma)M\alpha(0) + M^{-1}\mathscr{M}(\varsigma)Mz_1 + \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma - \psi)g(\psi)\,d\psi \\ & + \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma - \psi)BB^{*}M^{-1}\mathscr{M}^{*}(c - \varsigma)R(\delta, \aleph_0^c)\Bigg[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) \\ & - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c - \zeta)g(\zeta)\,d\zeta \\ & - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c - \varsigma_j)J_j\big(x(\varsigma_j^-) + \widehat{\alpha}(\varsigma_j^-)\big) \\ & - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c - \varsigma_j)\overline{J}_j\big(x(\varsigma_j^-) + \widehat{\alpha}(\varsigma_j^-)\big) \Bigg](\psi)\,d\psi \end{aligned}$$

$$+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)$$
$$+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J_j}\big(x\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big), \quad \varsigma \in V,$$

and $z(\varsigma)=\alpha(\varsigma)$, $\varsigma\in(-\infty,0]$.

Assume $\mathcal{P}''_\nu=\{x\in\mathcal{P}'_\nu : x_0=0\in\mathcal{P}_\nu\}$. For any $x\in\mathcal{P}''_\nu$,

$$\|x\|_c = \|x_0\|_{\mathcal{P}_\nu} + \sup\{|x(\psi)| : 0\le\psi\le c\} = \sup\{|x(\psi)| : 0\le\psi\le c\},$$

hence $(\mathcal{P}''_\nu, \|\cdot\|_c)$ is a Banach space. Fix $B_p=\{x\in\mathcal{P}''_\nu : |x|_c\le p\}$ for some $p>0$, then $B_p\subseteq\mathcal{P}''_\nu$ is uniformly bounded, and for $x\in B_p$, by referring to Lemma 3.3, we have

$$\begin{aligned}
\|x_\varsigma+\widehat{\alpha}_\varsigma\|_{\mathcal{P}_\nu} &\le \|x_\varsigma\|_{\mathcal{P}_\nu}+\|\widehat{\alpha}_\varsigma\|_{\mathcal{P}_\nu}\\
&\le l\sup_{\psi\in[0,\varsigma]}|x(\psi)| + \|x_0\|_{\mathcal{P}_\nu} + l\sup_{\psi\in[0,\varsigma]}|\widehat{\alpha}(\psi)| + \|\widehat{\alpha}_0\|_{\mathcal{P}_\nu}\\
&\le l\big(p+\widetilde{P}_m P_1\widehat{P}_m\|\alpha(0)|+\widetilde{P}_m P_2\widehat{P}_m|z_1|\big)+\|\widehat{\alpha}_0\|_{\mathcal{P}_\nu} = p'.
\end{aligned} \tag{3.3}$$

Considering Lemma 3.3, for each $\varsigma\in V$,

$$|x(\varsigma)+\widehat{\alpha}(\varsigma)| \le l^{-1}\|x_\varsigma+\widehat{\alpha}_\varsigma\|_{\mathcal{P}_\nu}.$$

For each $\varsigma\in V$, $x\in B_p$, from (3.3), $\mathbf{H_5}$ and $\mathbf{H_6}$, we have

$$\sup_{\varsigma\in V}|x(\varsigma)+\widehat{\alpha}(\varsigma)| \le l^{-1}\|x_\varsigma+\widehat{\alpha}_\varsigma\|_{\mathcal{P}_\nu}\le l^{-1}p',$$

hence

$$\begin{aligned}
\big|J_j\big(x\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\big| &\le P_m(|\big(x\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)|)\\
&\le P_m\Big(\sup_{\varsigma\in V}|x(\varsigma)+\widehat{\alpha}(\varsigma)|\Big)\\
&\le P_m\big(l^{-1}p'\big), \quad j=1,2,\dots,q,
\end{aligned}$$

and

$$\begin{aligned}
\big|\overline{J_j}\big(x\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\big| &\le \overline{P_m}(|\big(x\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)|)\\
&\le \overline{P_m}\Big(\sup_{\varsigma\in V}|x(\varsigma)+\widehat{\alpha}(\varsigma)|\Big)\\
&\le \overline{P_m}\big(l^{-1}p'\big), \quad j=1,2,\dots,q.
\end{aligned}$$

Define $\Phi : \mathcal{P}''_\nu \to \mathcal{P}''_\nu$ provided that $\Phi(x)$ is the set of $\overline{z} \in \mathcal{P}''_\nu$ such that

$$\overline{z}(\varsigma) = \begin{cases} 0, \quad \varsigma \in (-\infty, 0], \\ \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\zeta)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R(\delta,\aleph_0^c) \\ \quad \times [z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c-\zeta)g(\zeta)\,d\zeta \\ \quad - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)) \\ \quad - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-))](\psi)\,d\psi \\ \quad + \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)) \\ \quad + \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)), \quad \varsigma \in V. \end{cases}$$

Clearly, a fixed point of Φ^ϱ exists if a fixed point of Π exists. Hence, our focus is to verify that a fixed point of Π exists.

Lemma 3.4 *If hypotheses* $\mathbf{H_0}$–$\mathbf{H_5}$ *are satisfied, then* $\Phi : \mathcal{P}''_\nu \to \mathcal{P}''_\nu$ *is completely continuous multivalued, upper semicontinuous with a convex closed value.*

Proof To make things easier, we will divide our discussion into stages as follows:

Step 1. Φ is convex for each $x \in B_p$. Actually, if $\overline{z}_1, \overline{z}_2 \in \Phi(x)$, then there exist $g_1, g_2 \in T_{G,z}$ such that, for all $\varsigma \in V$, we have

$$\begin{aligned} \overline{z}(\varsigma) = {} & \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g_i(\psi)\,d\psi + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\zeta)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R(\delta,\aleph_0^c) \\ & \times \Biggl[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c-\zeta)g_i(\zeta)\,d\zeta \\ & - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\bigl(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\bigr) \\ & - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\bigl(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\bigr)\Biggr](\psi)\,d\psi \\ & + \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\bigl(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\bigr) \\ & + \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\bigl(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\bigr), \quad i = 1,2. \end{aligned}$$

Let $\beta \in [0,1]$. Then, for each $\varsigma \in V$, we have

$$\begin{aligned} & \bigl(\beta\overline{z}_1 + (1-\beta)\overline{z}_2\bigr)(\varsigma) \\ & \quad = \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)\bigl[\beta g_1(\psi) + (1-\beta)g_2(\psi)\bigr]\,d\psi \\ & \qquad + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\zeta)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R\bigl(\delta,\aleph_0^c\bigr) \\ & \qquad \times \Biggl[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 \\ & \qquad - \int_0^c M^{-1}\mathscr{M}(c-\zeta)\bigl[\beta g_1(\zeta) + (1-\beta)g_2(\zeta)\bigr]\,d\zeta \end{aligned}$$

$$\begin{aligned}
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big) \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big)\Big](\psi)\,d\psi \\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big) \\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big).
\end{aligned}$$

It is easy to verify that $T_{G,z}$ is convex and since G has convex values. Hence, $\beta g_1 + (1-\beta)g_2 \in T_{G,z}$. Consequently,

$$\beta\overline{z}_1 + (1-\beta)\overline{z}_2 \in \Phi(x).$$

Step 2. On bounded sets of $\mathcal{P}_v''$, $\Phi(x)$ is bounded.

In fact, this is sufficient to prove that there exists $\ell > 0$ such that for all $\overline{z}(x) \in \Phi(x)$, $x \in B_p$. In the above

$$B_p = \big\{x \in \mathcal{P}_v'' : |x|_c \le p\big\},$$

one possesses $\|\overline{z}\| \le \ell$.

