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New Hadamard-type inequalities for E -convex functions involving generalized fractional integrals

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Abstract

In this article, we establish some new Hadamard-type inequalities for E -convex functions involving generalized fractional integrals. These inequalities include a generalized Hadamard-type inequality and the corresponding right Hadamard-type inequalities for E -convex functions. The results presented here are generalizations of some of the results discussed in the recent literature.

MSC: 26A51; 39B62; 26D15; 26D10

Keywords: E -Convex functions; Generalized fractional integrals; Hölder's integral inequality; Power mean inequality

1 Introduction and basic definitions

The theory of convexity is not only important in itself but also it contributes to almost all areas of mathematics. Convexity gives rise to inequalities, the Hadamard inequality is the first consequence of convex functions. The book by Hardy [1] has played a key role in popularizing the subject of convex analysis. Over the years, the idea of convex sets and convex functions has been largely generalized. Today, the study of convex functions has evolved into a broader theory of functions including quasiconvex functions [2, 3], coordinated convex functions [4, 5], preinvex functions [6], GA -convex functions [7], strongly convex functions [8], (g, φ_h) -convex functions [9], E -convex functions [10] and so on. Youness [10] defined the E -convex set and the corresponding function as follows:

Definition 1 A set $S \subset \mathbb{R}$ is called E -convex if and only if there is a function $E: \mathbb{R} \rightarrow \mathbb{R}$ such that $tE(\zeta) + (1-t)E(\eta) \in S$ for each $\zeta, \eta \in S$ and $t \in [0, 1]$.

Definition 2 A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called E -convex on a set $S \subseteq \mathbb{R}$ if and only if there is a map $E: \mathbb{R} \rightarrow \mathbb{R}$ such that S is an E -convex set and

$$f(tE(\zeta) + (1-t)E(\eta)) \leq tf(E(\zeta)) + (1-t)f(E(\eta)) \quad (1.1)$$

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holds for each $\zeta, \eta \in S$ and $t \in [0, 1]$. On the other hand, if the inequality sign in the inequality (1.1) is reversed then f is called E -concave on the set S .

Every convex function f on a convex set S is an E -convex function provided that E is an identity function. For a detailed explanation of E -convex functions see [10]. The Hadamard-type inequality for E -convex given in [11] is as follows:

Theorem 1 *Let $E : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be an E -convex function on $[\zeta, \eta]$, then we have*

$$f\left(\frac{E(\zeta) + E(\eta)}{2}\right) \leq \frac{1}{E(\eta) - E(\zeta)} \int_{E(\zeta)}^{E(\eta)} f(E(t)) dE(t) \leq \frac{f(E(\zeta)) + f(E(\eta))}{2}, \tag{1.2}$$

inequality (1.2) is Hadamard's inequality for E -convex functions.

Convexity is mixed with other mathematical concepts such as; optimization [12], time scale [13, 14], quantum and postquantum calculus [15, 16], and fractional calculus [3, 11, 17–19]. Fractional calculus is basically a generalization of integer-order calculus. Strictly speaking, it is a generalization of operators beyond the integral order to real or complex order. Many fractional models have been proposed so far [20–27]. The key drivers behind such proposals are identified with the various real data corresponding to different systems under consideration requiring different kernels. Raina [27] and Agarwal [26] defined the following generalized fractional operators:

Definition 3 Let $f \in L(\zeta, \eta)$, then for $\sigma, \rho > 0, \omega \in \mathbb{R}$ the right-handed and left-handed generalized fractional integrals of f are, respectively, defined as follows:

$$\mathcal{J}_{\sigma, \rho, \zeta+; \omega}^\alpha f(s) = \int_\zeta^s (s-t)^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(s-t)^\sigma] f(t) dt \quad (s > \zeta),$$

and

$$\mathcal{J}_{\sigma, \rho, \eta-; \omega}^\alpha f(s) = \int_s^\eta (t-s)^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(t-s)^\sigma] f(t) dt \quad (s < \eta),$$

where $\mathcal{F}_{\sigma, \rho}^\alpha(s)$ is defined in [27] as follows:

$$\mathcal{F}_{\sigma, \rho}^\alpha(s) = \mathcal{F}_{\sigma, \rho}^{\alpha(0), \alpha(1), \alpha(2), \dots}(s) = \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho)} s^n \quad (\sigma, \rho > 0, |s| < R),$$

where R is a real positive constant. The coefficients $\alpha(n)$ ($n \in N_0 = N \cup \{0\}$) are terms of a bounded sequence of positive real numbers and \mathbb{R} is the set of real numbers. Moreover, the operators $\mathcal{J}_{\sigma, \rho, \zeta+; \omega}^\alpha f$ and $\mathcal{J}_{\sigma, \rho, \eta-; \omega}^\alpha f$ are bounded on $L(\zeta, \eta)$, i.e.,

$$\|\mathcal{J}_{\sigma, \rho, \zeta+; \omega}^\alpha f(s)\| \leq \mathfrak{P}(\eta - \zeta)^\rho \|f\|_1$$

and

$$\|\mathcal{J}_{\sigma, \rho, \eta-; \omega}^\alpha f(s)\| \leq \mathfrak{P}(\eta - \zeta)^\rho \|f\|_1,$$

where $\mathfrak{P} := \mathcal{F}_{\sigma, \rho}^\alpha[\omega(t-s)^\sigma] < \infty$ and $\|f\|_1 = \int_\zeta^\eta |f(t)| dt$.

