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On local spectral properties of operator matrices

Il Ju An[1], Eungil Ko[2] and Ji Eun Lee[3*]

*Correspondence:
jieunlee7@sejong.ac.kr;
jieun7@ewhain.net
[3]Department of Mathematics and Statistics, Sejong University, Seoul, 143-747, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we focus on a 2×2 operator matrix T_{ϵ_k} as follows:

$$T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix},$$

where ϵ_k is a positive sequence such that $\lim_{k\to\infty} \epsilon_k = 0$. We first explore how T_{ϵ_k} has several local spectral properties such as the single-valued extension property, the property (β), and decomposable. We next study the relationship between some spectra of T_{ϵ_k} and spectra of its diagonal entries, and find some hypotheses by which T_{ϵ_k} satisfies Weyl's theorem and a-Weyl's theorem. Finally, we give some conditions that such an operator matrix T_{ϵ_k} has a nontrivial hyperinvariant subspace.

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Keywords: 2×2 operator matrices; Hyperinvariant subspace; The single-valued extension property; The property (β); Decomposable; Weyl's theorem

1 Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on a separable Hilbert space $\mathcal{H}$. Let $\{T\}'$, the *commutant* of T, be the collection of all bounded linear operators such that commute with T. A subspace $\mathcal{G} \subset \mathcal{H}$ is *invariant* for $T \in \mathcal{L}(\mathcal{H})$ if an inclusion $T\mathcal{G} \subset \mathcal{G}$ holds, and is *hyperinvariant* for T if the inclusion $S\mathcal{G} \subset \mathcal{G}$ holds for all $S \in \{T\}'$. The *hyperinvariant subspace problem* is asking whether *every operator on a separable complex Hilbert space has a nontrivial hyperinvariant subspace*. It has been known that this is one of unresolved problems in operator theory and it has attracted a lot of interest by many authors.

For the study of this problem, in 2011, H. J. Kim [8] proved that, if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where T_1, T_2, and T_3 are arbitrary operators in $\mathcal{L}(\mathcal{H})$ such that T_1 is either a compact operator with $T_1 \neq 0$ or a normal operator with $T_1 \neq \lambda I$, then at least one of T and $\widehat{T}$, has a nontrivial hyperinvariant subspace where $\widehat{T} = \begin{pmatrix} T_3 & T_4 \\ 0 & T_1 \end{pmatrix}$ for an arbitrary operator $T_4 \in \mathcal{L}(\mathcal{H})$. In 2018, I. B. Jung, E. Ko, and C. Pearcy [7] showed if T_1 and T_3 are operators in $\mathcal{L}(\mathcal{H})$ such that either T_1 or T_3 has a nontrivial hyperinvariant subspace, then T and $\widehat{T}$ have a nontrivial hyperinvariant subspace where T_2 and T_4 are any operators in

$\mathcal{L}(\mathcal{H})$. As mentioned in the above results, in the case of a 2×2 upper triangular operator matrix, there are some known results, but in the case of a full 2×2 operator matrix, it is very difficult to solve the invariant subspace problem. So, we focus on the matrix T_{ϵ_k} as a variation of the 2×2 upper triangular operator matrix and we study some conditions so that a 2×2 operator matrix T_{ϵ_k} has a nontrivial hyperinvariant subspace.

We now provide a simple outline of the paper. We first study the local spectral theory of operator matrices (cf. [3] and [9]). In particular, we consider the case when the $(2,1)$-entry of a 2×2 operator matrix approaches zero. In addition, we give the relationship between some spectra of 2×2 operator matrices and spectra of their diagonal entries, and find some hypotheses by which such operator matrices T_{ϵ_k} entail Weyl's theorem and a-Weyl's theorem.

2 Preliminaries

We briefly review some notions of local spectral properties, which are used in this paper. We refer to [10] for more detailed information.

The operator $T \in \mathcal{L}(\mathcal{H})$ has *the single-valued extension property* if $f(\lambda) \equiv 0$ is the unique solution to $(T-\lambda)f(\lambda) \equiv 0$ on D for every open subset D of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on D. The *local resolvent set* $\rho_T(x)$ of $T \in \mathcal{L}(\mathcal{H})$ at $x \in \mathcal{H}$ is the union of all open subset D of $\mathbb{C}$ such that there is an analytic function $f : D \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on D. The set $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ is the *local spectrum* of T at x. The *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ is given by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for any $F \subset \mathbb{C}$. We say that $T \in \mathcal{L}(\mathcal{H})$ has *Bishop's property* (β) if every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on D for every open subset D of $\mathbb{C}$ such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of D, it follows that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. Notice that, if T has Bishop's property (β), then it has the single-valued extension property. The operator $T \in \mathcal{L}(\mathcal{H})$ is *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that

$$\mathcal{H} = \mathcal{M} + \mathcal{N}, \qquad \sigma(T|_{\mathcal{M}}) \subset \overline{U} \quad \text{and} \quad \sigma(T|_{\mathcal{N}}) \subset \overline{V}.$$

In general, it is known that T is decomposable if and only if T and its adjoint T^* possess property (β) [1, 10].