Provided that $\overline{z} \in \Phi(x)$, there exists $g \in T_{G,z}$ such that, for all $\varsigma \in V$,

$$\begin{aligned}
\overline{z}(\varsigma) = {}& \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R\big(\delta,\aleph_0^c\big) \\
&\times \Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c-\zeta)g(\zeta)\,d\zeta \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big) \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big)\Big](\psi)\,d\psi \\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big) \\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big).
\end{aligned} \tag{3.4}$$

By referring to $(\mathbf{H_2})$–$(\mathbf{H_5})$ and (3.4), we get

$$\begin{aligned}
\big|(\overline{z})(\varsigma)\big| \le {}& \left|\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi\right| + \left|\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)Bu_\delta(\psi,x+\widehat{\alpha})\,d\psi\right| \\
&+ \left|\sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big)\right| \\
&+ \left|\sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big)\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \widetilde{P}_m P_2 \int_0^{\varsigma} \rho(\psi)\Theta\big(\|x_\psi + \widehat{\alpha}_\psi\|_{\mathcal{P}_v}\big)\,d\psi + \frac{1}{\delta}\widetilde{P}_m^2 P_2^2 P_B^2 c\Big[\|z_c\| + \widetilde{P}_m\widehat{P}_m P_1\big\|\alpha(0)\big\| \\
&\quad + \widetilde{P}_m\widehat{P}_m P_2\|z_1\| \\
&\quad + \widetilde{P}_m P_2 \int_0^{c} \rho(\zeta)\Theta\big(\|x_\zeta + \widehat{\alpha}_\zeta\|_{\mathcal{P}_v}\big)\,d\zeta + \widetilde{P}_m P_1 \sum_{j=1}^{q} P_m\big(\big|x^p\big(\varsigma_j^-\big) + \widehat{\alpha}\big(\varsigma_j^-\big)\big|\big) \\
&\quad + \widetilde{P}_m P_2 \sum_{j=1}^{q} \overline{P_m}\big(\big|x^p\big(\varsigma_j^-\big) + \widehat{\alpha}\big(\varsigma_j^-\big)\big|\big)\Big] + \widetilde{P}_m P_1 \sum_{j=1}^{q} P_m\big(\big|x^p\big(\varsigma_j^-\big) + \widehat{\alpha}\big(\varsigma_j^-\big)\big|\big) \\
&\quad + \widetilde{P}_m P_2 \sum_{j=1}^{q} \overline{P_m}\big(\big|x^p\big(\varsigma_j^-\big) + \widehat{\alpha}\big(\varsigma_j^-\big)\big|\big) \\
&\le \widetilde{P}_m P_2 \sup_{x\in[0,p']} \Theta(x) \int_0^c \rho(\psi)\,d\psi \\
&\quad + \frac{1}{\delta}\widetilde{P}_m^2 P_2^2 P_B^2 c\Big[\|z_c\| + \widetilde{P}_m\widehat{P}_m P_1\big\|\alpha(0)\big\| + \widetilde{P}_m\widehat{P}_m P_2\|z_1\| \\
&\quad + \widetilde{P}_m P_2 \sup_{x\in[0,p']} \Theta(x) \int_0^c \rho(\zeta)\,d\zeta + \widetilde{P}_m P_1 \sum_{j=1}^{q} P_m\big(l^{-1}p'\big) + \widetilde{P}_m P_2 \sum_{j=1}^{q} \overline{P_m}\big(l^{-1}p'\big)\Big] \\
&\quad + \widetilde{P}_m P_1 \sum_{j=1}^{q} P_m\big(l^{-1}p'\big) + \widetilde{P}_m P_2 \sum_{j=1}^{q} \overline{P_m}\big(l^{-1}p'\big) = \ell.
\end{aligned}
$$

As a result, for all $\overline{z} \in \Phi(B_p)$, we get $\|\overline{z}\|_c \le \ell$.

Step 3. $\Phi(B_p)$ is equicontinuous. In fact, assume that $\varrho > 0$ is small, $0 < \omega_1 < \omega_2 \le c$. For each $x \in B_p$ and $\overline{z} \in \Phi_1(x)$, there exists $g \in T_{G,z}$ such that, for each $\varsigma \in V$, we have

$$
\begin{aligned}
&\big|\overline{z}(\omega_2) - \overline{z}(\omega_1)\big| \\
&\quad = \bigg|\int_{\omega_1}^{\omega_2} M^{-1}\mathscr{M}(\omega_2 - \psi)g(\psi)\,d\psi\bigg| \\
&\qquad + \bigg|\int_{\omega_1-\varrho}^{\omega_1} M^{-1}\big[\mathscr{M}(\omega_2 - \psi) - \mathscr{M}(\omega_1 - \psi)\big]g(\psi)\,d\psi\bigg| \\
&\qquad + \bigg|\int_0^{\omega_1-\varrho} M^{-1}\big[\mathscr{M}(\omega_2 - \psi) - \mathscr{M}(\omega_1 - \psi)\big]g(\psi)\,d\psi\bigg| \\
&\qquad + \bigg|\int_0^{\omega_1-\varrho} M^{-1}\big[\mathscr{M}(\omega_2 - \psi) - \mathscr{M}(\omega_1 - \psi)\big]Bu_\delta^p(\psi,x)\,d\psi\bigg| \\
&\qquad + \bigg|\int_{\omega_1-\varrho}^{\omega_1} M^{-1}\big[\mathscr{M}(\omega_2 - \psi) - \mathscr{M}(\omega_1 - \psi)\big]Bu_\delta^p(\psi,x)\,d\psi\bigg| \\
&\qquad + \bigg|\int_{\omega_1}^{\omega_2} M^{-1}\mathscr{M}(\omega_2 - \psi)Bu_\delta^p(\psi,x)\,d\psi\bigg| \\
&\qquad + \bigg|\sum_{0<\varsigma_j<\omega_1} M^{-1}\big[\mathscr{N}(\omega_2 - \varsigma_j) - \mathscr{N}(\omega_1 - \varsigma_j)\big]J_j\big(x\big(\varsigma_j^-\big) + \widehat{\alpha}\big(\varsigma_j^-\big)\big)\bigg| \\
&\qquad + \bigg|\sum_{\omega_1<\varsigma_j<\omega_2} M^{-1}\mathscr{N}(\omega_2 - \varsigma_j)J_j\big(x\big(\varsigma_j^-\big) + \widehat{\alpha}\big(\varsigma_j^-\big)\big)\bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\sum_{0<\varsigma_j<\omega_1} M^{-1}\bigl[\mathscr{M}(\omega_2-\varsigma_j)-\mathscr{M}(\omega_1-\varsigma_j)\bigr]\overline{J}_j\bigl(x\bigl(\varsigma_j^-\bigr)+\widehat{\alpha}\bigl(\varsigma_j^-\bigr)\bigr)\right| \\
&+\left|\sum_{\omega_1<\varsigma_j<\omega_2} M^{-1}\mathscr{M}(\omega_2-\varsigma_j)\overline{J}_j\bigl(x\bigl(\varsigma_j^-\bigr)+\widehat{\alpha}\bigl(\varsigma_j^-\bigr)\bigr)\right| \\
\leq{}& \widetilde{P}_m P_2\int_{\omega_1}^{\omega_2}\rho(\psi)\Theta\bigl(\|x_\psi+\widehat{\alpha}_\psi\|_{\mathcal{P}_v}\bigr)\,d\psi \\
&+\widetilde{P}_m\int_{\omega_1-\varrho}^{\omega_1}\bigl\|\mathscr{M}(\omega_2-\psi)-\mathscr{M}(\omega_1-\psi)\bigr\|\rho(\psi)\Theta\bigl(\|x_\psi+\widehat{\alpha}_\psi\|_{\mathcal{P}_v}\bigr)\,d\psi \\
&+\widetilde{P}_m\int_{0}^{\omega_1-\varrho}\bigl\|\mathscr{M}(\omega_2-\psi)-\mathscr{M}(\omega_1-\psi)\bigr\|\rho(\psi)\Theta\bigl(\|x_\psi+\widehat{\alpha}_\psi\|_{\mathcal{P}_v}\bigr)\,d\psi \\
&+\widetilde{P}_m P_B\int_{0}^{\omega_1-\varrho}\bigl\|\mathscr{M}(\omega_2-\psi)-\mathscr{M}(\omega_1-\psi)\bigr\| \\
&\times\Biggl[\|z_c\|+\widetilde{P}_m\widehat{P}_m P_1\bigl\|\alpha(0)\bigr\|+\widetilde{P}_m\widehat{P}_m P_2\|z_1\| \\
&+\widetilde{P}_m P_2\int_0^c\rho(\zeta)\Theta\bigl(\|x_\zeta+\widehat{\alpha}_\zeta\|_{\mathcal{P}_v}\bigr)\,d\zeta+\widetilde{P}_m P_1\sum_{j=1}^q P_m\bigl(l^{-1}p'\bigr) \\
&+\widetilde{P}_m P_2\sum_{j=1}^q\overline{P_m}\bigl(l^{-1}p'\bigr)\Biggr](\psi)\,d\psi \\
&+\widetilde{P}_m P_B\int_{\omega_1-\varrho}^{\omega_1}\bigl\|\mathscr{M}(\omega_2-\psi)-\mathscr{M}(\omega_1-\psi)\bigr\| \\
&\times\Biggl[\|z_c\|+\widetilde{P}_m\widehat{P}_m P_1\bigl\|\alpha(0)\bigr\|+\widetilde{P}_m\widehat{P}_m P_2\|z_1\| \\
&+\widetilde{P}_m P_2\int_0^c\rho(\zeta)\Theta\bigl(\|x_\zeta+\widehat{\alpha}_\zeta\|_{\mathcal{P}_v}\bigr)\,d\zeta+\widetilde{P}_m P_1\sum_{j=1}^q P_m\bigl(l^{-1}p'\bigr) \\
&+\widetilde{P}_m P_2\sum_{j=1}^q\overline{P_m}\bigl(l^{-1}p'\bigr)\Biggr](\psi)\,d\psi \\
&+\widetilde{P}_m P_2 P_B\int_{\omega_1}^{\omega_2}\Biggl[\|z_c\|+\widetilde{P}_m\widehat{P}_m P_1\bigl\|\alpha(0)\bigr\|+\widetilde{P}_m\widehat{P}_m P_2\|z_1\| \\
&+\widetilde{P}_m P_2\int_0^c\rho(\zeta)\Theta\bigl(\|x_\zeta+\widehat{\alpha}_\zeta\|_{\mathcal{P}_v}\bigr)\,d\zeta+\widetilde{P}_m P_1\sum_{j=1}^q P_m\bigl(l^{-1}p'\bigr) \\
&+\widetilde{P}_m P_2\sum_{j=1}^q\overline{P_m}\bigl(l^{-1}p'\bigr)\Biggr](\psi)\,d\psi \\
&+\widetilde{P}_m\sum_{0<\varsigma_j<\omega_1}\bigl\|\mathscr{N}(\omega_2-\varsigma_j)-\mathscr{N}(\omega_1-\varsigma_j)\bigr\|P_m\bigl(l^{-1}p'\bigr)+\widetilde{P}_m P_1\sum_{\omega_1<\varsigma_j<\omega_2}P_m\bigl(l^{-1}p'\bigr) \\
&+\widetilde{P}_m\sum_{0<\varsigma_j<\omega_1}\bigl\|\mathscr{M}(\omega_2-\varsigma_j)-\mathscr{M}(\omega_1-\varsigma_j)\bigr\|\overline{P_m}\bigl(l^{-1}p'\bigr) \\
&+\widetilde{P}_m P_2\sum_{\omega_1<\varsigma_j<\omega_2}\overline{P_m}\bigl(l^{-1}p'\bigr).
\end{aligned}
\tag{3.5}
$$