These fractional integrals are really important because of their generality. Many other fractional operators can be obtained by specifying the coefficients $\alpha(n)$. For instance, if we set $n = 0$, $\alpha(0) = 1$ and $\omega = 0$, we obtain the well-known Riemann–Liouville fractional operators

$$J_{\zeta+}^{\lambda} f(s) = \frac{1}{\Gamma(\lambda)} \int_{\zeta}^s (s-t)^{\lambda-1} f(t) dt \quad (s > \zeta)$$

and

$$J_{\eta-}^{\lambda} f(s) = \frac{1}{\Gamma(\lambda)} \int_s^{\eta} (t-s)^{\lambda-1} f(t) dt \quad (s < \eta).$$

Lemma 1 ([28, 29]) *For $0 < \alpha \leq 1$ and $0 \leq x < y$, we have*

$$|x^{\alpha} - y^{\alpha}| \leq (y - x)^{\alpha}.$$

Fractional calculus has useful applications in almost all areas of applied mathematics and other sciences, see [30] and the references therein. In the present work, notions of E -convexity and generalized fractional operators are joined together. These ideas are independently utilized before, however, in combined form we obtain even more generalized results.

2 Main outcomes

In this section, mainly the Hadamard inequality for E -convex function (1.2) is extended using Definition 3 of generalized fractional integrals. Then, an identity is established for differentiable functions that is used to develop right Hadamard-type inequalities for the said extended Hadamard-type inequality. Likewise, another important identity is developed for twice-differentiable functions that is further used to develop more right Hadamard-type inequalities for the said extended Hadamard-type inequality for E -convex functions.

In the following, we use J to represent the interval of nonnegative real numbers and I to represent the interval of real numbers. Moreover, we use the following notations for brevity;

$$\begin{aligned} \Delta &:= \eta - \zeta, & E(\Delta) &:= E(\eta) - E(\zeta), \\ \mathcal{M}(J) &:= \frac{f(\zeta) + f(\eta)}{2} - \frac{\Gamma(\lambda + 1)}{2(\Delta)^{\lambda}} [J_{\zeta+}^{\lambda} f(\eta) + J_{\eta-}^{\lambda} f(\zeta)], \\ \mathcal{M}(\mathcal{J}) &:= \frac{f(\zeta) + f(\eta)}{2} - \frac{1}{2(\Delta)^{\rho} \mathcal{F}_{\sigma, \rho+1}^{\alpha}[\omega(\Delta)^{\sigma}]} [\mathcal{J}_{\sigma, \rho, \zeta+; \omega}^{\alpha} f(\eta) + \mathcal{J}_{\sigma, \rho, \eta-; \omega}^{\alpha} f(\zeta)], \\ \mathcal{M}(\mathcal{J}_E) &:= \frac{f(E(\zeta)) + f(E(\eta))}{2} \\ &\quad - \frac{1}{2(E(\Delta))^{\rho} \mathcal{F}_{\sigma, \rho+1}^{\alpha}[\omega(E(\Delta))^{\sigma}]} [\mathcal{J}_{\sigma, \rho, E(\zeta)+; \omega}^{\alpha} f(E(\eta)) + \mathcal{J}_{\sigma, \rho, E(\eta)-; \omega}^{\alpha} f(E(\zeta))]. \end{aligned}$$

Theorem 2 *Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a function such that $f \in L[E(\zeta), E(\eta)]$, where $E(\zeta), E(\eta) \in I$. If f is an E -convex*

function on $[\zeta, \eta]$, then the following inequality holds for generalized fractional integral operators

$$\begin{aligned}
 & f\left(\frac{E(\zeta) + E(\eta)}{2}\right) \\
 & \leq \frac{1}{2(E(\Delta))^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\mathcal{J}_{\sigma, \rho, E(\zeta)+; \omega}^\alpha f(E(\eta)) + \mathcal{J}_{\sigma, \rho, E(\eta)-; \omega}^\alpha f(E(\zeta)) \right] \\
 & \leq \frac{f(E(\zeta)) + f(E(\eta))}{2},
 \end{aligned} \tag{2.1}$$

for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$.

Proof Since f is an E -convex function on $[\zeta, \eta]$, therefore for $E(x), E(y) \in I$ we have

$$f\left(\frac{E(x) + E(y)}{2}\right) \leq \frac{f(E(x)) + f(E(y))}{2}$$

and we let $E(x) = tE(\zeta) + (1 - t)E(\eta)$ and $E(y) = (1 - t)E(\zeta) + tE(\eta)$, so that we have

$$2f\left(\frac{E(\zeta) + E(\eta)}{2}\right) \leq f(tE(\zeta) + (1 - t)E(\eta)) + f((1 - t)E(\zeta) + tE(\eta)). \tag{2.2}$$

On multiplying both sides of inequality (2.2) by $t^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(E(\Delta))^\sigma t^\sigma]$ and then integrating the resultant inequality with respect to t over $[0, 1]$, we have

$$\begin{aligned}
 & 2f\left(\frac{E(\zeta) + E(\eta)}{2}\right) \int_0^1 t^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] dt \\
 & \leq \int_0^1 t^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] f(tE(\zeta) + (1 - t)E(\eta)) dt \\
 & \quad + \int_0^1 t^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] f((1 - t)E(\zeta) + tE(\eta)) dt.
 \end{aligned}$$

Further suppose that $u = tE(\zeta) + (1 - t)E(\eta)$ and $v = (1 - t)E(\zeta) + tE(\zeta)$ and using the definition of generalized fractional integrals

$$\begin{aligned}
 & 2f\left(\frac{E(\zeta) + E(\eta)}{2}\right) \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma] \\
 & \leq \frac{1}{(E(\Delta))^\rho} \int_{E(\zeta)}^{E(\eta)} (E(\eta) - u)^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(E(\Delta))^\sigma] f(u) du \\
 & \quad + \frac{1}{(E(\Delta))^\rho} \int_{E(\zeta)}^{E(\eta)} (v - E(\zeta))^{\rho-1} \mathcal{F}_{\sigma, \rho}^\alpha[\omega(E(\Delta))^\sigma] f(v) dv
 \end{aligned} \tag{2.3}$$

$$\begin{aligned}
 & f\left(\frac{E(\zeta) + E(\eta)}{2}\right) \\
 & \leq \frac{1}{2(E(\Delta))^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha((E(\Delta))^\sigma)} \left[\mathcal{J}_{\sigma, \rho, E(\zeta)+; \omega}^\alpha f(E(\eta)) + \mathcal{J}_{\sigma, \rho, E(\eta)-; \omega}^\alpha f(E(\zeta)) \right].
 \end{aligned}$$