Now, we introduce some Weyl type theorems related to definitions of various spectra (see [12] for more details). For these, we first take a look at some notions needed in this paper. If $T \in \mathcal{L}(\mathcal{H})$, we shall write $\ker(T)$ (or $N(T)$) and $\operatorname{ran}(T)$ (or $R(T)$) for the null space and the range of T, respectively. We know that the family $\{\ker(T^k)\}$ forms an ascending sequence of subspaces for $T \in \mathcal{L}(\mathcal{H})$ and $k \in \mathbb{N}$. So we call the *ascent* of T for the smallest nonnegative integer k for which $\ker(T^k) = \ker(T^{k+1})$ holds. We also see that the family $\{\operatorname{ran}(T^k)\}$ forms a descending sequence for $k \in \mathbb{N}$, and then the smallest nonnegative integer k for which $\operatorname{ran}(T^k) = \operatorname{ran}(T^{k+1})$ is said the *descent* of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* (resp., *lower semi-Fredholm*) if it has both finite dimensional kernel and closed range (resp., it has both finite dimensional co-kernel and closed range). Either upper or lower semi-Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is called *semi-Fredholm*, and its *index* is given by $\operatorname{ind}(T) := \dim\ker(T) - \dim\ker(T^*)$. When both $\dim\ker(T)$ and $\dim\ker(T^*)$ are finite, then T is called *Fredholm*. If $T \in \mathcal{L}(\mathcal{H})$ is a Fredholm operator satisfying $\operatorname{ind}(T) = 0$, then it is called *Weyl*, and if T is a Fredholm operator with finite descent and ascent, then it is called *Browder*.

If $T \in \mathcal{L}(\mathcal{H})$, we shall write $\sigma_p(T)$, $\sigma_s(T)$, $\sigma_a(T)$, $\sigma(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ *the point spectrum, the surjective spectrum, the approximate point spectrum, the spectrum, the essential spectrum, the left essential spectrum*, and *the right essential spectrum the left essential spectrum* of T, respectively. The *Weyl spectrum* $\sigma_w(T) := \{\mu \in \mathbb{C} : T - \mu I \text{ is not Weyl}\}$ and the *Browder spectrum* $\sigma_b(T) := \{\mu \in \mathbb{C} : T - \mu I \text{ is not Browder}\}$, where I is an identity operator on $\mathcal{H}$. We write $\mathcal{K}(\mathcal{H})$ for the set of all compact operators on $\mathcal{H}$ and review another spectra as follows: the *Weyl essential approximate point spectrum* $\sigma_{ea}(T) := \{\mu \in \mathbb{C} : T + C - \mu I \text{ is not bounded below for all } C \in \mathcal{K}(\mathcal{H})\}$ and the *Browder essential approximate point spectrum* $\sigma_{ab}(T) := \{\mu \in \mathbb{C} : T + C - \mu I \text{ is not bounded below for all } C \in \mathcal{K}(\mathcal{H}) \text{ and } TC = CT\}$. Evidently, we get the inclusions

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \quad \text{and} \quad \sigma_{ea}(T) \subseteq \sigma_{ab}(T).$$

Let $\operatorname{iso} K$ be the collection of all isolated points of a complex subset K. We write $\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \dim\ker(T - \lambda) < \infty\}$. And we denote $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$ which is the collection of *Riesz points of* T. We say that *Weyl's theorem is obeyed* for T provided $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and that *Browder's theorem is obeyed* for T provided $\sigma(T) \setminus \sigma_w(T) = p_{00}(T)$, equivalently, if $\sigma_w(T) = \sigma_b(T)$. We say that *a-Weyl's theorem is obeyed* for T provided $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$ and that *a-Browder's theorem is obeyed* for T provided $\sigma_a(T) \setminus \sigma_{ea}(T) = p_{00}^a(T)$, where $\pi_{00}^a(T) := \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \dim\ker(T - \lambda) < \infty\}$ and $p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ab}(T)$. Then it is well known that

$$\begin{array}{ccc} a\text{-Weyl's theorem} & \Longrightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Downarrow \\ \text{Weyl's theorem} & \Longrightarrow & \text{Browder's theorem} \end{array}$$

3 Main results

In this section, we study 2×2 operator matrices. In particular, we consider the case when their $(2,1)$-entry approaches zero. We begin our program with the following theorem.

Theorem 3.1 *Let* $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ *where* $A, B, C, D \in \mathcal{L}(\mathcal{H})$ *and* $\{\epsilon_k\}$ *is a positive sequence such that* $\lim_{k\to\infty} \epsilon_k = 0$. *Then the following statements hold.*

(i) *If both* A *and* B *have the single-valued extension property, then* T_{ϵ_k} *has the single-valued extension property.*

(ii) *If* T_{ϵ_k} *has the single-valued extension property,* $BC = CB$, *and* C *is nilpotent of order* m, *then* B *has the single-valued extension property.*

Proof (i) Suppose that A and B have the single-valued extension property. Let G be an open set in $\mathbb{C}$ and let $f : G \to \mathcal{H} \oplus \mathcal{H}$ be an analytic function with $f = f_1 \oplus f_2$ such that

$$(T_{\epsilon_k} - \lambda) \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} = 0. \tag{1}$$

Then

$$\begin{pmatrix} A - \lambda & C \\ \epsilon_k D & B - \lambda \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, we get

$$\begin{cases} (A-\lambda)f_1(\lambda)+Cf_2(\lambda)=0, \\ \epsilon_k Df_1(\lambda)+(B-\lambda)f_2(\lambda)=0. \end{cases} \tag{2}$$

Since $\|(B-\lambda)f_2(\lambda)\| \leq \|\epsilon_k Df_1(\lambda)+(B-\lambda)f_2(\lambda)\|+\epsilon_k\|Df_1(\lambda)\| = \epsilon_k\|Df_1(\lambda)\|$ and $\lim_{k\to\infty}\epsilon_k = 0$, $(B-\lambda)f_2(\lambda)=0$. Moreover, since B has the single-valued extension property, $f_2(\lambda)=0$. From (2), we have

$$(A-\lambda)f_1(\lambda)=0.$$

Since A has the single-valued extension property, $f_1(\lambda)=0$. Hence T_{ϵ_k} has the single-valued extension property.

