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Some parameterized Simpson-, midpoint- and trapezoid-type inequalities for generalized fractional integrals

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Abstract

In this paper, we first obtain an identity for differentiable mappings. Then, we establish some new generalized inequalities for differentiable convex functions involving some parameters and generalized fractional integrals. We show that these results reduce to several new Simpson-, midpoint- and trapezoid-type inequalities. The results given in this study are the generalizations of results proved in several earlier papers.

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1 Introduction

Simpson's inequality plays an important role in many areas of mathematics. The classical Simpson's inequality is expressed as follows for four-times continuously differentiable functions:

Theorem 1 *Suppose that* $F:[\kappa_1,\kappa_2]\to\mathbb{R}$ *is a four-times continuously differentiable mapping on* (κ_1,κ_2), *and let* $\|F^{(4)}\|_\infty=\sup_{x\in(\kappa_1,\kappa_2)}|F^{(4)}(x)|<\infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{F(\kappa_1)+F(\kappa_2)}{2}+2F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(x)\,dx\right|\leq\frac{1}{2880}\left\|F^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.$$

In recent years, many writers have focused on Simpson-type inequalities in various categories of work. Specifically, some mathematicians have worked on the results of the Simpson- and Newton-type inequalities by using convex mappings, because convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics. For example, Dragomir et al. [11] presented new Simpson-type results and their applications to quadrature formulas in numerical integration. Also, new Newton-type inequalities for functions whose local fractional derivatives are generalized convex are given by Iftikhar et al. in [20]. For more recent developments, one can consult [1–5, 9, 12–15, 19, 28–30, 35].

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The aim of this paper is to obtain several generalized inequalities for differentiable mappings by utilizing generalized fractional integrals and some nonnegative parameters. By special choice of parameters, the obtained results reduce some well-known Simpson-, midpoint- and trapezoid-type inequalities obtained by several authors in [10, 16, 17, 23, 33, 34].

2 Generalized fractional integral operators

In this section, we mention the generalized fractional integrals defined by Sarikaya and Ertuğral in [32].

Let $\varphi : [0,\infty) \to [0,\infty)$ be a function satisfying the following condition:

$$\int_0^1 \frac{\varphi(t)}{t}\,dt < \infty.$$

The left-sided and right-sided generalized fractional integral operators are defined, respectively, as follows:

$${}_{\kappa_1+}I_{\varphi}F(x) = \int_{\kappa_1}^{x} \frac{\varphi(x-t)}{x-t}F(t)\,dt, \quad x > \kappa_1, \tag{2.1}$$

$${}_{\kappa_2-}I_{\varphi}F(x) = \int_{x}^{\kappa_2} \frac{\varphi(t-x)}{t-x}F(t)\,dt, \quad x < \kappa_2. \tag{2.2}$$

These fractional operators genralize several fractional integrals such as Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc. These important special cases of the integral operators (2.1) and (2.2) are mentioned below.

(1) If we choose $\varphi(t) = t$, the operators (2.1) and (2.2) reduce to the Riemann integral.
(2) Considering $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, the operators (2.1) and (2.2) reduce to the Riemann–Liouville fractional integrals $J^{\alpha}_{\kappa_1+}F(x)$ and $J^{\alpha}_{\kappa_2-}F(x)$, respectively. Here, Γ is the Gamma function.
(3) For $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators (2.1) and (2.2) reduce to the k-Riemann–Liouville fractional integrals [27] $J^{\alpha}_{\kappa_1+,k}F(x)$ and $J^{\alpha}_{\kappa_2-,k}F(x)$, respectively. Here, Γ_k is the k-Gamma function.

Sarıkaya and Ertuğral also established the following Hermite–Hadamard inequality for the generalized fractional integral operators:

Theorem 2 ([32]) *Let $F : [\kappa_1,\kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ with $\kappa_1 < \kappa_2$, then the following inequalities for fractional integral operators hold*:

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_{\varphi}F(\kappa_2) + {}_{\kappa_2-}I_{\varphi}F(\kappa_1)\right] \le \frac{F(\kappa_1)+F(\kappa_2)}{2}, \tag{2.3}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ is defined by

$$\Lambda(t) = \int_0^t \frac{\varphi((\kappa_2-\kappa_1)u)}{u}\,du.$$

In the literature there are several papers on inequalities for generalized fractional integrals. For some of these please refer to [6, 7, 16, 18, 21, 22, 24–26, 31, 36].

3 An identity for generalized fractional integrals

In this section, we offer a parameterized identity involving an ordinary first derivative via generalized fractional integrals.

Lemma 1 *Let $F:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable function on (κ_1,κ_2). If F' is continuous and integrable on $[\kappa_1,\kappa_2]$, then for $\rho,\sigma\geq 0$, one has the identity*

$$\begin{aligned}&(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\\&\quad-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&=\frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\int_0^1\big(\Delta(t)-\Delta(1)\rho\big)F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)dt\right.\\&\quad\left.+\int_0^1\big(\Delta(1)\sigma-\Delta(t)\big)F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)dt\right],\end{aligned}\tag{3.1}$$

where the mapping $\Delta:[0,1]\to\mathbb{R}$ is defined by

$$\Delta(t)=\int_0^t\frac{\varphi((\frac{\kappa_2-\kappa_1}{2})u)}{u}\,du.$$

Proof Applying the fundamental rules of integration, we have

$$\begin{aligned}&\int_0^1\big(\Delta(t)-\Delta(1)\rho\big)F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)dt\\&=\frac{2}{\kappa_2-\kappa_1}\left[\Delta(1)\left((1-\rho)F(\kappa_2)+\rho F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)-{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}&\int_0^1\big(\Delta(1)\sigma-\Delta(t)\big)F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)dt\\&=\frac{2}{\kappa_2-\kappa_1}\left[\Delta(1)\left((1-\sigma)F(\kappa_1)+\sigma F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right)-{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right].\end{aligned}\tag{3.3}$$

By adding (3.2) and (3.3), we obtain the required equality (3.1). □

Corollary 1 *If we assume $\varphi(t)=t$ in Lemma 1, then we obtain the following equality*:

$$\begin{aligned}&\frac{1}{2}\left[(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\\&=\frac{\kappa_2-\kappa_1}{4}\left[\int_0^1(t-\rho)F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)dt\right.\\&\quad\left.+\int_0^1(\sigma-t)F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)dt\right].\end{aligned}$$

Corollary 2 *In Lemma 1, if we set* $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, *then we obtain the following new identity for the Riemann–Liouville fractional integral*:

$$\begin{aligned}
&(1-\sigma)F(\kappa_1) + (\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right) + (1-\rho)F(\kappa_2) \\
&\quad - \frac{2^{\alpha}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \\
&= \frac{\kappa_2-\kappa_1}{2}\left[\int_0^1 \left(t^{\alpha}-\rho\right)F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right)dt \right. \\
&\quad \left. + \int_0^1 \left(\sigma - t^{\alpha}\right)F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right)dt\right].
\end{aligned}$$

Corollary 3 *In Lemma 1, if we take* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then we obtain the following new identity for the k-Riemann–Liouville fractional integral*:

$$\begin{aligned}
&(1-\sigma)F(\kappa_1) + (\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right) + (1-\rho)F(\kappa_2) \\
&\quad - \frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1^+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2^-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \\
&= \frac{\kappa_2-\kappa_1}{2}\left[\int_0^1 \left(t^{\frac{\alpha}{k}}-\rho\right)F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right)dt \right. \\
&\quad \left. + \int_0^1 \left(\sigma - t^{\frac{\alpha}{k}}\right)F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right)dt\right].
\end{aligned}$$

4 Some parameterized inequalities for generalized fractional integral operators

In this section, we establish some new generalized inequalities for differentiable convex functions via generalized fractional integrals.

