A Jackson-type inequality associated with wavelet bases decomposition

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Abstract

Although wavelet decompositions of functions in Besov spaces have been extensively investigated, those involved with mild decay bases are relatively unexplored. In this paper, we study wavelet bases of Besov spaces and the relation between norms and wavelet coefficients. We establish the l^p-stability as a measure of how effectively the Besov norm of a function is evaluated by its wavelet coefficients and the L^p-completeness of wavelet bases. We also discuss wavelets with decay conditions and establish the Jackson inequality.

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1 Introduction

As any Besov space can be continuously imbedded into a Lebesgue space, we are interested in characterizing the L^p-completeness of certain bases and spaces. The L^p-completeness of a basis refers to the unconditional summability of the basic decomposition. In view of Theorem 2.1, we utilize either the Calderón–Zygmund operators (CZOs) or directly the Calderón–Zygmund decomposition theorem (CZD). Some authors use CZOs, asserting that unconditional summability can be obtained under smooth assumptions; see, for example, [1, Thm. 3.3], [6, 10, 11], [13, Thms. 9.1.5–9.1.6], [23, Ch. 5, Thms. 6.14 and 6.23], [29, Sect. 7.3, Thm. 1], and [28, Ch. 6]. Other authors (see, e.g., [20, 21, 31]) prefer CZD, wondering how the smoothness of wavelets characterizes Lebesgue spaces.

The unconditional summability is a key problem in characterizing Lebesgue spaces and thus Besov spaces. The authors of [2, 3, 18] suggest that the smoothness and regularity are necessary for the Besov unconditional summability. This evolves more technicality in the case of Lebesgue spaces. The present author contends that the decay on wavelets suffices for a positive answer. Our approach involves two parts: characterizing Lebesgue spaces by wavelet coefficients and the design for middle class (3.1), which depends only on wavelet coefficients. We will associate the L^p-unconditional summability to a special design for equivalent Besov norms. We will see that the l^p-stability for functions in a Lebesgue space can provide an alternative framework for Besov-unconditional summability, as well as Jackson inequality. We remark that the use of wavelet bases is motivated by [21, Rem. 2.2],

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[28, pp. 130, 136], and [30]. However, canonical dual bases without wavelet structures may not contribute to the unconditional summability.

This paper is divided into four sections. After this introduction, Sect. 2 collects some preliminary facts, especially about the L^p-boundedness of affine operators. Section 3 presents a number of applications of bounded affine operators, including completeness, l^p-stability of biorthogonal Riesz wavelet bases, middle class $\mathcal{K}'_{\tau,\tau}$, the Jackson inequality, and characterization of Besov spaces. Discussion and conclusions are given in Sect. 4.

2 Preliminaries

We use the following notation:

$$A \lesssim B \quad \text{if } A \leq cB \text{ for some positive constant } c, \text{ and}$$

$$A \sim B \quad \text{if } A \lesssim B, \text{ and } B \lesssim A.$$

Let X, Y be two quasi-normed spaces. By writing $X \hookrightarrow Y$ we mean that X is continuously embedded in Y; in other words, we can think of $X \subset Y$ and $\| \cdot \|_Y \lesssim \| \cdot \|_X$. A sequence $\{f_i\}_{i\in\mathbb{Z}} \in L^p(\mathbb{R})$ is said to be *l^p-stable* if

$$\left\| \sum_{i\in\mathbb{Z}} a_i f_i \right\|_p \sim \|a\|_{l^p(\mathbb{Z})} \quad \text{for some } a = \{a_i\}_{i\in\mathbb{Z}} \in l^\infty(\mathbb{Z}).$$

Recall that the Lorentz space $l^{p,q}(\mathbb{N})$, $0 < p,q < \infty$ [24, p. 955], consists of sequences $a = \{a_i\}_{i\in\mathbb{N}}$ satisfying $\|a\|_{l^{p,q}(\mathbb{N})} < \infty$, where

$$\|a\|_{l^{p,q}(\mathbb{N})} := \begin{cases} [\sum_{i\in\mathbb{N}}(i^{\frac{1}{p}-\frac{1}{q}}a_i^*)^q]^{\frac{1}{q}}, & 0 < q < \infty, \\ \sup_{i\geq 0} i^{\frac{1}{p}}a_i^*, & q = \infty, \end{cases}$$

where

$$a_1^* := \|a\|_{l^\infty(\mathbb{N})} \quad \text{and} \quad a_i^* := \max_{|J_i|=i} \sum_{j\in J_i} |x_j| - \sum_{j=1}^{n-1} x_j^* \quad \text{for } i > 1.$$

For $p = q$, we have $l^{p,q}(\mathbb{N}) = l^p(\mathbb{N})$ with equivalent norms. The spaces are ordered lexicographically. More precisely, let $0 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \leq \infty$. Then

$$l^{p_1,q_1} \subsetneq l^{p_2,q_2} \quad \text{if } p_1 < p_2, \text{ or if } p_1 = p_2 \text{ and } q_1 < q_2.$$

A sequence $\{f_i\}_{i\in\mathbb{Z}}$ is said to be *$l^{p,q}$-Hilbertian* [19, p. 59] in a Banach space X if $\sum_{i\in\mathbb{Z}} a_i f_i$ converges unconditionally in X and $\|\sum_{i\in\mathbb{Z}} a_i f_i\|_X \lesssim \|a\|_{l^{p,q}(\mathbb{Z})}$ for any $a = \{a_i\}_{i\in\mathbb{Z}} \in l^{p,q}(\mathbb{Z})$, $0 < p,q \leq \infty$.

For $f : \mathbb{R} \to \mathbb{C}$, $h \in \mathbb{R}$, and $r \in \mathbb{N}$, the *differences of rth order* and the *rth modulus of smoothness* of $f \in L^p(\mathbb{R})$, $0 < p \leq \infty$ [16, p. 44], are given inductively by

$$\bigwedge_h^r f(\cdot) := \bigwedge_h^1 \bigwedge_h^{r-1} f(\cdot) = \sum_{i=0}^r \binom{r}{i}(-1)^{r-i}f(\cdot + ih),$$

$$w_r(f,t)_p := \sup_{0<h\le t}\left\|\bigtriangleup_h^r f\right\|_p,$$

where $\bigtriangleup_h^1 f(\cdot) := f(\cdot+h)-f(\cdot)$.

