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Convergence and stability of an iteration process and solution of a fractional differential equation

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Abstract

In this paper, we prove that a three-step iteration process is stable for contractive-like mappings. It is also proved analytically and numerically that the considered process converges faster than some remarkable iterative processes for contractive-like mappings. Furthermore, some convergence results are proved for the mappings satisfying Suzuki's condition (C) in uniformly convex Banach spaces. A couple of nontrivial numerical examples are presented to support the main results and the visualization is showed by Matlab. Finally, by utilizing our main result the solution of a nonlinear fractional differential equation is approximated.

MSC: 47H09; 47H10; 54H25

Keywords: Suzuki's condition (C); Contractive-like mapping; Iteration processes; Fixed point; Fractional differential equation; Uniformly convex Banach space

1 Introduction

Throughout this paper, $\mathbb{Z}^+$ denotes the set of all nonnegative integers. We assume that $\mathcal{U}$ is a nonempty subset of a Banach space $\mathcal{W}$ and $F(\mathcal{F}) = \{t \in \mathcal{U} : \mathcal{F} : \mathcal{U} \to \mathcal{U} \text{ and } \mathcal{F}t = t\}$. A mapping $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ is called non-expansive if $\|\mathcal{F}x - \mathcal{F}y\| \leq \|x - y\|$, $\forall x, y \in \mathcal{U}$. It is said to be a quasi-non-expansive if $F(\mathcal{F}) \neq \emptyset$ and $\|\mathcal{F}x - t\| \leq \|x - t\|$, $\forall x \in \mathcal{U}$ and $\forall t \in F(\mathcal{F})$.

Hardy and Rogers [10] introduced generalized non-expansive mapping which is defined as follows:

A self-map $\mathcal{F}$ on $\mathcal{U}$ is called generalized non-expansive if for all $x, y \in \mathcal{U}$ there exist real numbers $a, b, c \geq 0$ with $a + 2b + 2c \leq 1$ such that

$$\|\mathcal{F}x - \mathcal{F}y\| \leq a\|x - y\| + b\big[\|x - \mathcal{F}x\| + \|y - \mathcal{F}y\|\big] + c\big[\|x - \mathcal{F}y\| + \|y - \mathcal{F}x\|\big]. \tag{1.1}$$

It can be easily verified that if $F(\mathcal{F}) \neq \emptyset$, then $\mathcal{F}$ is a quasi-non-expansive mapping but the converse is not true in general.

In 2008, Suzuki [22] defined a condition on the mappings, called condition (C); such mappings are also known as generalized non-expansive mappings.

A mapping $\mathcal{F}:\mathcal{U}\to\mathcal{U}$ is said to satisfy condition (C) if,

$$\frac{1}{2}\|x-\mathcal{F}x\|\le\|x-y\| \quad\Longrightarrow\quad \|\mathcal{F}x-\mathcal{F}y\|\le\|x-y\|,\quad \forall x,y\in\mathcal{U}.$$

Suzuki [22] proved existence and convergence theorems for such mappings. He also exhibited that every non-expansive mapping satisfies condition (C), but the reverse is not true in general. Moreover, if $F(\mathcal{F})\neq\emptyset$ and satisfies condition (C) then it is a quasi-non-expansive mapping. Recently, a number of researchers studied the fixed points of Suzuki generalized non-expansive mappings; e.g. see [3, 5, 8, 23, 24].

The generalized non-expansive mappings coined by Hardy and Rogers, and Suzuki are generalizations of non-expansive mappings. So, most recently, Ali et al. [2] compared the classes of mappings due to Suzuki, and Hardy and Rogers and showed that the two classes of mappings do not imply each other. They also presented two examples to verify their claim.

In 2003, Imoru and Olantiwo [11] defined the class of contractive-like mappings which is wider than the classes of contractions, Zamfirescu mappings, weak contractions, etc. They also proved that the Picard and Mann iteration processes are stable with respect to contractive-like mappings. The definition of contractive-like mapping runs as follows.

Definition 1.1 ([11]) Let $\varphi:[0,\infty)\to[0,\infty)$ be a strictly increasing and continuous function with $\varphi(0)=0$ and a constant $\delta\in[0,1)$. A self-map $\mathcal{F}$ on $\mathcal{U}$ is said to be contractive-like if, for all $x,y\in\mathcal{U}$, we have

$$\|\mathcal{F}x-\mathcal{F}y\|\le\delta\|x-y\|+\varphi\big(\|x-\mathcal{F}x\|\big).$$

During approximation of fixed points, the better speed of convergence of iteration process saves time. Berinde [7] gave the following definitions to compare the rate of convergence of iteration processes.

Definition 1.2 Let $(\mathcal{W},\|\cdot\|)$ be a normed space and $\mathcal{F}:\mathcal{W}\to\mathcal{W}$ a mapping. Suppose that the two fixed point iteration processes $\{\tau_n\}$ and $\{\sigma_n\}$ converge to the same point t. Furthermore, assume that the error estimates

$$\|\tau_n-t\|\le\alpha_n,$$
$$\|\sigma_n-t\|\le\beta_n,$$

are available (and these estimates are the best ones available), where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of nonnegative real numbers that converge to 0. Now, in order to compare the two fixed point sequences $\{\tau_n\}$ and $\{\sigma_n\}$ in $\mathcal{W}$, it suffices to compare the two sequences of real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ converging to 0. For this, one can use the following concept of rate of convergence of two sequences given by Berinde [7].

Definition 1.3 Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative real numbers that converge to x and y, respectively. Assume that

$$\ell=\lim_{n\to\infty}\frac{|\alpha_n-x|}{|\beta_n-y|}.$$

(i) If $\ell = 0$, then $\{\alpha_n\}$ converges to x faster than $\{\beta_n\}$ to y.
(ii) If $0 < \ell < \infty$, then $\{\alpha_n\}$ and $\{\beta_n\}$ have the same rate of convergence.

We say that the given fixed point iteration process is stable if and only if the approximate sequence $\{t_n\}$ still converges to the fixed point of $\mathcal{F}$. To approximate fixed points of the mappings, we consider an approximate sequence $\{t_n\}$ instead of the theoretical sequence $\{\tau_n\}$, because of rounding errors and numerical approximation of functions. In view of this fact, Ostrowski [18] was first to coin the concept of stability for a fixed point iteration process and proved that Picard iteration process is stable for contraction mapping. The definition of stability due to Ostrowski runs as follows.

Definition 1.4 ([18]) Consider an approximate sequence $\{t_n\}$ in a subset $\mathcal{U}$ of a Banach space $\mathcal{W}$. Then an iteration procedure $\tau_{n+1} = f(\mathcal{F}, \tau_n)$ is said to be $\mathcal{F}$-stable or stable with respect to $\mathcal{F}$ for some function f, converging to a fixed point t, if for $\epsilon_n = \|t_{n+1} - f(\mathcal{F}, t_n)\|$, $n \in \mathbb{Z}^+$, we have $\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = t$.

In the last three decades, the study of fixed point iteration processes has taken an eminent place in the fixed point theory and applied mathematics. The iteration processes are used to solve initial and boundary value problems, image recovery problems, image restoration problems, image processing problems, variational inequality problems, functional equations [19] etc. Thus, several authors introduced and studied remarkable iteration processes to approximate the fixed point of different nonlinear mappings. The following iteration processes have been introduced by Mann [15], Ishikawa [12], Agrawal et al. (S) [1], Gursoy and Karakaya (Picard-S) [9] and Noor [16], respectively. Here the sequence $\{\tau_n\}$ with an initial guess $\tau_0 \in \mathcal{U}$ for the self-mapping $\mathcal{F}$ on $\mathcal{U}$ is defined as follows:

$$\begin{cases} \tau_{n+1} = (1-\theta_n)\tau_n + \theta_n \mathcal{F}\tau_n, \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.2}$$

$$\begin{cases} \tau_{n+1} = (1-\theta_n)\tau_n + \theta_n \mathcal{F}\sigma_n, \\ \sigma_n = (1-\mu_n)\tau_n + \mu_n \mathcal{F}\tau_n, \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.3}$$

$$\begin{cases} \tau_{n+1} = (1-\theta_n)\mathcal{F}\tau_n + \theta_n \mathcal{F}\sigma_n, \\ \sigma_n = (1-\mu_n)\tau_n + \mu_n \mathcal{F}\tau_n, \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.4}$$

$$\begin{cases} \tau_{n+1} = \mathcal{F}\sigma_n, \\ \sigma_n = (1-\theta_n)\mathcal{F}\tau_n + \theta_n \mathcal{F}\xi_n, \\ \xi_n = (1-\mu_n)\tau_n + \mu_n \mathcal{F}\tau_n, \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.5}$$

$$\begin{cases} \tau_{n+1} = (1-\theta_n)\tau_n + \theta_n \mathcal{F}\sigma_n, \\ \sigma_n = (1-\mu_n)\tau_n + \mu_n \mathcal{F}\xi_n, \\ \xi_n = (1-\gamma_n)\tau_n + \gamma_n \mathcal{F}\tau_n, \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.6}$$

where the sequences $\{\theta_n\}$, $\{\mu_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$.