Considering again the E -convexity of f over the interval $[\zeta, \eta]$, we have

$$f(tE(\zeta) + (1 - t)E(\eta)) \leq tf(E(\zeta)) + (1 - t)f(E(\eta)), \tag{2.4}$$

$$f((1 - t)E(\zeta) + tE(\eta)) \leq (1 - t)f(E(\zeta)) + tf(E(\eta)), \tag{2.5}$$

and on adding inequality (2.4) and inequality (2.5), we have

$$f(tE(\zeta) + (1 - t)E(\eta)) + f((1 - t)E(\zeta) + tE(\eta)) \leq f(E(\zeta)) + f(E(\eta)). \tag{2.6}$$

On multiplying both sides of inequality (2.6) by $t^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha [\omega(E(\Delta))^\sigma t^\sigma]$, integrating with respect to t over the interval $[0, 1]$ and finally using the definition of generalized fractional integrals, we have

$$\begin{aligned} & \int_0^1 t^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha [\omega(E(\Delta))^\sigma t^\sigma] f(tE(\zeta) + (1 - t)E(\eta)) dt \\ & \quad + \int_0^1 t^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha [\omega(E(\Delta))^\sigma t^\sigma] f((1 - t)E(\zeta) + tE(\eta)) dt \\ & \leq [f(E(\zeta)) + f(E(\eta))] \int_0^1 t^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha [\omega(E(\Delta))^\sigma t^\sigma] dt, \end{aligned}$$

on letting $u = tE(\zeta) + (1 - t)E(\eta)$ and $v = (1 - t)E(\zeta) + tE(\eta)$ and then using the definition of generalized fractional integrals, we have

$$\begin{aligned} & \frac{1}{(E(\Delta))^\rho} [\mathcal{J}_{\sigma,\rho,E(\zeta)+;\omega}^\alpha f(E(\eta)) + \mathcal{J}_{\sigma,\rho,E(\eta)-;\omega}^\alpha f(E(\zeta))] \\ & \leq \mathcal{F}_{\sigma,\rho+1}^\alpha [\omega(E(\Delta))] [f(E(\zeta)) + f(E(\eta))]. \end{aligned} \tag{2.7}$$

On combining inequality (2.3) and inequality (2.7), we obtain the required result. Hence it is proved. \square

Remark 1 If in Theorem 2, the function E is chosen to be an identity function, then the following inequality holds for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$:

$$f\left(\frac{\zeta + \eta}{2}\right) \leq \frac{1}{2(\Delta)^\rho \mathcal{F}_{\sigma,\rho+1}^\alpha [\omega(\Delta)^\sigma]} [\mathcal{J}_{\sigma,\rho,\zeta+;\omega}^\alpha f(\eta) + \mathcal{J}_{\sigma,\rho,\eta-;\omega}^\alpha f(\zeta)] \leq \frac{f(\zeta) + f(\eta)}{2},$$

which was given in [31].

Remark 2 If in Theorem 2, the function E is chosen to be an identity function, $\alpha(0) = 1$, $\rho = \lambda$ and $\omega = 0$, then the following inequality holds:

$$f\left(\frac{\zeta + \eta}{2}\right) \leq \frac{\Gamma(\lambda + 1)}{2(\Delta)^\lambda} [J_{\zeta+}^\lambda f(\eta) + J_{\eta-}^\lambda f(\zeta)] \leq \frac{f(\zeta) + f(\eta)}{2},$$

which was given in [32].

Lemma 2 Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I° . If $f' \in L([E(\zeta), E(\eta)])$ for $E(\zeta), E(\eta) \in I$, then

the following identity holds for generalized fractional operators:

$$\begin{aligned} \mathcal{M}(\mathcal{J}_E) &= \frac{E(\Delta)}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \int_0^1 \{(1-t)^\rho \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] \\ &\quad - t^\rho \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma t^\sigma]\} f'(tE(\zeta) + (1-t)E(\eta)) dt. \end{aligned} \tag{2.8}$$

Proof Solving the subsequent integral by integration by parts, then using a change of variable and finally the definition of the left generalized fractional integral operator

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^\rho \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] f'(tE(\zeta) + (1-t)E(\eta)) dt \\ &= \frac{1}{E(\Delta)} \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma] f(E(\eta)) \\ &\quad - \frac{1}{E(\Delta)} \int_0^1 (1-t)^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] f(tE(\zeta) + (1-t)E(\eta)) dt \\ &= \frac{1}{E(\Delta)} \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma] f(E(\eta)) \\ &\quad - \frac{1}{(E(\Delta))^{\rho+1}} \int_{E(\zeta)}^{E(\eta)} (v-E(\zeta))^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha[\omega(E(\Delta))^\sigma (v-E(\zeta))^\sigma] f(v) dv \\ &= \frac{1}{E(\Delta)} \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma] f(E(\eta)) - \frac{1}{(E(\Delta))^{\rho+1}} \mathcal{J}_{\sigma,\rho,\eta;\omega}^\alpha f(E(\zeta)). \end{aligned} \tag{2.9}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 t^\rho \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] f'(tE(\zeta) + (1-t)E(\eta)) dt \\ &= -\frac{1}{E(\Delta)} \mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma] f(E(\zeta)) + \frac{1}{(E(\Delta))^{\rho+1}} \mathcal{J}_{\sigma,\rho,E(\zeta);\omega}^\alpha f(E(\eta)) \end{aligned} \tag{2.10}$$

and on subtracting inequality (2.9) and inequality (2.10), then multiplying by $\frac{E(\Delta)}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]}$, we obtain

$$\frac{E(\Delta)}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} [I_1 - I_2] = \mathcal{M}(\mathcal{J}_E),$$

and on submitting the expressions for I_1 and I_2 , we obtain the required result. □

