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Iterative algorithm for singularities of inclusion problems in Hadamard manifolds

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Abstract

The main purpose of this paper is to introduce a new iterative algorithm to solve inclusion problems in Hadamard manifolds. Moreover, applications to convex minimization problems and variational inequality problems are studied. A numerical example also is presented to support our main theorem.

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1 Introduction

Let H be a Hilbert space, $A : H \to H$ an operator and $B : H \to 2^H$ a multivalued operator. The inclusion problem is to find $p^* \in H$ such that

$$0 \in (A + B)\big(p^*\big). \tag{1}$$

If $A = 0$, then the problem (1) becomes the inclusion problem introduced by Rockafellar [1]. Several nonlinear problems such as optimization problems, variational inequality problems, DEs [2–6] and economics can be formulated to find a singularity of the problem (1). The problem (1) is highly considered by many authors who performed dedicated work to theoretical results as well as iterative procedures; see, for instance [7–10] and the references therein.

In 1979, Lions and Mercier [11] showed that the problem (1) is equivalent to find fixed points of the mapping $J_\lambda^B(I - \lambda A)$, that is, $p^* = J_\lambda^B(p^* - \lambda A(p^*)) \Leftrightarrow 0 \in (A + B)(p^*)$, where $J_\lambda^B = (I + \lambda B)^{-1}$. Owing to the fixed point formulation, Lions and Mercier [11] presented the following proximal point method: let $p_0 \in H$ be an initial point and

$$p_{n+1} = J_\lambda^B\big(p_n - \lambda A(p_n)\big), \quad \forall n \in \mathbb{N}, \tag{2}$$

where $\lambda > 0$. The proximal point method (2) has been extensively studied with many authors; see, e.g. [8, 12–17]. In particular, Chen and Rockafellar [14] studied convergence rates of the method (2). Afterwards, Tseng [15] proposed the modification for approximating singularities of the inclusion problem (1), also known as Tseng's splitting algorithm.

In 2012, Takahashi et al. [16] introduced an iterative scheme to solve the problem (1) by combining Mann-type and Halpern-type algorithms with the proximal point method. Recently, Lorenz and Pock [13] have defined an iterative algorithm related to the inertial extrapolation technique.

Over the past year, many significant techniques, concepts of nonlinear and analytical optimization that fit in Euclidean spaces have been extended to Riemannian manifolds. From the Riemannian geometry point of view, some non-convex constrained optimization problems can be viewed as convex unconstrained optimization problems by the introduction of a suitable Riemannian metric (see, e.g. [18–26] and the references therein).

In recent years, several researchers have extended the relevance of inclusion theory from linear spaces to the Riemannian context. For instance, Ferreira et al. [27] considered inclusion problem (1) in the setting of Hadamard manifolds. Later on, Ansari et al. [28] introduced Korpelevich's algorithm to solve the inclusion problem (1) and discussed its convergence. Moreover, they [28] obtained the relationship between the set of singularities of inclusion problems and fixed points of the resolvent of maximal monotone vector fields in Hadamard manifolds. In 2019, Al-Homidan et al. [29] presented Halpern-type and Mann-type iterative methods for approximating singularities of the inclusion problem (1) in the framework of Hadamard manifolds. Very recent, Ansari and Babu [30] presented the proximal point method for finding singularities of the inclusion problem (1) on Hadamard manifolds. The authors [30] also devoted their results to convex minimization problems and variational inequality problems.

Inspired by the work mentioned above, the purpose of this paper is to introduce a new class of inverse-strongly-monotone operators, and then develop a new class of iterative algorithms to solve the problem of finding singularities defined by the sum of an inverse-strongly-monotone vector field and a multivalued maximal monotone vector field in Hadamard manifolds.

The paper is organized as follows: In the next section, we give some fundamental concepts of geometry and nonlinear analysis in Riemannian manifolds. In Sect. 3, we construct the inclusion problem (1) in the setting of Hadamard manifolds and exhibit the concept of monotonicity for single-valued as well as for multivalued vector fields. Some fundamental realized results identified with the monotone vector fields are additionally mentioned. In Sect. 4, we present the Mann-type splitting method and establish convergence theorems of any sequence generated by the proposed algorithm converges to a solution of the proposed problem in Hadamard manifolds. In Sect. 5, an application of this results to solve the convex minimization problems and variational inequality in Hadamard manifolds were presented. In Sect. 6, we provide a numerical example to support the Mann-type splitting method.