Given $\alpha > 0$, $r := [\![\alpha]\!] + 1$, and $0 < p$, $\tau \le \infty$, the Besov space $\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))$ [16, p. 54] is the set of functions $f \in L^p(\mathbb{R})$ such that

$$|f|_{\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))} := \begin{cases} [\int_0^\infty (t^{-\alpha} w_r(f,t)_p)^\tau \frac{dt}{t}]^{\frac{1}{\tau}}, & 0 < \tau < \infty, \\ \sup_{t>0} t^{-\alpha} w_r(f,t)_p, & \tau = \infty, \end{cases}$$

is finite. The norm for $\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))$ is $\|f\|_{\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))} := |f|_{\mathcal{B}_\tau^\alpha(L^p(\mathbb{R}))} + \|f\|_p$.

In this paper, we study $\mathcal{B}^\alpha := \mathcal{B}_\tau^\alpha(L^\tau(\mathbb{R}))$ with Schauder basis, where $\frac{1}{\tau} = \alpha + \frac{1}{p}$, $1 < p < \infty$.

Theorem 2.1 ([22, Theorem 2.8]) *Let* $\{x_n\}$ *be a sequence in a Banach space. The following statements are equivalent*:

(1) $\sum x_n$ *converges unconditionally.*

(2) $\sum \epsilon_n x_n$ *converges for every choice of signs* $\epsilon_n = \pm 1$.

(3) $\sum \lambda_n x_n$ *converges for every bounded sequence of scalars* $\{\lambda_n\}$.

A sequence $\{\psi_i : i \in \mathbb{Z}\}$ (not necessarily linearly independent) is a *frame* for $L^2(\mathbb{R})$ if there exist two constants $0 < A \le B < \infty$ such that

$$A\|f\|_2^2 \le \sum_{i\in\mathbb{Z}} \left|\langle f, \psi_i \rangle\right|^2 \le B\|f\|_2^2 \quad \text{for all } f \in L^2(\mathbb{R}).$$

Here A and B are called *frame bounds*. If $A = B$, then we call this a *tight frame*. A *Riesz basis* is a frame that consists of linear independent basic vectors. A sequence $\{\psi_i : i \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$ if for any $\{c_i\} \in l^\infty$,

$$A\sum_{i\in\mathbb{Z}}|c_i|^2 \le \left\|\sum_{i\in\mathbb{Z}} c_i \psi_i\right\|_2^2 \le B\sum_{i\in\mathbb{Z}}|c_i|^2$$

for some positive constants A, B. The *frame operator* S of $\{\psi_i : i \in \mathbb{Z}\}$ is defined by

$$Sf := \sum_{i\in\mathbb{Z}}\langle f, \psi_i \rangle \psi_i, \tag{2.1}$$

and each $f \in L^2(\mathbb{R})$ has the decomposition

$$f = \sum_{i\in\mathbb{Z}}\langle f, S'\psi_i \rangle \psi_i = \sum_{i\in\mathbb{Z}}\langle f, \psi_i \rangle S'\psi_i,$$

which converges unconditionally in $L^2(\mathbb{R})$ [8, p. 90–91]. The sequence $\{S'\psi_i : i \in \mathbb{Z}\}$ is called the *canonical dual* of $\{\psi_i : i \in \mathbb{Z}\}$.

The *preframe operator* $T : l^2(\mathbb{N}) \to L^2(\mathbb{R})$ is a bounded linear operator defined by

$$T\{c_i\}_{i\in\mathbb{Z}} := \sum_{i\in\mathbb{Z}} c_i \psi_i.$$

Note that $S = TT^*$. Both S and S' are of type $(2,2)$, bounded, invertible, self-adjoint, and positive on $L^2(\mathbb{R})$. The sequence $\{S'\psi_i : i \in \mathbb{Z}\}$ is also a frame for $L^2(\mathbb{R})$, and its frame operator is S'. The canonical dual frame $\{S'\psi_i : i \in \mathbb{Z}\}$ of a tight frame is simply $\{\frac{1}{A}\psi_i : i \in \mathbb{Z}\}$. More details can be found in [8, 21].

The *affine wavelet frame system* $\{\psi_{j,k}\} = \{\psi_{j,k} : j, k \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$ generated by ψ is defined by

$$\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x + k\right).$$

The sequence $\{\psi_{j,k}^{(p)}\} = \{\psi_{j,k}^{(p)} : j, k \in \mathbb{Z}\}$ is the *primal wavelet frame* of $\{\psi_{j,k}\}$, where

$$\psi_{j,k}^{(p)} = 2^{j/p}\psi\left(2^j x + k\right).$$

In general,

$$S'D^j T_k \psi = D^j S' T_k \psi,$$

$$D^j S' T_k \psi \neq D^j T_k S' \psi,$$

where $D^j(\cdot)(x) := 2^{j/2}(\cdot)(2^j x)$ and $T_k(\cdot)(x) := (\cdot)(x + k)$ (see, e.g., [8, p. 276]). In this case, we say that the canonical dual frame $\{S'\psi_{j,k}\}$ of $\{\psi_{j,k}\}$ *does not have wavelet structure.*

Two sequences $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ form a pair of *dual wavelet frames/biframes* ([8, p. 277], [9, 14]) if both are frames for $L^2(\mathbb{R})$ and

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, \widetilde{\psi}_{j,k} \rangle \psi_{j,k} \quad \text{for all } f \in L^2(\mathbb{R}).$$

They form a pair of *biorthogonal (Riesz) wavelet bases* if they are also Riesz bases for $L^2(\mathbb{R})$. Obviously, orthogonal wavelet bases for $L^2(\mathbb{R})$ are also biorthogonal Riesz wavelet bases.

A wavelet has zero mean [10, p. 433] and is in L^2. In addition, compactly supported or exponential decay orthogonal wavelets cannot belong to $C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ ([17], [23, Sect. 4.6, 6]). More precisely, if such a wavelet exists, then it must be the zero function on $\mathbb{R}$.

We say that $\psi \in \mathcal{M}^\tau$ if

$$\int_0^\infty \left[\log_2(1 + x)\right]\Psi^\tau(x)\,dx < \infty, \tag{2.2}$$

where $\Psi(x) := \sup_{0 \le x \le |y|} |\psi|(y)$, and $\tau > 0$. Let L_ψ be a constant such that $\psi \in \mathcal{M}^\tau$ for all $\tau \in (L_\psi, \infty)$ and $\psi \notin \mathcal{M}^\tau$ for all $\tau \in [0, L_\psi]$. We have

$$0 < \sum_{r \in \mathbb{N}} \Psi^\tau(r) \lesssim \sum_{r \in \mathbb{N}} [\![\log_2 r]\!]\Psi^\tau(r) \sim \int_1^\infty \left[\log_2(1 + x)\right]\Psi^\tau(x)\,dx \tag{2.3}$$

$$\le \int_0^\infty \left[\log_2(1 + x)\right]\Psi^\tau(x)\,dx.$$

The infinite integrals either both converge or both diverge. If ψ is compactly supported, then ψ belongs to $\mathcal{M}^\tau$ for all $\tau \in (0, \infty)$.