(ii) Let T_{ϵ_k} have the single-valued extension property and $(B-\lambda)f_2(\lambda)=0$ where f_2 is an analytic function. Then

$$(T_{\epsilon_k}-\lambda)\begin{pmatrix} 0 \\ C^{m-1}f_2(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

Since T_{ϵ_k} has the single-valued extension property, it follows from (3) that $C^{m-1}f_2(\lambda)=0$. Thus

$$(T_{\epsilon_k}-\lambda)\begin{pmatrix} 0 \\ C^{m-2}f_2(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4}$$

Since T_{ϵ_k} has the single-valued extension property, $C^{m-2}f_2(\lambda)=0$. By induction, we have $f_2(\lambda)=0$. Hence B has the single-valued extension property. □

Corollary 3.2 *Let* $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ *where* $A,B,C,D \in \mathcal{L}(\mathcal{H})$ *and* $\{\epsilon_k\}$ *is a positive sequence such that* $\lim_{k\to\infty}\epsilon_k = 0$. *If* A *and* B *have the single-valued extension property, then the following inclusions hold.*

(i) $\sigma_B(x_2) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus x_2)$ *for all* $x_1, x_2 \in \mathcal{H}$ *and* $H_{T_{\epsilon_k}}(F) \subset \mathcal{H} \oplus H_B(F)$ *for any subset* F *of* $\mathbb{C}$.

(ii) $\sigma_A(x_1) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus 0)$ *for all* $x_1 \in \mathcal{H}$.

Proof (i) We know that T_{ϵ_k} has the single-valued extension property from Theorem 3.1. Let $\lambda_0 \notin \sigma_{T_{\epsilon_k}}(x_1 \oplus x_2)$ for all $x_1, x_2 \in \mathcal{H}$. Then there exists a neighborhood $\mathcal{D}$ of λ_0 and an analytic function $f = f_1 \oplus f_2 : \mathcal{D} \to \mathcal{H} \oplus \mathcal{H}$ such that $(T_{\epsilon_k}-\lambda)f(\lambda) = x_1 \oplus x_2$ for every $\lambda \in \mathcal{D}$. Then we have

$$\begin{cases} (A-\lambda)f_1(\lambda)+Cf_2(\lambda)=x_1, \\ \epsilon_k Df_1(\lambda)+(B-\lambda)f_2(\lambda)=x_2. \end{cases}$$

Letting $\epsilon_k \to 0$, $(B-\lambda)f_2(\lambda)=x_2$. Hence $\lambda_0 \notin \sigma_B(x_2)$ for all $x_2 \in \mathcal{H}$.

On the other hand, if $x_1 \oplus x_2 \in H_{T_{\epsilon_k}}(F)$, then $\sigma_{T_{\epsilon_k}}(x_1 \oplus x_2) \subset F$. Since

$$\sigma_B(x_2) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus x_2) \subset F,$$

it follows from (i) that $x_2 \in H_B(F)$. Thus $x_1 \oplus x_2 \in \mathcal{H} \oplus H_B(F)$. Hence

$$H_{T_{\epsilon_k}}(F) \subset \mathcal{H} \oplus H_B(F)$$

for any subset F of $\mathbb{C}$.

(ii) Let $\lambda_0 \in \rho_{T_{\epsilon_k}}(x_1 \oplus 0)$ for all $x_1 \in \mathcal{H}$. Then there exists a neighborhood $\mathcal{G}$ of λ_0 and an analytic function $f_1 \oplus f_2 : \mathcal{G} \to \mathcal{H} \oplus \mathcal{H}$ such that $(T_{\epsilon_k} - \lambda)\binom{f_1(\lambda)}{f_2(\lambda)} = \binom{x_1}{0}$ for every $\lambda \in \mathcal{G}$. Thus this implies that

$$\begin{cases} (A-\lambda)f_1(\lambda) + Cf_2(\lambda) = x_1, \\ \epsilon_k Df_1(\lambda) + (B-\lambda)f_2(\lambda) = 0. \end{cases}$$

Letting $\epsilon_k \to 0$, we get $(B-\lambda)f_2(\lambda) = 0$. Since B has the single-valued extension property, $f_2(\lambda) = 0$ for every $\lambda \in \mathcal{G}$. Thus $(A-\lambda)f_1(\lambda) = x_1$, and hence $\lambda_0 \in \rho_A(x_1)$. Therefore $\sigma_A(x_1) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus 0)$ for all $x_1 \in \mathcal{H}$. □

Example 3.3 In Corollary 3.2, if both A and B are substituted with the unilateral shift U on $\ell^2(\mathbb{N})$, then T_{ϵ_k} has the single-valued extension property on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Furthermore, we get the following inclusions:

$$\sigma_U(x_1) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus 0) \quad \text{and} \quad \sigma_U(x_2) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus x_2)$$

for all $x_1, x_2 \in \ell^2(\mathbb{N})$.

We next investigate some relations among the spectra, the point spectra and the approximate point spectra of A, B and T_{ϵ_k}, respectively.

Theorem 3.4 *Let* $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ *where* $A, B, C, D \in \mathcal{L}(\mathcal{H})$ *and* $\{\epsilon_k\}$ *is a positive sequence such that* $\lim_{k\to\infty} \epsilon_k = 0$.

