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Some remarks on α-admissibility in S-metric spaces

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Abstract

The concept of α-admissible mapping introduced by Samet et al. (Nonlinear Anal. 75:2154–2165, 2012) has various generalizations. In this paper, we introduce the concept of α_s-admissible mapping and its various forms by generalizing the concept of α-admissible mapping in the setting of S-metric spaces. Further, we also introduce generalized rational α_s-Geraghty contraction type mappings and study the existence of fixed point theorems in S-metric spaces. Examples are also given to verify the main results.

MSC: 47H10; 54H25

Keywords: α-admissible; α_s-admissible; Generalized rational α_s-Geraghty contraction type mapping; S-metric space

1 Introduction and preliminaries

The Banach contraction principle is one of the most interesting topics for many researchers because of its applications in various fields, simplicity, and easiness. They attempted to generalize the Banach contraction principle in different directions. Samet et al. [1] made an attempt by introducing the concept of α-admissible mappings and by further introducing the concept of α-ψ-contractive mappings in metric spaces. The results of Samet et al. [1] show that Banach's fixed point theorem and various other results are direct consequences of their results. On the other hand, as one result of the generalization of a metric space, Sedghi et al. [2] introduced the concept of S-metric space.

Definition 1.1 ([2]) Let X be a nonempty set. An S-metric on X is a function $S : X \times X \times X \to [0,+\infty)$ that satisfies the following conditions for each $x, y, z, a \in X$:

(1) $S(x,y,z) \geq 0$,
(2) $S(x,y,z) = 0$ if and only if $x = y = z$,
(3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called S-metric space.

Definition 1.2 ([2]) In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.

Definition 1.3 ([2]) Let (X,S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to +\infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$, and we denote this by $\lim_{n\to+\infty} x_n = x$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.
(3) The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

In this paper we introduce various concepts of α-admissible mappings in the context of S-metric spaces and name them α_s-admissible. Further, we prove various fixed point theorems based on different contractive types due to α_s-admissible mappings.

Here firstly, we recall the definition of α-admissible mappings and their generalizations in metric space, G-metric space, S-metric space, and S_b-metric space.

Definition 1.4 ([1]) Let S be a self-mapping on a metric space (X, d), and let $\alpha : X \times X \to [0, +\infty)$ be a function. It is said that S is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ imply $\alpha(Sx, Sy) \geq 1$.

Example 1 Consider $X = [0, +\infty)$, and define $S : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by $Sx = 5x$ for all $x, y \in X$ and

$$\alpha(x, y) = \begin{cases} e^{\frac{y}{x}} & \text{if } x \geq y, x \neq 0, \\ 0 & \text{if } x < y. \end{cases}$$

Then S is α-admissible.

Definition 1.5 ([3]) Let $S, T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. It is said that the pair (S, T) is α-admissible if $x, y \in X$ such that $\alpha(x, y) \geq 1$, then we have $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$.

Definition 1.6 ([4]) Let $S : X \to X$ and $\alpha : X \times X \to (-\infty, +\infty)$. It is said that S is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Sy) \geq 1, x, y \in X$,
(T2) $\alpha(x, z) \geq 1$, $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1, x, y, z \in X$.

Definition 1.7 ([3]) Let $S, T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. It is said that a pair (S, T) is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1, x, y \in X$,
(T2) $\alpha(x, z) \geq 1$, $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1, x, y, z \in X$.

Definition 1.8 ([5]) Let S be a self-mapping on a metric space (X, d), and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. It is said that T is an α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ imply $\alpha(Sx, Sy) \geq \eta(Sx, Sy)$.

It can be noted that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 1.4. Also, if we take $\alpha(x, y) = 1$, then S is said to be an η-subadmissible mapping.

Lemma 1.9 ([6]) *Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Lemma 1.10 ([7]) *Let $S, T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define sequences $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Alghamdi and Karapinar [8] generalized the concept of α-admissible mappings in the context of G-metric space and called them β-admissible. The definition of β-admissible given by Alghamdi and Karapinar is as follows.

Definition 1.11 ([8]) Let $T : X \to X$ and $\beta : X \times X \times X \to [0, +\infty)$, then T is said to be β-admissible if for all $x, y, z \in X$

$$\beta(x, y, z) \geq 1 \quad \text{implies } \beta(Tx, Ty, Tz) \geq 1.$$

They gave a suitable example for β-admissible mappings. Further, they also generalized the α-ψ contractive mappings by introducing generalized G-β-ψ contractive mappings of type I and II.

Hussain et al. [9] further generalized the concept of α-admissible mappings in G-metric space by introducing rectangular G-α-admissible. They also extended rectangular G-α-admissible for two mappings.

Ansari et al. [10] also studied α-admissible mappings in G-metric space by introducing a G-η-subadmissible mapping and an α-dominating map. They also introduced an η-subdominating map, α-regular in the context of G-metric space, partially weakly G-α-admissible, partially weakly G-η-subadmissible mappings, etc.

The concept of α-admissible mappings was extended to S-metric space by Zhou et al. [11] and was called γ-admissible. They defined it as follows.

Definition 1.12 ([11]) Let $T : X \to X$ and $\gamma : X^3 \to [0, +\infty)$. Then T is said to be γ-admissible if for all $x, y, z \in X$

$$\gamma(x, y, z) \geq 1 \quad \text{implies } \gamma(Tx, Ty, Tz) \geq 1.$$

They also extended γ-admissibility for two mappings. Further, they also introduced concepts of various contractive mappings viz. type A, type B, type C, type D, and type E.

Bulbul et al. [12] also introduced the concept of generalized S-β-ψ contractive type mappings on the line of generalized G-β-γ contractive type mappings. Nabil et al. [13] also introduced the concept of α-admissible mappings in S_b-metric space.

From these, what we observed is that β-admissible was for the first time used by Samet et al. [1] to represent α-admissible while dealing with coupled fixed point related problems. Phiangsungnoen et al. [14] also used the name β-admissible mapping in order to represent α-admissible for fuzzy mappings. On the other hand, β-admissible of Alghamdi and Karapinar [8] and γ-admissible of Zhou et al. [11] are all extended versions of α-admissible mappings in G-metric space and S-metric space, respectively. Thus, we can remark that α-admissible and its various forms can be extended to G-metric as well as S-metric spaces and further to G_b-metric and S_b-metric spaces. With this idea, we introduce various forms of α-admissible mappings in the context of S-metric space and present the following definitions. For more detailed information on the generalization of a metric space, one can see research papers in [11–24].