Denote $\mathcal{F}_{\psi,\mathcal{M}^\tau} := \{\psi_{j,k} : j, k \in \mathbb{Z}, \tau > 0\}$, which is a frame for $L^2(\mathbb{R})$ with $\psi \in \mathcal{M}^\tau$. Let $\mathcal{F}$ be a sequence in $L^p(\mathbb{R})$ and consider the set of all possible m-term expansions with elements from $\mathcal{F}$,

$$\Upsilon_{m,\mathcal{F}} := \left\{ \sum_{i \in \Gamma} c_i f_i : f_i \in \mathcal{F}, c_i \in \mathbb{C}, \operatorname{card} \Gamma \le m \right\}. \tag{2.4}$$

We denote the error of the best m-term approximation to $f \in L^p(\mathbb{R})$ by

$$\sigma_m(f, \mathcal{F}) := \inf_{f_m \in \Upsilon_{m,\mathcal{F}}} \|f - f_m\|_p. \tag{2.5}$$

Let $\mathcal{F}_\psi := \{\psi_{j,k} : j, k \in \mathbb{Z}\}$ be a frame for $L^2(\mathbb{R})$. For $p \in (1, \infty)$ and $\tau, q \in (0, p)$, we define

$$\mathcal{K}_{\tau,q}\big(L^p(\mathbb{R}), \mathcal{F}_\psi\big) := \left\{ f \in L^p(\mathbb{R}) : \exists \{c_{j,k}\} \in l^{\tau,q}, f = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k}^{(p)} \right\}, \tag{2.6}$$

where $|f|_{\mathcal{K}_{\tau,q}(L^p(\mathbb{R}),\mathcal{F}_\psi)}$ is the smallest norm $\|\{c_{j,k}\}\|_{l^{\tau,q}}$ such that $\sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k}^{(p)}$ is convergent unconditionally in $L^p(\mathbb{R})$ and $\{\psi_{j,k}^{(p)}\}$ is $l^{p,1}$-Hilbertian. See Theorem 3.2, [2, Rem. 4.3] and [19, Prop. 3] for more detail.

Lemma 2.2 *Let $\psi, \rho \in \mathcal{M}^1$ be wavelets. Let λ_m, λ'_m, and Λ be the operators given by*

$$\lambda_m : (b_k)_{k \in \mathbb{Z}} \mapsto \left(\sum_{k \in \mathbb{Z}} b_k \int_{\mathbb{R}} 2^{j'} \rho\big(2^j x + k\big) \bar{\psi}\big(2^{j'} x + k'\big) \, dx \right)_{k' \in \mathbb{Z}}, \tag{2.7}$$

$$\lambda'_m : (b_k)_{k \in \mathbb{Z}} \mapsto \left(\sum_{k \in \mathbb{Z}} b_k \int_{\mathbb{R}} 2^j \rho\big(2^j x + k\big) \bar{\psi}\big(2^{j'} x + k'\big) \, dx \right)_{k' \in \mathbb{Z}}, \tag{2.8}$$

for $j, j' \in \mathbb{Z}$, $m = j - j'$, and

$$\Lambda : (c_{j,k})_{j,k \in \mathbb{Z}} \mapsto \left(\sum_{j,k \in \mathbb{Z}} c_{j,k} \big\langle \rho_{j,k}^{(p)}, \psi_{j',k'}^{(p')} \big\rangle \right)_{j',k' \in \mathbb{Z}} \tag{2.9}$$

for $p \in [1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then λ_m and λ'_m are bounded on $l^\tau(\mathbb{Z})$ for any $1 \le \tau \le \infty$, and Λ is bounded on $l^\tau(\mathbb{Z} \times \mathbb{Z})$ for any $1 \le \tau \le \infty$. In particular, if ψ, ρ are compactly supported, then Λ is bounded on $l^\tau(\mathbb{Z} \times \mathbb{Z})$ for all $\tau \in (0, \infty]$.

Proof The boundedness of λ_m and λ'_m follows from the Riesz–Thorin interpolation theorem [16, p. 32]. Let $\varrho(\cdot) := \sup_{0 \le \cdot \le |y|} |\rho|(y)$. For $\{b_k\}_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$ and $2^j x + k = u + k'$, we have

$$\big\| \lambda_m(b_k) \big\|_{l^1(\mathbb{Z})}$$

$$= \sum_{k' \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} b_k \int_{\mathbb{R}} 2^{j'} \rho\big(2^j x + k\big) \bar{\psi}\big(2^{j'} x + k'\big) \, dx \right|$$

$$\le \sum_{k' \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |b_k| \left(\int_{-\infty}^0 + \int_0^\infty \right) 2^{-m} |\rho|\big(u + k'\big) |\bar{\psi}|\big(2^{-m}\big(u + k' - k\big) + k'\big) \, du$$

$$\leq \sum_{k\in\mathbb{Z}} |b_k| \left(\int_{-\infty}^{0} + \int_{0}^{\infty} \right) 2^{-m} \left[\sum_{k'\in\mathbb{Z}} |\rho|(u+k') \right] |\bar{\psi}| \left(2^{-m}(u+k'-k) + k' \right) du$$

$$\lesssim \|\psi\| \sum_{k\in\mathbb{Z}} |b_k|.$$

Indeed, for given $u \geq 0$ such that $u = N_u + l_u$, $N_u \in \mathbb{N} \cup \{0\}$, $0 \leq l_u < 1$,

$$\sum_{k'\in\mathbb{Z}} |\rho|(u+k') = \left(\sum_{\substack{k'\in\mathbb{Z} \\ u+k'\geq 0}} + \sum_{\substack{k'\in\mathbb{Z} \\ u+k'<0}} \right) |\rho|(N_u + l_u + k')$$

$$= \sum_{\substack{k'\in\mathbb{Z} \\ u+k'\geq 0}} |\rho|(N_u + l_u + k') + \sum_{\substack{k'\in\mathbb{Z} \\ u+k'<0}} |\rho|(N_u + k' + 1 + (l_u - 1))$$

$$\leq \left[\sum_{\substack{r_1\in\mathbb{N} \\ u+k'\geq 0, r_1 = N_u + k'}} \varrho(r_1) + \sum_{\substack{r_2\in\mathbb{N} \\ u+k'<0, r_2 = -(N_u+k'+1)}} \varrho(r_2) \right] + 2\varrho(0)$$

$$\leq 2\varrho(0) + 2 \sum_{r_3\in\mathbb{N}} \varrho(r_3) < \infty.$$

The finiteness of $\sum_{k'\in\mathbb{Z}} |\rho|(u+k')$, $u < 0$, can also be obtained by setting $u = N_u' + l_u'$, $N_u' \in \mathbb{Z}^- \cup \{0\}$, $-1 < l_u' \leq 0$.