2 Preliminaries

Let M be a connected finite-dimensional Riemannian manifold, ∇ a Levi-Civita connection, and χ a smooth curve on M. F is the unique vector field such that $\nabla_{\chi'}F = \mathbf{0}$ for all $t \in [a,b]$, where $\mathbf{0}$ is the zero section of the tangent bundle TM. Then the parallel transport $\mathrm{P}_{\chi,\chi(b),\chi(a)} : T_{\chi(a)}M \to T_{\chi(b)}M$ on the tangent bundle TM along $\chi : [a,b] \to \mathbb{R}$ is defined by

$$\mathrm{P}_{\chi,\chi(b),\chi(a)}(\nu) = F\big(\chi(b)\big), \quad \forall a,b \in \mathbb{R} \text{ and } \nu \in T_{\chi(a)}M.$$

If χ is a minimizing geodesic joining p to q, then we write $\mathrm{P}_{q,p}$ instead of $\mathrm{P}_{\chi,q,p}$. Note that, for every $a, b, b_1, b_2 \in \mathbb{R}$, we have

$$\mathrm{P}_{\chi(b_2),\chi(b_1)} \circ \mathrm{P}_{\chi(b_1),\chi(a)} = \mathrm{P}_{\chi(b_2),\chi(a)} \quad \text{and} \quad \mathrm{P}^{-1}_{\chi(b),\chi(a)} = \mathrm{P}_{\chi(a),\chi(b)}.$$

Also, $\mathrm{P}_{\chi(b),\chi(a)}$ is an isometry from $T_{\chi(a)}M$ to $T_{\chi(b)}M$, that is, the parallel transport preserves the inner product,

$$\left\langle \mathrm{P}_{\chi(b),\chi(a)}(\upsilon), \mathrm{P}_{\chi(b),\chi(a)}(\nu) \right\rangle_{\chi(b)} = \langle \upsilon, \nu \rangle_{\chi(a)}, \tag{3}$$

for all $\upsilon, \nu \in T_{\chi(a)}M$.

A Riemannian manifold is complete if for any $p \in M$ all geodesic emanating from p are defined for all $t \in \mathbb{R}$.

Let M be a complete Riemannian manifold and $p \in M$. The exponential map $\exp_p : T_pM \to M$ is defined as $\exp_p \nu = \chi_\nu(1, p)$, then, for any value of t, we have $\exp_p t\nu = \chi_\nu(t, p)$. Note that the mapping $\exp_p$ is differentiable on T_pM for every $p \in M$. The exponential map has inverse $\exp_p^{-1} : M \to T_pM$. Moreover, for any $p, q \in M$, we have $d(p, q) = \| \exp_p^{-1} q \|$, where $d(\cdot, \cdot)$ is a Riemannian distance.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is said to be an *Hadamard manifold.* Throughout, M always denotes a finite-dimensional Hadamard manifold. The following proposition is outstanding and will be helpful.

Proposition 1 ([31]) *Let $p \in M$. The $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is can be expressed by the formula*

$$\chi(t) = \exp_p t \exp_p^{-1} q, \quad \forall t \in [0, 1].$$

A geodesic triangle $\triangle(p_1, p_2, p_3)$ of a Riemannian manifold M is a set consisting of three points p_1, p_2 and p_3, and three minimizing geodesics joining these points.

Proposition 2 ([31]) *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle. Then*

$$d^2(p_1, p_2) + d^2(p_2, p_3) - 2\left\langle \exp_{p_2}^{-1} p_1, \exp_{p_2}^{-1} p_3 \right\rangle \leq d^2(p_3, p_1) \tag{4}$$

and

$$d^2(p_1, p_2) \leq \left\langle \exp_{p_1}^{-1} p_3, \exp_{p_1}^{-1} p_2 \right\rangle + \left\langle \exp_{p_2}^{-1} p_3, \exp_{p_2}^{-1} p_1 \right\rangle. \tag{5}$$

Moreover, if θ is the angle at p_1, then we have

$$\left\langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3 \right\rangle = d(p_2, p_1) d(p_1, p_3) \cos\theta.$$

The following relation between geodesic triangles in Riemannian manifolds and triangles in $\mathbb{R}^2$ can be found in [32].

Lemma 1 ([32]) *Let $\triangle(p_1,p_2,p_3)$ be a geodesic triangle in a Hadamard manifold M. Then there exists a triangle $\triangle(\overline{p_1},\overline{p_2},\overline{p_3})$ for $\triangle(p_1,p_2,p_3)$ such that $d(p_i,p_{i+1}) = \|\overline{p_i} - \overline{p_{i+1}}\|$, with the indices taken modulo* 3; *it is unique up to an isometry of $\mathbb{R}^2$.*

The triangle $\triangle(\overline{p_1},\overline{p_2},\overline{p_3})$ in Lemma 1 is said to be a *comparison triangle* for $\triangle(p_1,p_2,p_3)$. The points $\overline{p_1}$, $\overline{p_2}$, $\overline{p_3}$ are called comparison points to the points p_1, p_2, p_3, respectively.