Theorem 3 *We assume that the conditions of Lemma 1 hold. If the mapping* $|F'|$ *is convex on* $[\kappa_1,\kappa_2]$, *then the following inequality holds for generalized fractional integrals*:

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1) + (\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right) + (1-\rho)F(\kappa_2)\right. \\
&\quad \left. - \frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\le \frac{\kappa_2-\kappa_1}{4\Delta(1)}\left[\left|F'(\kappa_1)\right|\left(\Pi_1^{\varphi}(\rho)+\Pi_2^{\varphi}(\sigma)\right) + \left|F'(\kappa_2)\right|\left(\Pi_1^{\varphi}(\sigma)+\Pi_2^{\varphi}(\rho)\right)\right],
\end{aligned} \tag{4.1}$$

where

$$\Pi_1^{\varphi}(\tau) = \int_0^1 (1-t)\left|\Delta(t) - \Delta(1)\tau\right|dt$$

and

$$\Pi_2^{\varphi}(\tau) = \int_0^1 (1+t)\left|\Delta(t) - \Delta(1)\tau\right|dt.$$

Proof By taking the modulus in Lemma 1 and using the properties of the modulus, we obtain that

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right. \\
&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&=\frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\int_0^1\left|\Delta(t)-\Delta(1)\rho\right|\left|F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left|\Delta(1)\sigma-\Delta(t)\right|\left|F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)\right|dt\right].
\end{aligned} \tag{4.2}$$

Since the mapping $|F'|$ is convex on $[\kappa_1,\kappa_2]$, we have

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right. \\
&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\le\frac{\kappa_2-\kappa_1}{4\Delta(1)}\left[\left|F'(\kappa_1)\right|\left(\int_0^1(1-t)\left|\Delta(t)-\Delta(1)\rho\right|dt+\int_0^1(1+t)\left|\Delta(1)\sigma-\Delta(t)\right|dt\right)\right. \\
&\quad\left.+\left|F'(\kappa_2)\right|\left(\int_0^1(1+t)\left|\Delta(t)-\Delta(1)\rho\right|dt+\int_0^1(1-t)\left|\Delta(1)\sigma-\Delta(t)\right|dt\right)\right] \\
&=\frac{\kappa_2-\kappa_1}{4\Delta(1)}\left[\left|F'(\kappa_2)\right|\left(\Pi_1^{\varphi}(\sigma)+\Pi_2^{\varphi}(\rho)\right)+\left|F'(\kappa_1)\right|\left(\Pi_1^{\varphi}(\rho)+\Pi_2^{\varphi}(\sigma)\right)\right],
\end{aligned}$$

which ends the proof. □

Corollary 4 *Under the assumption of Theorem 3 with $\varphi(t)=t$, we obtain the following inequality*:

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\right| \\
&\le\frac{\kappa_2-\kappa_1}{8}\left[\left|F'(\kappa_1)\right|\left(\Pi_1(\rho)+\Pi_2(\sigma)\right)+\left|F'(\kappa_2)\right|\left(\Pi_1(\sigma)+\Pi_2(\rho)\right)\right],
\end{aligned}$$

where

$$\Pi_1(\tau)=\tau^2-\frac{\tau^3}{3}-\frac{\tau}{2}+\frac{1}{6},$$

and

$$\Pi_2(\tau)=\frac{\tau^3}{3}+\tau^2-\frac{3\tau}{2}+\frac{5}{6}.$$

Corollary 5 *Under the assumption of Theorem 3 with* $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\kappa_1)}$, *we obtain the following inequality for Riemann–Liouville fractional integrals*:

$$\begin{aligned}&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right.\\&\quad\left.-\frac{2^{\alpha}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\le\frac{\kappa_2-\kappa_1}{4}\left[\left|F'(\kappa_1)\right|\left(\Pi^{\alpha}_1(\rho)+\Pi^{\alpha}_2(\sigma)\right)+\left|F'(\kappa_2)\right|\left(\Pi^{\alpha}_1(\sigma)+\Pi^{\alpha}_2(\rho)\right)\right],\end{aligned}$$

where

$$\Pi^{\alpha}_1(\tau)=\frac{2\alpha}{\alpha+1}\tau^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{\alpha+2}\tau^{\frac{\alpha+2}{\alpha}}-\frac{\tau}{2}+\frac{1}{(\alpha+2)(\alpha+1)},$$

and

$$\Pi^{\alpha}_2(\tau)=\frac{\alpha}{\alpha+2}\tau^{\frac{\alpha+2}{\alpha}}-\frac{3\tau}{2}+\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\tau^{\frac{\alpha+1}{\alpha}}.$$

Corollary 6 *In Theorem 3, if we take* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\kappa_1)}$, *then we obtain the following inequality for k-Riemann–Liouville fractional integrals*:

$$\begin{aligned}&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right.\\&\quad\left.-\frac{2^{\frac{\alpha}{k}}\Gamma(\alpha+k)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1^+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2^-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\le\frac{\kappa_2-\kappa_1}{4}\left[\left|F'(\kappa_1)\right|\left(\Pi^{\frac{\alpha}{k}}_1(\rho)+\Pi^{\frac{\alpha}{k}}_2(\sigma)\right)+\left|F'(\kappa_2)\right|\left(\Pi^{\frac{\alpha}{k}}_1(\sigma)+\Pi^{\frac{\alpha}{k}}_2(\rho)\right)\right],\end{aligned}$$

where

$$\Pi^{\frac{\alpha}{k}}_1(\tau)=\frac{2\alpha}{\alpha+k}\tau^{\frac{\alpha+k}{\alpha}}-\frac{\alpha}{\alpha+2k}\tau^{\frac{\alpha+2k}{\alpha}}-\frac{\tau}{2}+\frac{k^2}{(\alpha+2k)(\alpha+k)},$$

and

$$\Pi^{\frac{\alpha}{k}}_2(\tau)=\frac{\alpha}{\alpha+2k}\tau^{\frac{\alpha+2k}{\alpha}}-\frac{3\tau}{2}+\frac{2\alpha k+3k^2}{(\alpha+2k)(\alpha+k)}+\frac{2\alpha}{\alpha+k}\tau^{\frac{\alpha+k}{\alpha}}.$$