Similarly, let us address $\{b_k\}_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, and let $v = 2^j x$. We obtain

$$\|\lambda_m(b_k)\|_{l^\infty(\mathbb{Z})}$$

$$= \sup_{k'\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} b_k \int_{\mathbb{R}} 2^{j'} \rho(2^j x + k) \bar{\psi}(2^{j'} x + k') dx \right|$$

$$\leq \sup_{k'\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} |b_k| \left(\int_{-\infty}^{0} + \int_{0}^{\infty} \right) 2^{-m} |\rho|(v+k) |\bar{\psi}| (2^{-m}v + k') dv \right]$$

$$\lesssim \|\{b_k\}\|_{l^\infty(\mathbb{Z})}.$$

So λ_m is also bounded on $l^1(\mathbb{Z})$ and $l^\infty(\mathbb{Z})$. By the Riesz–Thorin interpolation theorem, λ_m is bounded on $l^\tau(\mathbb{Z})$ for any $1 < \tau < \infty$. Similarly, λ_m' is bounded on $l^\tau(\mathbb{Z})$ for any $1 \leq \tau \leq \infty$ by using the method described above.

For the statement on $\Lambda, \tau \geq 1$, we need to show that

$$\sum_{\substack{j',k'\in\mathbb{Z}}} \left| \sum_{\substack{j,k\in\mathbb{Z} \\ j\leq j'}} c_{j,k} \langle \rho_{j,k}^{(p)}, \psi_{j',k'}^{(p')} \rangle + \sum_{\substack{j,k\in\mathbb{Z} \\ j>j'}} c_{j,k} \langle \rho_{j,k}^{(p)}, \psi_{j',k'}^{(p')} \rangle \right|^\tau \tag{2.10}$$

$$\lesssim \|\{c_{j,k}\}\|_{l^\tau(\mathbb{Z}\times\mathbb{Z})}^\tau.$$

The idea of proving (2.10) comes from [3, p. 25 Lemma B.2]. However, both the hypotheses and the strategies are quite different. A proof is included here. For $\tau \geq 1$, $\frac{1}{\tau} + \frac{1}{\tau'} = 1$, the

Hölder inequality for the summation over j, together with the results for λ_m and λ'_m, yields

$$
\sum_{j',k'\in\mathbb{Z}}\left|\sum_{\substack{j,k\in\mathbb{Z}\\ j\leq j'}}c_{j,k}\langle\rho_{j,k}^{(p)},\psi_{j',k'}^{(p')}\rangle\right|^{\tau}
$$

$$
=\sum_{j',k'\in\mathbb{Z}}\left|\sum_{\substack{j\in\mathbb{Z}\\ j\leq j'}}2^{\frac{m}{p}(\frac{1}{\tau}+\frac{1}{\tau'})}\sum_{k\in\mathbb{Z}}c_{j,k}\int_{\mathbb{R}}2^{j'}\rho\left(2^{j}x+k\right)\bar{\psi}\left(2^{j'}x+k'\right)dx\right|^{\tau}
$$

$$
\leq\sum_{j',k'\in\mathbb{Z}}\left\{\left(\sum_{\substack{j\in\mathbb{Z}\\ j\leq j'}}2^{\frac{m}{p}}\right)^{\frac{1}{\tau'}}\left[\sum_{\substack{j\in\mathbb{Z}\\ j\leq j'}}2^{\frac{m}{p}}\left|\sum_{k\in\mathbb{Z}}c_{j,k}\int_{\mathbb{R}}2^{j'}\rho\left(2^{j}x+k\right)\bar{\psi}\left(2^{j'}x+k'\right)dx\right|^{\tau}\right]^{\frac{1}{\tau}}\right\}^{\tau}
$$

$$
\lesssim\sum_{j',k'\in\mathbb{Z}}\sum_{\substack{j\in\mathbb{Z}\\ j\leq j'}}2^{\frac{m}{p}}\left|\sum_{k\in\mathbb{Z}}c_{j,k}\int_{\mathbb{R}}2^{j'}\rho\left(2^{j}x+k\right)\bar{\psi}\left(2^{j'}x+k'\right)dx\right|^{\tau}
$$

$$
\lesssim\sum_{j'\in\mathbb{Z}}\sum_{\substack{m\in\mathbb{Z}\\ m\leq 0}}2^{\frac{m}{p}}\sum_{k\in\mathbb{Z}}|c_{m+j',k}|^{\tau}\lesssim\left\|\{c_{j,k}\}\right\|_{l^{\tau}(\mathbb{Z}\times\mathbb{Z})}^{\tau}.
$$

Similar arguments give

$$
\sum_{j',k'\in\mathbb{Z}}\left|\sum_{\substack{j,k\in\mathbb{Z}\\ j>j'}}c_{j,k}\langle\rho_{j,k}^{(p)},\psi_{j',k'}^{(p')}\rangle\right|^{\tau}
$$

$$
=\sum_{j',k'\in\mathbb{Z}}\left|\sum_{\substack{j\in\mathbb{Z}\\ j>j'}}2^{\frac{-m}{p'}(\frac{1}{\tau}+\frac{1}{\tau'})}\sum_{k\in\mathbb{Z}}c_{j,k}\int_{\mathbb{R}}2^{j}\rho\left(2^{j}x+k\right)\bar{\psi}\left(2^{j'}x+k'\right)dx\right|^{\tau}
$$

$$
\leq\sum_{j',k'\in\mathbb{Z}}\left\{\left(\sum_{\substack{j\in\mathbb{Z}\\ j>j'}}2^{\frac{-m}{p'}}\right)^{\frac{1}{\tau'}}\left[\sum_{\substack{j\in\mathbb{Z}\\ j>j'}}2^{\frac{-m}{p'}}\left|\sum_{k\in\mathbb{Z}}c_{j,k}\int_{\mathbb{R}}2^{j}\rho\left(2^{j}x+k\right)\bar{\psi}\left(2^{j'}x+k'\right)dx\right|^{\tau}\right]^{\frac{1}{\tau}}\right\}^{\tau}
$$

$$
\lesssim\sum_{j',k'\in\mathbb{Z}}\sum_{\substack{j\in\mathbb{Z}\\ j>j'}}2^{\frac{-m}{p'}}\left|\sum_{k\in\mathbb{Z}}c_{j,k}\int_{\mathbb{R}}2^{j}\rho\left(2^{j}x+k\right)\bar{\psi}\left(2^{j'}x+k'\right)dx\right|^{\tau}
$$

$$
\lesssim\sum_{j'\in\mathbb{Z}}\sum_{m\in\mathbb{N}}2^{\frac{-m}{p'}}\sum_{k\in\mathbb{Z}}|c_{m+j',k}|^{\tau}\lesssim\left\|\{c_{j,k}\}\right\|_{l^{\tau}(\mathbb{Z}\times\mathbb{Z})}^{\tau}.
$$

By the Minkowski inequality, (2.10) holds.