Lemma 2 *Let $\triangle(p_1,p_2,p_3)$ be a geodesic triangle in M and $\triangle(\overline{p_1},\overline{p_2},\overline{p_3})$ be its comparison triangle.*

(i) *Let θ_1, θ_2, θ_3 (respectively, $\overline{\theta_1}$, $\overline{\theta_2}$, $\overline{\theta_3}$) be the angles of $\triangle(p_1,p_2,p_3)$ (respectively, $\triangle(\overline{p_1},\overline{p_2},\overline{p_3})$) at the vertices p_1, p_2, p_3 (respectively, $\overline{p_1}$, $\overline{p_2}$, $\overline{p_3}$). Then*

$$\theta_1 \le \overline{\theta_1}, \qquad \theta_2 \le \overline{\theta_2} \quad \text{and} \quad \theta_3 \le \overline{\theta_3}.$$

(ii) *Let q be a point on the geodesic joining p_1 to p_2 and $\overline{q}$ its comparison point in the interval $[\overline{p_1},\overline{p_2}]$. If $d(p_1,q) = \|\overline{p_1} - \overline{q}\|$ and $d(p_2,q) = \|\overline{p_2} - \overline{q}\|$, then $d(p_3,q) \le \|\overline{p_3} - \overline{q}\|$.*

Definition 1 A subset Γ in a Hadamard manifold M is called *geodesic convex* if for all p and q in Γ, and for any geodesic $\chi : [a,b] \to M$, $a,b \in \mathbb{R}$ such that $p = \chi(a)$ and $q = \chi(b)$, one has $\chi((1-t)a + tb) \in \Gamma$ for all $t \in [0,1]$.

Definition 2 A function $\phi : M \to \mathbb{R}$ is called *geodesic convex* if for any geodesic χ in M, the composition function $\phi \circ \chi : [a,b] \to \mathbb{R}$ is convex, that is,

$$(\phi \circ \chi)\big(ta + (1-t)b\big) \le t(\phi \circ \chi)(a) + (1-t)(\phi \circ \chi)(b), \quad a,b \in \mathbb{R}, \text{ and } \forall t \in [0,1].$$

Proposition 3 ([31]) *Let $d : M \times M \to \mathbb{R}$ be the distance function. Then $d(\cdot,\cdot)$ is a geodesic convex function with respect to the product Riemannian metric, that is, for any pair of geodesics $\chi_1 : [0,1] \to M$ and $\chi_2 : [0,1] \to M$ the following inequality holds*:

$$d\big(\chi_1(t),\chi_2(t)\big) \le (1-t)d\big(\chi_1(0),\chi_2(0)\big) + td\big(\chi_1(1),\chi_2(1)\big), \quad \forall t \in [0,1].$$

Particularly, for all $q \in M$, the function $d(\cdot,q) : M \to \mathbb{R}$ is a geodesic convex function.

We now present the results of parallel transport which will be helpful in the sequel.

Remark 1 ([24]) If $p,q \in M$ and $v \in T_pM$, then

$$\big\langle v, -\exp_p^{-1} q\big\rangle = \big\langle v, \mathrm{P}_{p,q} \exp_q^{-1} p\big\rangle = \big\langle \mathrm{P}_{q,p} v, \exp_q^{-1} p\big\rangle. \tag{6}$$

Remark 2 ([33]) Let $p,q,r \in M$ and $v \in T_pM$, and using (5) and Remark 1,

$$\big\langle v, \exp_p^{-1} q\big\rangle \le \big\langle v, \exp_p^{-1} r\big\rangle + \big\langle v, \mathrm{P}_{p,r} \exp_r^{-1} q\big\rangle. \tag{7}$$

Let us end the preliminary section with the following results, which are important in establishing our convergence theorem.