Theorem 4 *We assume that the conditions of Lemma 1 hold. If the mapping* $|F|^{p_1}$, $p_1>1$, *is convex on* $[\kappa_1,\kappa_2]$, *then we have the following inequality for generalized fractional integrals*:

$$\begin{aligned}&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right.\\&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\end{aligned}$$

$$\leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1 \left|\Delta(t)-\Delta(1)\rho\right| dt\right)^{1-\frac{1}{p_1}}\left(\frac{\Pi_1^{\varphi}(\rho)|F'(\kappa_1)|^{p_1}+\Pi_2^{\varphi}(\rho)|F'(\kappa_2)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(\int_0^1 \left|\Delta(1)\sigma-\Delta(t)\right| dt\right)^{1-\frac{1}{p_1}}\left(\frac{\Pi_1^{\varphi}(\sigma)|F'(\kappa_2)|^{p_1}+\Pi_2^{\varphi}(\sigma)|F'(\kappa_1)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right],$$

where $\Pi_1^{\varphi}(\tau)$ *and* $\Pi_2^{\varphi}(\tau)$ *are defined as in Theorem* 3.

Proof Reutilizing inequality (4.2) and from the power mean inequality, we have

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right.\\
&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1 \left|\Delta(t)-\Delta(1)\rho\right| dt\right)^{1-\frac{1}{p_1}}\right.\\
&\quad\times\left(\int_0^1 \left|\Delta(t)-\Delta(1)\rho\right|\left|F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)\right|^{p_1} dt\right)^{\frac{1}{p_1}}\\
&\quad+\left(\int_0^1 \left|\Delta(1)\sigma-\Delta(t)\right| dt\right)^{1-\frac{1}{p_1}}\\
&\quad\left.\times\left(\int_0^1 \left|\Delta(1)\sigma-\Delta(t)\right|\left|F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)\right|^{p_1} dt\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Using the convexity of $|F'|^{p_1}$, we have

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right.\\
&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1 \left|\Delta(t)-\Delta(1)\rho\right| dt\right)^{1-\frac{1}{p_1}}\right.\\
&\quad\times\left(\left|F'(\kappa_1)\right|^{p_1}\int_0^1\left(\frac{1-t}{2}\right)\left|\Delta(t)-\Delta(1)\rho\right| dt\right.\\
&\quad\left.+\left|F'(\kappa_2)\right|^{p_1}\int_0^1\left(\frac{1+t}{2}\right)\left|\Delta(t)-\Delta(1)\rho\right| dt\right)^{\frac{1}{p_1}}\\
&\quad+\left(\int_0^1 \left|\Delta(1)\sigma-\Delta(t)\right| dt\right)^{1-\frac{1}{p_1}}\\
&\quad\times\left(\left|F'(\kappa_1)\right|^{p_1}\int_0^1\left(\frac{1+t}{2}\right)\left|\Delta(1)\sigma-\Delta(t)\right| dt\right.\\
&\quad\left.\left.+\left|F'(\kappa_2)\right|^{p_1}\int_0^1\left(\frac{1-t}{2}\right)\left|\Delta(1)\sigma-\Delta(t)\right| dt\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

$$= \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1 \left|\Delta(t)-\Delta(1)\rho\right| dt\right)^{1-\frac{1}{p_1}}\left(\frac{\Pi_1^{\varphi}(\rho)|F'(\kappa_1)|^{p_1}+\Pi_2^{\varphi}(\rho)|F'(\kappa_2)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(\int_0^1 \left|\Delta(1)\sigma-\Delta(t)\right| dt\right)^{1-\frac{1}{p_1}}\left(\frac{\Pi_1^{\varphi}(\sigma)|F'(\kappa_2)|^{p_1}+\Pi_2^{\varphi}(\sigma)|F'(\kappa_1)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right],$$

which finishes the proof. □

Corollary 7 *If we assume that $\varphi(t)=t$ in Theorem 4, then we obtain the following inequality*:

$$\left|\frac{1}{2}\left[(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\right|$$
$$\le \frac{\kappa_2-\kappa_1}{8}\left[\left(\Pi_1(\rho)+\Pi_2(\rho)\right)^{1-\frac{1}{p_1}}\left(\Pi_1(\rho)\left|F'(\kappa_1)\right|^{p_1}+\Pi_2(\rho)\left|F'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(\Pi_1(\sigma)+\Pi_2(\sigma)\right)^{1-\frac{1}{p_1}}\left(\Pi_1(\sigma)\left|F'(\kappa_2)\right|^{p_1}+\Pi_2(\sigma)\left|F'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right],$$

where $\Pi_1(\tau)$ and $\Pi_2(\tau)$ are defined as in Corollary 4.

Corollary 8 *If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 4, then we have the following inequality for Riemann–Liouville fractional integrals*:

$$\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right.$$
$$\left.-\frac{2^{\alpha}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \tag{4.3}$$
$$\le \frac{\kappa_2-\kappa_1}{4}\left[\left(\Pi_1^{\alpha}(\rho)+\Pi_2^{\alpha}(\rho)\right)^{1-\frac{1}{p_1}}\left(\Pi_1^{\alpha}(\rho)\left|F'(\kappa_1)\right|^{p_1}+\Pi_2^{\alpha}(\rho)\left|F'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(\Pi_1^{\alpha}(\sigma)+\Pi_2^{\alpha}(\sigma)\right)^{1-\frac{1}{p_1}}\left(\Pi_1^{\alpha}(\sigma)\left|F'(\kappa_2)\right|^{p_1}+\Pi_2^{\alpha}(\sigma)\left|F'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right],$$

where $\Pi_1^{\alpha}(\tau)$ and $\Pi_2^{\alpha}(\tau)$ are defined as in Corollary 5.

Corollary 9 *If we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 4, then we have the following inequality for k-Riemann–Liouville fractional integrals*:

$$\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1^+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2^-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \tag{4.4}$$
$$\le \frac{\kappa_2-\kappa_1}{4}\left[\left(\Pi_1^{\frac{\alpha}{k}}(\rho)+\Pi_2^{\frac{\alpha}{k}}(\rho)\right)^{1-\frac{1}{p_1}}\left(\Pi_1^{\frac{\alpha}{k}}(\rho)\left|F'(\kappa_1)\right|^{p_1}+\Pi_2^{\frac{\alpha}{k}}(\rho)\left|F'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(\Pi_1^{\frac{\alpha}{k}}(\sigma)+\Pi_2^{\frac{\alpha}{k}}(\sigma)\right)^{1-\frac{1}{p_1}}\left(\Pi_1^{\frac{\alpha}{k}}(\sigma)\left|F'(\kappa_2)\right|^{p_1}+\Pi_2^{\frac{\alpha}{k}}(\sigma)\left|F'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right],$$

where $\Pi_1^{\frac{\alpha}{k}}(\tau)$ and $\Pi_2^{\frac{\alpha}{k}}(\tau)$ are described in Corollary 5.