For $0<\tau<1$, we have the following inequality:

$$
\sum_{j',k'\in\mathbb{Z}}\left|\sum_{j,k\in\mathbb{Z}}c_{j,k}\langle\rho_{j,k}^{(p)},\psi_{j',k'}^{(p')}\rangle\right|^{\tau}\leq\sum_{j',k'\in\mathbb{Z}}\sum_{j,k\in\mathbb{Z}}|c_{j,k}|^{\tau}\left|\langle\rho_{j,k}^{(p)},\psi_{j',k'}^{(p')}\rangle\right|^{\tau}
$$

$$
=\sum_{j',k'\in\mathbb{Z}}\left(\sum_{\substack{j,k\in\mathbb{Z}\\ j>j'}}+\sum_{\substack{j,k\in\mathbb{Z}\\ j\leq j'}}\right)|c_{j,k}|^{\tau}\left|\langle\rho_{j,k}^{(p)},\psi_{j',k'}^{(p')}\rangle\right|^{\tau}. \tag{2.11}
$$

Suppose that $w = 2^{j'}x$ and ψ is compactly supported on $E, 0 < |E| < \infty$. Then we can estimate the first sum appearing in (2.11) by

$$\sum_{\substack{j',k' \in \mathbb{Z} \\ j > j'}} \sum_{j,k \in \mathbb{Z}} |c_{j,k}|^{\tau} |\langle \rho_{j,k}^{(p)}, \psi_{j',k'}^{(p')} \rangle|^{\tau}$$

$$\leq \sum_{\substack{j',k' \in \mathbb{Z} \\ j > j'}} \sum_{j,k \in \mathbb{Z}} |c_{j,k}|^{\tau} \left\{ \int_{D} 2^{\frac{j}{p}-j(\frac{1}{p}+\frac{1}{p'})+\frac{j'}{p'}} \left[2^{j}2^{-j'} |\rho| (2^{m}w + k) \right] \cdot |\psi| (w + k') \, dw \right\}^{\tau}$$

$$\leq \|\rho\|_{1}^{\tau} \sum_{j' \in \mathbb{Z}} \sum_{m \in \mathbb{N}} 2^{\frac{-m\tau}{p'}} \left[\sum_{k' \in \mathbb{N}} \sup_{w \in E} |\psi|^{\tau} (w + k') \right] \sum_{k \in \mathbb{Z}} |c_{m+j',k}|^{\tau}$$

$$\lesssim \| \{c_{j,k}\} \|_{l^{\tau}(\mathbb{Z} \times \mathbb{Z})}^{\tau}.$$

For the second summation in (2.11), by a similar argument we have

$$\sum_{\substack{j',k' \in \mathbb{Z} \\ j \leq j'}} \sum_{j,k \in \mathbb{Z}} |c_{j,k}|^{\tau} |\langle \rho_{j,k}^{(p)}, \psi_{j',k'}^{(p')} \rangle|^{\tau}$$

$$\leq \sum_{\substack{j',k' \in \mathbb{Z} \\ j \leq j'}} \sum_{j,k \in \mathbb{Z}} |c_{j,k}|^{\tau} \left\{ \int_{\mathbb{R}} 2^{\frac{j}{p}-j'(\frac{1}{p}+\frac{1}{p'})+\frac{j'}{p'}} |\rho| (2^{j}x + k) \left[2^{j'} |\psi| (2^{j'}x + k') \right] dx \right\}^{\tau}$$

$$\lesssim \sum_{j' \in \mathbb{Z}} \sum_{\substack{m \in \mathbb{Z} \\ m \leq 0}} 2^{\frac{m\tau}{p}} \sum_{k \in \mathbb{Z}} |c_{m+j',k}|^{\tau}$$

$$\lesssim \| \{c_{j,k}\} \|_{l^{\tau}(\mathbb{Z} \times \mathbb{Z})}^{\tau}. \qquad \square$$

3 Biorthogonal Riesz wavelet bases in Besov spaces

Theorem 3.1 is established in our earlier work [21, Thm. 3.3]. It serves as a base for understanding the L^p-unconditional summability. There several consequences of Theorem 3.1. First, the frame and synthesis operators are L^p-bounded. Second, biorthogonal Riesz wavelet bases are unconditionally summable in Lebesgue spaces.

Theorem 3.1 *Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair of biorthogonal Riesz wavelet bases for $L^2(\mathbb{R})$.*

(1) The operator S associated with $\mathcal{F}_{\psi,\mathcal{M}^1}$ is L^p-bounded and bijective on $L^p(\mathbb{R})$ for all $1 < p < \infty$.

(2) Both $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ are bases for $L^p(\mathbb{R})$, $1 < p < \infty$.

Despite the feasibility of biframes, we paid the price of the linear independence, and we can deduce from Theorem 3.1 that the generator for the canonical dual bases requires mild decay to confirm the bijectivity of the frame operator on $L^p(\mathbb{R})$, $1 < p < \infty$. Secondly, the conditions for Theorem 3.1 and Theorem 3.2 have been improved significantly in comparison with the results in [2, 3, 6], [27, Thm. 1.1] and [26, Thm. 1.1].

Here we give two examples. Local commutant biorthogonal bases can be found in [5]. The last example refers to those discovered by Lemvig and Bownik [4, p. 219, Example] and [25] and is a family of band-limited wavelet frames. More information on (bi)orthonormal bases can be found in [21, Sect. 4].

Next, we characterize Lebesgue spaces by wavelet coefficients. For $f \in L^p(\mathbb{R})$, Theorem 3.2 guarantees $\{\langle f, \widetilde{\psi}_{j,k}^{(p')}\rangle\} \in l^p(\mathbb{Z} \times \mathbb{Z})$, which leads characterization of the middle class $\mathcal{K}_{\tau,\tau}(L^p(\mathbb{R}), \mathcal{F}.)$ depending only on wavelet coefficients. The middle class is designed to contribute to understanding the relationship between the best m-term approximation and Besov norms.