Hence, for $\varrho > 0$, we conclude that inequality (3.5) tends to zero as $\varsigma_2 \to \varsigma_1$. Then, the compactness of $\mathscr{M}(\varsigma)$ for $\varsigma > 0$ gives continuity in uniform operator topology. Therefore, Φ maps B_p into an equicontinuous family of functions.

Therefore, from *Step 2* and *Step 3*, and utilizing Arzela–Ascoli theorem, we can deduce that Φ is a compact multivalued function and, hence, a condensing map.

Step 4: Φ has a closed graph.

Assume $x_n \to x_*$ as $n \to \infty$, $\overline{z}_n \in \Phi(x_n)$ for each $x_n \in B_p$, and $\overline{z}_n \to \overline{z}_*$ as $n \to \infty$. Now, we demonstrate $\overline{z}_* \in \Phi(x_*)$. Because $\overline{z}_n \in \Phi(x_n)$, there exists $g_n \in T_{G,z_n}$ such that

$$\begin{aligned}
\overline{z}_n(\varsigma) = {}& \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g_n(\psi)\,d\psi + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R\big(\delta,\aleph_0^c\big)\\
&\times\Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c-\psi)g_n(\psi)\,d\psi\\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\Big](\psi)\,d\psi\\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big), \quad \varsigma \in V.
\end{aligned}$$

We must demonstrate that there exists $g_* \in T_{G,z_*}$ such that

$$\begin{aligned}
\overline{z}_*(\varsigma) = {}& \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g_*(\psi)\,d\psi + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R\big(\delta,\aleph_0^c\big)\\
&\times\Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c-\psi)g_*(\psi)\,d\psi\\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)J_j\big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\Big](\psi)\,d\psi\\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\\
&+ \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big), \quad \varsigma \in V.
\end{aligned}$$

For every $\varsigma \in V$, since G is continuous and from x^ϱ, we have

$$\begin{aligned}
&\bigg\|\bigg(\overline{z}_n(\varsigma) - \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\\
&\quad - \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma-\psi)BB^{*}M^{-1}\mathscr{M}^{*}(c-\varsigma)R\big(\delta,\aleph_0^c\big) \\
&\times \Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big) \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big(x_n\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\Big](\psi)\,d\psi\big) \\
&- (\overline{z}_*(\varsigma) - \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big) \\
&- \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big) \\
&\times - \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma-\psi)BB^{*}M^{-1}\mathscr{M}^{*}(c-\varsigma)R\big(\delta,\aleph_0^c\big) \\
&\times \Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\Big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big) \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big(x_*\big(\varsigma_j^-\big)+\widehat{\alpha}\big(\varsigma_j^-\big)\big)\Big](\psi)\,d\psi\Big)\Big\| \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Consider the continuous linear operator $\Theta : L^1(V,\mathscr{Z}) \to C(V,\mathscr{Z})$,

$$
\begin{aligned}
(\Theta g)(\varsigma) = {}& \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi \\
&- \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma-\psi)BB^{*}M^{-1}\mathscr{M}^{*}(c-\varsigma)R\big(\delta,\aleph_0^c\big) \\
&\times \left(\int_0^{c} M^{-1}\mathscr{M}(c-\zeta)g(\zeta)\,d\zeta\right)d\psi.
\end{aligned}
$$

Therefore, by referring to Lemma 2.4, $\Theta \circ T_G$ is a closed graph operator. Additionally, from Θ, we have

$$
\begin{aligned}
&\overline{z}_n(\varsigma) - M^{-1}\mathscr{N}(\varsigma)M\alpha(0) - M^{-1}\mathscr{M}(\varsigma)Mz_1 - \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big((x_n)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big) \\
&- \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big((x_n)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big) \\
&- \int_0^{\varsigma} M^{-1}\mathscr{M}(\varsigma-\psi)BB^{*}M^{-1}\mathscr{M}^{*}(c-\varsigma)R\big(\delta,\aleph_0^c\big) \\
&\times \Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1 \\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\big((x_n)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big)
\end{aligned}
$$

$$- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big((x_n)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big)\Big](\psi)\,d\psi \in \Theta(T_{G,z_n}).$$