Definition 1.13 Let $(\mathbb{U}, S)$ be an S-metric space, $A : \mathbb{U} \to \mathbb{U}$, and $\alpha_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to [0, +\infty)$. Then A is called α_s-admissible if $u, v, w \in \mathbb{U}$, $\alpha_s(u, v, w) \geq 1$ imply $\alpha_s(Au, Av, Aw) \geq 1$.

Example 2 Consider $\mathbb{U} = [0, +\infty)$ and define $A : \mathbb{U} \to \mathbb{U}$ and $\alpha_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to [0, +\infty)$ by $Au = 4u$ for all $u, v, w \in \mathbb{U}$ and

$$\alpha_s(u, v, w) = \begin{cases} e^{\frac{w}{uv}} & \text{if } u \geq v \geq w, u, v \neq 0, \\ 0 & \text{if } u < v < w. \end{cases}$$

Then A is α_s-admissible.

Definition 1.14 Let $(\mathbb{U}, S)$ be an S-metric space, $A, B : \mathbb{U} \to \mathbb{U}$, and $\alpha_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to [0, +\infty)$. We say that the pair (A, B) is α_s-admissible if $u, v, w \in \mathbb{U}$ such that $\alpha_s(u, v, w) \geq 1$, then we have $\alpha_s(Au, Av, Bw)) \geq 1$ and $\alpha_s(Bu, Bv, Aw) \geq 1$.

Definition 1.15 Let $(\mathbb{U}, S)$ be an S-metric space, $A : \mathbb{U} \to \mathbb{U}$, and $\alpha_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to [0, +\infty)$. We say that A is a triangular α_s-admissible mapping if

(i) $\alpha_s(u, v, w) \geq 1$ implies $\alpha_s(Au, Av, Aw) \geq 1$, $u, v, w \in \mathbb{U}$.

(ii) $\alpha_s(u, v, t) \geq 1$ and $\alpha_s(t, t, w) \geq 1$ imply $\alpha_s(u, v, w) \geq 1$, $u, v, w, t \in \mathbb{U}$.

Definition 1.16 Let $(\mathbb{U}, S)$ be an S-metric space, $A : \mathbb{U} \to \mathbb{U}$, and let $\alpha_s, \eta_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to [0, +\infty)$ be two functions. We say that A is an α_s-admissible mapping with respect to η_s if $u, v, w \in \mathbb{U}$,

$$\alpha_s(u, v, w) \geq \eta_s(u, v, w) \quad \text{implies } \alpha_s(Au, Av, Aw) \geq \eta_s(Au, Av, Aw).$$

Note that if we take $\eta_s(u, v, w) = 1$, then this definition reduces to Definition 1.13. Also, if we take $\alpha_s(u, v, w) = 1$, then we say that A is an η_s-subadmissible mapping.

Now we state the following two lemmas in the line of Lemma 1.9 and Lemma 1.10.

Lemma 1.17 *Let $(\mathbb{U}, S)$ be an S-metric space, $A : \mathbb{U} \to \mathbb{U}$ be a triangular α_s-admissible mapping. Assume that there exists $u_0 \in \mathbb{U}$ such that $\alpha_s(u_0, u_0, Au_0) \geq 1$. Define a sequence $\{u_n\}$ by $u_{n+1} = Au_n$. Then we have*

$$\alpha_s(u_n, u_n, u_m) \geq 1 \quad \textit{for all } m, n \in \mathbb{N} \cup \{0\}.$$

Lemma 1.18 *Let $(\mathbb{U}, S)$ be an S-metric space, $A, B : \mathbb{U} \to \mathbb{U}$ be a triangular α_s-admissible mapping. Assume that there exists $u_0 \in \mathbb{U}$ such that $\alpha_s(u_0, u_0, Au_0) \geq 1$. Define sequences*

$$u_{2i+1} = Au_{2i} \quad \textit{and} \quad u_{2i+2} = Bu_{2i+1}, \quad \textit{where } i = 0, 1, 2, \ldots.$$

Then we have $\alpha_s(u_n, u_n, u_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

We denote by $\mathcal{G}$ the family of all functions $g : [0,+\infty) \to [0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $g(t_n) \to 1$ implies $t_n \to 0$. Then the following theorem of Geraghty contraction can be stated in the context of S-metric spaces.

Theorem 1.19 *Let $(\mathbb{U},S)$ be a S-metric space. Let $A : \mathbb{U} \to \mathbb{U}$ be a self-mapping. Suppose that there exists $g \in \mathcal{G}$ such that, for all $u,v,w \in \mathbb{U}$,*

$$S(Au,Av,Aw) \le g\big(S(u,v,w)\big)S(u,v,w).$$

Then A has a unique fixed point $a \in \mathbb{U}$ and $\{A^n u\}$ converges to a for each $u \in \mathbb{U}$.

2 Main results

In this section, we prove some fixed point theorems satisfying generalized rational α_s-Geraghty contraction type mappings in complete S-metric spaces. Let $(\mathbb{U},S)$ be an S-metric, and let $\alpha_s : \mathbb{U}\times\mathbb{U}\times\mathbb{U} \to [0,+\infty)$ be a function. Mappings $A,B : \mathbb{U} \to \mathbb{U}$ are called a pair of generalized rational α_s-Geraghty contraction mappings of type I if there exists $g \in \mathcal{G}$ such that, for all $u,v,w \in \mathbb{U}$,

$$\alpha_s(u,v,w)S(Au,Av,Bw) \le g\big(\nabla_1(u,v,w)\big)\nabla_1(u,v,w), \tag{2.1}$$

where

$$\begin{aligned}\nabla_1(u,v,w) = \max\bigg\{& S(u,v,w), S(Au,Av,Bw), \frac{S(u,u,Au)S(v,v,Av)}{1+S(u,v,w)+S(Au,Av,Bw)}, \\ & \frac{S(v,v,Av)S(w,w,Bw)}{1+S(u,v,w)+S(Au,Av,Bw)}, \frac{S(w,w,Bw)S(u,u,Au)}{1+S(u,v,w)+S(Au,Av,Bw)}\bigg\}.\end{aligned}$$