Theorem 3.2 *Let $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\mathcal{F}_{\psi, \mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi}, \mathcal{M}^1}$ be a pair of biorthogonal Riesz wavelet bases for $L^2(\mathbb{R})$.*

(1) Both primal wavelet bases $\{\psi_{j,k}^{(p)}\}$ and $\{\widetilde{\psi}_{j,k}^{(p')}\}$ are unconditional and $l^{p,1}$-Hilbertian in $L^p(\mathbb{R})$ for all $1 < p < \infty$. Moreover, for any $f = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k}^{(p)} \in L^p(\mathbb{R})$,

$$\|f\|_p \lesssim \|c_{j,k}\|_{l^p} \lesssim \|c_{j,k}\|_{l^\tau} \quad \text{for all } 1 < p < \infty, \tau \in (0, p).$$

(2) For $\tau \in [1, p)$, $\mathcal{K}_{\tau,\tau}(L^p(\mathbb{R}), \mathcal{F}_{\psi, \mathcal{M}^1}) = \mathcal{K}'_{\tau,\tau}(L^p(\mathbb{R}), \mathcal{F}_{\psi, \mathcal{M}^1})$ with equivalent norms, where

$$\mathcal{K}'_{\tau,\tau}(L^p(\mathbb{R}), \mathcal{F}_{\psi, \mathcal{M}^1}) \tag{3.1}$$
$$:= \{f \in L^p(\mathbb{R}) : |f|_{\mathcal{K}'_{\tau,\tau}(L^p(\mathbb{R}), \mathcal{F}_{\psi, \mathcal{M}^1})} := \|\{\langle f, \widetilde{\psi}_{j,k}^{(p')}\rangle\}\|_{l^\tau} < \infty\}.$$

Moreover, if ψ and $\widetilde{\psi}$ are compactly supported, then $\mathcal{K}_{\tau,\tau}(L^p(\mathbb{R}), \mathcal{F}_\psi) = \mathcal{K}'_{\tau,\tau}(L^p(\mathbb{R}), \mathcal{F}_\psi)$ with equivalent norms for all $\tau \in (0, p)$.

Proof (1) We consider

$$R(\cdot) := \sum_{j,k \in \mathbb{Z}} \epsilon_{j,k} \langle \cdot, \widetilde{\psi}_{j,k}\rangle \psi_{j,k}$$
$$= \sum_{j,k \in \mathbb{Z}} \epsilon_{j,k} \langle \cdot, \widetilde{\psi}_{j,k}^{(p')}\rangle \psi_{j,k}^{(p)}, \quad \epsilon_{j,k} = \pm 1, \frac{1}{p} + \frac{1}{p'} = 1, 1 < p < \infty.$$

Utilizing Theorem 3.1 and the L^p-boundedness of R, we obtain the unconditional summability for $\{\psi_{j,k}^{(p)}\}$ in $L^p(\mathbb{R})$. Note that $l^p = l^{p,p}$ with equivalent norms and $l^\tau \subset l^{p,1} \subset l^{p,p}$, $\tau \in (0, p)$. Let $f = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi_{j,k}^{(p)} \in L^p(\mathbb{R})$, and let $t = 2^j x$ for a given j. The Hölder inequality for the summation over k yields

$$\int_{\mathbb{R}} \left[\sum_{k \in \mathbb{Z}} |c_{j,k}| 2^{\frac{j}{p}} |\psi|^{\frac{1}{p}} (2^j x + k) |\psi|^{\frac{1}{p'}} (2^j x + k) \right]^p dx$$
$$\leq \int_{\mathbb{R}} \left[\sum_{k \in \mathbb{Z}} |c_{j,k}|^p |\psi|(t + k) \right] \left[\sum_{k \in \mathbb{Z}} |\psi|(t + k) \right]^{\frac{p}{q}} dt$$
$$\lesssim \|\psi\|_1 \sum_{k \in \mathbb{Z}} |c_{j,k}|^p.$$

Thus, by the Minkowski inequality, $\|f\|_p \lesssim \|c_{j,k}\|_{l^p} \lesssim \|c_{j,k}\|_{l^{p,1}} \lesssim \|c_{j,k}\|_{l^\tau}$. Indeed, the finiteness of $\sum_{k \in \mathbb{Z}} |\psi|(t + k)$ can be found in Lemma 2.2. Similar arguments can be made for $\{\widetilde{\psi}_{j,k}^{(p')}\}$.

(2) Again, applying Theorem 3.1, we see that both $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ are bases for $L^p(\mathbb{R})$, $1 < p < \infty$. By (1) and (2.6) we have

$$\mathcal{K}'_{\tau,\tau}\left(L^p(\mathbb{R}),\mathcal{F}_{\psi,\mathcal{M}^1}\right) \hookrightarrow \mathcal{K}_{\tau,\tau}\left(L^p(\mathbb{R}),\mathcal{F}_{\psi,\mathcal{M}^1}\right) \quad \text{for all } \tau > 0.$$

By the admissible hypotheses and the range of τ we have

$$f = \sum_{j',k'\in\mathbb{Z}} c_{j',k'}\psi^{(p)}_{j',k'} \in L^p(\mathbb{R})$$

with $\{c_{j',k'}\} \in l^{\tau,\tau}$. It follows that

$$\langle f, \widetilde{\psi}^{(p')}_{j,k}\rangle = \sum_{j',k'\in\mathbb{Z}} c_{j',k'}\langle \psi^{(p)}_{j',k'}, \widetilde{\psi}^{(p')}_{j,k}\rangle, \quad j,k\in\mathbb{Z}.$$

From Lemma 2.2 we obtain $\|\{\langle f, \widetilde{\psi}^{(p')}_{j,k}\rangle\}\|_{l^{\tau,\tau}} \lesssim \|\{c_{j',k'}\}\|_{l^{\tau,\tau}}$. Thus

$$\mathcal{K}_{\tau,\tau}\left(L^p(\mathbb{R}),\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}\right) \hookrightarrow \mathcal{K}'_{\tau,\tau}\left(L^p(\mathbb{R}),\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}\right).$$

Finally, the compactly supported cases can be done similarly and are skipped. □

We next characterize Besov spaces and the (general) Jackson inequality with respect to the wavelet bases $(\rho, \widetilde{\rho})$, which already gain Besov-unconditional summability ([2, Cor. 4.9], [15, Lemma 4.2], [18, Thm. 3.8]). Fortunately, such wavelets do exist (e.g., biorthogonal compactly supported basis [7, Sect. 6.6, Thm. 4], [12]; Daubechies and Meyer wavelets).