Theorem 5 *We assume that the conditions of Lemma 1 hold. If the mapping $|F'|^{r_1}$, $r_1>1$, is convex on $[\kappa_1,\kappa_2]$, then we have the following inequality for generalized fractional inte-*

grals:

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right. \\
&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1\left|\Delta(t)-\Delta(1)\rho\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\quad\left.+\left(\int_0^1\left|\Delta(1)\sigma-\Delta(t)\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right],
\end{aligned} \tag{4.5}$$

where $\frac{1}{p_1}+\frac{1}{r_1}=1$.

Proof Reutilizing inequality (4.2) and from the well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right. \\
&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1\left|\Delta(t)-\Delta(1)\rho\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\int_0^1\left|F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)\right|^{r_1}dt\right)^{\frac{1}{r_1}}\right. \\
&\quad\left.+\left(\int_0^1\left|\Delta(1)\sigma-\Delta(t)\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\int_0^1\left|F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)\right|^{r_1}dt\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

Using the fact that $|F'|^{r_1}$ is convex, we have

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right. \\
&\quad\left.-\frac{1}{\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1\left|\Delta(t)-\Delta(1)\rho\right|^{p_1}dt\right)^{\frac{1}{p_1}}\right. \\
&\quad\times\left.\left(\left|F'(\kappa_1)\right|^{r_1}\int_0^1\left(\frac{1-t}{2}\right)dt+\left|F'(\kappa_2)\right|^{r_1}\int_0^1\left(\frac{1+t}{2}\right)dt\right)^{\frac{1}{r_1}}\right. \\
&\quad+\left.\left(\int_0^1\left|\Delta(1)\sigma-\Delta(t)\right|^{p_1}dt\right)^{\frac{1}{p_1}}\right. \\
&\quad\times\left.\left(\left|F'(\kappa_1)\right|^{r_1}\int_0^1\left(\frac{1+t}{2}\right)dt+\left|F'(\kappa_2)\right|^{r_1}\int_0^1\left(\frac{1-t}{2}\right)dt\right)^{\frac{1}{r_1}}\right] \\
&=\frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\left(\int_0^1\left|\Delta(t)-\Delta(1)\rho\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\quad\left.+\left(\int_0^1\left|\Delta(1)\sigma-\Delta(t)\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right],
\end{aligned}$$

which completes the proof. □

Corollary 10 *In Theorem 5, if we set* $\varphi(t) = t$, *then we obtain the following inequality*:

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\right| \\
&\quad\le\frac{\kappa_2-\kappa_1}{4}\left[\left(\Pi_3(\rho)\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\qquad\left.+\left(\Pi_3(\sigma)\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right],
\end{aligned}$$

where

$$\Pi_3(\tau)=\frac{\tau^{p_1+1}+(1-\tau)^{p_1+1}}{p_1+1}.$$

Corollary 11 *In Theorem 5, if we take* $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, *then we obtain the following inequality for Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right. \\
&\qquad\left.-\frac{2^\alpha\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^\alpha_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\quad\le\frac{\kappa_2-\kappa_1}{2}\left[\left(\int_0^1\left|t^\alpha-\rho\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\qquad\left.+\left(\int_0^1\left|\sigma-t^\alpha\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

Corollary 12 *In Theorem 5, if we set* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then we obtain the following inequality for k-Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\left|(1-\sigma)F(\kappa_1)+(\sigma+\rho)F\left(\frac{\kappa_1+\kappa_2}{2}\right)+(1-\rho)F(\kappa_2)\right. \\
&\qquad\left.-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^\alpha_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^\alpha_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\quad\le\frac{\kappa_2-\kappa_1}{2}\left[\left(\int_0^1\left|t^{\frac{\alpha}{k}}-\rho\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\qquad\left.+\left(\int_0^1\left|\sigma-t^{\frac{\alpha}{k}}\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

5 Special cases

In this section, we give some special cases of our main results.

Remark 1 From Lemma 1, we give the following identities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following identity:

$$\begin{aligned}&\frac{1}{6}\left[F(\kappa_1)+4F\left(\frac{\kappa_1+\kappa_2}{2}\right)+F(\kappa_2)\right]-\frac{1}{2\Delta(1)}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&=\frac{\kappa_2-\kappa_1}{2\Delta(1)}\left[\int_0^1\left(\frac{\Delta(t)}{2}-\frac{\Delta(1)}{3}\right)F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)dt\right.\\&\quad\left.+\int_0^1\left(\frac{\Delta(1)}{3}-\frac{\Delta(t)}{2}\right)F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)dt\right],\end{aligned}$$

which is given by Ertuğral and Sarikaya in [16].

2. For $\rho = \sigma = 1$, we have the following identity:

$$\begin{aligned}&F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Delta(1)}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&=\frac{\kappa_2-\kappa_1}{4\Delta(1)}\int_0^1\left[\left(\Delta(t)-\Delta(1)\right)\right]\left[F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)-F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)\right]dt,\end{aligned}$$

which is given by Ertuğral et al. in [17].

3. For $\rho = \sigma = 0$, we have the following identity:

$$\begin{aligned}&\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Delta(1)}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&=\frac{\kappa_2-\kappa_1}{4\Delta(1)}\int_0^1\left[\Delta(t)F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)dt-\int_0^1\Delta(t)F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)dt\right],\end{aligned}$$

which is given by Ertuğral et al. in [17].

Remark 2 From Corollary 1, we have the following identities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following new identity:

$$\begin{aligned}&\frac{1}{6}\left[F(\kappa_1)+4F\left(\frac{\kappa_1+\kappa_2}{2}\right)+F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\\&=\frac{\kappa_2-\kappa_1}{2}\left[\int_0^1\left(\frac{t}{2}-\frac{2}{3}\right)F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)dt\right.\\&\quad\left.+\int_0^1\left(\frac{2}{3}-\frac{t}{2}\right)F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)dt\right].\end{aligned}$$

2. For $\rho = \sigma = 0$, we have the following identity:

$$\begin{aligned}&\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\\&=\frac{\kappa_2-\kappa_1}{4}\int_0^1 t\left[F'\left(\frac{1-t}{2}\kappa_1+\frac{1+t}{2}\kappa_2\right)-F'\left(\frac{1+t}{2}\kappa_1+\frac{1-t}{2}\kappa_2\right)\right]dt.\end{aligned}$$

3. For $\rho = \sigma = 1$, we have the following identity:

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(t)\,dt = \frac{\kappa_2-\kappa_1}{4}\int_0^1 (t-1)\left[F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right) - F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right)\right]dt.$$

Remark 3 From Corollary 2, we have the following identities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following new identity:

$$\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^\alpha_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] = \frac{\kappa_2-\kappa_1}{2}\left[\int_0^1 \left(\frac{t^\alpha}{2} - \frac{1}{3}\right)F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right) - \int_0^1 \left(\frac{t^\alpha}{2} - \frac{1}{3}\right)F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right)dt\right],$$

which is given by Chen and Huang in [8].