Mappings $A,B : \mathbb{U} \to \mathbb{U}$ are called a pair of generalized rational α_s-Geraghty contraction mappings of type-II if there exists $g \in \mathcal{G}$ such that, for all $u,v \in \mathbb{U}$,

$$\alpha_s(u,u,v)S(Au,Au,Bv) \le g\big(\nabla_2(u,u,v)\big)\nabla_2(u,u,v), \tag{2.2}$$

where

$$\begin{aligned}\nabla_2(u,u,v) = \max\bigg\{& S(u,u,v), S(Au,Au,Bv), \\ & \frac{S(u,u,Au)S(u,u,Au)}{1+S(u,u,v)+S(Au,Au,Bv)}, \frac{S(u,u,Au)S(v,v,Bv)}{1+S(u,u,v)+S(Au,Au,Bv)}\bigg\}.\end{aligned}$$

Let $A = B$, then B is called a generalized rational α_s-Geraghty contraction mapping of type-I if there exists $g \in \mathcal{G}$ such that, for all $u,v,w \in \mathbb{U}$,

$$\alpha_s(u,v,w)S(Bu,Bv,Bw) \le g\big(\nabla_1(u,v,w)\big)\nabla_1(u,v,w), \tag{2.3}$$

where

$$\nabla_1(u,v,w) = \max\bigg\{S(u,v,w), S(Bu,Bv,Bw), \frac{S(u,u,Bu)S(v,v,Bv)}{1+S(u,v,w)+S(Bu,Bv,Bw)},$$

$$\left.\frac{S(v,v,Bv)S(w,w,Bw)}{1+S(u,v,w)+S(Bu,Bv,Bw)}, \frac{S(w,w,Bw)S(u,u,Bu)}{1+S(u,v,w)+S(Bu,Bv,Bw)}\right\}.$$

$B : \mathbb{U} \to \mathbb{U}$ is called a generalized rational α_s-Geraghty contraction mapping of type-II if there exists $g \in \mathcal{G}$ such that, for all $u, v \in \mathbb{U}$,

$$\alpha_s(u,u,v)S(Bu,Bu,Bv) \le g\big(\nabla_2(u,u,v)\big)\nabla_2(u,u,v), \tag{2.4}$$

where

$$\begin{aligned}\nabla_2(u,u,v) = \max\bigg\{&S(u,u,v), S(Bu,Bu,Bv),\\ &\frac{S(u,u,Bu)S(u,u,Bu)}{1+S(u,u,v)+S(Bu,Bu,Bv)}, \frac{S(u,u,Bu)S(v,v,Bv)}{1+S(u,u,v)+S(Bu,Bu,Bv)}\bigg\}.\end{aligned}$$

Theorem 2.1 *Let* $(\mathbb{U}, S)$ *be a complete S-metric space,* $\alpha_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to [0,+\infty)$ *be a function. Let* $A, B : \mathbb{U} \to \mathbb{U}$ *be two mappings, then suppose that the following hold*:

(i) (A,B) *is a pair of generalized rational* α_s*-Geraghty contraction mappings of type I,*
(ii) (A,B) *is triangular* α_s*-admissible,*
(iii) *There exists* $u_0 \in \mathbb{U}$ *such that* $\alpha_s(u_0,u_0,Au_0) \ge 1$,
(iv) *A and B are continuous.*

Then (A,B) *has a common fixed point.*

Proof Let $u_1 \in \mathbb{U}$ be such that $u_1 = Au_0$ and $u_2 = Bu_1$. Continuing this process, we construct a sequence u_n of points in $\mathbb{U}$ such that

$$u_{2i+1} = Au_{2i} \quad \text{and} \quad u_{2i+2} = Bu_{2i+1}, \tag{2.5}$$

where $i = 0, 1, 2, 3, \ldots$.

By the assumption $\alpha_s(u_0,u_0,u_1) \ge 1$ and the pair (A,B) is α_s-admissible, by Lemma 1.18, we have

$$\alpha_s(u_n,u_n,u_{n+1}) \ge 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.6}$$

Then

$$\begin{aligned}S(u_{2i+1},u_{2i+1},u_{2i+2}) &= S(Au_{2i},Au_{2i},Bu_{2i+1})\\ &\le \alpha_s(u_{2i},u_{2i},u_{2i+1})S(Au_{2i},Au_{2i},Bu_{2i+1})\\ &\le g\big(\nabla_1(u_{2i},u_{2i},u_{2i+1})\big)\nabla_1(u_{2i},u_{2i},u_{2i+1})\end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned}\nabla_1(u_{2i},u_{2i},u_{2i+1}) = \max\bigg\{&S(u_{2i},u_{2i},u_{2i+1}), S(Au_{2i},Au_{2i},Bu_{2i+1}),\\ &\frac{S(u_{2i},u_{2i},Au_{2i})S(u_{2i},u_{2i},Au_{2i})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(Au_{2i},Au_{2i},Bu_{2i+1})},\\ &\frac{S(u_{2i},u_{2i},Au_{2i})S(u_{2i+},u_{2i+1},Bu_{2i+1})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(Au_{2i},Au_{2i},Bu_{2i+1})},\end{aligned}$$

$$\left.\frac{S(u_{2i+1},u_{2i+1},Bu_{2i+1})S(u_{2i},u_{2i},Au_{2i})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(Au_{2i},Au_{2i},Bu_{2i+1})}\right\}$$
$$\begin{aligned}&= \max\biggl\{S(u_{2i},u_{2i},u_{2i+1}),S(u_{2i+1},u_{2i+1},u_{2i+2}),\\ &\quad\frac{S(u_{2i},u_{2i},u_{2i+1})S(u_{2i},u_{2i},u_{2i+1})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(u_{2i+1},u_{2i+1},u_{2i+2})},\\ &\quad\frac{S(u_{2i},u_{2i},u_{2i+1})S(u_{2i+1},u_{2i+1},u_{2i+2})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(u_{2i+1},u_{2i+1},u_{2i+2})},\\ &\quad\frac{S(u_{2i+1},u_{2i+1},u_{2i+2})S(u_{2i},u_{2i},u_{2i+1})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(u_{2i+1},u_{2i+1},u_{2i+2})}\biggr\}\\ &= \max\bigl\{S(u_{2i},u_{2i},u_{2i+1}),S(u_{2i+1},u_{2i+1},u_{2i+2})\bigr\}.\end{aligned}$$