Because $x_n \to x_*$, for some $x_* \in T_{G,z_*}$, by referring to Lemma 2.4, we have

$$\begin{aligned}
&\overline{z}_*(\varsigma) - M^{-1}\mathscr{N}(\varsigma)M\alpha(0) - M^{-1}\mathscr{M}(\varsigma)Mz_1 - \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big((x_*)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big)\\
&- \sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big((x_*)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big)\\
&- \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R\big(\delta,\aleph_0^c\big)\\
&\times \Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0) - M^{-1}\mathscr{M}(c)Mz_1\\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\big((x_*)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big)\\
&- \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big((x_*)_{\varsigma_j}+\widehat{\alpha}_{\varsigma_j}\big)\Big](\psi)\,d\psi \in \Theta(T_{G,z_*})
\end{aligned}$$

for some $g_* \in (T_{G,z_*})$. Hence, Φ has a closed graph. □

Therefore Φ is a completely continuous multivalued function with convex closed values and upper semicontinuity. Now, by using Theorem 2.5, we determine a parameter $\lambda > 1$ and define the following auxiliary system:

$$\frac{d^2}{d\varsigma^2}\big(Mz(\varsigma)\big) \in Az(\varsigma) + \frac{1}{\lambda}G(\varsigma,z_\varsigma) + \frac{1}{\lambda}Bu(\varsigma),$$

$$\varsigma \in V = [0,c], \varsigma \neq \varsigma_j, j = 1,2,\dots,q, \tag{3.6}$$

$$z(\varsigma) = \alpha(\varsigma) \in \mathcal{P}_v, \quad \varsigma \in (-\infty,0], z'(0) = z_1 \in \mathscr{Z}, \tag{3.7}$$

$$\Delta z|_{\varsigma=\varsigma_j} = \frac{1}{\lambda}J_j\big(z(\varsigma_j^-)\big), \quad j = 1,2,\dots,q, \tag{3.8}$$

$$\Delta z'|_{\varsigma=\varsigma_j} = \frac{1}{\lambda}\overline{J}_j\big(z(\varsigma_j^-)\big), \quad j = 1,2,\dots,q. \tag{3.9}$$

As a result, from Definition 3.1, the mild solution of system (3.6)–(3.9) is given by

$$z(\varsigma) = \begin{cases} \alpha(\varsigma), \quad \varsigma \in (-\infty,0], \\ M^{-1}\mathscr{N}(\varsigma)M\alpha(0) + M^{-1}\mathscr{M}(\varsigma)Mz_1 \\ \quad + \frac{1}{\lambda}\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi \\ \quad + \frac{1}{\lambda}\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)Bu_\delta(\psi,z)\,d\psi \\ \quad + \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j(z_{\varsigma_j}) \\ \quad + \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j(z_{\varsigma_j}), \quad \varsigma \in V, \end{cases} \tag{3.10}$$

where $g \in T_{G,z} = \{g \in L^1(V,\mathscr{Z}) : g(\varsigma) \in G(\varsigma,z_\varsigma) \text{ for } \varsigma \in V\}$.

Lemma 3.5 *Consider* $(\mathbf{H_1})$–$(\mathbf{H_6})$ *to be satisfied. Assume that* $z(\varsigma)$ *is a mild solution of system* (3.9). *In addition, there exists a priori bounds* $J > 0$ *such that* $\|z_\varsigma\|_{\mathcal{P}_\nu} \le J$, $\varsigma \in V$, *where* J *depends only on* μ *and on the* $\Theta(\cdot)$, *and* ρ.

Proof By referring to system (3.10), we get

$$\begin{aligned}
|z(\varsigma)| &\le \left|M^{-1}\mathscr{N}(\varsigma)M\alpha(0)\right| + \left|M^{-1}\mathscr{M}(\varsigma)Mz_1\right| + \left|\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi\right| \\
&\quad + \left|\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\zeta)BB^*M^{-1}\mathscr{M}^*(c-\varsigma)R(\delta,\aleph_0^c)\Big[z_c - M^{-1}\mathscr{N}(c)M\alpha(0)\right. \\
&\quad - M^{-1}\mathscr{M}(c)Mz_1 - \int_0^c M^{-1}\mathscr{M}(c-\zeta)g(\zeta)\,d\zeta - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(c-\varsigma_j)J_j\big(z(\varsigma_j^-)\big) \\
&\quad \left. + \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(c-\varsigma_j)\overline{J}_j\big(z(\varsigma_j^-)\big)\Big](\psi)\,d\psi\right| + \left|\sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(z(\varsigma_j^-)\big)\right| \\
&\quad + \left|\sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(z(\varsigma_j^-)\big))\right| \\
&\le \widetilde{P}_m\widehat{P}_mP_1\|\alpha(0)\| + \widetilde{P}_m\widehat{P}_mP_2\|z_1\| + \widetilde{P}_mP_2\int_0^\varsigma \rho(\psi)\Theta\big(\|z_\psi\|_{\mathcal{P}_\nu}\big)\,d\psi \\
&\quad + \frac{1}{\delta}\widetilde{P}_m^2P_2^2P_B^2c\Big[\|z_c\| + \widetilde{P}_m\widehat{P}_mP_1\|\alpha(0)\| + \widetilde{P}_m\widehat{P}_mP_2\|z_1\| \\
&\quad + \widetilde{P}_mP_2\int_0^c \rho(\zeta)\Theta\big(\|z_\zeta\|_{\mathcal{P}_\nu}\big)\,d\zeta \\
&\quad + \widetilde{P}_mP_1\sum_{j=1}^q P_m\big(|z^p(\varsigma_j^-)|\big) + \widetilde{P}_mP_2\sum_{j=1}^q \overline{P_m}\big(|z^p(\varsigma_j^-)|\big)\Big] \\
&\quad + \widetilde{P}_mP_1\sum_{j=1}^q P_m\big(|z^p(\varsigma_j^-)|\big) + \widetilde{P}_mP_2\sum_{j=1}^q \overline{P_m}\big(|z^p(\varsigma_j^-)|\big) \\
&\le \Lambda + \Lambda_2 + \Lambda_1\int_0^\varsigma \rho(\psi)\Theta\big(\|z_\psi\|_{\mathcal{P}_\nu}\big)\,d\psi + \Lambda_1\eta\int_0^c \rho(\zeta)\Theta\big(\|z_\zeta\|_{\mathcal{P}_\nu}\big)\,d\zeta \\
&\le \Lambda + \Lambda_2 + \Lambda_1(1+\eta)\int_0^\varsigma \rho(\psi)\Theta\big(\|z_\psi\|_{\mathcal{P}_\nu}\big)\,d\psi, \quad \varsigma \in V.
\end{aligned}$$

As a result, by Lemma 3.3, we have

$$\begin{aligned}
\|z_\varsigma\|_{\mathcal{P}_\nu} &\le l\sup\big\{|z(\psi)| : 0 \le \psi \le \varsigma\big\} + \|\wp\|_{\mathcal{P}_\nu} \\
&\le l\Lambda + l\Lambda_2 + l\Lambda_1(1+\eta)\int_0^\varsigma \rho(\psi)\Theta\big(\|z_\psi\|_{\mathcal{P}_\nu}\big)\,d\psi + \|\wp\|_{\mathcal{P}_\nu}.
\end{aligned}$$

Assume that $\nu(\varsigma) = \sup\{\|z_\psi\|_{\mathcal{P}_\nu} : 0 \le \psi \le \varsigma\}$. In addition, the function $\nu(\varsigma) \in V$ is increasing, then

$$\nu(\varsigma) \le l\Lambda + l\Lambda_2 + l\Lambda_1(1+\eta)\int_0^\varsigma \rho(\psi)\Theta\big(\nu(\psi)\big)\,d\psi + \|\wp\|_{\mathcal{P}_\nu}.$$

Assume that $\nu(\varsigma)$ is the right-hand side of the above inequality. Since we get

$$a = \nu(0) = l\Lambda + l\Lambda_2 + \|\wp\|_{\mathcal{P}_\nu}, \qquad \nu(\varsigma) \le y(\varsigma), \quad \varsigma \in V,$$

and

$$y'(\varsigma) \le \varpi\rho(\varsigma)\Theta\big(\nu(\varsigma)\big), \quad \varsigma \in V.$$

Applying the nondecreasing properties of Θ, we have

$$y'(\varsigma) \le \varpi\rho(\varsigma)\Theta\big(y(\varsigma)\big), \quad \varsigma \in V.$$

The above inequality implies, for all $\varsigma \in V$, that

$$\int_a^{y(\varsigma)} \frac{d\psi}{\Theta(\psi)} \le \varpi \int_0^c \rho(\psi)\,d\psi \le \int_a^\infty \frac{d\psi}{\Theta(\psi)}. \qquad \square$$

This implies that $y(\varsigma) < \infty$. Hence, there exists $J > 0$ such that $y(\varsigma) \le J$, $\varsigma \in V$, and then

$$\|z_\varsigma\|_{\mathscr{B}_j} \le \nu(\varsigma) \le y(\varsigma) \le J, \quad \varsigma \in V.$$

Here, J depends only on c, the functions Θ and ρ. The proof has been completed.