(i) *If both A and B have the single-valued extension property, then*

$$\sigma(A) \cup \sigma(B) = \sigma(T_{\epsilon_k}) \quad \text{and} \quad \sigma_p(A) \cup \sigma_p(B) = \sigma_p(T_{\epsilon_k}).$$

(ii) *If both* A^* *and* B^* *have the single-valued extension property, then*

$$\sigma_a(A) \cup \sigma_a(B) = \sigma_a(T_{\epsilon_k}).$$

Proof (i) From Theorem 3.1, we know that T_{ϵ_k} has the single-valued extension property. Since $\sigma_B(x_2) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus x_2)$ from Corollary 3.2, we have

$$\sigma(B) = \bigcup_{x_2 \in \mathcal{H}} \sigma_B(x_2) \subset \bigcup_{x_1 \oplus x_2 \in \mathcal{H} \oplus \mathcal{H}} \sigma_{T_{\epsilon_k}}(x_1 \oplus x_2) = \sigma(T_{\epsilon_k}).$$

Since $\sigma_A(x_1) \subset \sigma_{T_{\epsilon_k}}(x_1 \oplus 0)$ for all $x_1 \in \mathcal{H}$, we get

$$\sigma(A) = \bigcup_{x_1 \in \mathcal{H}} \sigma_A(x_1) \subset \bigcup_{x_1 \oplus x_2 \in \mathcal{H} \oplus \mathcal{H}} \sigma_{T_{\epsilon_k}}(x_1 \oplus x_2) = \sigma(T_{\epsilon_k}).$$

For the converse, we suppose that $\gamma \notin \sigma(A) \cup \sigma(B)$. If $\lim_{n\to\infty} \|(T_{\epsilon_k} - \gamma)\binom{x_n}{y_n}\| = 0$, then

$$\begin{cases} \lim_{n\to\infty} \|(A-\gamma)x_n + Cy_n\| = 0, \\ \lim_{n\to\infty} \|\epsilon_k D x_n + (B-\gamma)y_n\| = 0. \end{cases} \tag{5}$$

Since $B - \gamma$ is invertible,

$$\begin{aligned} \|y_n\| &\le \left\|(B-\gamma)^{-1}\right\| \left\|(B-\gamma)y_n\right\| \\ &\le \left\|(B-\gamma)^{-1}\right\| \left\|(B-\gamma)y_n + \epsilon_k D x_n\right\| + \epsilon_k \|D\| \left\|(B-\gamma)^{-1}\right\| \|x_n\|. \end{aligned}$$

Then $\limsup_{n\to\infty} \|y_n\| \le \epsilon_k \|D\| \|(B-\gamma)^{-1}\| (\limsup_{n\to\infty} \|x_n\|)$. Taking $\lim_{k\to\infty} \epsilon_k = 0$, we have $\limsup_{n\to\infty} \|y_n\| = 0$ and so

$$\lim_{n\to\infty} \|y_n\| = 0.$$

From (5), $\lim_{n\to\infty} \|(A-\gamma)x_n\| = 0$. Since $A - \gamma$ is invertible, $\lim_{n\to\infty} \|x_n\| = 0$. Therefore, $T_{\epsilon_k} - \gamma$ is bounded below. If $(T_{\epsilon_k} - \gamma)^* \binom{x}{y} = \binom{0}{0}$, then

$$\begin{cases} (A^* - \overline{\gamma})x + \epsilon_k D^* y = 0, \\ C^* x + (B^* - \overline{\gamma})y = 0. \end{cases}$$

Since $\lim_{k\to\infty} \epsilon_k = 0$, we have $(A^* - \overline{\gamma})x = 0$. Since $A^* - \overline{\gamma}$ is invertible, $x = 0$ and so $(B^* - \overline{\gamma})y = 0$. Since $B^* - \overline{\gamma}$ is invertible, $y = 0$. Thus $\ker(T_{\epsilon_k} - \gamma)^* = \{0\}$ and so $\operatorname{ran}(T_{\epsilon_k} - \gamma)$ is dense in $\mathcal{H} \oplus \mathcal{H}$. Hence $T_{\epsilon_k} - \gamma$ is invertible. So $\gamma \notin \sigma(T_{\epsilon_k})$. Consequently, the first equation is established. Moreover, if A has the single-valued extension property, then it is well known that the surjective spectrum $\sigma_s(A)$ of A identifies with the spectrum of A (see [10]), so that $\sigma_p(A) = \sigma(A) \setminus \sigma_s(A) = \emptyset$. Similarly, $\sigma_p(B) = \emptyset$. Hence $\sigma_p(A) \cup \sigma_p(B) = \emptyset \subset \sigma_p(T_{\epsilon_k})$. Since T_{ϵ_k} has the single-valued extension property by Theorem 3.1, we have $\sigma_p(T_{\epsilon_k}) = \sigma(T_{\epsilon_k}) \setminus \sigma_s(T_{\epsilon_k}) = \emptyset$. From these arguments, the second equality trivially holds.

(ii) Suppose that both A^* and B^* have the single-valued extension property. Then we can prove that $T_{\epsilon_k}^*$ also has the single-valued extension property using a similar method from the proof of Theorem 3.1. It is known that $\sigma_a(T) = \sigma(T)$ provided T^* has the single-valued extension property for every $T \in \mathcal{L}(\mathcal{H})$. This means that the equality $\sigma_a(A) \cup \sigma_a(B) = \sigma_a(T_{\epsilon_k})$ holds by (i). □

It is well known that, if A and B have the property (β), then $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ has the property (β) without any conditions. However, 2×2 operator matrices which their all entries are nonzero, in addition, their $(2,1)$-entries are either μI for some nonzero constant μ, or $\epsilon_k I$ for a positive sequence $\{\epsilon_k\}$ with $\lim_{k\to\infty} \epsilon_k = 0$ may not have the property (β) even though their diagonal entries have the property (β) (see (8)). We now study the property (β) and decomposability of such a 2×2 operator matrix T_{ϵ_k}.

Theorem 3.5 *Let* $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ *where* $A, B, C, D \in \mathcal{L}(\mathcal{H})$ *and* $\{\epsilon_k\}$ *is a positive sequence such that* $\lim_{k\to\infty} \epsilon_k = 0$. *If* $\sup_n \|f_{n,1}\|_K < \infty$ *whenever*

$$\left\| (T_{\epsilon_k} - \lambda) \begin{pmatrix} f_{n,1}(\lambda) \\ f_{n,2}(\lambda) \end{pmatrix} \right\|_K \to 0 \quad \text{as } n \to \infty, \tag{6}$$

then the following statements hold.