Theorem 3 *Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I° and $f' \in L([E(\zeta), E(\eta)])$ for $E(\zeta), E(\eta) \in I$. If $|f'|$ is an E -convex function on $[\zeta, \eta]$, then the following inequality holds for generalized fractional integral operators:*

$$|\mathcal{M}(\mathcal{J}_E)| \leq \frac{E(\Delta)}{2} \frac{\mathcal{F}_{\sigma,\rho+1}^{\alpha_1}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} [|f'(E(\zeta))| + |f'(E(\eta))|],$$

for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$, where

$$\alpha_1(n) = \frac{\alpha(n)}{(\sigma n + \rho + 1)} \left(1 - \frac{1}{2^{\sigma n + \rho}}\right) \text{ for } n = 0, 1, 2, \dots \tag{2.11}$$

Proof Using Lemma 2, the properties of modulus, and the E -convexity of $|f'|$, respectively, we have

$$\begin{aligned}
 & \frac{2\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]}{E(\Delta)} |\mathcal{M}(\mathcal{J}_E)| \\
 &= \left| \int_0^1 \{ (1-t)^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] \right. \\
 &\quad \left. - t^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] \} f'(tE(\zeta) + (1-t)E(\eta)) dt \right| \\
 &= \left| \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \right. \\
 &\quad \left. \times \int_0^1 [(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}] f'(tE(\zeta) + (1-t)E(\eta)) dt \right| \\
 &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \\
 &\quad \times \int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}| |f'(tE(\zeta) + (1-t)E(\eta))| dt \\
 &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \\
 &\quad \times \left[\int_0^{\frac{1}{2}} [(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}] [t|f'(E(\zeta))| + (1-t)|f'(E(\eta))|] dt \right. \\
 &\quad \left. + \int_{\frac{1}{2}}^1 [t^{\sigma n + \rho} - (1-t)^{\sigma n + \rho}] [t|f'(E(\zeta))| + (1-t)|f'(E(\eta))|] dt \right] \\
 &= \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n [I_3 + I_4]. \tag{2.12}
 \end{aligned}$$

Consider the following integral

$$\begin{aligned}
 I_3 &= \int_0^{\frac{1}{2}} [(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}] [t|f'(E(\zeta))| + (1-t)|f'(E(\eta))|] dt \\
 &= \int_0^{\frac{1}{2}} [t(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho + 1}] |f'(E(\zeta))| dt \\
 &\quad + \int_0^{\frac{1}{2}} [(1-t)^{\sigma n + \rho + 1} - t^{\sigma n + \rho + 1}(1-t)] |f'(E(\eta))| dt \\
 &= \left[\frac{1}{(\sigma n + \rho + 1)(\sigma n + \rho + 2)} - \frac{1}{2^{\sigma n + \rho + 1}(\sigma n + \rho + 1)} \right] |f'(E(\zeta))| \\
 &\quad + \left[\frac{1}{(\sigma n + \rho + 2)} - \frac{1}{2^{\sigma n + \rho + 1}(\sigma n + \rho + 1)} \right] |f'(E(\eta))|.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 I_4 &= \int_{\frac{1}{2}}^1 [t^{\sigma n + \rho} - (1-t)^{\sigma n + \rho}] [t|f'(E(\zeta))| + (1-t)|f'(E(\eta))|] dt \\
 &= \left[\frac{1}{(\sigma n + \rho + 2)} - \frac{1}{2^{\sigma n + \rho + 1}(\sigma n + \rho + 1)} \right] |f'(E(\zeta))| \\
 &\quad + \left[\frac{1}{(\sigma n + \rho + 1)(\sigma n + \rho + 2)} - \frac{1}{2^{\sigma n + \rho + 1}(\sigma n + \rho + 1)} \right] |f'(E(\eta))|
 \end{aligned}$$

and on submitting values of integrals I_3 and I_4 into inequality (2.12), we have

$$\begin{aligned}
 &\frac{2\mathcal{F}_{\sigma, \rho + 1}^\alpha [\omega(E(\Delta))^\sigma]}{E(\Delta)} |\mathcal{M}(\mathcal{J}_E)| \\
 &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \\
 &\quad \times \left[\frac{1}{\sigma n + \rho + 1} \left(1 - \frac{1}{2^{\sigma n + \rho}} \right) \{ |f'(E(\zeta))| + |f'(E(\eta))| \} \right] \\
 &= \mathcal{F}_{\sigma, \rho + 1}^{\alpha_1} [\omega(E(\Delta))^\sigma] [|f'(E(\zeta))| + |f'(E(\eta))|],
 \end{aligned}$$

where α_1 is as defined in (2.11). On rearranging we obtain the required result.

Hence it is proved. □

Remark 3 If in Theorem 3, the function E is chosen to be an identity function, then the following inequality holds for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{\Delta}{2} \frac{\mathcal{F}_{\sigma, \rho + 1}^{\alpha_1} [\omega(\Delta)^\sigma]}{\mathcal{F}_{\sigma, \rho + 1}^\alpha [\omega(\Delta)^\sigma]} [|f'(\zeta)| + |f'(\eta)|],$$

which was given in [31].

Remark 4 If in Theorem 3, the function E is chosen to be an identity function, $\alpha(0) = 1$, $\rho = \lambda$ and $\omega = 0$, then the following inequality holds:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{\Delta}{2} \left[\frac{1}{\lambda + 1} \left(1 - \frac{1}{2^\lambda} \right) \right] [|f'(\zeta)| + |f'(\eta)|],$$

which was given in [32].