Theorem 3.6 *If* $\mathbf{H_0}$–$\mathbf{H_6}$ *are satisfied. In addition,* (1.1)–(1.4) *admits at least one mild solution on* V.

Proof Suppose that $\Omega = \{x \in \mathcal{P}''_\nu : \lambda x \in \Phi x \text{ for some } \lambda > 1\}$. Since for all $x \in \Omega$, we have

$$\begin{aligned} x(\varsigma) ={}& \frac{1}{\lambda}M^{-1}\mathscr{N}(\varsigma)M\alpha(0) + \frac{1}{\lambda}M^{-1}\mathscr{M}(\varsigma)Mz_1 + \frac{1}{\lambda}\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi \\ &+ \frac{1}{\lambda}\int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\zeta)Bu(\psi)\,d\psi + \frac{1}{\lambda}\sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big) \\ &+ \frac{1}{\lambda}\sum_{0<\varsigma_j<\varsigma} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j\big(x(\varsigma_j^-)+\widehat{\alpha}(\varsigma_j^-)\big), \end{aligned}$$

which implies $z = x + \widehat{\alpha}$ is a mild solution of (3.10), that we demonstrated in Lemma 3.3 and Lemma 3.5

$$\begin{aligned} \|x\|_\varsigma &= \|x_0\|_{\mathcal{P}_\nu} + \sup\big\{\big|x(\varsigma)\big| : 0 \le \varsigma \le c\big\} \\ &= \sup\big\{\big|x(\varsigma)\big| : 0 \le \varsigma \le c\big\} \\ &\le \sup\big\{\big|z(\varsigma)\big| : 0 \le \varsigma \le c\big\} + \sup\big\{\big|\widehat{\alpha}(\varsigma)\big| : 0 \le \varsigma \le c\big\} \\ &\le \sup\big\{l^{-1}\|z_\varsigma\|_{\mathcal{P}_\nu} : 0 \le \varsigma \le c\big\} \\ &\quad + \sup\big\{\big|M^{-1}\mathscr{N}(\varsigma)M\alpha(0) + M^{-1}\mathscr{M}(\varsigma)Mz_1\big| : 0 \le \varsigma \le c\big\} \\ &\le 7l^{-1}J + \widetilde{P}_m\widehat{P}_mP_1\big|\alpha(0)\big| + \widetilde{P}_m\widehat{P}_mP_2|z_1|, \end{aligned}$$

which implies Ω is bounded on V.

Accordingly, by referring to Theorem 2.5 and Lemma 3.4, then Φ has a fixed point $x^* \in \mathcal{P}''_\nu$. Assume that $z(\varsigma) = x^*(\varsigma) + \widehat{\alpha}(\varsigma)$, $\varsigma \in (-\infty, c]$. As a result, z is a fixed point of $\bigwedge^\varrho$ that is a mild solution of system (1.1)–(1.4). □

Definition 3.7 The second-order Sobolev system (1.1)—(1.4) is said to be approximately controllable on V provided that $\overline{R(c,z_0)} = \mathscr{Z}$, where $R(c,z_0) = \{z_c(z_0;u) : z(\cdot) \in L^2(V,U)\}$ is a mild solution of system (1.1)–(1.4).

Theorem 3.8 *If hypotheses* $\mathbf{H_0}$–$\mathbf{H_6}$ *are satisfied. In addition, there exists* $N \in L^1(V,[0,\infty))$ *such that*

$$\sup_{z\in\mathcal{P}_\nu} \|G(\varsigma,z)\| \le N(\varsigma),$$

for almost everywhere $\varsigma \in V$, *then system* (1.1)–(1.4) *is approximately controllable on* V.

Proof Assume $\widehat{z}^\delta(\cdot)$ to be a fixed point of $\bigwedge^\varrho$ in B_p. In view of Theorem 3.4, any fixed point of $\bigwedge^\varrho$ is a mild solution of system (1.1)–(1.4) under

$$\widehat{u}^\varsigma(\varsigma) = B^* M^{-1}\mathscr{M}^*(c-\varsigma)R(\delta,\aleph_0^c)\sigma(\widehat{z}^\delta),$$

and satisfies the following:

$$\widehat{z}^\delta(c) = z_c + \delta R(\delta,\aleph_0^c)\sigma(\widehat{z}^\delta). \tag{3.11}$$

Further, by using the facts about G and Dunford–Pettis theorem, we know that $\{g^\varsigma(\psi)\}$ is weakly compact in $L^1(V,\mathscr{Z})$; accordingly, there is a subsequence $\{g^\delta(\psi)\}$, which converges weakly to say $g(\psi)$ in $L^1(V,\mathscr{Z})$. Define

$$\begin{aligned} w = {}& z_c - M^{-1}\mathscr{N}(\varsigma)M\alpha(0) - M^{-1}\mathscr{M}(\varsigma)Mz_1 - \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)g(\psi)\,d\psi \\ & - \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)Bu(\psi)\,d\psi - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(\varsigma-\varsigma_j)J_j(z_{\varsigma_j}) \\ & - \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(\varsigma-\varsigma_j)\overline{J}_j(z_{\varsigma_j}). \end{aligned}$$

Now, we have

$$\begin{aligned} \|\sigma(\widehat{z}^\delta) - w\| &= \left\| \int_0^c M^{-1}\mathscr{M}(c-\psi)\big[g(\psi,\widehat{z}^\delta(\psi)) - g(\psi)\big]\,d\psi \right\| \\ &\le \sup_{\varsigma\in V} \left\| \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma-\psi)\big[g(\psi,\widehat{z}^\delta(\psi)) - g(\psi)\big]\,d\psi \right\|. \end{aligned} \tag{3.12}$$

By using the Ascoli–Arzela theorem of infinite-dimensional version, we prove $l(\cdot) \to \int_0^\cdot M^{-1}\mathscr{M}(\cdot-\psi)l(\psi)\,d\psi : L^1(V,\mathscr{Z}) \to C(V,\mathscr{Z})$ is compact. Thus, $\|\sigma(\widehat{z}^\delta) - w\| \to 0$ as $\delta \to 0^+$. Additionally, from (3.11), we have

$$\begin{aligned} \|\widehat{z}^\delta(c) - z_c\| &\le \|\delta R(\delta,\aleph_0^c)(w)\| + \|\delta R(\delta,\aleph_0^c)\| \|\sigma(\widehat{z}^\delta) - w\| \\ &\le \|\delta R(\delta,\aleph_0^c)(w)\| + \|\sigma(\widehat{z}^\delta) - w\|. \end{aligned}$$

In view of $\mathbf{H_0}$ and from (3.12), $\|\widehat{z}^{\delta}(c) - z_c\| \to 0$ as $\delta \to 0^+$, which shows the approximate controllability of system (1.1)–(1.4). □

4 Nonlocal conditions

Byszewski has presented the concept of nonlocal conditions for the extension of problems based on classical conditions. When comparing nonlocal initial conditions with the classical initial condition, which is more accurate to depict the nature marvels, since more information is considered, along these lines we lessen the negative impacts initiated by a potential incorrect single estimation taken toward the beginning time. The researchers recently established the nonlocal fractional differential systems with or without delay by referring to nondense domain, semigroup, cosine families, several fixed point techniques, and measure noncompactness. It is a very useful discussion about differential systems, including nonlocal conditions, and one can refer to [4, 5, 19–21, 50].