(i) *If A and B have the property (β), then T_{ϵ_k} has the property (β).*

(ii) *If A and B are decomposable, then T_{ϵ_k} is decomposable.*

Proof (i) Suppose that A and B have the property (β). Let G be an open set in $\mathbb{C}$ and let $f_n : G \to \mathcal{H} \oplus \mathcal{H}$ be a sequence of analytic functions with $f_n = f_{n,1} \oplus f_{n,2}$ such that

$$\lim_{n\to\infty} \left\| (T_{\epsilon_k} - \lambda) \begin{pmatrix} f_{n,1}(\lambda) \\ f_{n,2}(\lambda) \end{pmatrix} \right\|_K = 0 \tag{7}$$

for every compact set K in G, where $\|f\|_K = \sup_{\lambda\in K} \|f(\lambda)\|$ for an $\mathcal{H} \oplus \mathcal{H}$-valued function $f(\lambda)$. Then

$$\lim_{n\to\infty} \left\| \begin{pmatrix} A-\lambda & C \\ \epsilon_k D & B-\lambda \end{pmatrix} \begin{pmatrix} f_{n,1}(\lambda) \\ f_{n,2}(\lambda) \end{pmatrix} \right\|_K = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, we get

$$\begin{cases} \lim_{n\to\infty} \|(A-\lambda)f_{n,1}(\lambda) + Cf_{n,2}(\lambda)\|_K = 0, \\ \lim_{n\to\infty} \|\epsilon_k D f_{n,1}(\lambda) + (B-\lambda)f_{n,2}(\lambda)\|_K = 0. \end{cases} \tag{8}$$

We observe that

$$\begin{aligned} \left\|(B-\lambda)f_{n,2}(\lambda)\right\|_K &\le \left\|(B-\lambda)f_{n,2}(\lambda) + \epsilon_k D f_{n,1}(\lambda) - \epsilon_k D f_{n,1}(\lambda)\right\|_K \\ &\le \left\|(B-\lambda)f_{n,2}(\lambda) + \epsilon_k D f_{n,1}(\lambda)\right\|_K + \left\|\epsilon_k D f_{n,1}(\lambda)\right\|_K. \end{aligned}$$

From (8), we get

$$\limsup_{n\to\infty} \left\|(B-\lambda)f_{n,2}(\lambda)\right\|_K \le \epsilon_k \|D\| \limsup_{n\to\infty} \left\|f_{n,1}(\lambda)\right\|_K.$$

Since $\sup_n \|f_{n,1}\|_K < \infty$, letting $\epsilon_k \to 0$,

$$\limsup_{n\to\infty} \left\|(B-\lambda)f_{n,2}(\lambda)\right\|_K = 0$$

and so $\lim_{n\to\infty} \|(B-\lambda)f_{n,2}(\lambda)\|_K = 0$. Moreover, since B has the property (β), $\lim_{n\to\infty} \|f_{n,2}(\lambda)\|_K = 0$. From (8), we have

$$\lim_{n\to\infty} \left\|(A-\lambda)f_{n,1}(\lambda)\right\|_K = 0.$$

Since A has the property (β), $\lim_{n\to\infty} \|f_{n,1}(\lambda)\|_K = 0$. Thus

$$\lim_{n\to\infty} \left\|f_n(\lambda)\right\|_K = \lim_{n\to\infty} \left\|f_{n,1}(\lambda) \oplus f_{n,2}(\lambda)\right\|_K = 0.$$

Hence T_{ϵ_k} has the property (β).

(ii) If A and B are decomposable, then A, A^*, B, and B^* have the property (β). Since A and B have the property (β), it follows from (1) that T_{ϵ_k} has the property (β). Note that

$T^*_{\epsilon_k} = \begin{pmatrix} A^* & \epsilon_k D^* \\ C^* & B^* \end{pmatrix}$ and $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} B^* & C^* \\ \epsilon_k D^* & A^* \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = T^*_{\epsilon_k}$. Since A^* and B^* also have property (β), it follows from (i) that $\begin{pmatrix} B^* & C^* \\ \epsilon_k D^* & A^* \end{pmatrix}$ has the property (β). Since $\begin{pmatrix} B^* & C^* \\ \epsilon_k D^* & A^* \end{pmatrix}$ is unitarily equivalent to $T^*_{\epsilon_k}$, we see that $T^*_{\epsilon_k}$ has the property (β). Therefore T_{ϵ_k} is decomposable. □

From these arguments for some local spectral properties of the operator matrices T_{ϵ_k}, we get more corollaries.

Corollary 3.6 *Let $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ where $A, B, C, D \in \mathcal{L}(\mathcal{H})$ and $\{\epsilon_k\}$ is a positive sequence such that $\lim_{k\to\infty} \epsilon_k = 0$. If $\sup_n \|f_{n,1}\|_K < \infty$ whenever (6), then the following statements are satisfied.*

(i) *If A and B have the property (β) and $\sigma(T_{\epsilon_k})$ has nonempty interior, then T_{ϵ_k} has a nontrivial invariant subspace.*
(ii) *If A and B are hyponormal, then T_{ϵ_k} has the property (β).*
(iii) *If A and B are compact or normal, then T_{ϵ_k} is decomposable.*

Proof (i) If A and B have the property (β), then it follows from Theorem 3.5 that T_{ϵ_k} has the property (β). Since $\sigma(T_{\epsilon_k})$ has nonempty interior, T_{ϵ_k} has a nontrivial invariant subspace by [5, Theorem 2.1].

(ii) If A and B are hyponormal, then they are subscalar by [14], and so it is known that they have the property (β). Hence it is obvious that T_{ϵ_k} has the property (β) from Theorem 3.5.