Theorem 4 Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I° and $f' \in L([E(\zeta), E(\eta)])$ for $E(\zeta), E(\eta) \in I$. If $|f'|^q, q > 1$, is an E -convex function on $[\zeta, \eta]$, then the following inequality holds for generalized fractional integral operators:

$$|\mathcal{M}(\mathcal{J}_E)| \leq \frac{E(\Delta)}{2} \frac{\mathcal{F}_{\sigma, \rho + 1}^{\alpha_2} [\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma, \rho + 1}^\alpha [\omega(E(\Delta))^\sigma]} \left[\frac{|f'(E(\zeta))|^q + |f'(E(\eta))|^q}{2} \right]^{\frac{1}{q}}$$

for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$, where p, q are conjugate indices and

$$\alpha_2(n) = \alpha(n) \left(\frac{2}{p(\sigma n + \rho) + 1} \left(1 - \frac{1}{2^{p(\sigma n + \rho)}} \right) \right)^{\frac{1}{p}} \quad \text{for } n = 0, 1, 2, \dots \tag{2.13}$$

Proof Using Lemma 2, the properties of modulus, and the well-known Hölder’s inequality, respectively,

$$\begin{aligned} & \frac{2\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]}{E(\Delta)} |\mathcal{M}(\mathcal{J}_E)| \\ &= \left| \int_0^1 \{ (1-t)^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] \right. \\ & \quad \left. - t^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] \} f'(tE(\zeta) + (1-t)E(\eta)) dt \right| \\ &= \left| \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \right. \\ & \quad \left. \times \int_0^1 [(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}] f'(tE(\zeta) + (1-t)E(\eta)) dt \right| \\ &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \\ & \quad \times \int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}| |f'(tE(\zeta) + (1-t)E(\eta))| dt \\ &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \\ & \quad \times \left(\int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tE(\zeta) + (1-t)E(\eta))|^q dt \right)^{\frac{1}{q}} \\ &= \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n (I_5)^{\frac{1}{p}} (I_6)^{\frac{1}{q}}. \end{aligned} \tag{2.14}$$

Solving the first integral from the right side of inequality (2.14) and using Lemma 1, we have

$$\begin{aligned} I_5 &= \int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}|^p dt \\ &= \int_0^{\frac{1}{2}} [(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}]^p dt + \int_{\frac{1}{2}}^1 [t^{\sigma n + \rho} - (1-t)^{\sigma n + \rho}]^p dt \\ &= \int_0^{\frac{1}{2}} [(1-t)^{p(\sigma n + \rho)} - t^{p(\sigma n + \rho)}] dt + \int_{\frac{1}{2}}^1 [t^{p(\sigma n + \rho)} - (1-t)^{p(\sigma n + \rho)}] dt \\ &= \frac{2}{p(\sigma n + \rho) + 1} \left(1 - \frac{1}{2^{p(\sigma n + \rho)}} \right). \end{aligned} \tag{2.15}$$

Solving the second integral from the right side of inequality (2.14) by using the fact that $|f'|^q$, for any $q > 1$, is E -convex, therefore we have

$$\begin{aligned} I_6 &= \int_0^1 |f'(tE(\zeta) + (1-t)E(\eta))|^q dt \\ &\leq \int_0^1 [t|f'(E(\zeta))|^q + (1-t)|f'(E(\eta))|^q] dt \\ &= \frac{|f'(E(\zeta))|^q + |f'(E(\eta))|^q}{2}. \end{aligned}$$

On submitting the values of integrals I_5 and I_6 on the right side of inequality (2.14), we have

$$\begin{aligned} |\mathcal{M}(\mathcal{J}_E)| &\leq \frac{E(\Delta)}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^\alpha \\ &\quad \times \left(\frac{2}{p(\sigma n + \rho) + 1} \left(1 - \frac{1}{2^{p(\sigma n + \rho)}} \right) \right)^{\frac{1}{p}} \left(\frac{|f'(E(\zeta))|^q + |f'(E(\eta))|^q}{2} \right)^{\frac{1}{q}} \\ &= \frac{E(\Delta)}{2} \frac{\mathcal{F}_{\sigma,\rho+1}^{\alpha_2}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f'(E(\zeta))|^q + |f'(E(\eta))|^q}{2} \right]^{\frac{1}{q}}, \end{aligned}$$

where α_2 is as defined in (2.13). Hence it is proved. □

Remark 5 If in Theorem 4, the function E is chosen to be an identity function, then the following inequality holds for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{\Delta}{2} \frac{\mathcal{F}_{\sigma,\rho+1}^{\alpha_2}[\omega(\Delta)^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(\Delta)^\sigma]} \left[\frac{|f'(\zeta)|^q + |f'(\eta)|^q}{2} \right]^{\frac{1}{q}},$$

which was given in [33].

Remark 6 If in Theorem 4, the function E is chosen to be an identity function, $\alpha(0) = 1$, $\rho = \lambda$ and $\omega = 0$, then the following inequality holds:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{\Delta}{2} \left(\frac{2}{p\lambda + 1} \left(1 - \frac{1}{2^{p\lambda}} \right) \right)^{\frac{1}{p}} \left[\frac{|f'(\zeta)|^q + |f'(\eta)|^q}{2} \right]^{\frac{1}{q}},$$

which was given in [33].