2. For $\rho = \sigma = 0$, we have the following identity:

$$\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^\alpha_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] = \frac{\kappa_2-\kappa_1}{4}\left[\int_0^1 t^\alpha F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right)dt - \int_0^1 t^\alpha F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right)dt\right],$$

which is given by Ertuğral et al. in [17].

3. For $\rho = \sigma = 1$, we have the following identity:

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^\alpha_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] = \frac{\kappa_2-\kappa_1}{4}\left[\int_0^1 (t^\alpha-1)F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right)dt - \int_0^1 (t^\alpha-1)F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right)dt\right],$$

which is given by Ertuğral et al. in [17].

Remark 4 From Corollary 3, we have the following identities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following identity:

$$\frac{1}{6}\left[F(\kappa_1) + 2F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{2^{\frac{\alpha}{2}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^\alpha_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^\alpha_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]$$

$$= \frac{\kappa_2 - \kappa_1}{2}\left[\int_0^1 \left(\frac{t^{\frac{\alpha}{k}}}{2} - \frac{1}{3}\right) F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right) dt - \int_0^1 \left(\frac{t^{\frac{\alpha}{k}}}{2} - \frac{1}{3}\right) F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right) dt\right],$$

which is given by Ertuğral and Sarikaya in [16].

2. For $\rho = \sigma = 0$, we have the following identity:

$$\frac{F(\kappa_1) + F(\kappa_2)}{2} - \frac{2^{\frac{\alpha}{2}-1}\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1 + \kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1 + \kappa_2}{2}\right)\right]$$
$$= \frac{\kappa_2 - \kappa_1}{4}\left[\int_0^1 \left(t^{\frac{\alpha}{k}}\right) F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right) dt - \int_0^1 \left(t^{\frac{\alpha}{k}}\right) F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right) dt\right],$$

which is given by Ertuğral et al. in [17].

3. For $\rho = \sigma = 1$, we have the following identity:

$$F\left(\frac{\kappa_1 + \kappa_2}{2}\right) - \frac{2^{\frac{\alpha}{2}-1}\Gamma_k(\alpha + k)}{(\kappa_2 - \kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1 + \kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1 + \kappa_2}{2}\right)\right]$$
$$= \frac{\kappa_2 - \kappa_1}{4}\left[\int_0^1 \left(t^{\frac{\alpha}{k}} - 1\right) F'\left(\frac{1-t}{2}\kappa_1 + \frac{1+t}{2}\kappa_2\right) dt - \int_0^1 \left(t^{\frac{\alpha}{k}} - 1\right) F'\left(\frac{1+t}{2}\kappa_1 + \frac{1-t}{2}\kappa_2\right) dt\right],$$

which is given by Ertuğral et al. in [17].

Remark 5 From Theorem 3, we have the following new inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following inequality:

$$\left|\frac{1}{6}\left[F(\kappa_1) + 2F\left(\frac{\kappa_1 + \kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{1}{2\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1 + \kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1 + \kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2 - \kappa_1}{8\Delta(1)}\left[\Pi_1^{\varphi}\left(\frac{2}{3}\right) + \Pi_2^{\varphi}\left(\frac{2}{3}\right)\right]\left[\left|F'(\kappa_2)\right| + \left|F'(\kappa_1)\right|\right],$$

which is given by Ertuğral and Sarikaya in [16].

2. For $\rho = \sigma = 0$, we have the following inequality:

$$\left|\frac{F(\kappa_1) + F(\kappa_2)}{2} - \frac{1}{2\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1 + \kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1 + \kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2 - \kappa_1}{4\Delta(1)}\left(\int_0^1 \left|\Delta(t)\right| dt\right)\left[\left|F'(\kappa_2)\right| + \left|F'(\kappa_1)\right|\right],$$

which is given by Ertuğral et al. in [17].

3. For $\rho = \sigma = 1$, we have the following inequality:

$$\left| F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{2\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| = \frac{\kappa_2-\kappa_1}{4\Delta(1)}\left(\int_0^1 \left|\Delta(t)-\Delta(1)\right| dt\right)\left[\left|F'(\kappa_2)\right| + \left|F'(\kappa_1)\right|\right],$$

which is given by Ertuğral et al. in [17].

Remark 6 From Corollary 4, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson's inequality for Riemann integrals:

$$\left|\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(t)\,dt\right| \le \frac{5(\kappa_2-\kappa_1)}{72}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right],$$

which is given by Sarikaya et al. in [33, 34].

2. For $\rho = \sigma = 0$, we have the following trapezoid inequality for Riemann integrals:

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(t)\,dt\right| \le \frac{\kappa_2-\kappa_1}{8}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right],$$

which is proved by Dragomir and Agarwal [10].

3. For $\rho = \sigma = 1$, we have the following midpoint inequality for Riemann integrals:

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(t)\,dt\right| \le \frac{\kappa_2-\kappa_1}{8}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right],$$

which is given by Kirmaci in [23].

Remark 7 From Corollary 5, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson's inequality for Riemann–Liouville fractional integrals:

$$\left|\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \le \frac{\kappa_2-\kappa_1}{2}\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} + \frac{1}{2(\alpha+1)} - \frac{1}{3}\right)\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right],$$

which is given by Ertuğral and Sarikaya in [16].

2. For $\rho = \sigma = 0$, we have the following trapezoidal-type inequality for Riemann–Liouville fractional integrals:

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \le \frac{\kappa_2-\kappa_1}{4(\alpha+1)}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right].$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \leq \frac{\alpha(\kappa_2-\kappa_1)}{4(\alpha+1)}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right],$$

which is given by Ertuğral et al. in [17].

Remark 8 From Corollary 6, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for k-Riemann–Liouville fractional integrals:

$$\left|\frac{1}{6}\left[F(\kappa_1)+4F\left(\frac{\kappa_1+\kappa_2}{2}\right)+F(\kappa_2)\right]-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \leq \frac{\kappa_2-\kappa_1}{2}\left(\frac{\alpha}{\alpha+k}\left(\frac{2}{3}\right)^{\frac{\alpha+k}{\alpha}}+\frac{k}{2(\alpha+k)}-\frac{1}{3}\right)\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right].$$

2. For $\rho = \sigma = 0$, we have the following trapezoidal-type inequality for k-Riemann–Liouville fractional integrals:

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \leq \frac{k(\kappa_2-\kappa_1)}{4(\alpha+k)}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right].$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for k-Riemann–Liouville fractional integrals:

$$\left|2F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \leq \frac{\alpha(\kappa_2-\kappa_1)}{4(\alpha+k)}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right],$$

which is given by Ertuğral et al. in [17].