If $\max\{S(u_{2i},u_{2i},u_{2i+1}),S(u_{2i+1},u_{2i+1},u_{2i+2})\}=S(u_{2i+1},u_{2i+1},u_{2i+2})$, then

$$\begin{aligned}S(u_{2i+1},u_{2i+1},u_{2i+2}) &\le g\bigl(S(u_{2i+1},u_{2i+1},u_{2i+2})\bigr)S(u_{2i+1},u_{2i+1},u_{2i+2})\\ &< S(u_{2i+1},u_{2i+1},u_{2i+2}),\end{aligned}$$

which is a contradiction. Hence,

$$S(u_{2i+1},u_{2i+1},u_{2i+2}) < S(u_{2i},u_{2i},u_{2i+1}). \tag{2.7}$$

This implies that

$$S(u_{n+1},u_{n+1},u_{n+2}) < S(u_n,u_n,u_{n+1}) \tag{2.8}$$

for all $n\in\mathbb{N}\cup\{0\}$.

So, the sequence $\{S(u_n,u_n,u_{n+1})\}$ is nonnegative and nonincreasing. Now, we prove that $S(u_n,u_n,u_{n+1})\to 0$. It is clear that $\{S(u_n,u_n,u_{n+1})\}$ is a decreasing sequence. Therefore, there exists some positive number r such that $\lim_{n\to+\infty}S(u_n,u_n,u_{n+1})=r$.

From (2.7), we have

$$\frac{S(u_{n+1},u_{n+1},u_{n+2})}{S(u_n,u_n,u_{n+1})}\le g\bigl(S(u_n,u_n,u_{n+1})\bigr)\le 1.$$

Now, by taking limit $n\to+\infty$, we have

$$1\le g\bigl(S(u_n,u_n,u_{n+1})\bigr)\le 1,$$

that is,

$$\lim_{n\to+\infty} g\bigl(S(u_n,u_n,u_{n+1})\bigr)=1.$$

By the property of g, we have

$$\lim_{n\to+\infty} S(u_n,u_n,u_{n+1})=0. \tag{2.9}$$

Now, we show that the sequence $\{u_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{u_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and sequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ such that, for all positive integers k, we have $m_k > n_k > k$,

$$S(u_{m_k}, u_{m_k}, u_{n_k}) \geq \varepsilon, \tag{2.10}$$

and m_k is the smallest number such that (2.10) holds. From (2.10), we get

$$S(u_{m_k-1}, u_{m_k-1}, u_{n_k}) < \varepsilon. \tag{2.11}$$

Using the triangle inequality and (2.11),

$$\begin{aligned}\varepsilon &\leq S(u_{m_k}, u_{m_k}, u_{n_k})\\ &\leq 2S(u_{m_k}, u_{m_k}, u_{m_{k-1}}) + S(u_{m_{k-1}}, u_{m_{k-1}}, u_{n_k})\\ &< 2S(u_{m_k}, u_{m_k}, u_{m_{k-1}}) + \varepsilon.\end{aligned}$$

Letting $k \to +\infty$ in the above inequality and using (2.9), we obtain

$$\lim_{n\to+\infty} S(u_{m_k}, u_{m_k}, u_{n_k}) = \varepsilon. \tag{2.12}$$

Also, from the triangular inequality, we have

$$\left|S(u_{n_k+1}, u_{n_k+1}, u_{m_k}) - S(u_{m_k}, u_{m_k}, u_{n_k})\right| \leq 2S(u_{n_k}, u_{n_k}, u_{n_k+1})$$

and

$$\left|S(u_{m_k+1}, u_{m_k+1}, u_{n_k+1}) - S(u_{n_k+1}, u_{n_k+1}, u_{m_k})\right| \leq 2S(u_{m_k+1}, u_{m_k+1}, u_{m_k}).$$

Taking limit as $k \to +\infty$ and using (2.9) and (2.12), we obtain

$$\lim_{k\to+\infty} S(u_{n_k+1}, u_{n_k+1}, u_{m_k}) = \varepsilon$$

and

$$\lim_{k\to+\infty} S(u_{m_k+1}, u_{m_k+1}, u_{n_k+1}) = \varepsilon. \tag{2.13}$$

Using (2.13), we have that $\lim_{k\to+\infty} S(u_{n_k}, u_{n_k}, u_{m_k+1}) = \varepsilon$.
By Lemma 1.18, $\alpha(u_{n_k}, u_{n_k}, u_{m_k+1}) \geq 1$, we have

$$\begin{aligned}S(u_{n_k+1}, u_{n_k+1}, u_{m_k+2}) &= S(Au_{n_k}, Au_{n_k}, Bu_{m_k+1})\\ &\leq \alpha_s(u_{n_k}, u_{n_k}, u_{m_k+1})S(Au_{n_k}, Au_{n_k}, Bu_{m_k+1})\\ &\leq g\bigl(\nabla_1(u_{n_k}, u_{n_k}, u_{m_k+1})\bigr)\nabla_1(u_{n_k}, u_{n_k}, u_{m_k+1}).\end{aligned}$$

We know that

$$\nabla_1(u_{n_k}, u_{n_k}, u_{m_k+1}) = \max\bigl\{S(u_{n_k}, u_{n_k}, u_{n_k+1}), S(u_{n_k+1}, u_{n_k+1}, u_{m_k+2})\bigr\}$$

$$= S(u_{n_k}, u_{n_k}, u_{m_k+1}).$$

Finally, we conclude that

$$\frac{S(u_{n_k+1}, u_{n_k+1}, u_{m_k+2})}{\nabla_1(u_{n_k}, u_{n_k}, u_{m_k+1})} \leq g\big(\nabla_1(u_{n_k}, u_{n_k}, u_{m_k+1})\big).$$