(iii) Since A and B are compact or normal, then they are decomposable (see [10]) and this implies from Theorem 3.5 that T_{ϵ_k} is also decomposable. □

From [2], if $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ and $R(C)$ is closed, then we have the following matrix representation:

$$T_{\epsilon_k} = \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & 0 & C_1 \\ \epsilon_k D & B_1 & B_2 \end{pmatrix}, \tag{9}$$

which maps from $\mathcal{H} \oplus \mathcal{H} = \mathcal{H} \oplus N(C) \oplus N(C)^\perp$ into $\mathcal{H} \oplus \mathcal{H} = R(C)^\perp \oplus R(C) \oplus \mathcal{H}$ where $C_1 = C|_{N(C)^\perp}$, $A_1 = P_{R(C)^\perp}A$, $A_2 = P_{R(C)}A$, $B_1 = B|_{N(C)}$ and $B_2 = B|_{N(C)^\perp}$. Here, $P_{N(C)}$ (resp. $P_{N(C)^\perp}$) denotes the projection of $\mathcal{K}$ onto $N(C)$ (resp. $N(C)^\perp$). We now study the next theorem in the sense of the representation (9) and mention that a sequence $\{\epsilon_k\}$ need not converge to 0.

Theorem 3.7 *Let $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ where $\{\epsilon_k\}$ is a bounded sequence and $R(C)$ is closed. Suppose that $A_1 = P_{R(C)^\perp}A|_{\mathcal{H}}$ and $B_1 = B|_{N(C)}$. If A_1 has the property (β) and B_1 is decomposable, then T_{ϵ_k} is decomposable. Moreover, if 0 is not an eigenvalue of C^*, then T_{ϵ_k} is decomposable if and only if B_1 is decomposable.*

Proof Since B_1 is decomposable, both B_1 and ${B_1}^*$ have the property (β). Moreover, A_1 has the property (β), thus T_{ϵ_k} and its adjoint operator have the property (β) from [3, Theorem 3.3]. Therefore T_{ϵ_k} is decomposable. On the other hand, if T_{ϵ_k} is decomposable, then T_{ϵ_k} and its adjoint operator have the property (β). Thus, by [3, Theorem 3.3], both B_1 and ${B_1}^*$ have the property (β). Hence B_1 is decomposable. The converse implication holds by a similar method. □

Corollary 3.8 *Let $T_{\epsilon_k} = \left(\begin{smallmatrix} A & C \\ \epsilon_k D & B \end{smallmatrix}\right)$ where $\{\epsilon_k\}$ is a bounded sequence. If $C = \epsilon_k I$ and A is self-adjoint, then T_{ϵ_k} is decomposable.*

Proof Since A is self-adjoint, so is $A_1 = P_{R(C)^\perp} A|_{\mathcal{H}}$. Thus it has the property (β). Since $\{\epsilon_k\}$ is a bounded sequence, $B_1 = B|_{N(C)}$ is decomposable, so it follows from Theorem 3.7 that T_{ϵ_k} is decomposable. □

Example 3.9 Let $T_{\epsilon_k} = \left(\begin{smallmatrix} A & U^* \\ \epsilon_k D & B \end{smallmatrix}\right)$ on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ where $\{\epsilon_k\}$ is a bounded sequence and U is the unilateral shift given by $Ue_n = e_{n+1}$ on $\ell^2(\mathbb{N})$ for $n \in \mathbb{N}$. Then $B_1 = B|_{N(U)}$ is decomposable and this is equivalent to T_{ϵ_k} being decomposable by Theorem 3.7.

Now, we address Weyl type theorems for T_{ϵ_k}. We start with the following lemma.

Lemma 3.10 *Let $T_{\epsilon_k} = \left(\begin{smallmatrix} A & C \\ \epsilon_k D & B \end{smallmatrix}\right)$ where $A, B, C, D \in \mathcal{L}(\mathcal{H})$ and $\{\epsilon_k\}$ is a positive sequence such that $\lim_{k\to\infty} \epsilon_k = 0$. Assume that A and B have the single-valued extension property. Then (a-)Browder's theorem holds for T_{ϵ_k}.*

Proof By Theorem 3.1, we know that T_{ϵ_k} has the single-valued extension property. Then it is obvious that $\sigma_w(T_{\epsilon_k}) = \sigma_b(T_{\epsilon_k})$ and $\sigma_{ea}(T_{\epsilon_k}) = \sigma_{ab}(T_{\epsilon_k})$ (see [1]). Hence this means that (a-)Browder's theorem holds for T_{ϵ_k}. □

Theorem 3.11 *Let $T_{\epsilon_k} = \left(\begin{smallmatrix} A & C \\ \epsilon_k D & B \end{smallmatrix}\right)$ where $A, B, C, D \in \mathcal{L}(\mathcal{H})$ and $\{\epsilon_k\}$ is a positive sequence such that $\lim_{k\to\infty} \epsilon_k = 0$. Suppose that Weyl's theorem holds for A and B. Then the following statements hold.*

(i) *If A and B have the single-valued extension property, then Weyl's theorem holds for T_{ϵ_k}.*
(ii) *If A^* and B^* have the single-valued extension property, then a-Weyl's theorem holds for T_{ϵ_k}.*

Proof (i) If A and B have the single-valued extension property, then it follows from Lemma 3.10 that $\sigma(T_{\epsilon_k}) \setminus \sigma_w(T_{\epsilon_k}) = p_{00}(T_{\epsilon_k}) \subseteq \pi_{00}(T_{\epsilon_k})$. To show the reverse, we suppose that $0 \in \pi_{00}(T_{\epsilon_k})$ without loss of generality. It follows from Theorem 3.4 that

$$0 \in \left[\pi_{00}(A) \setminus \sigma(B)\right] \cup \left[\pi_{00}(B) \setminus \sigma(A)\right] \cup \left[\pi_{00}(A) \cap \pi_{00}(B)\right].$$