Remark 9 From Theorem 4, we have the following new inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for generalized fractional integrals:

$$\left|\frac{1}{6}\left[F(\kappa_1)+4F\left(\frac{\kappa_1+\kappa_2}{2}\right)+F(\kappa_2)\right]-\frac{1}{2\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left(\int_0^1\left|\Delta(t)-\Delta(1)\frac{2}{3}\right|dt\right)^{1-\frac{1}{p_1}}$$

$$\times \left[\left(\frac{\Pi_1^{\varphi}(\frac{2}{3})|F'(\kappa_1)|^{p_1} + \Pi_2^{\varphi}(\frac{2}{3})|F'(\kappa_2)|^{p_1}}{2} \right)^{\frac{1}{p_1}} + \left(\frac{\Pi_1^{\varphi}(\frac{2}{3})|F'(\kappa_2)|^{p_1} + \Pi_2^{\varphi}(\frac{2}{3})|F'(\kappa_1)|^{p_1}}{2} \right)^{\frac{1}{p_1}} \right].$$

2. For $\rho = \sigma = 0$, we have the following trapezoidal-type inequality for generalized fractional integrals:

$$\left| \frac{F(\kappa_1) + F(\kappa_2)}{2} - \frac{1}{2\Delta(1)} \left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) \right] \right|$$
$$\leq \frac{\kappa_2 - \kappa_1}{4\Delta(1)} \left(\int_0^1 \left|\Delta(t)\right| dt \right)^{1-\frac{1}{p_1}}$$
$$\times \left[\left(\frac{\Pi_1^{\varphi}(0)|F'(\kappa_1)|^{p_1} + \Pi_2^{\varphi}(0)|F'(\kappa_2)|^{p_1}}{2} \right)^{\frac{1}{p_1}} + \left(\frac{\Pi_1^{\varphi}(0)|F'(\kappa_2)|^{p_1} + \Pi_2^{\varphi}(0)|F'(\kappa_1)|^{p_1}}{2} \right)^{\frac{1}{p_1}} \right].$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\left| F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{2\Delta(1)} \left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) \right] \right|$$
$$\leq \frac{\kappa_2 - \kappa_1}{4\Delta(1)} \left(\int_0^1 \left|\Delta(t) - \Delta(1)\right| dt \right)^{1-\frac{1}{p_1}} \left[\left(\frac{\Pi_1^{\varphi}(1)|F'(\kappa_1)|^{p_1} + \Pi_2^{\varphi}(1)|F'(\kappa_2)|^{p_1}}{2} \right)^{\frac{1}{p_1}} + \left(\frac{\Pi_1^{\varphi}(1)|F'(\kappa_2)|^{p_1} + \Pi_2^{\varphi}(1)|F'(\kappa_1)|^{p_1}}{2} \right)^{\frac{1}{p_1}} \right],$$

which is given by Ertuğral et al. in [17].

Remark 10 From Corollary 7, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for Riemann integrals:

$$\left| \frac{1}{6} \left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2) \right] - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(t)\,dt \right|$$
$$\leq \frac{\kappa_2 - \kappa_1}{8} \left(\frac{5}{9} \right)^{1-\frac{1}{p_1}} \left[\left(\frac{29|F'(\kappa_1)|^{p_1} + 61|F'(\kappa_2)|^{p_1}}{162} \right)^{\frac{1}{p_1}} + \left(\frac{29|F'(\kappa_2)|^{p_1} + 61|F'(\kappa_1)|^{p_1}}{324} \right)^{\frac{1}{p_1}} \right],$$

which is given by Sarikaya et al. in [33, 34]

2. For $\rho = \sigma = 0$, we have the following trapezoid-type inequality for Riemann integrals:

$$\left| \frac{F(\kappa_1) + F(\kappa_2)}{2} - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(t)\,dt \right|$$
$$\leq \frac{\kappa_2 - \kappa_1}{8} \left[\left(\frac{|F'(\kappa_1)|^{p_1} + 5|F'(\kappa_2)|^{p_1}}{6} \right)^{\frac{1}{p_1}} + \left(\frac{|F'(\kappa_2)|^{p_1} + 5|F'(\kappa_1)|^{p_1}}{6} \right)^{\frac{1}{p_1}} \right].$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for Riemann integrals:

$$\begin{aligned}&\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\right|\\&\leq\frac{\kappa_2-\kappa_1}{8}\left[\left(\frac{|F'(\kappa_1)|^{p_1}+2|F'(\kappa_2)|^{p_1}}{3}\right)^{\frac{1}{p_1}}+\left(\frac{|F'(\kappa_2)|^{p_1}+2|F'(\kappa_1)|^{p_1}}{3}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

Remark 11 From Corollary 8, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for Riemann–Liouville fractional integrals:

$$\begin{aligned}&\left|\frac{1}{6}\left[F(\kappa_1)+4F\left(\frac{\kappa_1+\kappa_2}{2}\right)+F(\kappa_2)\right]\right.\\&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{4\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}+\frac{2}{\alpha+1}-\frac{4}{3}\right)^{1-\frac{1}{p_1}}\\&\quad\times\left[\left(\Pi_1^{\alpha}\left(\frac{2}{3}\right)\left|F'(\kappa_1)\right|^{p_1}+\Pi_2^{\alpha}\left(\frac{2}{3}\right)\left|F'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.\\&\quad\left.+\left(\Pi_1^{\alpha}\left(\frac{2}{3}\right)\left|F'(\kappa_2)\right|^{p_1}+\Pi_2^{\alpha}\left(\frac{2}{3}\right)\left|F'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

2. For $\rho = \sigma = 0$, we have the following trapezoidal-type inequality for Riemann–Liouville fractional integrals:

$$\begin{aligned}&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{2\alpha}{\alpha+1}\right)^{1-\frac{1}{p_1}}\\&\quad\times\left[\left(\frac{|F'(\kappa_1)|^{p_1}+(2\alpha+3)|F'(\kappa_2)|^{p_1}}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{p_1}}+\left(\frac{|F'(\kappa_2)|^{p_1}+(2\alpha+3)|F'(\kappa_1)|^{p_1}}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\begin{aligned}&\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1^+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2^-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{2\alpha}{\alpha+1}\right)^{1-\frac{1}{p_1}}\\&\quad\times\left[\left(\frac{\alpha(\alpha+3)|F'(\kappa_1)|^{p_1}+\alpha(3\alpha+5)|F'(\kappa_2)|^{p_1}}{2(\alpha+1)(\alpha+2)}\right)^{\frac{1}{p_1}}\right.\\&\quad\left.+\left(\frac{\alpha(\alpha+3)|F'(\kappa_2)|^{p_1}+\alpha(3\alpha+5)|F'(\kappa_1)|^{p_1}}{2(\alpha+1)(\alpha+2)}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

Remark 12 From Corollary 9, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for k-Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right]\right.\\
&\quad\left. - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\le \frac{\kappa_2-\kappa_1}{4}\left(\frac{4\alpha}{\alpha+k}\left(\frac{2}{3}\right)^{\frac{\alpha+k}{\alpha}} + \frac{2k}{\alpha+k} - \frac{4}{3}\right)^{1-\frac{1}{p_1}}\\
&\quad\times\left[\left(\Pi_1^{\frac{\alpha}{k}}\left(\frac{2}{3}\right)\left|F'(\kappa_1)\right|^{p_1} + \Pi_2^{\frac{\alpha}{k}}\left(\frac{2}{3}\right)\left|F'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.\\
&\quad\left.+\left(\Pi_1^{\frac{\alpha}{k}}\left(\frac{2}{3}\right)\left|F'(\kappa_2)\right|^{p_1} + \Pi_2^{\frac{\alpha}{k}}\left(\frac{2}{3}\right)\left|F'(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