Assume the nonlocal impulsive differential systems of the following form:

$$\frac{d^2}{d\varsigma^2}\big(Mz(\varsigma)\big) \in Az(\varsigma) + G(\varsigma, z_\varsigma) + Bu(\varsigma), \quad \varsigma \in V = [0,c], \varsigma \neq \varsigma_j, j = 1,2,\dots,q, \tag{4.1}$$

$$z(\varsigma) = \alpha(\varsigma) + h(z_{\varsigma_1}, z_{\varsigma_2}, z_{\varsigma_3}, \dots, z_{\varsigma_n}) \in \mathcal{P}_\nu, \quad \varsigma \in (-\infty, 0], z'(0) = z_1 \in \mathscr{X}, \tag{4.2}$$

$$\Delta z|_{\varsigma=\varsigma_j} = J_j\big(z\big(\varsigma_j^-\big)\big), \quad j = 1,2,\dots,q, \tag{4.3}$$

$$\Delta z'|_{\varsigma=\varsigma_j} = J_j\big(z\big(\varsigma_j^-\big)\big), \quad j = 1,2,\dots,q, \tag{4.4}$$

where $0 < \varsigma_1 < \varsigma_2 < \varsigma_3 < \cdots < \varsigma_j \leq c$, $h : \mathcal{P}_\nu^j \to \mathcal{P}_\nu$ which satisfies the following hypothesis:

$\mathbf{H_7}$ $h : \mathcal{P}^q \to \mathcal{P}$ is continuous and $P_j(h) > 0$ such that

$$\big\|h(u_1, u_2, u_3, \dots, u_q) - h(v_1, v_2, v_3, \dots, v_q)\big\| \leq \sum_{j=1}^{q} P_j(h)\|u - v\|_{\mathcal{P}},$$

for each $u, v \in \mathcal{P}_\nu$ and $N_h = \sup\{\|h(u_{\varsigma_1}, u_{\varsigma_2}, u_{\varsigma_3}, \dots, u_{\varsigma_n})\| : u \in \mathcal{P}_\nu\}$.

Definition 4.1 A function $z : (-\infty, c] \to \mathscr{X}$ is said to be a mild solution of system (1.1)–(1.4) provided that $z_0 = \alpha \in \mathcal{P}_\nu, z'(0) = z_1 \in \mathscr{X}$ on $(-\infty, 0]$, $\Delta z|_{\varsigma=\varsigma_j} = J_j(z(\varsigma_j^-)), \Delta z'|_{\varsigma=\varsigma_j} = \overline{J}_j(z(\varsigma_j^-)), j = 1,2,\dots,q$; $z(\cdot)$ to J_j $(j = 0,1,\dots,q)$ is continuous and

$$\begin{aligned} z(\varsigma) = {}& M^{-1}\mathscr{N}(\varsigma)M\big[\alpha(0) + q(z_{\varsigma_1}, z_{\varsigma_2}, z_{\varsigma_3}, \dots, z_{\varsigma_n})(0)\big] + M^{-1}\mathscr{M}(\varsigma)Mz_1 \\ & + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma - \psi)g(\psi)\,d\psi + \int_0^\varsigma M^{-1}\mathscr{M}(\varsigma - \psi)Bu(\psi)\,d\psi \\ & + \sum_{0<\varsigma_j<c} M^{-1}\mathscr{N}(\varsigma - \varsigma_j)J_j(z_{\varsigma_j}) + \sum_{0<\varsigma_j<c} M^{-1}\mathscr{M}(\varsigma - \varsigma_j)\overline{J}_j(z_{\varsigma_j}), \quad \varsigma \in V, \end{aligned}$$

is satisfied.

Theorem 4.2 *If hypotheses* $\mathbf{H_0}$–$\mathbf{H_7}$ *are satisfied, then system* (4.1)–(4.4) *is approximately controllable on* V.

Remark 4.3 Many integral and differential equations related to integrals and derivatives of integer order have proven to be a powerful tool in describing various phenomena of engineering systems, advancement of the calculus of variations and optimal control to fractional dynamic systems, bioengineering, biomedical applications, image and signal processing, and other fields in recent years. Inspired by the above theory and the research articles [16, 17, 54, 56], one can extend the current study to the second-order integrodifferential systems and Volterra–Fredholm integrodifferential systems with impulses by using the way of the approach presented in this article.

5 An example

Consider the second-order Sobolev-type impulsive differential control system of the following form:

$$\frac{\partial^2}{\partial\varsigma^2}\big[y(\varsigma,z)-y_{zz}(\varsigma,z)\big]\in y_{zz}(\varsigma,z)+\widehat{\mu}(\varsigma,z)+\widehat{G}\big(\varsigma,y(\varsigma-p,z)\big),\quad \varsigma\in[0,c],p>0, \tag{5.1}$$

$$y(\varsigma,0)=y(\varsigma,\pi)=0,\quad \varsigma\in[0,c], \tag{5.2}$$

$$y(\varsigma,z)=\alpha(\varsigma,z),\quad z\in[0,\pi],\varsigma\in(-\infty,0],\qquad \frac{\partial}{\partial\varsigma}y(0,z)=y_1,\quad z\in[0,\pi], \tag{5.3}$$

$$y\big(\varsigma_j^+,z\big)-y\big(\varsigma_j^-,z\big)=\int_{-\infty}^{\varsigma_j}\gamma_j(\varsigma_j-\psi)y(\psi,z)\,d\psi,\quad j=1,2,\ldots,q, \tag{5.4}$$

$$y'\big(\varsigma_j^+,z\big)-y'\big(\varsigma_j^-,z\big)=\int_{-\infty}^{\varsigma_j}\widetilde{\gamma}_k(\varsigma_j-\psi)y(\psi,z)\,d\psi,\quad j=1,2,\ldots,q. \tag{5.5}$$

To change this framework into abstract structure (1.1)–(1.4), assume $\mathscr{X}=L^2([0,\pi])$ and let $A:D(A)\subset\mathscr{X}\to\mathscr{X}$, $M:D(M)\subset\mathscr{X}\to\mathscr{X}$ given by $Av=v''$ and $Mv=v-A$, where $D(A)$ and $D(M)$ is given by $\{v\in\mathscr{X}: v,v'$ are absolutely continuous, $v(0)=v(\pi)=0\}$. Additionally, A and M can be given by

$$Av=\sum_{j=1}^{\infty}j^2\langle v,y_j\rangle y_j,$$

$w\in D(A)$,

$$Mv=\sum_{j=1}^{\infty}\big(1+j^2\big)\langle v,y_j\rangle y_j,$$

$v\in D(M)$, where $y_j(z)=\sqrt{\frac{2}{\pi}}\sin(jz)$, $j=1,2,3,\ldots$, is the orthonormal of vectors of A. Additionally, for $y\in\mathscr{X}$, we have

$$M^{-1}y=\sum_{j=1}^{\infty}\frac{1}{(1+j^2)}\langle v,y_j\rangle y_j,$$

$$AM^{-1}y=\sum_{j=1}^{\infty}\frac{j^2}{(1+j^2)}\langle v,y_j\rangle y_j,$$

$$\mathscr{N}(\varsigma)y=\sum_{j=1}^{\infty}\cos j\varsigma\langle v,y_j\rangle y_j,$$

and

$$\mathscr{M}(\varsigma)y = \sum_{j=1}^{\infty} \frac{\sin j\varsigma}{j} \langle v, y_j\rangle y_j.$$

Phase space $\mathcal{P}_v$ along the norm is given by

$$\|\varphi\|_{\mathcal{P}_v} = \int_{-\infty}^{0} g(\psi) \sup_{\psi \le \theta \le 0} \left(\|\alpha(\theta)\|\right)_{L^2} d\psi,$$

where $g(\psi) = e^{2\psi}$, $\psi < 0$, and $j = \int_{-\infty}^{0} g(\psi)\,d\psi = \frac{1}{2}$.

Consider $y(\varsigma)(z) = y(\varsigma,z)$ and define $G(\varsigma,y)(\cdot) = \widehat{G}(\varsigma,y(\cdot))$. $B: U \to \mathscr{X}$ is interpreted as $Bu(\varsigma)(z) = \widehat{\mu}(\varsigma,z)$. Hence, AM^{-1} is compact and bounded with $\|M^{-1}\| \le 1$ and $\|\mathscr{N}(\varsigma)\| = \|\mathscr{M}(\varsigma)\| \le 1$ for all $\varsigma \in \mathbb{R}$, and $\mathscr{M}(\varsigma)$ is compact for all $\varsigma \in \mathbb{R}$.