Theorem 5 Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I° and $f' \in L([E(\zeta), E(\eta)])$ for $E(\zeta), E(\eta) \in I$. If $|f'|^q, q \geq 1$, is an E -convex function on $[\zeta, \eta]$, then the following inequality holds for generalized fractional integral operators:

$$|\mathcal{M}(\mathcal{J}_E)| \leq \frac{E(\Delta)}{2} \frac{\mathcal{F}_{\sigma,\rho+1}^{\alpha_3}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f'(E(\zeta))|^q + |f'(E(\eta))|^q}{2} \right]^{\frac{1}{q}}$$

for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$, where

$$\alpha_3(n) = \alpha(n) \frac{2}{\sigma n + \rho + 1} \left(1 - \frac{1}{2^{\sigma n + \rho}}\right) \quad \text{for } n = 0, 1, 2, \dots \tag{2.16}$$

Proof Using Lemma 2, the well-known power mean inequality, and the E -convexity of $|f'|^q$, respectively, we have

$$\begin{aligned} & \frac{2\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]}{E(\Delta)} |\mathcal{M}(\mathcal{J}_E)| \\ &= \left| \int_0^1 \{(1-t)^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] \right. \\ & \quad \left. - t^\rho \mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma t^\sigma]\} f'(tE(\zeta) + (1-t)E(\eta)) dt \right| \\ &= \left| \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \right. \\ & \quad \left. \times \int_0^1 [(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}] f'(tE(\zeta) + (1-t)E(\eta)) dt \right| \\ &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \\ & \quad \times \int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}| |f'(tE(\zeta) + (1-t)E(\eta))| dt \\ &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \left(\int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}| dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}| |f'(tE(\zeta) + (1-t)E(\eta))|^q dt \right)^{\frac{1}{q}} \\ &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \left[\int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}| dt \right]^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 |(1-t)^{\sigma n + \rho} - t^{\sigma n + \rho}| [t|f'(E(\zeta))|^q + (1-t)|f'(E(\eta))|^q] dt \right)^{\frac{1}{q}} \\ &\leq \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \left[\frac{2}{\sigma n + \rho + 1} \left(1 - \frac{1}{2^{\sigma n + \rho}}\right) \right]^{1-\frac{1}{q}} \\ & \quad \times \left[\frac{1}{\sigma n + \rho + 1} \left(1 - \frac{1}{2^{\sigma n + \rho}}\right) \{ |f'(E(\zeta))|^q + |f'(E(\eta))|^q \} \right]^{\frac{1}{q}} \\ &= \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 1)} [\omega(E(\Delta))^\sigma]^n \\ & \quad \times \frac{2}{\sigma n + \rho + 1} \left(1 - \frac{1}{2^{\sigma n + \rho}}\right) \left[\frac{|f'(\zeta)|^q + |f'(\eta)|^q}{2} \right]^{\frac{1}{q}} \tag{2.17} \end{aligned}$$

and on rearranging we obtain

$$|\mathcal{M}(\mathcal{J}_E)| = \frac{E(\Delta)}{2} \frac{\mathcal{F}_{\sigma,\rho+1}^{\alpha_3}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f'(\zeta)|^q + |f'(\eta)|^q}{2} \right]^{\frac{1}{q}},$$

where α_3 is as defined in (2.16).

Hence it is proved. □

Remark 7 If in Theorem 5, the function E is chosen to be an identity function, then the following inequality holds for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{\Delta}{2} \frac{\mathcal{F}_{\sigma,\rho+1}^{\alpha_3}[\omega(\Delta)^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(\Delta)^\sigma]} \left[\frac{|f'(\zeta)|^q + |f'(\eta)|^q}{2} \right]^{\frac{1}{q}},$$

which was given in [33].

Remark 8 If in Theorem 5, the function E is chosen to be an identity function, $\alpha(0) = 1$, $\rho = \lambda$ and $\omega = 0$, then the following inequality holds:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{\Delta}{\lambda + 1} \left(1 - \frac{1}{2^\lambda} \right) \left[\frac{|f'(\zeta)|^q + |f'(\eta)|^q}{2} \right]^{\frac{1}{q}},$$

which was given in [33].

Lemma 3 Let $E : J \subset \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function on I° . If $f'' \in L([E(\zeta), E(\eta)])$ for $E(\zeta), E(\eta) \in I$, then the following identity holds for generalized fractional operators:

$$\begin{aligned} \mathcal{M}(\mathcal{J}_E) &= \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \\ &\quad \times \int_0^1 \{ \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma] - (1-t)^{\rho+1} \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] \\ &\quad - t^{\rho+1} \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] \} f''(tE(\zeta) + (1-t)E(\eta)) dt \end{aligned}$$

for all $\sigma, \rho > 0$ and $\omega \geq 0$.

Proof Solving the following integral by simple integration

$$\begin{aligned} I_7 &= \int_0^1 \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma] f''(tE(\zeta) + (1-t)E(\eta)) dt \\ &= \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma] \int_0^1 f''(tE(\zeta) + (1-t)E(\eta)) dt \\ &= \frac{1}{E(\Delta)} \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma] [f'(E(\eta)) - f'(E(\zeta))]. \end{aligned}$$

Solving the next integral by applying integration by parts twice, we have

$$I_8 = \int_0^1 (1-t)^{\rho+1} \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] f''(tE(\zeta) + (1-t)E(\eta)) dt,$$

$$\begin{aligned}
 I_8 &= \frac{\mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma]}{E(\Delta)} f'(E(\eta)) - \frac{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]}{(E(\Delta))^2} f(E(\eta)) \\
 &\quad + \frac{1}{(E(\Delta))^2} \int_0^1 (1-t)^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha[\omega(E(\Delta))^\sigma (1-t)^\sigma] f(tE(\zeta) + (1-t)E(\eta)) dt \\
 &= \frac{\mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma]}{E(\Delta)} f'(E(\eta)) - \frac{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]}{(E(\Delta))^2} f(E(\eta)) \\
 &\quad + \frac{1}{(E(\Delta))^{\rho+2}} \int_{E(\zeta)}^{E(\eta)} (x-E(\zeta))^{\rho-1} \mathcal{F}_{\sigma,\rho}^\alpha[\omega(E(x-E(\zeta)))^{\rho-1}] f(x) dx \\
 &= \frac{\mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma]}{E(\Delta)} f'(E(\eta)) - \frac{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]}{(E(\Delta))^2} f(E(\eta)) \\
 &\quad + \frac{1}{(E(\Delta))^{\rho+2}} \mathcal{J}_{\sigma,\rho,\eta^+;\omega}^\alpha f(E(\zeta)),
 \end{aligned}$$