2. For $\rho = \sigma = 0$, we have the following trapezoidal-type inequality for k-Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|F(\kappa_1) + F(\kappa_2) - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\le \frac{\kappa_2-\kappa_1}{4}\left(\frac{2\alpha}{\alpha+k}\right)^{1-\frac{1}{p_1}}\left[\left(\frac{k^2|F'(\kappa_1)|^{p_1} + k(2\alpha+3k)|F'(\kappa_2)|^{p_1}}{(\alpha+k)(\alpha+2k)}\right)^{\frac{1}{p_1}}\right.\\
&\quad\left.+\left(\frac{k^2|F'(\kappa_2)|^{p_1} + k(2\alpha+3k)|F'(\kappa_1)|^{p_1}}{(\alpha+k)(\alpha+2k)}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for k-Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|2F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\le \frac{\kappa_2-\kappa_1}{4}\left(\frac{2\alpha}{\alpha+1}\right)^{1-\frac{1}{p_1}}\left[\left(\frac{\alpha(\alpha+3k)|F'(\kappa_1)|^{p_1} + \alpha(3\alpha+5k)|F'(\kappa_2)|^{p_1}}{2(\alpha+k)(\alpha+2k)}\right)^{\frac{1}{p_1}}\right.\\
&\quad\left.+\left(\frac{\alpha(\alpha+3k)|F'(\kappa_2)|^{p_1} + \alpha(3\alpha+5k)|F'(\kappa_1)|^{p_1}}{2(\alpha+k)(\alpha+2k)}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Remark 13 From Theorem 5, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for generalized fractional integrals:

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right]\right.\\
&\quad\left. - \frac{1}{2\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|
\end{aligned}$$

$$\leq \frac{\kappa_2-\kappa_1}{2\Delta(1)}\left(\int_0^1\left|\Delta(t)-\frac{2}{3}\Delta(1)\right|^{p_1}dt\right)^{\frac{1}{p_1}}$$
$$\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].$$

2. For $\rho=\sigma=0$, we have the following trapezoidal-type inequality for generalized fractional integrals:

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{2\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4\Delta(1)}\left(\int_0^1\left|\Delta(t)\right|^{p_1}dt\right)^{\frac{1}{p_1}}$$
$$\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right],$$

which is given by Ertuğral et al. in [17].

3. For $\rho=\sigma=1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{2\Delta(1)}\left[{}_{\kappa_1+}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_{\varphi}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4\Delta(1)}\left(\int_0^1\left|\Delta(t)-\Delta(1)\right|^{p_1}dt\right)^{\frac{1}{p_1}}$$
$$\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right],$$

which is given by Ertuğral et al. in [17].

Remark 14 From Corollary 10, we have the following inequalities:

1. For $\rho=\sigma=\frac{2}{3}$, we have the following Simpson-type inequality for Riemann integrals:

$$\left|\frac{1}{6}\left[F(\kappa_1)+4F\left(\frac{\kappa_1+\kappa_2}{2}\right)+F(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{2^{p_1+1}+1}{(p_1+1)3^{p_1+1}}\right)^{\frac{1}{p_1}}\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.$$
$$\left.+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].$$

2. For $\rho=\sigma=0$, we have the following trapezoid-type inequality for Riemann integrals:

$$\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{1}{p_1+1}\right)^{\frac{1}{p_1}}\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.$$
$$\left.+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for Riemann integrals:

$$\begin{aligned}&\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(t)\,dt\right|\\&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{1}{p_1+1}\right)^{\frac{1}{p_1}}\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.\\&\quad\left.+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right],\end{aligned}$$

which is proved by Kirmaci in [23].

Remark 15 From Corollary 11, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for Riemann–Liouville fractional integrals:

$$\begin{aligned}&\left|\frac{1}{6}\left[F(\kappa_1)+4F\left(\frac{\kappa_1+\kappa_2}{2}\right)+F(\kappa_2)\right]\right.\\&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{4}\left(\int_0^1\left|t^{\alpha}-\frac{2}{3}\right|^{p_1}dt\right)^{\frac{1}{p_1}}\\&\quad\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

2. For $\rho = \sigma = 0$, we have the following trapezoidal-type inequality for Riemann–Liouville fractional integrals:

$$\begin{aligned}&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{1}{\alpha p_1+1}\right)^{\frac{1}{p_1}}\\&\quad\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for generalized fractional integrals:

$$\begin{aligned}&\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{4}\left(\int_0^1\left(1-t^{\alpha}\right)^{p_1}dt\right)^{\frac{1}{p_1}}\\&\quad\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1}+|F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{3|F'(\kappa_1)|^{r_1}+|F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

Remark 16 From Corollary 12, we have the following inequalities:

1. For $\rho = \sigma = \frac{2}{3}$, we have the following Simpson-type inequality for k-Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(\kappa_1) + 4F\left(\frac{\kappa_1+\kappa_2}{2}\right) + F(\kappa_2)\right]\right.\\
&\quad\left. - \frac{2^{\frac{\alpha}{2}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\le \frac{\kappa_2-\kappa_1}{4}\left(\int_0^1\left|t^{\frac{\alpha}{k}} - \frac{2}{3}\right|^{p_1}dt\right)^{\frac{1}{p_1}}\\
&\quad\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1} + |F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}} + \left(\frac{3|F'(\kappa_1)|^{r_1} + |F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

2. For $\rho = \sigma = 0$, we have the following trapezoidal-type inequality for k-Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{2^{\frac{\alpha}{2}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\le \frac{\kappa_2-\kappa_1}{4}\left(\frac{k}{\alpha p_1 + k}\right)^{\frac{1}{p_1}}\\
&\quad\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1} + |F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}} + \left(\frac{3|F'(\kappa_1)|^{r_1} + |F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

3. For $\rho = \sigma = 1$, we have the following midpoint-type inequality for k-Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{2^{\frac{\alpha}{2}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_1+,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^{\alpha}_{\kappa_2-,k}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\le \frac{\kappa_2-\kappa_1}{4}\left(\int_0^1\left(1 - t^{\frac{\alpha}{k}}\right)^{p_1}dt\right)^{\frac{1}{p_1}}\\
&\quad\times\left[\left(\frac{3|F'(\kappa_2)|^{r_1} + |F'(\kappa_1)|^{r_1}}{4}\right)^{\frac{1}{r_1}} + \left(\frac{3|F'(\kappa_1)|^{r_1} + |F'(\kappa_2)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

6 Concluding remarks

In this study, we present some generalized inequalities for differentiable convex functions via generalized fractional integrals. It is also shown that the results proved here are the strong generalizations of some already published ones. It is an interesting and new problem that future researchers can use the techniques of this study and obtain similar inequalities for different kinds of convexity in their work.

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The authors declare that they have no competing interests.