Next, we verify hypotheses $\mathbf{H_1}$–$\mathbf{H_6}$ for system (5.1)–(5.5) one by one.

Verification of $\mathbf{H_1}$*:*

The operator $\mathscr{N}(\varsigma,0), \varsigma > 0$ is compact. Thus, clearly, $\|\mathscr{N}(\varsigma,\psi)\|^2 \le 1$ and $\|\mathscr{M}(\varsigma,\psi)\|^2 \le 1$ for $\varsigma \in \mathbb{R}$ and $\mathscr{M}(\varsigma,\psi)$ is compact for all $\varsigma \in \mathbb{R}$.

From the above conditions, hypothesis $\mathbf{H_1}$ is satisfied.

Verification of $\mathbf{H_2}$ *and* $\mathbf{H_3}$*:*

Set

$$G(\varsigma,y) = \widehat{G}\big(\varsigma,y(\varsigma-p,z)\big) = \big\{\mathscr{G} \in \mathscr{X}; g_1\big(\varsigma,y(\varsigma-p,z)\big) \le \mathscr{G} \le g_2\big(\varsigma,y(\varsigma-p,z)\big)\big\}, \quad (5.6)$$

where $g_1, g_2 : V \times \mathcal{P}_v \to BCC(\mathscr{X})$. We assume that, for each $\varsigma \in V$, g_1 is lower semi-continuous and g_2 is upper semi-continuous. Assume that $p: V \to [0,\infty)$ is an integrable function and $\theta_2 : [0,\infty) \to (0,\infty)$ is a continuous increasing function such that

$$\begin{aligned} &\max\left\{\int_0^{\varsigma} \big\|g_1\big(\varsigma,y(\varsigma-p,z)\big)\big\|, \int_0^{\varsigma} \big\|g_2\big(\varsigma,y(\varsigma-p,z)\big)\big\|\right\} \\ &\quad \le p(\varsigma)\theta_2\big(\big\|y(\varsigma-p,z)\big\|_{\mathcal{P}_v}\big). \end{aligned} \quad (5.7)$$

From equations (5.6) and (5.7), $\widehat{G}$ satisfies conditions $\mathbf{H_2}$ and $\mathbf{H_3}$.

Verification of $\mathbf{H_4}$ *and* $\mathbf{H_5}$:

From system (5.1)–(5.5),

$$\Delta y|_{\varsigma=\varsigma_j} = \Delta y(\varsigma_j)(z) = \int_{-\infty}^{\varsigma_j} \gamma_j(\varsigma_j-\psi)y(\psi,z)\,d\psi, \quad j = 1,2,\dots,q,$$

and we consider the function $J_j : \mathscr{X} \to \mathscr{X}$ is given by

$$\begin{aligned} &J_j(y_{\varsigma_j}) = \int_{-\infty}^{\varsigma_j} \gamma_j(\varsigma_j-\psi)y(\psi,z)\,d\psi, \\ &\big\|J_j(y_{\varsigma_j})\big\| \le P_m. \end{aligned} \quad (5.8)$$

Similarly,

$$\Delta y'|_{\varsigma=\varsigma_j} = \Delta y'(\varsigma_j)(z) = \int_{-\infty}^{\varsigma_j} \widetilde{\gamma}_j(\varsigma_j-\psi)y(\psi,z)\,d\psi, \quad j = 1,2,\dots,q,$$

and we consider the function $\overline{J}_j : \mathscr{X} \to \mathscr{X}$ is given by

$$\begin{aligned} &\overline{J}_j(y'_{\varsigma_j}) = \int_{-\infty}^{\varsigma_j} \widetilde{\gamma}_j(\varsigma_j - \psi) y(\psi, z)\, d\psi, \\ &\|\overline{J}_j(y'_{\varsigma_j})\| \le \overline{P}_m. \end{aligned} \tag{5.9}$$

From equations (5.8) and (5.9), we observe that hypotheses $\mathbf{H_4}$ and $\mathbf{H_5}$ are satisfied. By using hypotheses $\mathbf{H_1}$–$\mathbf{H_5}$ and Lemma 3.5, we realize that hypothesis $\mathbf{H_6}$ is also satisfied.

Clearly, all the hypotheses of Theorem 3.8 and Lemma 3.5 are satisfied. Hence, by the conclusion of Theorem 3.8 and Lemma 3.5, it follows that system (5.1)–(5.5) has a solution, and we conclude that system (5.1)–(5.5) is approximately controllable.

6 Conclusion

This work focused on the approximate controllability of second-order impulsive delay differential inclusions. Our key tasks are dictated by the usage of the outcomes, facts related to operators' cosine and sine functions, Martelli's fixed point theorem, and their results when paired with the features of differential inclusions. Finally, we proposed an illustration of the hypothesis that had been proven. In the future, we will focus our study on the existence and approximate controllability of second-order Sobolev-type neutral stochastic differential inclusions by employing Martelli's fixed point theorem.

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References

1. Bainov, D.D., Simeonov, P.S.: Impulsive Differential Equations: Periodic Solutions and Applications. Longman Scientific and Technical Group, England (1993)
2. Banas, J., Goebel, K.: Measures of Noncompactness in Banach Spaces. Dekker, New York (1980)
3. Brill, H.: A semilinear Sobolev evolution equation in Banach space. J. Differ. Equ. **24**, 412–425 (1977)
4. Byszewski, L.: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**, 494–505 (1991)
5. Byszewski, L., Akca, H.: On a mild solution of a semilinear functional-differential evolution nonlocal problem. J. Appl. Math. Stoch. Anal. **10**(3), 265–271 (1997)