Since Weyl's theorem holds for both A and B, we have $0 \notin \sigma_w(A) \cup \sigma_w(B)$. Set $T_0 := \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$. Then T_0 is Weyl by [11, Lemma 3]. This implies from [13, Theorem 1] that, if $T_{\epsilon_k} \to T_0$ in norm, then $\limsup_{k\to\infty} \sigma_w(T_{\epsilon_k}) \subset \sigma_w(T_0)$. Hence $0 \notin \limsup_{k\to\infty} \sigma_w(T_{\epsilon_k})$. So there exists $\delta_1 > 0$ such that, for $\mu \in D(0, \frac{\delta_1}{2})$, open disc with center 0 and radius $\frac{\delta_1}{2}$, such that $T_{\epsilon_k} - \mu I$ is Weyl. Since Weyl operators form an open set, there exists $\delta_2 > 0$ such that $\|T_{\epsilon_k} - \mu I - T_0\| < \frac{\delta_2}{2}$. We choose $\delta := \min\{\delta_1, \delta_2\} > 0$. Then

$$\|T_{\epsilon_k} - T_0\| \le \|T_{\epsilon_k} - \mu I - T_0\| + \|\mu I\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Therefore T_{ϵ_k} is Weyl but is not invertible. Consequently, Weyl's theorem holds for T_{ϵ_k}.

(ii) By Lemma 3.10, we have $\sigma_a(T_{\epsilon_k}) \setminus \sigma_{ea}(T_{\epsilon_k}) = p_{00}^a(T_{\epsilon_k}) \subseteq \pi_{00}^a(T_{\epsilon_k})$. We now suppose that $0 \in \pi_{00}^a(T_{\epsilon_k})$. From Theorem 3.4, we get

$$0 \in \left[\pi_{00}^a(A) \setminus \sigma_a(B)\right] \cup \left[\pi_{00}^a(B) \setminus \sigma_a(A)\right] \cup \left[\pi_{00}^a(A) \cap \pi_{00}^a(B)\right].$$

It is known that $\sigma(S) = \sigma_a(S)$ and $\sigma_w(S) = \sigma_{ea}(S)$ provided $S^* \in \mathcal{L}(\mathcal{H})$ has the single-valued extension property by [1]. On the other hand, A^* and B^* have the single-valued extension property, and satisfy Weyl's theorem. This implies that $0 \notin \sigma_w(A) \cup \sigma_w(B)$. Then T_0 is Weyl. Hence we get $0 \notin \limsup_{k\to\infty} \sigma_w(T_{\epsilon_k})$, so that $0 \in \sigma_a(T_{\epsilon_k}) \setminus \sigma_{ea}(T_{\epsilon_k})$. Therefore a-Weyl's theorem holds for T_{ϵ_k}. □

We say that $T \in \mathcal{L}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, *algebraically hyponormal* if there exists a nonconstant polynomial p such that $p(T)$ is hyponormal, respectively. It is known that normal operators imply hyponormal operators, and hyponormal operators imply algebraically hyponormal operators. From these notions, we have the following corollary.

Corollary 3.12 *Let* $T_{\epsilon_k} = \begin{pmatrix} A & C \\ \epsilon_k D & B \end{pmatrix}$ *where* $A, B, C, D \in \mathcal{L}(\mathcal{H})$ *and* $\{\epsilon_k\}$ *is a positive sequence such that* $\lim_{k\to\infty} \epsilon_k = 0$.

(i) *If A and B are normal, then a-Weyl's theorem holds for* T_{ϵ_k},

(ii) *If A and B are algebraically hyponormal, then Weyl's theorem holds for* T_{ϵ_k}.

Proof (i) It is obvious that normal operators are decomposable by [10]. So if A and B are normal, then their adjoint operators have the single-valued extension property. Moreover, A and B are hyponormal, hence it follows from [4] that they satisfy Weyl's theorem. Consequently, this means that a-Weyl's theorem holds for T_{ϵ_k} from Theorem 3.11.

(ii) If A and B are algebraically hyponormal, then so are their translation, and then it follows from [6, Lemma 1] that $A - \lambda I$ and $B - \lambda I$ have finite ascent for all complex number λ, so that they have the single-valued extension property. On the other hand, Weyl's theorem holds for both A and B from [6, Corollary 4]. Thus this implies from Theorem 3.11 that T_{ϵ_k} satisfies Weyl's theorem. □

Finally, we study 2×2 operator matrices

$$T_\gamma = \begin{pmatrix} A & C \\ \gamma I & B \end{pmatrix},$$

where γ is any scalar in $\mathbb{C}$. Let $\{T_\gamma\}'$ be the collection of operators commuting with T_γ as follows:

$$\{T_\gamma\}' = \left\{ \begin{pmatrix} L_\sigma & M_\sigma \\ N_\sigma & P_\sigma \end{pmatrix} : \sigma \in \Sigma \right\},$$

and let

$$\{T_\gamma\}_0' = \left\{ \begin{pmatrix} L_\sigma & M_\sigma \\ N_\sigma & P_\sigma \end{pmatrix} : \sigma \in \Sigma \text{ and } \sup_{\sigma\in\Sigma} \|L_\sigma - P_\sigma\| < \infty \right\}.$$

We recall that a transitive subalgebra of $\mathcal{L}(\mathcal{H})$ has the property that it has no nontrivial invariant subspace.