Most recently, Ali et al. [2] introduced a new iteration process, called JF iteration process and approximated the fixed points of Hardy and Rogers generalized non-expansive mappings in uniformly convex Banach spaces. In this process, the sequence $\{\tau_n\}$ is generated by an initial guess $\tau_0 \in \mathcal{U}$ and defined as follows:

$$\begin{cases} \tau_{n+1} = \mathcal{F}((1-\theta_n)\sigma_n + \theta_n \mathcal{F}\sigma_n), \\ \sigma_n = \mathcal{F}\xi_n, \\ \xi_n = \mathcal{F}((1-\mu_n)\tau_n + \mu_n \mathcal{F}\tau_n), \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.7}$$

where $\{\theta_n\}$ and $\{\mu_n\}$ are in $(0,1)$. They claimed numerically that JF iteration process converges to the fixed point of Hardy and Rogers mappings faster than some well-known iteration processes. They also approximated the solution of a delay differential equation via JF iteration process.

Motivated by the above, we prove the stability and rate of convergence of the JF iteration process for contractive-like mappings. We also prove some convergence results for Suzuki generalized non-expansive mappings via the JF iteration process in uniformly convex Banach spaces. In the last section, we estimate the solution of a nonlinear fractional differential equation via the JF iteration process. A couple of illustrative numerical examples are presented to validate the results. The results of this paper are remarkable from the point of view of the results of Ali et al. [2] and extend several relevant results in the literature.

2 Preliminaries

This section contains some lemmas, propositions and definitions that will be used in the main results.

Lemma 2.1 ([6]) *Let $\{\epsilon_n\}$ and $\{u_n\}$ be sequences of positive real numbers satisfying $u_{n+1} \leq \delta u_n + \epsilon_n$, $n \in \mathbb{Z}^+$, where $\delta \in [0,1)$. If $\lim_{n\to\infty} \epsilon_n = 0$ then $\lim_{n\to\infty} u_n = 0$.*

Lemma 2.2 ([22]) *Let $\mathcal{U}$ be a weakly compact convex subset of a uniformly convex Banach space $\mathcal{W}$ and $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ be a mapping satisfying Suzuki's condition (C). Then $\mathcal{F}$ has a fixed point.*

Lemma 2.3 ([22]) *Let $\mathcal{U}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{W}$ and $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ a mapping satisfying Suzuki's condition (C). If $\{\tau_n\}$ converges weakly to $t \in \mathcal{U}$ and $\lim_{n\to\infty} \|\tau_n - \mathcal{F}\tau_n\| = 0$, then $\mathcal{F}t = t$ implies $I - \mathcal{F}$ is demiclosed at zero.*

Lemma 2.4 ([20]) *Let $\mathcal{W}$ be a uniformly convex Banach space and $0 < a \leq s_n \leq b < 1$ for all $n \geq 1$. Let $\{\tau_n\}$ and $\{\sigma_n\}$ be two sequences in $\mathcal{W}$ such that $\limsup_{n\to\infty} \|\tau_n\| \leq w$, $\limsup_{n\to\infty} \|\sigma_n\| \leq w$ and $\lim_{n\to\infty} \|s_n\tau_n + (1-s_n)\sigma_n\| = w$ holds, for some $w \geq 0$. Then $\lim_{n\to\infty} \|\tau_n - \sigma_n\| = 0$.*

Proposition 2.5 ([22]) *Let $\mathcal{U}$ be a nonempty subset of a Banach space $\mathcal{W}$ and $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ be a mapping satisfying condition (C). Then*

$$\|x - \mathcal{F}y\| \leq 3\|\mathcal{F}x - x\| + \|x - y\|, \quad \forall x, y \in \mathcal{U}.$$

Definition 2.6 A Banach space $\mathcal{W}$ is said to satisfy Opial's property [17] if for any sequence $\{\tau_n\} \subset \mathcal{W}$ with $\{\tau_n\} \rightharpoonup x$ ($\rightharpoonup$ denotes weak convergence) it implies that

$$\liminf_{n\to\infty} \|\tau_n - x\| < \liminf_{n\to\infty} \|\tau_n - y\|$$

holds, for all $y \in \mathcal{W}$ with $y \neq x$.

Definition 2.7 ([21]) A self-map $\mathcal{F}$ on $\mathcal{U}$ is said to satisfy condition (I), if there exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\psi(z) > 0$, $\forall z > 0$ such that $d(x, \mathcal{F}x) \geq \psi(d(x, F(\mathcal{F})))$, $\forall x \in \mathcal{U}$.

Definition 2.8 Let $\mathcal{W}$ be a Banach space and let $\mathcal{U}$ a nonempty, closed and convex subset of $\mathcal{W}$, $\{\tau_n\}$ a bounded sequence in $\mathcal{W}$ and for $x \in \mathcal{U}$,

$$r\bigl(x, \{\tau_n\}\bigr) = \limsup_{n\to\infty} \|\tau_n - x\|.$$

The asymptotic radius of $\{\tau_n\}$ relative to $\mathcal{U}$ is defined by

$$r\bigl(\mathcal{U}, \{\tau_n\}\bigr) = \inf\bigl\{r\bigl(x, \{\tau_n\}\bigr) : x \in \mathcal{U}\bigr\}.$$

The asymptotic center of $\{\tau_n\}$ relative to $\mathcal{U}$ is defined by

$$A\bigl(\mathcal{U}, \{\tau_n\}\bigr) = \bigl\{x \in \mathcal{U} : r\bigl(x, \{\tau_n\}\bigr) = r\bigl(\mathcal{U}, \{\tau_n\}\bigr)\bigr\}.$$

It is known that if $\mathcal{W}$ is a uniformly convex Banach space, then $A(\mathcal{U}, \{\tau_n\})$ contains only one point.

3 Rate of convergence and stability results for contractive-like mappings

Throughout this section, we presume that $\mathcal{U}$ is a nonempty, closed and convex subset of a Banach space $\mathcal{W}$ and $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ a contractive-like mapping. The purpose of this section is to prove stability and convergence results for contractive-like mappings via JF iteration process.

Theorem 3.1 *Let $\{\tau_n\}$ be an iteration process defined by* (1.7). *Then iteration process* (1.7) *is $\mathcal{F}$-stable.*

Proof Suppose $\{t_n\}$ is an arbitrary sequence in $\mathcal{U}$ and $t_{n+1} = f(\mathcal{F}, t_n)$ is the sequence generated by (1.7) and $\epsilon_n = \|t_{n+1} - f(\mathcal{F}, t_n)\|$ for all $n \in \mathbb{Z}^+$. We have to prove that $\lim_{n\to\infty} \epsilon_n = 0 \iff \lim_{n\to\infty} t_n = t$.

Suppose $\lim_{n\to\infty} \epsilon_n = 0$, then by iteration process (1.7), we have

$$\begin{aligned} \|t_{n+1} - t\| &\leq \bigl\|t_{n+1} - f(\mathcal{F}, t_n)\bigr\| + \bigl\|f(\mathcal{F}, t_n) - t\bigr\| \\ &\leq \epsilon_n + \bigl\|f(\mathcal{F}, t_n) - t\bigr\| \\ &\leq \epsilon_n + \delta^2\bigl(1 - (1-\delta)\theta_n\bigr)\bigl(1 - (1-\delta)\mu_n\bigr)\|t_n - t\|. \end{aligned} \tag{3.1}$$

Since $0 < (1-(1-\delta)\theta_n) \leq 1$ and $0 < (1-(1-\delta)\mu_n) \leq 1$ and using (3.1), we get

$$\|t_{n+1} - t\| \leq \epsilon_n + \delta^2 \|t_n - t\|.$$

Define $u_n = \|t_n - t\|$, then

$$u_{n+1} \leq \delta^2 u_n + \epsilon_n.$$

Since $\lim_{n\to\infty} \epsilon_n = 0$, so by Lemma 2.1, we have $\lim_{n\to\infty} u_n = 0$ *i.e.*, $\lim_{n\to\infty} t_n = t$.