By using (2.9), taking limit as $k \to +\infty$ in the above inequality, we obtain

$$\lim_{k\to+\infty} g\big(S(u_{n_k}, u_{n_k}, u_{m_k+1})\big) = 1. \tag{2.14}$$

So, $\lim_{k\to+\infty} S(u_{n_k}, u_{n_k}, u_{m_k+1}) = 0 < \varepsilon$, which is a contradiction. Hence $\{u_n\}$ is a Cauchy sequence. Since $\mathbb{U}$ is complete, there exists $a \in \mathbb{U}$ such that $u_n \to a$ implies that $u_{2i+1} \to a$ and $u_{2i+2} \to a$. As A and B are continuous, so we get $Bu_{2i+1} \to Ba$ and $Au_{2i+2} \to Aa$. Thus $a = Aa$. Similarly, $a = Ba$, we have $Aa = Ba = a$. Then (A, B) has a common fixed point. □

In the following theorem, we dropped continuity.

Theorem 2.2 *Let* $(\mathbb{U}, S)$ *be a complete S-metric space,* $\alpha_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to \mathbb{R}$ *be a function. Let* $A, B : \mathbb{U} \to \mathbb{U}$ *be two mappings, then suppose that the following hold*:

- (i) (A, B) *is a pair of generalized rational* α_s*-Geraghty contraction mappings of type-I,*
- (ii) (A, B) *is triangular* α_s*-admissible,*
- (iii) *There exists* $u_0 \in \mathbb{U}$ *such that* $\alpha_s(u_0, u_0, Au_0) \geq 1$,
- (iv) *If* $\{u_n\}$ *is a sequence in* $\mathbb{U}$ *such that* $\alpha_s(u_n, u_n, u_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $u_n \to a \in \mathbb{U}$ *as* $n \to +\infty$, *then there exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that* $\alpha_s(u_{n_k}, u_{n_k}, a) \geq 1$ *for all* k.

Then (A, B) *has a common fixed point.*

Proof Follows similar lines of Theorem 2.1. Define a sequence $u_{2i+1} = Au_{2i}$ and $u_{2i+2} = Bu_{2i+1}$, where $i = 0, 1, 2, \dots$ converges to $a \in \mathbb{U}$. By the hypothesis of (iv), there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha_s(u_{2n_k}, u_{2n_k}, a) \geq 1$ for all k. Now, by using (2.1) for all k, we have

$$\begin{aligned}
S(u_{2n_k+1}, u_{2n_k+1}, Ba) &= S(Au_{2n_k}, Au_{2n_k}, Ba) \\
&\leq \alpha_s(u_{2n_k}, u_{2n_k}, a) S(Au_{2n_k}, Au_{2n_k}, Ba) \\
&\leq g\big(\nabla_1(u_{2n_k}, u_{2n_k}, a)\big) \nabla_1(u_{2n_k}, u_{2n_k}, a)
\end{aligned}$$

so that

$$S(u_{2n_k+1}, u_{2n_k+1}, Ba) \leq g\big(\nabla_1(u_{2n_k}, u_{2n_k}, a)\big) \nabla_1(u_{2n_k}, u_{2n_k}, a). \tag{2.15}$$

On the other hand, we obtain

$$\begin{aligned}
\nabla_1(u_{2n_k}, u_{2n_k}, a) = \max\bigg\{ & S(u_{2n_k}, u_{2n_k}, a), S(Au_{2n_k}, Au_{2n_k}, Ba), \\
& \frac{S(u_{2n_k}, u_{2n_k}, Au_{2n_k}) S(u_{2n_k}, u_{2n_k}, Au_{2n_k})}{1 + S(u_{2n_k}, u_{2n_k}, a) + S(Au_{2n_k}, Au_{2n_k}, Ba)},
\end{aligned}$$

$$\begin{aligned}
&\frac{S(u_{2n_k},u_{2n_k},Au_{2n_k})S(a,a,Ba)}{1+S(u_{2n_k},u_{2n_k},a)+S(Au_{2n_k},Au_{2n_k},Ba)},\\
&\left.\frac{S(a,a,Ba)S(u_{2n_k},u_{2n_k},Au_{2n_k})}{1+S(u_{2n_k},u_{2n_k},a)+S(Au_{2n_k},Au_{2n_k},Ba)}\right\}\\
={}&\max\Bigg\{S(u_{2n_k},u_{2n_k},a),S(u_{2n_k+1},u_{2n_k+1},Ba),\\
&\frac{S(u_{2n_k},u_{2n_k},u_{2n_k+1})S(u_{2n_k},u_{2n_k},u_{2n_k+1})}{1+S(u_{2n_k},u_{2n_k},a)+S(u_{2n_k+1},u_{2n_k+1},Ba)},\\
&\frac{S(u_{2n_k}u_{2n_k},u_{2n_k+1})S(a,a,Ba)}{1+S(u_{2n_k},u_{2n_k},a)+S(u_{2n_k+1},u_{2n_k+1},Ba)},\\
&\left.\frac{S(a,a,Ba)S(u_{2n_k},u_{2n_k},u_{2n_k+1})}{1+S(u_{2n_k},u_{2n_k},a)+S(u_{2n_k+1},u_{2n_k+1},Ba)}\right\}.
\end{aligned}$$

Letting $k\to+\infty$, we have

$$\begin{aligned}
\lim_{k\to+\infty}\nabla_1(u_{2n_k},u_{2n_k},a) &= \max\Bigg\{S(a,a,a),S(Aa,Aa,Ba),\frac{S(a,a,Aa)S(a,a,Aa)}{1+S(a,a,a)+S(Aa,Aa,Ba)},\\
&\qquad\left.\frac{S(a,a,Aa)S(a,a,Ba)}{1+S(a,a,a)+S(Aa,Aa,Ba)},\frac{S(a,a,Ba)S(a,a,Aa)}{1+S(a,a,a)+S(Aa,Aa,Ba)}\right\}\\
&= \max\bigl\{S(a,a,Aa),S(a,a,Ba)\bigr\}. \qquad (2.16)
\end{aligned}$$