Authors' contributions
HB: conceptualization, computation, writing, reviewing and editing. SKY: computation, writing original draft. MZS: computation, writing, reviewing and editing. HY: supervision and editing. All authors read and approved the final manuscript.

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References

1. Abdeljawad, T., Rashid, S., Hammouch, Z., İşcan, İ., Chu, Y.M.: Some new Simpson-type inequalities for generalized p-convex function on fractal sets with applications. Adv. Differ. Equ. **2020**(1), 496 (2020)
2. Ali, M.A., Abbas, M., Budak, H., Agarwal, P., Murtaza, G., Chu, Y.M.: New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. Adv. Differ. Equ. **2021**(1), 64 (2021)
3. Ali, M.A., Budak, H., Zhang, Z., Yildrim, H.: Some new Simpson-type inequalities for co-ordinated convex functions in quantum calculus. Math. Methods Appl. Sci. **44**(6), 4515–4540 (2021)
4. Alomari, M., Darus, M., Dragomir, S.S.: New inequalities of Simpson's type for *s*-convex functions with applications. RGMIA Res. Rep. Collect. **12**(4) (2009)
5. Budak, H., Erden, S., Ali, M.A.: Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. Math. Methods Appl. Sci. **44**(1), 378–390 (2021)
6. Budak, H., Kara, H., Kapucu, R.: New midpoint type inequalities for generalized fractional integral. Comput. Methods Differ. Equ. **10**(1), 93–108 (2022)
7. Budak, H., Pehlivan, E., Kösem, P.: On new extensions of Hermite–Hadamard inequalities for generalized fractional integrals. Sahand Commun. Math. Anal. **18**(1), 73–88 (2021)
8. Chen, J., Huang, X.: Some new inequalities of Simpson's type for *s*-convex functions via fractional integrals. Filomat **31**(15), 4989–4997 (2017)
9. Chu, Y.M., Awan, M.U., Javad, M.Z., Khan, A.G.: Bounds for the remainder in Simpson's inequality via *n*-polynomial convex functions of higher order using Katugampola fractional integrals. J. Math., **2020**, Article ID 4189036 (2020)
10. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998)
11. Dragomir, S.S., Agarwal, R.P., Cerone, P.: On Simpson's inequality and applications. J. Inequal. Appl. **5**, 533–579 (2000)
12. Du, T.S., Awan, M.U., Kashuri, A., Zhao, S.: Some *k*-fractional extensions of the trapezium inequalities through generalized relative semi-(m,h)-preinvexity. Appl. Anal. **100**(3), 642–662 (2021)
13. Du, T.S., Li, Y., Yang, Z.: A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions. Appl. Math. Comput. **293**, 358–369 (2017)
14. Du, T.S., Wang, H., Khan, M.A., Zhang, Y.: Certain integral inequalities considering generalized *m*-convexity on fractal sets and their applications. Fractals **27**(7), 1–17 (2019)
15. Erden, S., Iftikhar, S., Delavar, R.M., Kumam, P., Thounthong, P., Kumam, W.: On generalizations of some inequalities for convex functions via quantum integrals. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**(3), 110 (2020). https://doi.org/10.1007/s13398-020-00841-3
16. Ertuğral, F., Sarikaya, M.Z.: Simpson type integral inequalities for generalized fractional integral. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(4), 3115–3124 (2019)
17. Ertuğral, F., Sarikaya, M.Z., Budak, H.: On Hermite–Hadamard type inequalities associated with the generalized fractional integrals (2019). https://www.researchgate.net/publication/334634529
18. Han, J., Mohammed, P.O., Zeng, H.: Generalized fractional integral inequalities of Hermite–Hadamard-type for a convex function. Open Math. **18**(1), 794–806 (2020)
19. Hussain, S., Khalid, J., Chu, Y.M.: Some generalized fractional integral Simpson-type inequalities with applications. AIMS Math. **5**(6), 5859–5883 (2020)
20. Iftikhar, S., Komam, P., Erden, S.: Newton's type integral inequalities via local fractional integrals. Fractals **28**(3), 2050037 (2020). https://doi.org/10.1142/S0218348X20500371
21. Kashuri, A., Ali, M.A., Abbas, M., Budak, H.: New inequalities for generalized *m*-convex functions via generalized fractional integral operators and their applications. Int. J. Nonlinear Anal. Appl. **10**(2), 275–299 (2019)
22. Kashuri, A., Liko, R.: On Fejér type inequalities for convex mappings utilizing generalized fractional integrals. Appl. Appl. Math. **15**(1), 135–150 (2020)
23. Kırmacı, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comput. **147**(1), 137–146 (2004)

24. Mohammed, P.O.: On new trapezoid-type inequalities for *h*-convex functions via generalized fractional integral. Turk. J. Anal. Number Theory **6**, 125–128 (2018)
25. Mohammed, P.O., Abdeljawad, T.: Modification of certain fractional integral inequalities for convex functions. Adv. Differ. Equ. **2020**, 69 (2020)
26. Mohammed, P.O., Sarikaya, M.Z.: On generalized fractional integral inequalities for twice differentiable convex functions. J. Comput. Appl. Math. **372**, 112740 (2020)
27. Mubeen, S., Habibullah, G.M.: *k*-Fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(2), 89–94 (2012)
28. Noor, M.A., Noor, K.I., Iftikhar, S.: Some Newton's type inequalities for harmonic convex functions. J. Adv. Math. Stud. **9**(1), 07-16 (2016)
29. Noor, M.A., Noor, K.I., Iftikhar, S.: Newton inequalities for *p*-harmonic convex functions. Honam Math. J. **40**(2), 239–250 (2018)
30. Park, J.: On Simpson-like type integral inequalities for differentiable preinvex functions. Appl. Math. Sci. **7**(121), 6009–6021 (2013)
31. Qi, F., Mohammed, P.O., Yao, J.-C., Yao, Y.-H.: Generalized fractional integral inequalities of Hermite–Hadamard type for (α, m)-convex functions. J. Inequal. Appl. **2019**, 135 (2019)
32. Sarikaya, M.Z., Ertuğral, F.: On the generalized Hermite–Hadamard inequalities. An. Univ. Craiova-Math. Comput. Sci. Ser. **47**(1), 193–213 (2020)
33. Sarikaya, M.Z., Set, E., Özdemir, M.E.: On new inequalities of Simpson's type for convex functions. RGMIA Res. Rep. Collect. **13**(2), Article 2 (2010)
34. Sarikaya, M.Z., Set, E., Özdemir, M.E.: On new inequalities of Simpson's type for *s*-convex functions. Comput. Math. Appl. **60**(8), 2191–2199 (2020)
35. Vivas-Cortez, M., Ali, M.A., Kashuri, A., Sial, I.B., Zhang, Z.: Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus. Symmetry **12**(9), 1476 (2020)
36. Zhao, D., Ali, M.A., Kashuri, A., Budak, H., Sarikaya, M.Z.: Hermite–Hadamard-type inequalities for the interval-valued approximately *h*-convex functions via generalized fractional integrals. J. Inequal. Appl. **2020**(1), 222 (2020)