Conversely, suppose $\lim_{n\to\infty} t_n = t$, we have

$$\begin{aligned}
\epsilon_n &= \bigl\|t_{n+1} - f(\mathcal{F}, t_n)\bigr\| \\
&\leq \|t_{n+1} - t\| + \bigl\|f(\mathcal{F}, t_n) - t\bigr\| \\
&\leq \|t_{n+1} - t\| + \delta^2\bigl(1-(1-\delta)\theta_n\bigr)\bigl(1-(1-\delta)\mu_n\bigr)\|t_n - t\| \\
&\leq \|t_{n+1} - t\| + \delta^2\|t_n - t\|.
\end{aligned}$$

This implies that $\lim_{n\to\infty} \epsilon_n = 0$. Hence iteration process (1.7) is $\mathcal{F}$- stable. □

Theorem 3.2 *Let $F(\mathcal{F}) \neq \emptyset$ and $\{\tau_n\}$ be a sequence defined by* (1.7), *then $\{\tau_n\}$ converges faster than the iteration processes* (1.2)-(1.6).

Proof From (1.5), for any $t \in F(\mathcal{F})$, we have

$$\begin{aligned}
\|\xi_n - t\| &= \bigl\|(1-\mu_n)\tau_n + \mu_n \mathcal{F}\tau_n - t\bigr\| \\
&\leq (1-\mu_n)\|\tau_n - t\| + \mu_n\delta\|\tau_n - t\| \\
&= \bigl(1-(1-\delta)\mu_n\bigr)\|\tau_n - t\|.
\end{aligned} \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned}
\|\sigma_n - t\| &= \bigl\|(1-\theta_n)\mathcal{F}\tau_n + \theta_n \mathcal{F}\xi_n - t\bigr\| \\
&\leq (1-\theta_n)\delta\|\tau_n - t\| + \theta_n\delta\|\xi_n - t\| \\
&\leq \delta\bigl(1-(1-\delta)\theta_n\mu_n\bigr)\|\tau_n - t\|.
\end{aligned} \tag{3.3}$$

Using (3.3), we get

$$\|\tau_{n+1} - t\| = \|\mathcal{F}\sigma_n - t\| \leq \delta\|\sigma_n - t\| \leq \delta^2\bigl(1-(1-\delta)\theta_n\mu_n\bigr)\|\tau_n - t\|. \tag{3.4}$$

By using the fact $0 < (1-(1-\delta)\theta_n\mu_n) \leq 1$, we get

$$\|\tau_{n+1} - t\| \leq \delta^2\|\tau_n - t\|.$$

Inductively, we get

$$\|\tau_n - t\| \leq \delta^{2(n+1)}\|\tau_0 - t\|. \tag{3.5}$$

Now from (1.7), we have

$$\begin{aligned}
\|\xi_n - t\| &= \big\|\mathcal{F}\big((1-\mu_n)\tau_n + \mu_n\mathcal{F}\tau_n\big) - t\big\| \\
&\leq \delta\big\|(1-\mu_n)\tau_n + \mu_n\mathcal{F}\tau_n - t\big\| \\
&\leq \delta\big[(1-\mu_n)\|\tau_n - t\| + \mu_n\delta\|\tau_n - t\|\big] \\
&\leq \delta\big(1-(1-\delta)\mu_n\big)\|\tau_n - t\|. \qquad (3.6)
\end{aligned}$$

Using (3.6), we get

$$\begin{aligned}
\|\sigma_n - t\| &= \|\mathcal{F}\xi_n - t\| \\
&\leq \delta\|\xi_n - t\| \\
&\leq \delta^2\big(1-(1-\delta)\mu_n\big)\|\tau_n - t\|. \qquad (3.7)
\end{aligned}$$

Using (3.7), we get

$$\begin{aligned}
\|\tau_{n+1} - t\| &= \big\|\mathcal{F}\big((1-\theta_n)\sigma_n + \theta_n\mathcal{F}\sigma_n\big) - t\big\| \\
&\leq \delta\big\|(1-\theta_n)\sigma_n + \theta_n\mathcal{F}\sigma_n - t\big\| \\
&\leq \delta^3\big(1-(1-\delta)\theta_n\big)\big(1-(1-\delta)\mu_n\big)\|\tau_n - t\|.
\end{aligned}$$

By using the fact $0 < (1-(1-\delta)\theta_n) \leq 1$ and $0 < (1-(1-\delta)\mu_n) \leq 1$, we have

$$\|\tau_{n+1} - t\| \leq \delta^3\|\tau_n - t\|.$$

Inductively, we get

$$\|\tau_{n+1} - t\| \leq \delta^{3(n+1)}\|\tau_0 - t\|. \qquad (3.8)$$

Let $\alpha_n = \delta^{3(n+1)}\|\tau_0 - t\|$ and $\beta_n = \delta^{2(n+1)}\|\tau_0 - t\|$, then

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n} = \lim_{n\to\infty}\frac{\delta^{3(n+1)}\|\tau_0 - t\|}{\delta^{2(n+1)}\|\tau_0 - t\|} = 0.$$

Hence iteration process (1.7) converges faster than iteration process (1.5). □

Similarly we can show that iteration process (1.7) converges faster than (1.2)–(1.4) and (1.6) iteration processes.

In support of Theorem 3.2, we construct the following example.

Example 3.3 Let $\mathcal{W} = \mathbb{R}^2$ be a Banach space with taxicab norm and $\mathcal{U} = [0,6] \times [0,6]$ a subset of $\mathbb{R}^2$. Let $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ be defined by

$$\mathcal{F}(x_1,x_2) = \begin{cases} (\frac{x_1}{3}, \frac{x_2}{3}), & \text{if } (x_1,x_2) \in [0,3) \times [0,3), \\ (\frac{x_1}{6}, \frac{x_2}{6}), & \text{if } (x_1,x_2) \in [3,6] \times [3,6]. \end{cases}$$

Clearly, $(0,0)$ is the fixed point of $\mathcal{F}$. Now, we show that $\mathcal{F}$ is a contractive-like mapping but not a contraction. For this, we define a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(x) = \frac{x}{4}$. Then, φ is strictly increasing and continuous function. Also, $\varphi(0) = 0$. We show that

$$\|\mathcal{F}x - \mathcal{F}y\| = \delta\|x-y\| + \varphi\bigl(\|x - \mathcal{F}x\|\bigr) \tag{3.9}$$

for all $x,y \in \mathcal{U}$ and $\delta \in (0,1)$. Before going ahead, let us note the following. When $x = (x_1,x_2) \in [0,3) \times [0,3)$, then

$$\|x - \mathcal{F}x\| = \left\|(x_1,x_2) - \left(\frac{x_1}{3},\frac{x_2}{3}\right)\right\| = \left\|\left(\frac{2x_1}{3},\frac{2x_2}{3}\right)\right\|$$

and

$$\varphi\bigl(\|x - \mathcal{F}x\|\bigr) = \varphi\left(\left\|\left(\frac{2x_1}{3},\frac{2x_2}{3}\right)\right\|\right) = \left\|\left(\frac{x_1}{6},\frac{x_2}{6}\right)\right\| = \left|\frac{x_1}{6}\right| + \left|\frac{x_2}{6}\right|. \tag{3.10}$$

Similarly, when $x = (x_1,x_2) \in [3,6] \times [3,6]$, then

$$\|x - \mathcal{F}x\| = \left\|(x_1,x_2) - \left(\frac{x_1}{6},\frac{x_2}{6}\right)\right\| = \left\|\left(\frac{5x_1}{6},\frac{5x_2}{6}\right)\right\|$$

and

$$\begin{aligned}\varphi\bigl(\|x - \mathcal{F}x\|\bigr) &= \varphi\left(\left\|\left(\frac{5x_1}{6},\frac{5x_2}{6}\right)\right\|\right) = \left\|\left(\frac{5x_1}{24},\frac{5x_2}{24}\right)\right\| \\ &= \left|\frac{5x_1}{24}\right| + \left|\frac{5x_2}{24}\right|.\end{aligned} \tag{3.11}$$

Now, we have the following cases:

Case (i): Let $x,y \in [0,3) \times [0,3)$, using (3.10) we get

$$\begin{aligned}\|\mathcal{F}x - \mathcal{F}y\| &= \left\|\left(\frac{x_1}{3},\frac{x_2}{3}\right) - \left(\frac{y_1}{3},\frac{y_2}{3}\right)\right\| \\ &= \left|\frac{x_1}{3} - \frac{y_1}{3}\right| + \left|\frac{x_2}{3} - \frac{y_2}{3}\right| \\ &= \frac{1}{3}|x_1 - y_1| + \frac{1}{3}|x_2 - y_2| \\ &= \frac{1}{3}\bigl\|(x_1,x_2) - (y_1,y_2)\bigr\| \\ &\le \frac{1}{3}\|x-y\| + \left|\frac{x_1}{6}\right| + \left|\frac{x_2}{6}\right| \\ &= \frac{1}{3}\|x-y\| + \varphi\bigl(\|x - \mathcal{F}x\|\bigr).\end{aligned} \tag{3.12}$$