Case I:

$$\lim_{k\to+\infty}\nabla_1(u_{2n_k},u_{2n_k},a)=S(a,a,Ba).$$

Suppose that $S(a,a,Ba)>0$. From (2.16), for large k, we have $\nabla_1(u_{2n_k},u_{2n_k},a)>0$, which implies that

$$g\bigl(\nabla_1(u_{2n_k},u_{2n_k},a)\bigr)<\nabla_1(u_{2n_k},u_{2n_k},a).$$

Then we have

$$S(u_{2n_k},u_{2n_k},Ba)<(u_{2n_k},u_{2n_k},a). \qquad (2.17)$$

Letting $k\to+\infty$ in (2.17), we claim that

$$S(a,a,Ba)<S(a,a,Ba),$$

which is a contradiction. Thus, we find that $S(a,a,Ba)=0$ implies $a=Ba$.

Case II:

$$\lim_{k\to+\infty}\nabla_1(u_{2n_k},u_{2n_k},a)=S(a,a,Aa).$$

Similarly, $a=Aa$. Thus $a=Ba=Aa$. □

3 Consequences

$$\textit{If } \nabla_1(u,v,w) = \max\left\{S(u,v,w), S(Au,Av,Aw), \frac{S(u,u,Au)S(v,v,Av)}{1+S(u,v,w)+S(Au,Av,Aw)}, \right.$$
$$\left. \frac{S(v,v,Av)S(w,w,Aw)}{1+S(u,v,w)+S(Au,Av,Aw)}, \frac{S(w,w,Aw)S(u,u,Au)}{1+S(u,v,w)+S(Au,Av,Aw)}\right\}$$

and $A = B$ in Theorem 2.1 and Theorem 2.2, we have the following corollaries.

Corollary 3.1 *Let $(\mathbb{U}, S)$ be a complete S-metric space, and let A be an α_s-admissible mapping such that the following hold*:

(i) *A is a generalized rational α_s-Geraghty contraction mapping of type-I,*
(ii) *A is triangular α_s-admissible,*
(iii) *There exists $u_0 \in \mathbb{U}$ such that $\alpha_s(u_0, u_0, Au_0) \geq 1$,*
(iv) *A is continuous.*

Then A has a fixed point $a \in \mathbb{U}$, and A is a Picard operator, that is, $\{A^n u_0\}$ converges to a.

Corollary 3.2 *Let $(\mathbb{U}, S)$ be a complete S-metric space, and let A be an α_s-admissible mapping such that the following hold*:

(i) *A is a generalized rational α_s-Geraghty contraction mapping of type-I,*
(ii) *A is triangular α_s-admissible,*
(iii) *There exists $u_0 \in \mathbb{U}$ such that $\alpha_s(u_0, u_0, Au_0) \geq 1$,*
(iv) *If $\{u_n\}$ is a sequence in $\mathbb{U}$ such that $\alpha_s(u_n, u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to a \in \mathbb{U}$ as $n \to +\infty$, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha_s(u_{n_k}, u_{n_k}, a) \geq 1$ for all k.*

Then A has a fixed point $a \in \mathbb{U}$ and A is a Picard operator, that is, $\{A^n u_0\}$ converges to a.

$$\textit{If } \nabla_1(u,v,w) = \max\{S(u,v,w), S(u,u,Au), S(v,v,Av), S(w,w,Bw)\}$$

in Theorem 2.1 and Theorem 2.2, we can have another result.

Let $(\mathbb{U}, S)$ be a S-metric space, and let $\alpha_s, \eta_s : \mathbb{U} \times \mathbb{U} \times \mathbb{U} \to [0, +\infty)$ be a function. Mappings $A, B : \mathbb{U} \to \mathbb{U}$ are called a pair of generalized rational α_s-Geraghty contraction type mappings with respect to η_s if there exists $g \in \mathcal{G}$ such that, for all $u, v, w \in \mathbb{U}$,

$$\begin{aligned} &\alpha_s(u,v,w) \geq \eta_s(u,v,w) \\ &\quad \Rightarrow \quad S(Au,Av,Bw) \leq g\bigl(\nabla_1(u,v,w)\bigr)\nabla_1(u,v,w), \end{aligned} \tag{3.1}$$

where

$$\nabla_1(u,v,w) = \max\left\{S(u,v,w), S(Au,Av,Bw), \frac{S(u,u,Au)S(v,v,Av)}{1+S(u,v,w)+S(Au,Av,Bw)}, \right.$$
$$\left. \frac{S(v,v,Av)S(w,w,Bw)}{1+S(u,v,w)+S(Au,Av,Bw)}, \frac{S(w,w,Bw)S(u,u,Au)}{1+S(u,v,w)+S(Au,Av,Bw)}\right\}.$$

Theorem 3.3 *Let $(\mathbb{U}, S)$ be a complete S-metric space. Let A be an α_s-admissible mapping with respect to η_s such that the following hold*:

(i) (A,B) *is a generalized rational* α_s*-Geraghty contraction type mapping,*
(ii) (A,B) *is triangular* α_s*-admissible,*
(iii) *There exists* $u_0 \in \mathbb{U}$ *such that* $\alpha_s(u_0,u_0,Au_0) \geq \eta_s(u_0,u_0,Au_0)$*,*
(iv) *A and B are continuous.*
Then (A,B) *has a common fixed point.*

Proof Let $u_1 \in \mathbb{U}$ be such that $u_1 = Au_0$ and $u_2 = Bu_1$. Continuing this process, we construct a sequence $\{u_n\}$ of points in $\mathbb{U}$ such that

$$u_{2i+1} = Au_{2i} \quad \text{and} \quad u_{2i+2} = Bu_{2i+1}, \tag{3.2}$$

where $i = 0,1,2,3,\ldots$.