Theorem 3.13 *Let $T_{\epsilon_k} = \left(\begin{smallmatrix} A & C \\ \epsilon_k I & B \end{smallmatrix}\right)$ where $A, B, C \in \mathcal{L}(\mathcal{H})$ and $\{\epsilon_k\}$ is a positive sequence such that $\lim_{k\to\infty} \epsilon_k = 0$ and there is $X \in \mathcal{L}(\mathcal{H})$ such that $AX = XB$. If there exists a nontrivial hyperinvariant subspace $\mathcal{N}$ for B such that $\mathcal{N} \not\subset \ker X$, then $S \in \{T_{\epsilon_k}\}'_0$ has a nontrivial invariant subspace.*

Proof Assume that there exists a nontrivial hyperinvariant subspace $\mathcal{N}$ for B such that $\mathcal{N} \not\subset \ker X$. Let $S \in \{T_{\epsilon_n}\}'_0$. Then we put $S = \left(\begin{smallmatrix} L_\sigma & M_\sigma \\ N_\sigma & P_\sigma \end{smallmatrix}\right)$ where $\sigma \in \Sigma$ and $\sup_{\sigma\in\Sigma} \|L_\sigma - P_\sigma\| < \infty$. Since $S \in \{T_{\epsilon_n}\}'_0$, we get

$$\begin{pmatrix} L_\sigma A + \epsilon_n M_\sigma & L_\sigma C + M_\sigma B \\ N_\sigma A + \epsilon_n P_\sigma & N_\sigma C + P_\sigma B \end{pmatrix} = \begin{pmatrix} AL_\sigma + CN_\sigma & AM_\sigma + CP_\sigma \\ \epsilon_n L_\sigma + BN_\sigma & \epsilon_n M_\sigma + BP_\sigma \end{pmatrix}.$$

Then we have $BN_\sigma - N_\sigma A = \epsilon_n(P_\sigma - L_\sigma)$ and so

$$\|BN_\sigma - N_\sigma A\| = \epsilon_n \|P_\sigma - L_\sigma\| \le \epsilon_n \sup_{\sigma\in\Sigma} \|P_\sigma - L_\sigma\|.$$

Since $\lim_{n\to\infty} \epsilon_n = 0$, $BN_\sigma = N_\sigma A$ for $\sigma \in \Sigma$. Hence $BN_\sigma X = N_\sigma AX = N_\sigma XB$ for $\sigma \in \Sigma$. So $N_\sigma X\mathcal{N} \subset \mathcal{N}$ for $\sigma \in \Sigma$. On the other hand, assume, to obtain a contradiction, that $\{T_{\epsilon_n}\}'_0$ is transitive. Then, for arbitrary $z \in \mathcal{H}$, it follows from [7, Proposition 2.2] and the hypothesis that there exist $\sigma_0 \in \Sigma$ and $y \in \mathcal{N}$ with $Xy \neq 0$ such that, for every $\epsilon > 0$,

$$\|N_{\sigma_0} Xy - z\| < \epsilon,$$

which means that $\overline{\{N_\sigma Xy : \sigma \in \Sigma\}} = \mathcal{H}$ for some $\sigma_0 \in \Sigma$. But this is a contradiction from $N_\sigma Xy \in \mathcal{N}$ for all $\sigma \in \Sigma$. Hence $\{T_{\epsilon_n}\}'_0$ is not transitive. Thus $S \in \{T_{\epsilon_n}\}'_0$ has nontrivial invariant subspace. □

We easily see that there exists a nontrivial hyperinvariant subspace $\mathcal{N}$ for B such that $\mathcal{N} \nsubseteq \ker X$ as the following example.

Example 3.14 Let $N \in \mathcal{L}(\mathcal{H})$ be a normal operator with $N \neq \lambda I$ for $\lambda \in \mathbb{C}$. Consider an operator matrix $\left(\begin{smallmatrix} N & C \\ \epsilon_k I & N \end{smallmatrix}\right)$ and $\{\epsilon_k\}$ is a positive sequence such that $\lim_{k\to\infty} \epsilon_k = 0$. Then there exists a nontrivial hyperinvariant subspace $\mathcal{N}$ for N but $\mathcal{N} \nsubseteq \ker I$. Hence $S \in \left\{\left(\begin{smallmatrix} N & C \\ \epsilon_k I & N \end{smallmatrix}\right)\right\}'_0$ has a nontrivial invariant subspace by Theorem 3.13.

Corollary 3.15 *Let $T_{\delta_k} = \left(\begin{smallmatrix} A & \delta_k I \\ Z & B \end{smallmatrix}\right)$ where $A, B, Z \in \mathcal{L}(\mathcal{H})$ and δ_k is a positive sequence such that $\lim_{n\to\infty} \delta_k = 0$ and there is $X \in \mathcal{L}(\mathcal{H})$ such that $BX = XA$. If there exists a nontrivial hyperinvariant subspace $\mathcal{M}$ for A such that $\mathcal{M} \not\subset \ker X$, then $S \in \{T_{\delta_k}\}'_0$ has nontrivial invariant subspace.*

Proof Set $R_{\delta_k} = \left(\begin{smallmatrix} B & Z \\ \delta_k I & A \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) R_{\delta_k} \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) = T_{\delta_k}$, R_{δ_k} and T_{δ_k} are unitarily equivalent. Since $W \in \{R_{\delta_k}\}'_0$ has a nontrivial invariant subspace by Theorem 3.13,

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} W \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} T_{\delta_k} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} WR_{\delta_k} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} R_{\delta_k} W \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
$$= T_{\delta_k} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} W \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Thus $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) W \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \in \{T_{\delta_k}\}'$. Since $W \in \{R_{\delta_k}\}'_0$,

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} W \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \{T_{\delta_k}\}'_0.$$

Since $W \in \{R_{\delta_k}\}'_0$ has a nontrivial invariant subspace, we conclude that

$$S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} W \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \{T_{\delta_k}\}'_0$$

has a nontrivial invariant subspace. □

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Author details
[1]Department of Applied Mathematics, Kyung Hee University, Yongin, Gyeonggi-do 17104, Korea. [2]Department of Mathematics, Ewha Womans University, Seoul, 120-750, Korea. [3]Department of Mathematics and Statistics, Sejong University, Seoul, 143-747, Korea.

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