Case (ii): Let $x,y \in [3,6] \times [3,6]$, using (3.10) we get

$$\begin{aligned}\|\mathcal{F}x - \mathcal{F}y\| &= \left\|\left(\frac{x_1}{6},\frac{x_2}{6}\right) - \left(\frac{y_1}{6},\frac{y_2}{6}\right)\right\| \\ &= \left|\frac{x_1}{6} - \frac{y_1}{6}\right| + \left|\frac{x_2}{6} - \frac{y_2}{6}\right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}|x_1 - y_1| + \frac{1}{6}|x_2 - y_2| \\
&= \frac{1}{6}\big\|(x_1, x_2) - (y_1, y_2)\big\| \\
&\leq \frac{1}{3}\|x - y\| + \left|\frac{5x_1}{24}\right| + \left|\frac{5x_2}{24}\right| \\
&= \frac{1}{3}\|x - y\| + \varphi\big(\|x - \mathcal{F}x\|\big).
\end{aligned}
\tag{3.13}
$$

Case (iii): Let $x \in [0,3)$ and $y \in [3,6]$, then using (3.10) we get

$$
\begin{aligned}
\|\mathcal{F}x - \mathcal{F}y\| &= \left\|\left(\frac{x_1}{3}, \frac{x_2}{3}\right) - \left(\frac{y_1}{6}, \frac{y_2}{6}\right)\right\| \\
&= \left\|\left(\frac{x_1}{3} - \frac{y_1}{6}\right), \left(\frac{x_2}{3} - \frac{y_2}{6}\right)\right\| \\
&= \left\|\left(\frac{x_1}{6} + \frac{x_1}{6} - \frac{y_1}{6}\right), \left(\frac{x_2}{6} + \frac{x_2}{6} - \frac{y_2}{6}\right)\right\| \\
&= \left|\frac{x_1}{6} + \frac{x_1}{6} - \frac{y_1}{6}\right| + \left|\frac{x_2}{6} + \frac{x_2}{6} - \frac{y_2}{6}\right| \\
&\leq \left|\frac{x_1}{6}\right| + \left|\frac{x_2}{6}\right| + \left|\frac{x_1}{6} - \frac{y_1}{6}\right| + \left|\frac{x_2}{6} - \frac{y_2}{6}\right| \\
&= \frac{1}{6}\big(|x_1 - y_1| + |x_2 - y_2|\big) + \varphi\big(\|x - \mathcal{F}x\|\big) \\
&\leq \frac{1}{3}\big\|(x_1, x_2) - (y_1, y_2)\big\| + \varphi\big(\|x - \mathcal{F}x\|\big) \\
&= \frac{1}{3}\|x - y\| + \varphi\big(\|x - \mathcal{F}x\|\big).
\end{aligned}
\tag{3.14}
$$

Case (iv): Let $x \in [3,6]$ and $y \in [0,3)$, then using (3.10) we get

$$
\begin{aligned}
\|\mathcal{F}x - \mathcal{F}y\| &= \left\|\left(\frac{x_1}{6}, \frac{x_2}{6}\right) - \left(\frac{y_1}{3}, \frac{y_2}{3}\right)\right\| \\
&= \left\|\left(\frac{x_1}{6} - \frac{y_1}{3}\right), \left(\frac{x_2}{6} - \frac{y_2}{3}\right)\right\| \\
&\leq \left\|\left(\frac{x_1}{3} - \frac{x_1}{6} - \frac{y_1}{3}\right), \left(\frac{x_2}{3} - \frac{x_2}{6} - \frac{y_2}{3}\right)\right\| \\
&= \left|\frac{x_1}{3} - \frac{x_1}{6} - \frac{y_1}{3}\right| + \left|\frac{x_2}{3} - \frac{x_2}{6} - \frac{y_2}{3}\right| \\
&\leq \left|\frac{x_1}{6}\right| + \left|\frac{x_2}{6}\right| + \left|\frac{x_1}{3} - \frac{y_1}{3}\right| + \left|\frac{x_2}{3} - \frac{y_2}{3}\right| \\
&= \frac{1}{3}\big(|x_1 - y_1| + |x_2 - y_2|\big) + \varphi\big(\|x - \mathcal{F}x\|\big) \\
&= \frac{1}{3}\big\|(x_1, x_2) - (y_1, y_2)\big\| + \varphi\big(\|x - \mathcal{F}x\|\big) \\
&= \frac{1}{3}\|x - y\| + \varphi\big(\|x - \mathcal{F}x\|\big).
\end{aligned}
$$

So, (3.9) is satisfied with $\delta = \frac{1}{3}$. Thus, $\mathcal{F}$ is a contractive-like mapping.

Table 1 Computational table of different iteration processes

Iter.	Mann	Ishikawa	S
1	(1.000000, 2.500000)	(1.000000, 2.500000)	(1.000000, 2.500000)
2	(0.666667, 1.666667)	(0.622222, 1.555556)	(0.288889, 0.722222)
⋮	⋮	⋮	⋮
14	(0.005138, 0.012846)	(0.002096, 0.005239)	(0.000000, 0.000000)
16	(0.002284, 0.005709)	(0.000811, 0.002028)	(0.000000, 0.000000)
⋮	⋮	⋮	⋮
33	(0.000002, 0.000006)	(0.000000, 0.000001)	(0.000000, 0.000000)
34	(0.000002, 0.000004)	(0.000000, 0.000000)	(0.000000, 0.000000)
⋮	⋮	⋮	⋮
39	(0.000000, 0.000001)	(0.000000, 0.000000)	(0.000000, 0.000000)
40	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)

Table 2 Computational table of different iteration processes

Iter.	Picard-S	Noor	JF
1	(1.000000, 2.500000)	(1.000000, 2.500000)	(1.000000, 2.500000)
2	(0.096296, 0.240741)	(0.617778, 1.544444)	(0.069136, 0.172840)
⋮	⋮	⋮	⋮
7	(0.000001, 0.000002)	(0.055590, 0.138974)	(0.000000, 0.000000)
8	(0.000000, 0.000000)	(0.034342, 0.085855)	(0.000000, 0.000000)
⋮	⋮	⋮	⋮
33	(0.000000, 0.000000)	(0.000000, 0.000001)	(0.000000, 0.000000)
34	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
⋮	⋮	⋮	⋮
40	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)

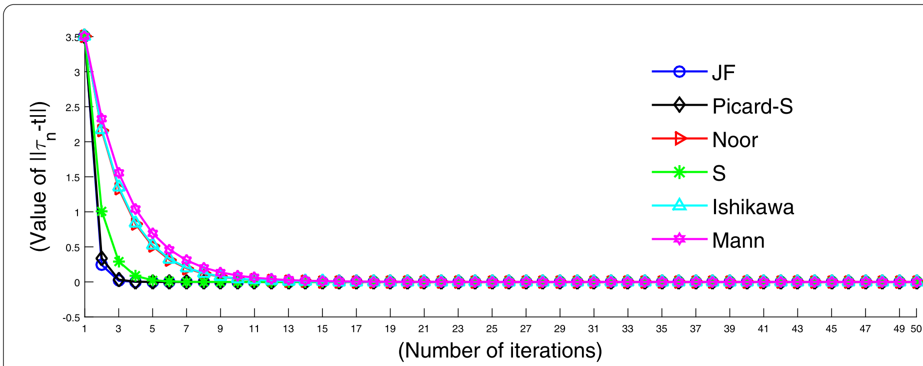

Figure 1 Convergence behavior of the sequences defined by distinct iteration processes

It can be easily seen in Tables 1–2 and Fig. 1 that JF iteration process converges to a fixed point $(0,0)$ of the mapping $\mathcal{F}$ faster than the leading iteration processes with initial point $(1,2.5)$ and control sequences $\theta_n = 0.5$, $\mu_n = 0.4$ and $\gamma_n = 0.3$, $n \in \mathbb{Z}^+$.

4 Convergence results for Suzuki's generalized non-expansive mappings

Throughout this section, we presume that $\mathcal{U}$ is a nonempty closed and convex subset of a uniformly convex Banach space $\mathcal{W}$ and let $\mathcal{F}:\mathcal{U} \to \mathcal{U}$ be a mapping satisfying Suzuki's condition (C).