By assumption $\alpha_s(u_0,u_0,u_1) \geq \eta_s(u_0,u_0,u_1)$ and the pair (A,B) is α_s-admissible with respect to η_s, we have $\alpha_s(Au_0,Au_0,Bu_1) \geq \eta_s(Au_0,Au_0,Bu_1)$, from which we deduce that $\alpha_s(u_1,u_1,u_2) \geq \eta_s(u_1,u_1,u_2)$, which also implies that $\alpha_s(Bu_1,Bu_1,Au_2) \geq \eta_s(Bu_1,Bu_1,Au_2)$. Continuing in this way, we obtain $\alpha_s(u_n,u_n,u_{n+1}) \geq \eta_s(u_n,u_n,u_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned} S(u_{2i+1},u_{2i+1},u_{2i+2}) &= S(Au_{2i},Au_{2i},Bu_{2i+1}) \\ &\leq \alpha_s(u_{2i},u_{2i},u_{2i+1})S(Au_{2i},Au_{2i},Bu_{2i+1}) \\ &\leq g\bigl(\nabla_1(u_{2i},u_{2i},u_{2i+1})\bigr)\nabla_1(u_{2i},u_{2i},u_{2i+1}). \end{aligned}$$

Therefore,

$$S(u_{2i+1},u_{2i+1},u_{2i+2}) \leq \alpha_s(u_{2i},u_{2i},u_{2i+1})S(Au_{2i},Au_{2i},Bu_{2i+1}) \tag{3.3}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} \nabla_1(u_{2i},u_{2i},u_{2i+1}) = {} & \max\biggl\{S(u_{2i},u_{2i},u_{2i+1}), S(Au_{2i},Au_{2i},Bu_{2i+1}), \\ & \frac{S(u_{2i},u_{2i},Au_{2i})S(u_{2i},u_{2i},Au_{2i})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(Au_{2i},Au_{2i},Bu_{2i+1})}, \\ & \frac{S(u_{2i},u_{2i},Au_{2i})S(u_{2i+1},u_{2i+1},Bu_{2i+1})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(Au_{2i},Au_{2i},Bu_{2i+1})}, \\ & \frac{S(u_{2i+1},u_{2i+1},Bu_{2i+1})S(u_{2i},u_{2i},Au_{2i})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(Au_{2i},Au_{2i},Bu_{2i+1})}\biggr\} \\ = {} & \max\biggl\{S(u_{2i},u_{2i},u_{2i+1}), S(u_{2i+1},u_{2i+1},u_{2i+2}), \\ & \frac{S(u_{2i},u_{2i},u_{2i+1})S(u_{2i},u_{2i},u_{2i+1})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(u_{2i+1},u_{2i+1},u_{2i+2})}, \\ & \frac{S(u_{2i},u_{2i},u_{2i+1})S(u_{2i+1},u_{2i+1},u_{2i+2})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(u_{2i+1},u_{2i+1},u_{2i+2})}, \\ & \frac{S(u_{2i+1},u_{2i+1},u_{2i+2})S(u_{2i},u_{2i},u_{2i+1})}{1+S(u_{2i},u_{2i},u_{2i+1})+S(u_{2i+1},u_{2i+1},u_{2i+2})}\biggr\} \\ \leq {} & \max\bigl\{S(u_{2i},u_{2i},u_{2i+1}), S(u_{2i+1},u_{2i+1},u_{2i+2})\bigr\}. \end{aligned}$$

From the definition of g, the case $\nabla_1(u_{2i}, u_{2i}, u_{2i+1}) = S(u_{2i+1}, u_{2i+1}, u_{2i+2})$ is impossible.

$$\begin{aligned} S(u_{2i+1}, u_{2i+1}, u_{2i+2}) &\le g\big(\nabla_1(u_{2i}, u_{2i}, u_{2i+1})\big)\nabla_1(u_{2i}, u_{2i}, u_{2i+1}) \\ &\le g\big(S(u_{2i+1}, u_{2i+1}, u_{2i+2})\big)S(u_{2i+1}, u_{2i+1}, u_{2i+2}) \\ &< S(u_{2i+1}, u_{2i+1}, u_{2i+2}), \end{aligned}$$

which is a contradiction. Otherwise, in the other case

$$\begin{aligned} S(u_{2i+1}, u_{2i+1}, u_{2i+2}) &\le g(\nabla_1(u_{2i}, u_{2i}, u_{2i+1})\nabla_1(u_{2i}, u_{2i}, u_{2i+1}) \\ &\le g\big(S(u_{2i}, u_{2i}, u_{2i+1})\big)S(u_{2i}, u_{2i}, u_{2i+1}) \\ &< S(u_{2i}, u_{2i}, u_{2i+1}). \end{aligned}$$

This implies that

$$S(u_{n+1}, u_{n+1}, u_{n+2}) < S(u_n, u_n, u_{n+1}) \tag{3.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. □

Following similar lines of Theorem 2.1, we can prove that A and B have a common fixed point.

Theorem 3.4 *Let $(\mathbb{U}, S)$ be a complete S-metric space, and let (A, B) be an α_s-admissible mapping with respect to η_s such that the following hold*:

(i) *(A, B) is a generalized rational α_s-Geraghty contraction type mapping,*
(ii) *(A, B) is triangular α_s-admissible,*
(iii) *There exists $u_0 \in \mathbb{U}$ such that $\alpha_s(u_0, u_0, Au_0) \ge \eta_s(u_0, u_0, Au_0)$,*
(iv) *If $\{u_n\}$ is a sequence in $\mathbb{U}$ such that $\alpha_s(u_n, u_n, u_{n+1}) \ge \eta_s(u_n, u_n, u_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to a \in \mathbb{U}$ as $n \to +\infty$, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha_s(u_{n_k}, u_{n_k}, a) \ge \eta_s(u_{n_k}, u_{n_k}, a)$ for all k.*

Then A and B have a common fixed point.

Proof Follows similar lines of Theorem 2.2. □

$$\begin{aligned} \textit{If } \nabla_1(u, v, w) = \max\biggl\{ & S(u, v, w), S(Au, Av, Aw), \frac{S(u, u, Au)S(v, v, Av)}{1 + S(u, v, w) + S(Au, Av, Aw)}, \\ & \frac{S(v, v, Av)S(w, w, Aw)}{1 + S(u, v, w) + S(Au, Av, Aw)}, \frac{S(w, w, Aw)S(u, u, Au)}{1 + S(u, v, w) + S(Au, Av, Aw)}\biggr\} \end{aligned}$$

and $A = B$ in Theorem 3.3 and Theorem 3.4, we get the following corollaries.