6. Ceng, L.C., Fu, Y.X., Yin, J., He, L., He, L., Hu, H.Y.: The solvability of generalized systems of time-dependent hemivariational inequalities enjoying symmetric structure in reflexive Banach spaces. Symmetry **13**, 1–14 (2021)
7. Ceng, L.C., Huang, N.J., Wen, C.F.: On generalized global fractional-order composite dynamical systems with set-valued perturbations. J. Nonlinear Sci. Appl. **6**, 149–163 (2022)
8. Ceng, L.C., Liou, Y.C., Yao, J.C., Yao, Y., Lo, C.H.: Well-posedness for systems of generalized mixed quasivariational inclusion problems and optimization problems with constraints. J. Nonlinear Sci. Appl. **10**, 5373–5392 (2017)
9. Ceng, L.C., Liou, Y.C., Yao, J.C., Yao, Y.H.: Well-posedness for systems of time-dependent hemivariational inequalities in Banach spaces. J. Nonlinear Sci. Appl. **10**, 4318–4336 (2017)
10. Ceng, L.C., Liu, Z.H., Yao, J.C., Yao, Y.: Optimal control of feedback control systems governed by systems of evolution hemivariational inequalities. Filomat **32**, 5205–5220 (2018)
11. Ceng, L.C., Petrusel, A., Yao, J.C., Yao, Y.: Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. Fixed Point Theory **20**, 113–133 (2019)
12. Ceng, L.C., Wen, C.F., Liou, Y.C., Yao, J.C.: A general class of differential hemivariational inequalities systems in reflexive Banach spaces. Mathematics **9**, 1–21 (2021)
13. Ceng, L.C., Wen, C.F., Yao, J.C., Yao, Y.H.: A system of evolutionary problems driven by a system of hemivariational inequalities. J. Nonlinear Sci. Appl. **11**, 342–357 (2018)
14. Chang, Y.K.: Controllability of impulsive functional differential systems with infinite delay in Banach spaces. Chaos Solitons Fractals **33**, 1601–1609 (2007)
15. Deimling, K.: Multivalued Differential Equations. de Gruyter, Berlin (1992)
16. Dineshkumar, C., Udhayakumar, R., Vijayakumar, V., Nisar, K.S., Shukla, A.: A note on the approximate controllability of Sobolev type fractional stochastic integro-differential delay inclusions with order $1 < r < 2$. Math. Comput. Simul. **190**, 1003–1026 (2021)
17. Dineshkumar, C., Udhayakumar, R., Vijayakumar, V., Shukla, A., Nisar, K.S.: A note on approximate controllability for nonlocal fractional evolution stochastic integrodifferential inclusions of order $r \in (1,2)$ with delay. Chaos Solitons Fractals **153**, 111565 (2021)
18. Henríquez, H.R.: Existence of solutions of non-autonomous second order functional differential equations with infinite delay. Nonlinear Anal., Theory Methods Appl. **74**, 3333–3352 (2011)
19. Henríquez, H.R., Hernández, E.: Existence of solutions of a second order abstract functional Cauchy problem with nonlocal conditions. Ann. Pol. Math. **88**(2), 141–159 (2006)
20. Henríquez, H.R., Poblete, V., Pozo, J.C.: Mild solutions of non-autonomous second order problems with nonlocal initial conditions. J. Math. Anal. Appl. **412**, 1064–1083 (2014)
21. Hernández, E.: Existence of solutions for an abstract second-order differential equation with nonlocal conditions. Electron. J. Differ. Equ. **96**, 1 (2009)
22. Hernández, E., Henríquez, H.R., McKibben, M.A.: Existence results for abstract impulsive second-order neutral functional differential equations. Nonlinear Anal., Theory Methods Appl. **70**, 2736–2751 (2009)
23. Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis (Theory). Kluwer Academic, Dordrecht (1997)
24. Kavitha, K., Vijayakumar, V., Shukla, A., Nisar, K.S., Udhayakumar, R.: Results on approximate controllability of Sobolev-type fractional neutral differential inclusions of Clarke subdifferential type. Chaos Solitons Fractals **151**, 111264 (2021)
25. Kisyński, J.: On cosine operator functions and one parameter group of operators. Stud. Math. **49**, 93–105 (1972)
26. Laksmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations. Series in Modern Applied Mathematics, vol. 6. World Scientific, Teaneck (1989)
27. Lasota, A., Opial, Z.: An application of the Kakutani–Ky–Fan theorem in the theory of ordinary differential equations or noncompact acyclic-valued map. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781–786 (1965)
28. Lightbourne, J.H., Rankin, S.: A partial functional differential equation of Sobolev type. J. Math. Anal. Appl. **93**, 328–337 (1983)
29. Mahmudov, N.I.: Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces. Abstr. Appl. Anal. **2013**, Article ID 502839 (2013)
30. Mahmudov, N.I., Denker, A.: On controllability of linear stochastic systems. Int. J. Control **73**, 144–151 (2000)
31. Mahmudov, N.I., Vijayakumar, V., Murugesu, R.: Approximate controllability of second-order evolution differential inclusions in Hilbert spaces. Mediterr. J. Math. **13**(5), 3433–3454 (2016)
32. Martelli, M.: A Rothe's type theorem for non-compact acyclic-valued map. Boll. Unione Mat. Ital. **2**, 70–76 (1975)
33. Mohan Raja, M., Vijayakumar, V., Huynh, L.N., Udhayakumar, R., Nisar, K.S.: Results on the approximate controllability of fractional hemivariational inequalities of order $1 < r < 2$. Adv. Differ. Equ. **2021**(237), 1 (2021)
34. Nisar, K.S., Vijayakumar, V.: Results concerning to approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral delay differential system. Math. Methods Appl. Sci. **44**(17), 13615–13632 (2021)
35. Papageorgiou, N.: Boundary value problems for evolution inclusions. Comment. Math. Univ. Carol. **29**, 355–363 (1988)
36. Patel, R., Shukla, A., Jadon, S.S.: Existence and optimal control problem for semilinear fractional order $(1,2)$ control system. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6662
37. Pierri, M., O'Regan, D., Prokopczyk, A.: On recent developments treating the exact controllability of abstract control problems. Electron. J. Differ. Equ. **2016**(160), 1 (2016)
38. Sakthivel, R., Ganesh, R., Anthoni, S.M.: Approximate controllability of fractional nonlinear differential inclusions. Appl. Math. Comput. **225**, 708–717 (2013)
39. Shukla, A., Patel, R.: Existence and optimal control results for second-order semilinear system in Hilbert spaces. Circuits Syst. Signal Process. **40**, 4246–4258 (2021)
40. Shukla, A., Sukavanam, N.: Complete controllability of semi-linear stochastic system with delay. Rend. Circ. Mat. Palermo **64**, 209–220 (2015)
41. Shukla, A., Sukavanam, N., Pandey, D.N.: Controllability of semilinear stochastic system with multiple delays in control. IFAC Proc. Vol. **47**(1), 306–312 (2014)
42. Shukla, A., Sukavanam, N., Pandey, D.N.: Approximate controllability of semilinear fractional control systems of order $\alpha \in (1,2]$. In: Proceedings of the Conference on Control and Its Applications, pp. 175–180. SIAM, Philadelphia (2015). https://doi.org/10.1137/1.9781611974072.25

43. Shukla, A., Sukavanam, N., Pandey, D.N.: Approximate controllability of semilinear stochastic control system with nonlocal conditions. Nonlinear Dyn. Syst. Theory **15**(3), 321–333 (2015)
44. Shukla, A., Vijayakumar, V., Nisar, K.S.: A new exploration on the existence and approximate controllability for fractional semilinear impulsive control systems of order $r \in (1,2)$. Chaos Solitons Fractals **154**, 111615 (2022)
45. Sivasankaran, S., Mallika Arjunan, M., Vijayakumar, V.: Existence of global solutions for second order impulsive abstract partial differential equations. Nonlinear Anal., Theory Methods Appl. **74**(17), 6747–6757 (2011)
46. Travis, C.C., Webb, G.F.: Compactness, regularity, and uniform continuity properties of strongly continuous cosine families. Houst. J. Math. **3**(4), 555–567 (1977)
47. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Sci. Hung. **32**, 76–96 (1978)
48. Vijayakumar, V.: Approximate controllability for a class of second-order stochastic evolution inclusions of Clarke's subdifferential type. Results Math. **73**(42), 1–23 (2018)
49. Vijayakumar, V.: Approximate controllability results for impulsive neutral differential inclusions of Sobolev-type with infinite delay. Int. J. Control **91**(10), 2366–2386 (2018)
50. Vijayakumar, V., Henríquez, H.R.: Existence of global solutions for a class of abstract second order nonlocal Cauchy problem with impulsive conditions in Banach spaces. Numer. Funct. Anal. Optim. **39**(6), 704–736 (2018)
51. Vijayakumar, V., Murugesu, R.: Controllability for a class of second order evolution differential inclusions without compactness. Appl. Anal. **98**(7), 1367–1385 (2019)
52. Vijayakumar, V., Panda, S.K., Nisar, K.S., Baskonus, H.M.: Results on approximate controllability results for second-order Sobolev-type impulsive neutral differential evolution inclusions with infinite delay. Numer. Methods Partial Differ. Equ. (2020). https://doi.org/10.1002/num.22573
53. Vijayakumar, V., Panda, S.K., Nisar, K.S., Baskonus, H.M.: Results on approximate controllability results for second-order Sobolev-type impulsive neutral differential evolution inclusions with infinite delay. Numer. Methods Partial Differ. Equ. **37**(2), 1200–1221 (2021)
54. Vijayakumar, V., Ravichandran, C., Nisar, K.S., Kucche, K.D.: New discussion on approximate controllability results for fractional Sobolev type Volterra–Fredholm integro-differential systems of order $1 < r < 2$. Numer. Methods Partial Differ. Equ. (2021). https://doi.org/10.1002/num.22772
55. Vijayakumar, V., Udhayakumar, R., Dineshkumar, C.: Approximate controllability of second order nonlocal neutral differential evolution inclusions. IMA J. Math. Control Inf. **38**(1), 192–210 (2021)
56. Williams, W.K., Vijayakumar, V., Udhayakumar, R., Panda, S.K., Nisar, K.S.: Existence and controllability of nonlocal mixed Volterra–Fredholm type fractional delay integro-differential equations of order $1 < r < 2$. Numer. Methods Partial Differ. Equ. (2020). https://doi.org/10.1002/num.22697
57. Yan, B.: Boundary value problems on the half-line with impulses and infinite delay. J. Math. Anal. Appl. **259**(1), 94–114 (2001)