Lemma 4.1 *Let $\mathcal{U}$ be a nonempty closed and convex subset of a uniformly convex Banach space $\mathcal{W}$ and $\mathcal{F}:\mathcal{U}\to\mathcal{U}$ a mapping satisfying Suzuki's condition (C). Suppose $F(\mathcal{F})\neq\emptyset$ and $\{\tau_n\}$ is a sequence developed by iteration process (1.7), then $\lim_{n\to\infty}\|\tau_n-t\|$ exists for all $t\in F(\mathcal{F})$.*

Proof Let $t\in F(\mathcal{F})$ and $x\in\mathcal{U}$. Since $\mathcal{F}$ satisfies Suzuki's condition (C), we have

$$\|\mathcal{F}x-t\|\leq\|x-t\|,\quad\text{for all } x\in\mathcal{U} \text{ and for all } t\in F(\mathcal{F}).$$

Now from iteration process (1.7), we get

$$\begin{aligned}\|\xi_n-t\| &= \big\|\mathcal{F}(1-\mu_n)\tau_n+\mu_n\mathcal{F}\tau_n-t\big\|\\ &\leq \big\|(1-\mu_n)\tau_n+\mu_n\mathcal{F}\tau_n-t\big\|\\ &\leq (1-\mu_n)\|\tau_n-t\|+\mu_n\|\tau_n-t\|\\ &= \|\tau_n-t\|\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\|\sigma_n-t\| &= \|\mathcal{F}\xi_n-t\|\\ &\leq \|\xi_n-t\|\\ &\leq \|\tau_n-t\|.\end{aligned}\tag{4.2}$$

Using (4.1) and (4.2), we have

$$\begin{aligned}\|\tau_{n+1}-t\| &= \big\|\mathcal{F}\big((1-\theta_n)\sigma_n+\theta_n\mathcal{F}\sigma_n\big)-t\big\|\\ &\leq \big\|(1-\theta_n)\sigma_n+\theta_n\mathcal{F}\sigma_n-t\big\|\\ &\leq (1-\theta_n)\|\sigma_n-t\|+\theta_n\|\sigma_n-t\|\\ &= \|\sigma_n-t\|\\ &\leq \|\tau_n-t\|.\end{aligned}\tag{4.3}$$

This shows that the sequence $\{\|\tau_n-t\|\}$ is non-increasing and bounded below for all $t\in F(\mathcal{F})$. Thus $\lim_{n\to\infty}\|\tau_n-t\|$ exists. □

Lemma 4.2 *Let $\{\tau_n\}$ be a sequence developed by iteration process (1.7) and sequence $\{\mu_n\}$ satisfying condition $0<a\leq\mu_n\leq b<1$ for all $n\geq 1$. Then $F(\mathcal{F})\neq\emptyset$ if and only if $\{\tau_n\}$ is bounded and $\lim_{n\to\infty}\|\tau_n-\mathcal{F}\tau_n\|=0$.*

Proof By Lemma 4.1, it follows that $\lim_{n\to\infty}\|\tau_n-t\|$ exists.

Presume that $\lim_{n\to\infty}\|\tau_n-t\|=c$.

By the inequalities (4.1) and (4.2), we get

$$\limsup_{n\to\infty}\|\xi_n-t\|\leq c\tag{4.4}$$

and

$$\limsup_{n\to\infty}\|\sigma_n - t\| \le c, \tag{4.5}$$

respectively. Since $\mathcal{F}$ satisfies Suzuki's condition (C), we have

$$\|\mathcal{F}\tau_n - t\| \le \|\tau_n - t\|, \qquad \|\mathcal{F}\sigma_n - t\| \le \|\sigma_n - t\|, \qquad \|\mathcal{F}\xi_n - t\| \le \|\xi_n - t\|.$$

$$\limsup_{n\to\infty}\|\mathcal{F}\tau_n - t\| \le c, \tag{4.6}$$

$$\limsup_{n\to\infty}\|\mathcal{F}\sigma_n - t\| \le c, \tag{4.7}$$

$$\limsup_{n\to\infty}\|\mathcal{F}\xi_n - t\| \le c. \tag{4.8}$$

Since

$$\begin{aligned}\|\tau_{n+1} - t\| &= \bigl\|\mathcal{F}\bigl((1-\theta_n)\sigma_n + \theta_n\mathcal{F}\sigma_n\bigr) - t\bigr\| \\ &\le \bigl\|(1-\theta_n)\sigma_n + \theta_n\mathcal{F}\sigma_n - t\bigr\| \\ &\le (1-\theta_n)\|\sigma_n - t\| + \theta_n\|\sigma_n - t\| \\ &= \|\sigma_n - t\|.\end{aligned}$$

Taking the lim inf on both sides, we get

$$c = \liminf_{n\to\infty}\|\tau_{n+1} - t\| \le \liminf_{n\to\infty}\|\sigma_n - t\|. \tag{4.9}$$

Thus, (4.5) and (4.9) give

$$\lim_{n\to\infty}\|\sigma_n - t\| = c.$$

We have

$$\begin{aligned}c &= \liminf_{n\to\infty}\|\sigma_n - t\| = \liminf_{n\to\infty}\|\mathcal{F}\xi_n - t\| \\ &\le \liminf_{n\to\infty}\|\xi_n - t\|.\end{aligned} \tag{4.10}$$

From (4.4) and (4.10), we have

$$\lim_{n\to\infty}\|\xi_n - t\| = c.$$

So,

$$\begin{aligned}c &= \lim_{n\to\infty}\|\xi_n - t\| \\ &= \lim_{n\to\infty}\bigl\|\mathcal{F}\bigl((1-\mu_n)\tau_n + \mu_n\mathcal{F}\tau_n\bigr) - t\bigr\| \\ &\le \lim_{n\to\infty}\bigl\|(1-\mu_n)\tau_n + \mu_n\mathcal{F}\tau_n - t\bigr\|\end{aligned}$$

$$\begin{aligned}
&\le \lim_{n\to\infty} \left\| (1-\mu_n)(\tau_n - t) + \mu_n(\mathcal{F}\tau_n - t) \right\| \\
&\le \lim_{n\to\infty} \left[(1-\mu_n)\|\tau_n - t\| + \mu_n\|\tau_n - t\| \right] \\
&\le \lim_{n\to\infty} \|\tau_n - t\| = c,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\| (1-\mu_n)(\tau_n - t) + \mu_n(\mathcal{F}\tau_n - t) \right\| = c. \tag{4.11}$$

From (4.11) and Lemma 2.4, we have

$$\lim_{n\to\infty} \|\tau_n - \mathcal{F}\tau_n\| = 0.$$

On the contrary, assume that $\{\tau_n\}$ is bounded and $\lim_{n\to\infty}\|\tau_n - \mathcal{F}\tau_n\| = 0$. Suppose $t \in A(\mathcal{U}, \{\tau_n\})$, so by Proposition 2.5, we have

$$\begin{aligned}
r(\mathcal{F}t, \{\tau_n\}) &= \limsup_{n\to\infty} \|\tau_n - \mathcal{F}t\| \\
&\le \limsup_{n\to\infty} \left(3\|\mathcal{F}\tau_n - \tau_n)\| + \|\tau_n - t\|\right) \\
&= \limsup_{n\to\infty} \|\tau_n - t\| \\
&= r(t, \{\tau_n\}) = r(\mathcal{U}, \{\tau_n\}).
\end{aligned}$$

This implies $\mathcal{F}t \in A(\mathcal{U}, \{\tau_n\})$. Since $\mathcal{W}$ is uniformly convex, $A(\mathcal{U}, \{\tau_n\})$ is singleton, hence we have $\mathcal{F}t = t$. □

Theorem 4.3 *Assume that $\mathcal{W}$ satisfies Opial's condition, then the sequence $\{\tau_n\}$ developed by iteration process (1.7) converge weakly to a point of $F(\mathcal{F})$.*

Proof From Lemma 4.1, we see that $\lim_{n\to\infty}\|\tau_n - t\|$ exists. In order to show the weak convergence of the iteration process (1.7) to a fixed point of $\mathcal{F}$, we will prove that $\{\tau_n\}$ has a unique weak subsequential limit in $F(\mathcal{F})$. For this, let $\{\tau_{n_j}\}$ and $\{\tau_{n_k}\}$ be two subsequences of $\{\tau_n\}$ which converges weakly to x and y, respectively. From Lemma 4.2, $\lim_{n\to\infty}\|\tau_n - \mathcal{F}\tau_n\| = 0$ and $I - \mathcal{F}$ is demiclosed at zero by Lemma 2.3. Thus $(I - \mathcal{F})x = 0$, that is, $x = \mathcal{F}x$. Similarly $y = \mathcal{F}y$.