Theorem 3.3 *Let $\mathcal{F}_{\psi,\mathcal{M}^1}$ and $\mathcal{F}_{\widetilde{\psi},\mathcal{M}^1}$ be a pair of biorthogonal Riesz wavelet bases for $L^2(\mathbb{R})$. Then, given $1 < p < \infty$, $\tau \in (0,p)$, and $\alpha = \frac{1}{\tau} - \frac{1}{p}$, we have:*
(1) For all $m \in \mathbb{N}$, $q \in (0,p)$, and $f \in \mathcal{K}_{\tau,q}(L^p(\mathbb{R}),\mathcal{F}_{\psi,\mathcal{M}^1})$,

$$\sigma_m(f,\mathcal{F}_{\psi,\mathcal{M}^1}) \lesssim m^{-\alpha}|f|_{\mathcal{K}_{\tau,q}(L^p(\mathbb{R}),\mathcal{F}_{\psi,\mathcal{M}^1})}. \tag{3.2}$$

(2) For all $m \in \mathbb{N}$, $\tau \in [1,p)$, and $g \in \mathcal{B}^\alpha$,

$$\sigma_m(g,\mathcal{F}_{\psi,\mathcal{M}^1}) \lesssim m^{-\alpha}|g|_{\mathcal{B}^\alpha}. \tag{3.3}$$

Moreover, if ψ and $\widetilde{\psi}$ are compactly supported, then

$$\sigma_m(g,\mathcal{F}_\psi) \lesssim m^{-\alpha}|g|_{\mathcal{B}^\alpha}, \quad \tau \in (0,p). \tag{3.4}$$

(3) Let the hypotheses and the admissible range of τ in (2) hold. For all $h \in \mathcal{B}^\alpha$, the wavelet frame expansion

$$h = \sum_{j,k\in\mathbb{Z}} \langle h, \widetilde{\psi}^{(p')}_{j,k}\rangle \psi^{(p)}_{j,k} \tag{3.5}$$

converges unconditionally in $L^p(\mathbb{R})$, and

$$|h|_{\mathcal{B}^\alpha} \sim \left\| \langle h, \widetilde{\psi}_{j,k}^{(p')} \rangle \right\|_{l^\tau(\mathbb{Z}\times\mathbb{Z})}, \qquad \mathcal{K}_{\tau,\tau}'\left(L^p(\mathbb{R}), \mathcal{F}\cdot\right) = \mathcal{B}^\alpha, \tag{3.6}$$

with equivalent norms.

Proof (1) Let $C := \{c_{j,k}\}$, and let $f = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}^{(p)} \in \mathcal{K}_{\tau,q}(L^p(\mathbb{R}), \mathcal{F}_{\psi,\mathcal{M}^1})$ with $\|C\|_{l^{\tau,q}} < \infty$. Let Γ be a finite set with card $\Gamma = m$, and let $C_\Gamma := \{c_{j,k} : j,k \in \Gamma\}$ be the m largest elements from $\{|c_{j,k}|\}$. By Theorem 3.2(1), for $\alpha = \frac{1}{\tau} - \frac{1}{p} > 0$, we have

$$\left\| f - \sum_{j,k\in\Gamma} c_{j,k}\psi_{j,k}^{(p)} \right\|_p \lesssim \|C_{\Gamma^c}\|_p \lesssim \|C_{\Gamma^c}\|_{p,p} \lesssim \|C_{\Gamma^c}\|_{p,q}$$

$$= \left[\sum_{i=m+1}^\infty \left(i^{\frac{1}{p}-\frac{1}{q}} c_i^* \right)^q \right]^{\frac{1}{q}} = \left[\sum_{i=m+1}^\infty \left(i^{\frac{1}{\tau}-\alpha-\frac{1}{q}} c_i^* \right)^q \right]^{\frac{1}{q}}$$

$$\le m^{-\alpha} \left[\sum_{i=m+1}^\infty \left(i^{\frac{1}{\tau}-\frac{1}{q}} c_i^* \right)^q \right]^{\frac{1}{q}} \le m^{-\alpha} \|C\|_{l^{\tau,q}}.$$

By (2.5) and (2.6) we get (3.2).

(2) Let the reference wavelets $(\rho, \widetilde{\rho})$ satisfy the hypotheses with the following properties. For all $g \in \mathcal{B}^\alpha \subset L^p(\mathbb{R})$, $\alpha = \frac{1}{\tau} - \frac{1}{p}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$, the expansion of g in the reference wavelet system is given by

$$g = \sum_{j',k'\in\mathbb{Z}} c_{j',k'}\rho_{j',k'}^{(p)} \in L^p(\mathbb{R}),$$

which converges unconditionally with coefficients $D' := \{d_{j',k'} : j',k' \in \mathbb{Z}\}$ satisfying $|g|_{\mathcal{B}^\alpha} \sim \|D'\|_{l^\tau(\mathbb{Z}\times\mathbb{Z})}$ for all $\tau \in (0,p)$. Note that

$$\langle g, \widetilde{\psi}_{j,k}^{(p')} \rangle = \sum_{j',k'\in\mathbb{Z}} d_{j',k'} \langle \rho_{j',k'}^{(p)}, \widetilde{\psi}_{j,k}^{(p')} \rangle, \quad j,k \in \mathbb{Z}. \tag{3.7}$$

With the admissible hypotheses and the range of τ, applying Lemma 2.2 and (3.7), we deduce that

$$\left\| \{ \langle g, \widetilde{\psi}_{j,k}^{(p')} \rangle \} \right\|_{l^\tau(\mathbb{Z}\times\mathbb{Z})} \lesssim \|D'\|_{l^\tau(\mathbb{Z}\times\mathbb{Z})} \lesssim |g|_{\mathcal{B}^\alpha}. \tag{3.8}$$

Combining (1), (3.8), and Theorem 3.2(2), we see that the Jackson inequality holds:

$$\sigma_m(g,\cdot) \lesssim m^{-\alpha} |g|_{\mathcal{K}_{\tau,\tau}(L^p(\mathbb{R}),\cdot)} \lesssim m^{-\alpha} \left\| \{ \langle g, \widetilde{\psi}_{j,k}^{(p')} \rangle \} \right\|_{l^\tau(\mathbb{Z}\times\mathbb{Z})} \lesssim |g|_{\mathcal{B}^\alpha}. \tag{3.9}$$