and similarly

$$\begin{aligned}
 I_9 &= \int_0^1 t^{\rho+1} \mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma t^\sigma] f''(tE(\zeta) + (1-t)E(\eta)) dt \\
 &= -\frac{\mathcal{F}_{\sigma,\rho+2}^\alpha[\omega(E(\Delta))^\sigma]}{E(\Delta)} f'(E(\zeta)) - \frac{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]}{(E(\Delta))^2} f(E(\zeta)) \\
 &\quad + \frac{1}{(E(\Delta))^{\rho+2}} \mathcal{J}_{\sigma,\rho,E(\zeta)^+;\omega}^\alpha f(E(\eta)),
 \end{aligned}$$

on subtracting I_8 and I_9 from I_7 , then multiplying by $\frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]}$, we obtain

$$\frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} [I_7 - I_8 - I_9] = \mathcal{M}(\mathcal{J}_E)$$

and on submitting the values of I_7, I_8 and I_9 we obtain the required result. □

Theorem 6 *Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I° and $f'' \in L([E(\zeta), E(\eta)])$ for $E(\zeta), E(\eta) \in I$. If $|f''|$ is an E -convex function on $[\zeta, \eta]$, then the following inequality holds for generalized fractional integral operators:*

$$|\mathcal{M}(\mathcal{J}_E)| \leq \frac{(E(\Delta))^2}{2} \frac{\mathcal{F}_{\sigma,\rho+3}^{\alpha_4}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f''(E(\zeta))| + |f''(E(\eta))|}{2} \right],$$

for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$, where

$$\alpha_4(n) = \alpha(n)(\sigma n + \rho) \quad \text{for } n = 0, 1, 2, \dots \tag{2.18}$$

Proof Using Lemma 3, the properties of modulus, and the E -convexity of $|f''|$, respectively,

$$\begin{aligned}
 &|\mathcal{M}(\mathcal{J}_E)| \\
 &\leq \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n
 \end{aligned}$$

$$\begin{aligned}
 & \times \int_0^1 [1 - (1-t)^{\sigma n + \rho + 1} - t^{\sigma n + \rho + 1}] |f''(tE(\zeta) + (1-t)E(\eta))| dt \\
 \leq & \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma, \rho + 1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \\
 & \times \int_0^1 [1 - (1-t)^{\sigma n + \rho + 1} - t^{\sigma n + \rho + 1}] [t|f''(E(\zeta))| + (1-t)|f''(tE(\eta))|] dt \\
 \leq & \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma, \rho + 1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \\
 & \times \left[\frac{\sigma n + \rho}{2(\sigma n + \rho + 2)} \{|f''(E(\zeta))| + |f''(E(\eta))|\} \right] \\
 \leq & \frac{(E(\Delta))^2}{2} \frac{\mathcal{F}_{\sigma, \rho + 3}^{\alpha_4}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma, \rho + 1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f''(E(\zeta))| + |f''(E(\eta))|}{2} \right],
 \end{aligned}$$

where $\alpha_4(n)$ is as defined in (2.18).

Hence it is proved. □

Remark 9 If in Theorem 6, the function E is chosen to be an identity function, then the following inequality holds for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{(\Delta)^2}{2} \frac{\mathcal{F}_{\sigma, \rho + 3}^{\alpha_4}[\omega(\Delta)^\sigma]}{\mathcal{F}_{\sigma, \rho + 1}^\alpha[\omega(\Delta)^\sigma]} \left[\frac{|f''(E(\zeta))| + |f''(E(\eta))|}{2} \right],$$

which was given in [33].

Remark 10 If in Theorem 6, the function E is chosen to be an identity function, $\alpha(0) = 1$, $\rho = \lambda$ and $\omega = 0$, then the following inequality holds:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{(\Delta)^2}{2} \frac{\lambda}{(\lambda + 1)(\lambda + 2)} \left[\frac{|f''(E(\zeta))| + |f''(E(\eta))|}{2} \right],$$

which was given in [33].

Theorem 7 Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I° and $f'' \in L([E(\zeta), E(\eta)])$ for $E(\zeta), E(\eta) \in I$. If $|f''|^q, q > 1$, is an E -convex function on $[\zeta, \eta]$, then the following inequality holds for generalized fractional integral operators:

$$|\mathcal{M}(\mathcal{J}_E)| \leq \frac{(E(\Delta))^2}{2} \frac{\mathcal{F}_{\sigma, \rho + 2}^{\alpha_5}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma, \rho + 1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}},$$

for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$, where

$$\alpha_5(n) = \alpha(n) \left(1 - \frac{2}{p(\sigma n + \rho + 1) + 1} \right)^{\frac{1}{p}} \quad \text{for } n = 0, 1, 2, 3, \dots \tag{2.19}$$