Corollary 3.5 *Let $(\mathbb{U}, S)$ be a complete S-metric space, and let A be an α_s-admissible mapping with respect to η_s such that the following hold*:

(i) *A is a generalized rational α_s-Geraghty contraction type mapping,*
(ii) *A is triangular α_s-admissible,*

(iii) *There exists* $u_0 \in \mathbb{U}$ *such that* $\alpha_s(u_0, u_0, Au_0) \geq \eta_s(u_0, u_0, Au_0)$,
(iv) *A is continuous.*

Then A has a fixed point $a \in \mathbb{U}$ *and A is a Picard operator, that is,* $\{A^n u_0\}$ *converges to* a.

Corollary 3.6 *Let* $(\mathbb{U}, S)$ *be a complete S-metric space, and let A be an* α_s*-admissible mapping with respect to* η_s *such that the following hold*:

(i) *A is a generalized rational* α_s*-Geraghty contraction type mapping,*
(ii) *A is triangular* α_s*-admissible,*
(iii) *There exists* $u_0 \in \mathbb{U}$ *such that* $\alpha_s(u_n, u_n, Au_{n+1}) \geq \eta_s(u_n, u_n, Au_{n+1})$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $u_n \to a \in \mathbb{U}$ *as* $n \to +\infty$, *then there exists a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that* $\alpha_s(u_{n_k}, u_{n_k}, a) \geq \eta_s(u_{n_k}, u_{n_k}, a)$ *for all* k.

Then A has a fixed point $a \in \mathbb{U}$, *and A is a Picard operator, that is,* $\{A^n u_0\}$ *converges to* a.

Example 3 Let $\mathbb{U} = \{1, 2, 3\}$ and S be an S-metric. Let $S(1,1,3) = S(3,3,1) = \frac{5}{7}$, $S(1,1,1) = S(2,2,2) = S(3,3,3) = 0$, $S(1,1,2) = S(2,2,1) = 1$, $S(2,2,3) = S(3,3,2) = \frac{4}{7}$. Also, let

$$\alpha_s(u,u,v) = \begin{cases} 1 & \text{if } u, v \in \mathbb{U}, \\ 0 & \text{otherwise.} \end{cases}$$

Define the mappings $A, B : \mathbb{U} \to \mathbb{U}$ as follows: $Au = 1$ for each $u \in \mathbb{U}$, $B(1) = B(3) = 1$, $B(2) = 3$, and $g : [0, +\infty) \to [0, 1)$, then

$$\alpha_s(u,u,v) S(Bu, Bu, Bv) \leq g\bigl(\nabla_2(u,u,v)\bigr) \nabla_2(u,u,v).$$

Let $u = 2$ and $v = 3$, then condition (i) of Theorem 2.1 is not satisfied as

$$S\bigl(B(2), B(2), B(3)\bigr) = S(3,3,1) = \frac{5}{7},$$

where

$$\begin{aligned} \nabla_2(u,u,v) = {} & \max\biggl\{ S(2,2,3), S(A2, A2, B3), \\ & \frac{S(2,2,A2)S(2,2,A2)}{1 + S(2,2,3) + S(A2,A2,B3)}, \frac{S(2,2,A2)S(3,3,B3)}{1 + S(2,2,3) + S(A2,A2,B3)} \biggr\} \\ = {} & \max\biggl\{ S(2,2,3), S(1,1,1), \\ & \frac{S(2,2,1)S(2,2,1)}{1 + S(2,2,3) + S(1,1,1)}, \frac{S(2,2,1)S(3,3,1)}{1 + S(2,2,3) + S(1,1,1)} \biggr\} \\ = {} & \max\biggl\{ \frac{4}{7}, 0, \frac{1}{11}, \frac{5}{11} \biggr\} = \frac{4}{7}. \end{aligned}$$

We prove that Theorem 2.1 can be applied to A and B. Let $u, v \in \mathbb{U}$, clearly (A, B) is α_s-admissible such that $\alpha_s(u,u,v) \geq 1$. Let $u, v \in \mathbb{U}$ so that $Au, Bv \in \mathbb{U}$ and $\alpha_s(Au, Au, Bv) = 1$. Hence (A, B) is α_s-admissible. We know that condition (i) of Theorem 2.1 is satisfied.

If $u, v \in \mathbb{U}$, then $\alpha_s(u,u,v) = 1$, we have

$$\alpha_s(u,u,v) S(Au, Au, Bv) \leq g\bigl(\nabla_2(u,u,v)\bigr) \nabla_2(u,u,v),$$

where

$$\begin{aligned}\nabla_2(u,u,v) = {} & \max\Bigg\{S(2,2,3), S(A2,A2,B3),\\ & \frac{S(2,2,A2)S(2,2,A2)}{1+S(2,2,3)+S(A2,A2,B3)}, \frac{S(2,2,A2)S(3,3,B3)}{1+S(2,2,3)+S(A2,A2,B3)}\Bigg\}\\ = {} & \max\Bigg\{S(2,2,3), S(1,1,1),\\ & \frac{S(2,2,1)S(2,2,1)}{1+S(2,2,3)+S(1,1,1)}, \frac{S(2,2,1)S(3,3,1)}{1+S(2,2,3)+S(1,1,1)}\Bigg\}\\ = {} & \max\left\{\frac{4}{7}, 0, \frac{1}{11}, \frac{5}{11}\right\} = \frac{4}{7},\end{aligned}$$

and $S(A2,A2,B3) = S(1,1,1) = 0$.

$$\alpha_s(u,u,v)S(Au,Au,Bv) \le g\big(\nabla_2(u,u,v)\big)\nabla_2(u,u,v).$$

Hence all the hypotheses of Theorem 2.1 are satisfied. So, A and B have a common fixed point.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
NP, YR, and ST together studied and prepared the manuscript. SR analyzed all the results and made necessary improvements. YR is the major contributor in writing the paper. All authors read and approved the final manuscript.

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