(3) With the admissible hypotheses and the range of τ, applying Theorem 3.2(1) and (3.8), we see that the synthesis operator U and the analysis operator U' given by

$$U : l^\tau(\mathbb{Z}\times\mathbb{Z}) \mapsto L^p(\mathbb{R}), \quad \{c_{j,k}\} \in l^\tau(\mathbb{Z}\times\mathbb{Z}) \mapsto \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}^{(p)}$$

and

$$U' : \mathcal{B}^\alpha \mapsto l^\tau(\mathbb{Z} \times \mathbb{Z}), \quad h \in \mathcal{B}^\alpha \mapsto \{\langle h, \widetilde{\psi}_{j,k}^{(p')}\rangle\}$$

are bounded, and thus UU' is bounded. Since $\mathcal{B}^\alpha$ are dense in both $L^p(\mathbb{R})$ and $L^2(\mathbb{R})$, any function h has a wavelet frame expansion

$$h = \sum_{j,k \in \mathbb{Z}} \langle h, \widetilde{\psi}_{j,k}^{(p')}\rangle \psi_{j,k}^{(p)} \in L^p(\mathbb{R}).$$

The boundedness of UU' guarantees that the sum converges unconditionally in $L^p(\mathbb{R})$.

By (3.8) we have both $\|\{\langle h, \widetilde{\psi}_{j,k}^{(p')}\rangle\}\|_{l^\tau(\mathbb{Z} \times \mathbb{Z})} \lesssim |h|_{\mathcal{B}^\alpha}$ and $\mathcal{B}^\alpha \hookrightarrow \mathcal{K}_{\tau,\tau}'(L^p(\mathbb{R}), \mathcal{F}.)$. Employing the reference wavelets $(\rho, \widetilde{\rho})$ again, for all $h \in \mathcal{B}^\alpha \subset L^p(\mathbb{R})$, $\alpha = \frac{1}{\tau} - \frac{1}{p}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$, with the admissible hypotheses and the range of τ, we see that

$$h = \sum_{j',k' \in \mathbb{Z}} \langle h, \widetilde{\rho}_{j',k'}^{(p')}\rangle \rho_{j',k'}^{(p)} \in L^p(\mathbb{R})$$

with coefficients satisfying $|h|_{\mathcal{B}^\alpha} \sim \|\{\langle h, \widetilde{\rho}_{j',k'}^{(p')}\rangle\}\|_{l^\tau(\mathbb{Z} \times \mathbb{Z})}$.

For $\{\langle h, \widetilde{\rho}_{j',k'}^{(p')}\rangle\} \in l^\tau(\mathbb{Z} \times \mathbb{Z})$, by the same argument as in the proof of (2) we can obtain. It follows that

$$\langle h, \widetilde{\rho}_{j',k'}^{(p')}\rangle = \sum_{j,k \in \mathbb{Z}} \langle \psi_{j,k}^{(p)}, \widetilde{\rho}_{j',k'}^{(p')}\rangle \langle h, \widetilde{\psi}_{j,k}^{(p')}\rangle, \quad j', k' \in \mathbb{Z}.$$

Applying Lemma 2.2, we have

$$|h|_{\mathcal{B}^\alpha} \lesssim \|\{\langle h, \widetilde{\rho}_{j',k'}^{(p')}\rangle\}\|_{l^\tau(\mathbb{Z} \times \mathbb{Z})} \lesssim \|\langle h, \widetilde{\psi}_{j,k}^{(p')}\rangle\|_{l^\tau(\mathbb{Z} \times \mathbb{Z})}.$$

This leads to $\mathcal{K}_{\tau,\tau}'(L^p(\mathbb{R}) \text{ and } \mathcal{F}.) \hookrightarrow \mathcal{B}^\alpha$. Consequently, $\mathcal{K}_{\tau,\tau}'(L^p(\mathbb{R}), \mathcal{F}.) = \mathcal{B}^\alpha$, and (3.6) is proved. $\qquad\square$

4 Discussion and conclusions

1. We compare Lemma 2.2 with [3, Prop. 5.2] and [18, Thm. 3.1]. With regard to the existence of a lower bound for τ, our findings confirm those of Borup et al., although there are important differences in the approaches. This study makes *no* assumption on the smoothness or regularity. However, the authors of [3] seem to suggest that the smoothness and regularity conditions are necessary. It may be due to the unconditional summability in certain function spaces and technical requirement in [3, Lemma B.3]. Similar situations happen in [2, 18]. Aforementioned papers reported that τ has a lower bound, which cannot be improved in general. However, some statements there are not clear. Special attention is given to the following:

(1) We mention the example in [3, p. 27, Prop. B.4]. The reason is that this proposition may not be in line with Lemma 2.2. The arguments emerge. The lower bound of τ in Proposition B.4 is inapplicable to [3, p. 11, Prop. 5.4]. It is not proper to assume that the wavelet η is in $C^\infty(\mathbb{R})$ and compactly supported. The wavelet should have zero mean [10, p. 433] and belong to L^2. In addition, compactly supported or exponential

decay orthogonal wavelets cannot belong to $C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ ([17], [23, Sect. 4.6, 6]). Otherwise, such a wavelet must be the zero function on $\mathbb{R}$.

(2) ψ, ρ satisfying $\mathcal{M}^1$ are sufficient for the boundedness of Λ. In contrast, we consider a stronger condition, the compact supportedness on Λ, $0 < \tau < 1$.

2. It is worth pointing out that $L := 0$ for both wavelets is compactly supported. In such cases, Lemma 2.2 provides not only an easy way to get a lower bound of τ, but also a "better" lower bound in comparison with the aforementioned papers.

3. The following problems are open to us.

(1) No specific value of L for more general wavelets is given in this paper. Thus the condition $L = 0$ for reference wavelets in Theorem 3.3 is critical.

(2) How to relax the condition of being compactly supported, and a possible point lies on the finiteness for $\sum_{k' \in \mathbb{N}} \sup_{w \in E} |\psi|^\tau (w + k')$ in Lemma 2.2.

4. If both ψ and $\widetilde{\psi}$ are compactly supported, then the admissible range for τ in Theorem 3.3 will be extended to $(0, \infty)$. One of the best examples is the Haar wavelet, which has no decay.

Finally, we point out that many authors assume the smoothness/regularity/CZOs, making it difficult to apply the results to wavelets with decay only. Few studies so far have attempted to account for the feasibility of a wavelet with decay, but only to Besov norms and the Jackson inequality. Our Theorem 3.3 closes this gap.

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