Proof Using Lemma 3 and applying Hölder’s inequality, respectively,

$$\begin{aligned}
 & |\mathcal{M}(\mathcal{J}_E)| \\
 & \leq \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \\
 & \quad \times \left[\int_0^1 |1 - (1-t)^{\sigma n + \rho + 1} - t^{\sigma n + \rho + 1}|^p dt \right]^{\frac{1}{p}} \left[\int_0^1 |f''(tE(\zeta) + (1-t)E(\eta))|^q dt \right]^{\frac{1}{q}} \\
 & \leq \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \\
 & \quad \times \left[\int_0^1 [1 - (1-t)^{p(\sigma n + \rho + 1)} - t^{p(\sigma n + \rho + 1)}] dt \right]^{\frac{1}{p}} \\
 & \quad \times \left[\int_0^1 [t|f''(E(\zeta))|^q + (1-t)|f''(E(\eta))|^q] dt \right]^{\frac{1}{q}} \\
 & \leq \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \\
 & \quad \times \left[1 - \frac{2}{p(\sigma n + \rho + 1) + 1} \right]^{\frac{1}{p}} \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}} \\
 & = \frac{(E(\Delta))^2}{2} \frac{\mathcal{F}_{\sigma,\rho+2}^{\alpha_5}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}},
 \end{aligned}$$

where α_5 is as defined in (2.19). Hence it is proved. □

Remark 11 If in Theorem 7, the function E is chosen to be an identity function, then the following inequality holds for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$:

$$|\mathcal{M}(\mathcal{J})| \leq \frac{(\Delta)^2}{2} \frac{\mathcal{F}_{\sigma,\rho+2}^{\alpha_5}[\omega(\Delta)^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(\Delta)^\sigma]} \left[\frac{|f''(\zeta)|^q + |f''(\eta)|^q}{2} \right]^{\frac{1}{q}},$$

which was given in [33].

Remark 12 If in Theorem 7, the function E is chosen to be an identity function, $\alpha(0) = 1$, $\rho = \lambda$ and $\omega = 0$, then the following inequality holds:

$$|\mathcal{M}(J)| \leq \frac{(\Delta)^2}{2} \frac{1}{\lambda + 1} \left(1 - \frac{2}{p(\lambda + 1) + 1} \right)^{\frac{1}{p}} \left[\frac{|f''(\zeta)|^q + |f''(\eta)|^q}{2} \right]^{\frac{1}{q}},$$

which was given in [33].

Theorem 8 Let $E : J \rightarrow \mathbb{R}$ be a continuous increasing function and $\zeta, \eta \in J$ with $\zeta < \eta$. Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I^o . If $|f''|^q, q \geq 1$, is an E -convex function on $[\zeta, \eta]$, then the following inequality holds for generalized fractional integral operators:

$$|\mathcal{M}(\mathcal{J}_E)| \leq \frac{(E(\Delta))^2}{2} \frac{\mathcal{F}_{\sigma,\rho+2}^{\alpha_6}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma,\rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}}$$

for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$, where p, q are conjugate indices and

$$\alpha_6(n) = \alpha(n) \left(\frac{\sigma n + \rho}{\sigma n + \rho + 2} \right) \quad \text{for } n = 0, 1, 2, \dots \tag{2.20}$$

Proof Using Lemma 3 and applying the well-known power mean inequality

$$\begin{aligned} & |\mathcal{M}(\mathcal{J}_E)| \\ & \leq \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \\ & \quad \times \left(\int_0^1 [1 - (1-t)^{\sigma n + \rho + 1} - t^{\sigma n + \rho + 1}] dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 [1 - (1-t)^{\sigma n + \rho + 1} - t^{\sigma n + \rho + 1}] |f''(tE(\zeta) + (1-t)E(\eta))|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \left[1 - \frac{2}{\sigma n + \rho + 2} \right]^{1-\frac{1}{q}} \\ & \quad \times \left[\frac{\sigma n + \rho}{2(\sigma n + \rho + 2)} \{ |f''(E(\zeta))|^q + |f''(E(\eta))|^q \} \right]^{\frac{1}{q}} \\ & \leq \frac{(E(\Delta))^2}{2\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \sum_{n=0}^\infty \frac{\alpha(n)}{\Gamma(\sigma n + \rho + 2)} [\omega(E(\Delta))^\sigma]^n \\ & \quad \times \left(\frac{\sigma n + \rho}{\sigma n + \rho + 2} \right) \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}} \\ & = \frac{(E(\Delta))^2}{2} \frac{\mathcal{F}_{\sigma, \rho+2}^{\alpha_6}[\omega(E(\Delta))^\sigma]}{\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(E(\Delta))^\sigma]} \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}}, \end{aligned}$$

where α_6 is as defined in (2.20). Hence it is proved. □

Corollary 1 *If in Theorem 8, the function E is chosen to be an identity function, then the following inequality holds for all $\sigma, \rho \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$:*

$$|\mathcal{M}(\mathcal{J})| \leq \frac{(\Delta)^2}{2} \frac{\mathcal{F}_{\sigma, \rho+2}^{\alpha_6}[\omega(\Delta)^\sigma]}{\mathcal{F}_{\sigma, \rho+1}^\alpha[\omega(\Delta)^\sigma]} \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}}.$$

Corollary 2 *If in Theorem 8, the function E is chosen to be an identity function, $\alpha(0) = 1$, $\rho = \lambda$ and $\omega = 0$, then the following inequality holds:*

$$|\mathcal{M}(\mathcal{J})| \leq \frac{(\Delta)^2}{2} \left(\frac{\lambda}{\lambda + 2} \right) \left[\frac{|f''(E(\zeta))|^q + |f''(E(\eta))|^q}{2} \right]^{\frac{1}{q}}.$$

Acknowledgements

The authors are grateful to the Mirpur University of Science & Technology for allowing study leave and utilization of educational resources to undertake this work.

Funding

There is no special funding from any source for this work.

Availability of data and materials

This work does not involve any supplementary data or material, however, all related calculations can be supplied on demand.

Declarations

Competing interests

The publication of this work is approved by all authors, and the authors declare that they have no competing interests.

Authors' contributions

AL: conceptualization, methodology, investigation, writing the original draft, writing, reviewing and editing. RH: problem statement, investigation, supervision, provision of study resources, reviewing and editing. The authors read and approved the final manuscript.

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Received: 3 May 2021 Accepted: 21 February 2022 Published online: 25 March 2022

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