Now we show uniqueness. If $x \neq y$, by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|\tau_n - x\| &= \lim_{n_j\to\infty} \|\tau_{n_j} - x\| \\
&< \lim_{n_j\to\infty} \|\tau_{n_j} - y\| \\
&= \lim_{n\to\infty} \|\tau_n - y\| \\
&= \lim_{n_k\to\infty} \|\tau_{n_k} - y\| \\
&< \lim_{n_k\to\infty} \|\tau_{n_k} - x\| \\
&= \lim_{n\to\infty} \|\tau_n - x\|,
\end{aligned}$$

which is a contradiction; hence $x = y$. Consequently, the $\{\tau_n\}$ converge weakly to a point of $F(\mathcal{F})$. □

Theorem 4.4 *Let $\mathcal{F}, \mathcal{U}$ and $\mathcal{W}$ be defined as in Lemma 4.1. Then the sequence $\{\tau_n\}$ developed by iteration process (1.7) converges to a point of $F(\mathcal{F})$ if and only if $\liminf_{n\to\infty} d(\tau_n, F(\mathcal{F})) = 0$, where $d(\tau_n, F(\mathcal{F})) = \inf\{\|\tau_n - t\| : t \in F(\mathcal{F})\}$.*

Proof The first part is trivial. So, we prove the converse part. Presume that $\liminf_{n\to\infty} d(\tau_n, F(\mathcal{F})) = 0$. From Lemma 4.1, $\lim_{n\to\infty} \|\tau_n - t\|$ exists, for all $t \in F(\mathcal{F})$ therefore $\lim_{n\to\infty} d(\tau_n, F(\mathcal{F})) = 0$ by hypothesis.

Now our assertion is that $\{\tau_n\}$ is a Cauchy sequence in $\mathcal{U}$. Since $\lim_{n\to\infty} d(\tau_n, F(\mathcal{F})) = 0$, and for a given $\lambda > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$

$$
\begin{aligned}
& d\bigl(\tau_n, F(\mathcal{F})\bigr) < \frac{\lambda}{2} \\
& \quad\implies\quad \inf\bigl\{\|\tau_n - t\| : t \in F(\mathcal{F})\bigr\} < \frac{\lambda}{2}.
\end{aligned}
$$

In particular, $\inf\{\|\tau_{m_0} - t\| : t \in F(\mathcal{F})\} < \frac{\lambda}{2}$. Therefore there exists $t \in F(\mathcal{F})$ such that

$$
\|\tau_{m_0} - t\| < \frac{\lambda}{2}.
$$

Now, for $m, n \geq m_0$,

$$
\begin{aligned}
\|\tau_{n+m} - \tau_n\| &\leq \|\tau_{n+m} - t\| + \|\tau_n - t\| \\
&\leq \|\tau_{m_0} - t\| + \|\tau_{m_0} - t\| \\
&= 2\|\tau_{m_0} - t\| < \lambda.
\end{aligned}
$$

Thus $\{\tau_n\}$ is a Cauchy sequence in $\mathcal{U}$. As $\mathcal{U}$ is closed, then there exists a point $q \in \mathcal{U}$ such that $\lim_{n\to\infty} \tau_n = q$. Now, $\lim_{n\to\infty} d(\tau_n, F(\mathcal{F})) = 0$ implies $d(q, F(\mathcal{F})) = 0$, hence we get $q \in F(\mathcal{F})$. □

Theorem 4.5 *Let $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ be a mapping satisfying Suzuki's condition (C), where $\mathcal{U}$ is a nonempty, compact and convex subset of a uniformly convex Banach space $\mathcal{W}$. Then the sequence $\{\tau_n\}$ developed by iteration process (1.7) converges strongly to a fixed point of $\mathcal{F}$.*

Proof By Lemma 2.2, $F(\mathcal{F}) \neq \emptyset$, so by Lemma 4.2, we have $\lim_{n\to\infty} \|\mathcal{F}\tau_n - \tau_n\| = 0$. Since $\mathcal{U}$ is compact, there exists a subsequence $\{\tau_{n_j}\}$ of $\{\tau_n\}$ such that $\tau_{n_j} \to t$ strongly for some $t \in \mathcal{U}$. By Proposition 2.5, we have

$$
\|\tau_{n_j} - \mathcal{F}t\| \leq 3\|\mathcal{F}\tau_{n_j} - \tau_{n_j}\| + \|\tau_{n_j} - t\|, \quad \forall j \geq 1.
$$

As $j \to \infty$, we get $\tau_{n_j} \to \mathcal{F}t$, implies $\mathcal{F}t = t$, *i.e.* $t \in F(\mathcal{F})$. Also, $\lim_{n\to\infty} \|\tau_n - t\|$ exists by Lemma 4.1. Thus t is the strong limit of the sequence $\{\tau_n\}$ itself. □

Applying condition (I) we now prove a strong convergence result.

Theorem 4.6 *Let $\mathcal{F}$, $\mathcal{U}$ and $\mathcal{W}$ be defined as in Lemma 4.1. Assume that the mapping $\mathcal{F}$ also satisfies condition (I). Then the sequence $\{\tau_n\}$ developed by iteration process (1.7) converges strongly to a fixed point of $\mathcal{F}$.*

Proof We proved in Lemma 4.2 that

$$\lim_{n\to\infty} \|\tau_n - \mathcal{F}\tau_n\| = 0. \tag{4.12}$$

Applying condition (I) and (4.12), we get

$$\begin{aligned} 0 \le \lim_{n\to\infty} \psi\big(d\big(\tau_n, F(\mathcal{F})\big)\big) &\le \lim_{n\to\infty} \|\tau_n - \mathcal{F}\tau_n\| = 0 \\ \implies \quad \lim_{n\to\infty} \psi\big(d\big(\tau_n, F(\mathcal{F})\big)\big) &= 0. \end{aligned}$$

And hence

$$\lim_{n\to\infty} d\big(\tau_n, F(\mathcal{F})\big) = 0.$$

So by Theorem 4.4, the sequence $\{\tau_n\}$ converge strongly to a fixed point of $\mathcal{F}$. □

5 An illuminate numerical example

The purpose of this section is to present a numerical example to compare the rate of convergence for a mapping satisfying Suzuki's condition (C).

Example 5.1 Let $\mathcal{F} : [0,2] \to [0,2]$ be a mapping defined by

$$\mathcal{F}(x) = \begin{cases} 2 - x, & \text{if } x \in [0, \frac{1}{9}), \\ \frac{x+16}{9}, & \text{if } x \in [\frac{1}{9}, 2]. \end{cases}$$

Here $\mathcal{F}$ satisfies Suzuki's condition (C), but $\mathcal{F}$ is not a non-expansive mapping.

Verification For $x = \frac{1}{10}$ and $y = \frac{1}{9}$, we obtain

$$\|x - y\| = \left\| \frac{1}{10} - \frac{1}{9} \right\| = \frac{1}{90}.$$

We have

$$\begin{aligned} \|\mathcal{F}x - \mathcal{F}y\| &= \left\| 2 - \frac{1}{10} - \frac{145}{81} \right\| \\ &= \frac{89}{810} > \frac{1}{90} = \|x - y\|. \end{aligned}$$

Hence $\mathcal{F}$ is not a non-expansive mapping.

We now show that $\mathcal{F}$ satisfies Suzuki's condition (C).

We have the following cases:

Case I If either $x, y \in [0, \frac{1}{9})$ or $x, y \in [\frac{1}{9}, 2]$, then obviously $\mathcal{F}$ satisfies Suzuki's condition (C).

Case II Let $x \in [0, \frac{1}{9})$. Then $\frac{1}{2}\|x - \mathcal{F}x\| = \frac{1}{2}\|x - (2 - x)\| = \frac{1}{2}\|2x - 2\| = \|x - 1\| \in (\frac{8}{9}, 1]$. For $\frac{1}{2}\|x - \mathcal{F}x\| \leq \|x - y\|$, we should have $1 - x \leq y - x$ implying $y \geq 1$ and $y \in [1, 2]$. Now,

$$\|\mathcal{F}x - \mathcal{F}y\| = \left\|\frac{y + 16}{9} - 2 + x\right\| = \left\|\frac{y + 9x - 2}{9}\right\| < \frac{1}{9}$$

and

$$\|x - y\| = |x - y| > \left|1 - \frac{1}{9}\right| = \left|\frac{8}{9}\right| = \frac{8}{9} > \frac{1}{9}.$$

Hence, $\frac{1}{2}\|x - \mathcal{F}x\| \leq \|x - y\| \implies \|\mathcal{F}x - \mathcal{F}y\| \leq \|x - y\|$.

Case III Let $x \in [\frac{1}{9}, 2]$. Then $\frac{1}{2}\|x - \mathcal{F}x\| = \frac{1}{2}\|\frac{x+16}{9} - x\| = \|\frac{16-8x}{18}\| \in [0, \frac{136}{162}]$. For $\frac{1}{2}\|x - \mathcal{F}x\| \leq \|x - y\|$, we should have $\frac{16-8x}{18} \leq |x - y|$, which indicates two possibilities:

(a) Let $x < y$, then $\frac{16-8x}{18} \leq y - x$, *i.e.* $\frac{10x+16}{18} \leq y \implies y \in [\frac{154}{162}, 2] \subset [\frac{1}{9}, 2]$. So

$$\|\mathcal{F}x - \mathcal{F}y\| = \left\|\frac{x + 16}{9} - \frac{y + 16}{9}\right\| = \frac{1}{9}\|x - y\| \leq \|x - y\|.$$

Hence, $\frac{1}{2}\|x - \mathcal{F}x\| \leq \|x - y\| \implies \|\mathcal{F}x - \mathcal{F}y\| \leq \|x - y\|$.

(b) Let $x > y$, then $\frac{16-8x}{18} \leq x - y$, *i.e.* $y \leq \frac{26x-16}{18} \implies y \leq \frac{-118}{162}$ and $y \leq 2$, so $y \in [0, 2]$. Since $y \in [0, 2]$ and $y \leq \frac{26x-16}{18} \implies \frac{18y+16}{26} \leq x$. Since the case $x \in [\frac{16}{26}, 2]$ and $y \in [\frac{1}{9}, 2]$ is already discussed in Case I. Now consider, $x \in [\frac{16}{26}, 2]$ and $y \in [0, \frac{1}{9})$. Then

$$\|\mathcal{F}x - \mathcal{F}y\| = \left\|\frac{x + 16}{9} - 2 + y\right\| = \left\|\frac{x + 9y - 2}{9}\right\| < \frac{1}{9}$$

and

$$\|x - y\| = |x - y| > \left|\frac{16}{26} - \frac{1}{9}\right| = \left|\frac{144 - 26}{234}\right| = \frac{118}{234} > \frac{1}{9}.$$

Hence, $\frac{1}{2}\|x - \mathcal{F}x\| \leq \|x - y\| \implies \|\mathcal{F}x - \mathcal{F}y\| \leq \|x - y\|$. Thus $\mathcal{F}$ satisfies Suzuki's condition (C).

It can be easily seen from Table 3 and Fig. 2 that the JF iteration process converges to a fixed point $t = 2$ of the mapping $\mathcal{F}$ faster than the leading iteration processes with initial point $\tau_0 = 0.11$ and control sequences $\theta_n = 0.75$, $\mu_n = 0.65$ and $\gamma_n = 0.55$, $n \in \mathbb{Z}^+$.

6 Application to a nonlinear fractional differential equation

In this section, by using iteration process (1.7) we approximate the solution of a nonlinear fractional differential equation. Consider the following nonlinear fractional differential equation:

$$\begin{cases} D^{\alpha}x(t) + D^{\beta}x(t) = f(t, x(t)) & (0 \leq t \leq 1, 0 < \beta < \alpha < 1), \\ x(0) = x(1) = 0, \end{cases} \tag{6.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and D^{α} and D^{β} denote the Caputo fractional derivatives of order α and β, respectively.

Table 3 Numerical comparison of iteration processes

Iter.	Mann	Ishikawa	S	Picard-S	Noor	JF
1	0.110000	0.110000	0.110000	0.110000	0.110000	0.110000
2	1.370000	1.461000	1.881000	1.986778	1.466561	1.999635
3	1.790000	1.846285	1.992507	1.999907	1.849441	2.000000
4	1.930000	1.956163	1.999528	1.999999	1.957506	2.000000
5	1.976667	1.987498	1.999970	2.000000	1.988006	2.000000
6	1.992222	1.996435	1.999998	2.000000	1.996615	2.000000
7	1.997407	1.998983	2.000000	2.000000	1.999045	2.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
13	1.999996	1.999999	2.000000	2.000000	2.000000	2.000000
14	1.999999	2.000000	2.000000	2.000000	2.000000	2.000000
15	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000

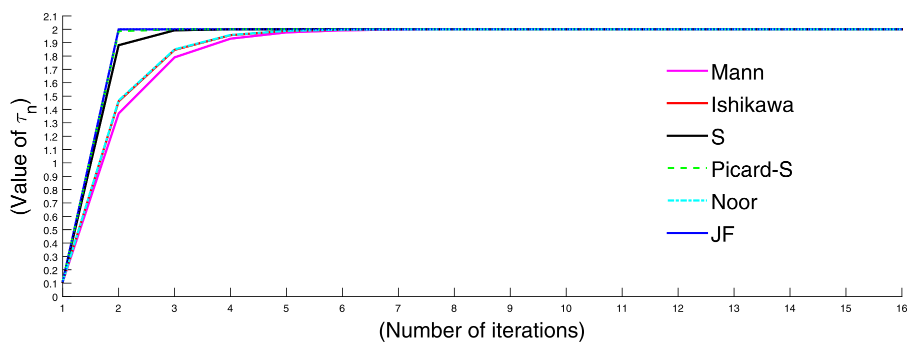

Figure 2 Graphical representation of the convergence of iteration processes

Let $\mathcal{W} = C[0,1]$ be a Banach space of continuous function from $[0,1]$ into $\mathbb{R}$ endowed with the supremum norm. The Green's function associated to (6.1) is defined by

$$G(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}\left(-t^{\alpha-\beta}\right),$$

where $E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}) = \sum_{k=0}^{\infty} \frac{(-t^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k+\alpha)}$ is the Mittag–Leffler function. Many authors studied the existence of the solution of problem (6.1) [e.g. see [4, 13, 14]]. Now we approximate the solution of problem (6.1) by utilizing iteration process (1.7) with the following assumption:

(C_1) Assume that

$$\left|f(t,a) - f(t,b)\right| \leq c|a-b|$$

for all $t \in [0,1]$, $a, b \in \mathbb{R}$ and $c \leq \alpha$.

Theorem 6.1 *Let $\mathcal{W} = C[0,1]$ be a Banach space with supremum norm. Let $\{\tau_n\}$ be a sequence defined by JF iteration process (1.7) for the operator $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ defined by*

$$\mathcal{F}\left(x(t)\right) = \int_0^t G(t-w) f\left(w, x(w)\right) dw, \tag{6.2}$$

$\forall t \in [0,1]$, $\forall x \in \mathcal{W}$. Assume that the condition (C_1) is satisfied. Then the sequence defined by JF iteration process (1.7) converges to a solution, say $x^* \in \mathcal{W}$ of the problem (6.1).

Proof Observe that $x^* \in \mathcal{W}$ is a solution of (6.1) if and only if x^* is a solution of the integral equation

$$x(t) = \int_0^t G(t-w)f\big(w,x(w)\big)\,dw.$$

Now, let $x, y \in \mathcal{W}$ and for all $t \in [0,1]$. Using (C_1), we get

$$\begin{aligned}\big|\mathcal{F}\big(x(t)\big) - \mathcal{F}\big(y(t)\big)\big| &= \left|\int_0^t G(t-w)f\big(w,x(w)\big)\,dw - \int_0^t G(t-w)f\big(w,y(w)\big)\,dw\right| \\ &\le \int_0^t G(t-w)\big|f\big(w,x(w)\big) - f\big(w,y(w)\big)\big|\,dw \\ &\le \int_0^t G(t-w)c\big|x(w)-y(w)\big|\,dw \\ &\le \left(\sup_{t\in[0,1]}\int_0^t G(t-w)\,dw\right)c\|x-y\| \\ &\le \frac{c}{\alpha}\|x-y\|.\end{aligned}$$

Note that $G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}) \le t^{\alpha-1}\frac{1}{1+|-t^{\alpha-\beta}|} \le t^{\alpha-1}$ for all $t \in [0,1]$. Thus, $\sup_{t\in[0,1]}\int_0^t G(t-w)\,dw \le \frac{1}{\alpha}$. Hence, for $x, y \in \mathcal{W}$ and for all $t \in [0,1]$, we have

$$\|\mathcal{F}x - \mathcal{F}y\| \le \|x-y\|.$$

Thus $\mathcal{F}$ is a Suzuki generalized non-expansive mapping. Hence the JF iteration process converges to the solution of (6.1). □

Now, we present the following example for the validity of Theorem 6.1.

Example 6.2 Consider the following fractional differential equation:

$$\begin{cases} D^{0.5}x(t) + D^{0.25}x(t) = t^3 + 1 \quad (0 \le t \le 1), \\ x(0) = x(1) = 0. \end{cases} \tag{6.3}$$

The exact solution of problem (6.3) is given by

$$x(t) = \int_0^t G(t-w)f\big(w,x(w)\big)\,dw.$$

The operator $\mathcal{F} : C[0,1] \to C[0,1]$ is defined by

$$\mathcal{F}x(t) = \int_0^t G(t-w)f\big(w,x(w)\big)\,dw. \tag{6.4}$$

For the initial guess $\tau_0(t) = t(1-t)$, $t \in [0,1]$ and control sequences $\theta_n = 0.85$, $\mu_n = 0.65$, $n \in \mathbb{Z}^+$, we observe that JF iteration process converges to the exact solution of problem (6.3) for the operator defined in (6.4) which is shown in Tables 4–5 and Figs. 3–6. Furthermore, we consider a Mittag-Leffler series expansion at $k = 3$, $k = 11$, $k = 31$ and $k = 500$

Table 4 Comparison between exact solution and approximate solution by using the JF iteration process for $k = 3$ and $k = 11$

S. No.	t	$k = 3$			$k = 11$		
		$x(t)$	τ_1	τ_{10}	$x(t)$	τ_1	τ_{10}
1	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
2	0.1	0.07127290	0.07127290	0.07127290	0.09420159	0.09420159	0.09420159
3	0.2	0.06081985	0.06081985	0.06081985	0.12109470	0.12109470	0.12109470
4	0.3	0.03349861	0.03349861	0.03349861	0.13856971	0.13856971	0.13856971
5	0.4	−0.00465639	−0.00465639	−0.00465639	0.15014913	0.15014913	0.15014913
6	0.5	−0.05171526	−0.05171526	−0.05171526	0.15632606	0.15632606	0.15632606
7	0.6	−0.10740643	−0.10740643	−0.10740643	0.15643118	0.15643118	0.15643118
8	0.7	−0.17250057	−0.17250057	−0.17250057	0.14894918	0.14894918	0.14894918
9	0.8	−0.24859763	−0.24859763	−0.24859763	0.13146728	0.13146728	0.13146728
10	0.9	−0.33804423	−0.33804423	−0.33804423	0.10046696	0.10046696	0.10046696
11	1.0	−0.44390251	−0.44390251	−0.44390251	0.05101478	0.05101478	0.05101478

Table 5 Comparison between exact solution and approximate solution by using the JF iteration process for $k = 31$ and $k = 500$

S. No.	t	$k = 31$			$k = 500$		
		$x(t)$	τ_1	τ_{10}	$x(t)$	τ_1	τ_{10}
1	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
2	0.1	0.09426838	0.09426838	0.09426838	0.09426838	0.09426838	0.09426838
3	0.2	0.12181107	0.12181107	0.12181107	0.12181107	0.12181107	0.12181107
4	0.3	0.14144144	0.14144144	0.14144144	0.14144144	0.14144144	0.14144144
5	0.4	0.15787367	0.15787367	0.15787367	0.15787369	0.15787369	0.15787369
6	0.5	0.17308385	0.17308385	0.17308385	0.17308398	0.17308398	0.17308398
7	0.6	0.18827138	0.18827138	0.18827138	0.18827202	0.18827202	0.18827202
8	0.7	0.20433052	0.20433052	0.20433052	0.20433294	0.20433294	0.20433294
9	0.8	0.22201428	0.22201428	0.22201428	0.22202200	0.22202200	0.22202200
10	0.9	0.24200310	0.24200310	0.24200310	0.24202490	0.24202490	0.24202490
11	1.0	0.26493588	0.26493588	0.26493588	0.26499176	0.26499176	0.26499176

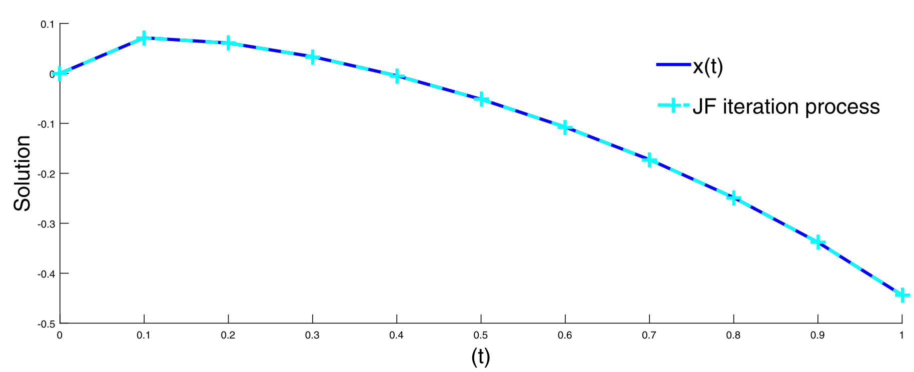

Figure 3 Graphical representation of exact solution and approximated solution of problem (6.3) by using the JF iteration process at $k = 3$

and show that JF iteration process converges to the solution of fractional differential equation. We also observe that the behavior of JF iteration process will remain identical after $k = 500$.

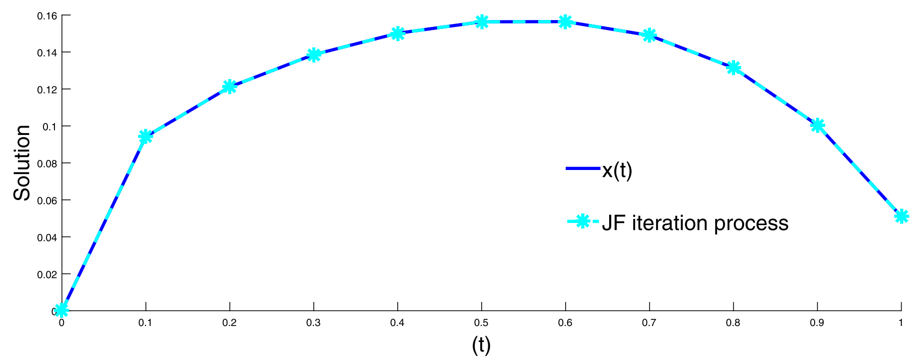

Figure 4 Graphical representation of exact solution and approximated solution of problem (6.3) by using the JF iteration process at $k = 11$

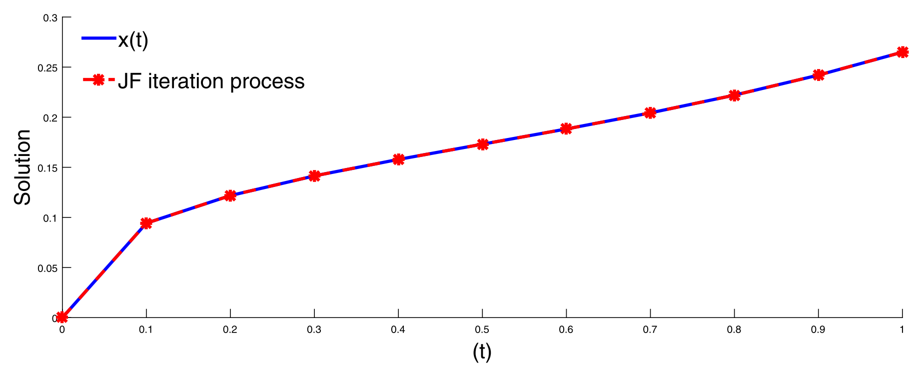

Figure 5 Graphical representation of exact solution and approximated solution of problem (6.3) by using the JF iteration process at $k = 31$

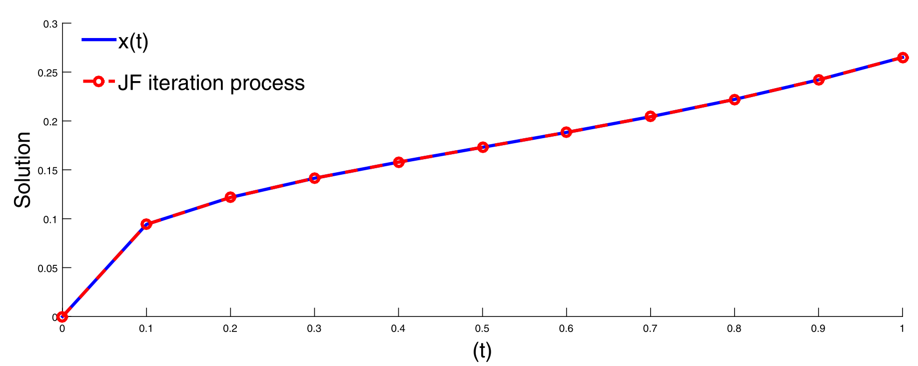

Figure 6 Graphical representation of exact solution and approximated solution of problem (6.3) by using the JF iteration process at $k = 500$

7 Conclusion

In this paper, convergence and stability of the JF iteration process have been studied for nonlinear mappings. Furthermore, the proposed iteration process has been successfully operating for the solution of boundary value problem of fractional differential equation. We compared the rate of convergence of remarkable iteration processes analytically and numerically for nonlinear mappings. High rates of convergence have been achieved. We also showed that the presented process (1.7) outperforms other iteration processes and resulted in very high accuracy via numerical examples.

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