Definition 3 ([19]) Let Γ be a nonempty subset of M and $\{p_n\}$ be a sequence in M. Then $\{p_n\}$ is said to be *Fejér monotone* with respect to Γ if for all $q \in \Gamma$ and $n \in \mathbb{N}$,

$$d(p_{n+1}, q) \leq d(p_n, q).$$

Lemma 3 ([19]) *Let Γ be a nonempty subset of M and $\{p_n\} \subset M$ be a sequence in M such that $\{p_n\}$ is a Fejér monotone with respect to Γ. Then the following hold*:

(i) *for every $q \in \Gamma$, $\{d(p_n, q)\}$ converges*;
(ii) *$\{p_n\}$ is bounded*;
(iii) *assume that any cluster point of $\{p_n\}$ belongs to Γ, then $\{p_n\}$ converges to a point in Γ.*

3 Problem formulations

Given Γ is a nonempty subset of a Hadamard manifold M. Let $\Psi(\Gamma)$ denote the set of all single-valued vector fields $A : \Gamma \to TM$ such that $A(p) \in T_pM$, for each $p \in \Gamma$. $\mathfrak{X}(\Gamma)$ denote the set of all multivalued vector fields $B : \Gamma \to 2^{TM}$ such that $B(p) \subseteq T_pM$ for all $p \in \Gamma$, and denote $D(B)$ the domain of B defined by $D(B) = \{p \in \Gamma : B(p) \neq \emptyset\}$.

Let a vector field $A \in \Psi(\Gamma)$ and a vector field $B \in \mathfrak{X}(\Gamma)$. In this paper, we consider the following inclusion problem: find $p^* \in \Gamma$ such that

$$\mathbf{0} \in (A + B)\big(p^*\big). \tag{8}$$

We denote by $(A + B)^{-1}(\mathbf{0})$ the set of singularities of the problem (8).

In this article we work mainly with specific classes of vector fields which are defined in the following.

Definition 4 ([34, 35]) A vector field $A \in \Psi(\Gamma)$ is called

(i) *monotone* if

$$\big\langle A(p), \exp_p^{-1} q\big\rangle \leq \big\langle A(q), -\exp_q^{-1} p\big\rangle, \quad \forall p, q \in \Gamma;$$

(ii) *β-strongly monotone* if there is $\beta > 0$ such that

$$\big\langle A(p), \exp_p^{-1} q\big\rangle + \big\langle A(q), \exp_q^{-1} p\big\rangle \leq -\beta d^2(p, q), \quad \forall p, q \in \Gamma;$$

(iii) *K-Lipschitz continuous* if there is $K > 0$ such that

$$\big\| \mathrm{P}_{p,q} A(q) - A(p) \big\| \leq K d(p, q), \quad \forall p, q \in \Gamma.$$

Definition 5 ([36]) A vector field $B \in \mathfrak{X}(\Gamma)$ is called

(i) *monotone* if for all $p, q \in D(B)$

$$\big\langle \upsilon, \exp_p^{-1} q\big\rangle \leq \big\langle \nu, -\exp_q^{-1} p\big\rangle, \quad \forall \upsilon \in B(p) \text{ and } \forall \nu \in B(q);$$

(ii) *maximal monotone* if it is monotone and for all $p \in \Gamma$ and $\upsilon \in T_p\Gamma$, the condition

$$\big\langle \upsilon, \exp_p^{-1} q\big\rangle \leq \big\langle \nu, -\exp_q^{-1} p\big\rangle, \quad \forall q \in D(B) \text{ and } \forall \nu \in B(q),$$

implies that $\upsilon \in B(p)$.

The concept of the resolvent for multivalued vector fields and firmly nonexpansive mappings on Hadamard manifolds was introduce by Li et al. [24] and reads as follows.

Definition 6 ([37]) Let a vector field $B \in \mathfrak{X}(\Gamma)$ and $\lambda \in (0,\infty)$. The λ-*resolvent* of B is multivalued map $J_\lambda^B : \Gamma \to 2^\Gamma$ defined by

$$J_\lambda^B(p) := \big\{r \in \Gamma : p \in \exp_r \lambda B(r)\big\}, \quad \forall p \in \Gamma.$$

Definition 7 ([37]) Let $T : \Gamma \subseteq M \to M$ be a mapping. Then T is said to be *firmly nonexpansive* if for any two points $p, q \in \Gamma$, the function $\Theta : [0,1] \to [0,+\infty]$ defined by

$$\Theta(t) := d\big(\exp_p t \exp_p^{-1} T(p), \exp_q t \exp_q^{-1} T(q)\big), \quad \forall t \in [0,1],$$

is nonincreasing.

Let $T : \Gamma \to \Gamma$ be a nonexpansive mapping, i.e., $d(T(p), T(q)) \le d(p,q)$ for all $p, q \in \Gamma$. By Definition 7, it is clear that any firmly nonexpansive mapping T is nonexpansive. In particular, the monotonicity and nonexpansivity are firmly related.

Theorem 1 ([37]) *Let a vector field $B \in \mathfrak{X}(\Gamma)$ is monotone if and only if J_λ^B is single-valued and firmly nonexpansive*.

Let Γ be a nonempty closed geodesic convex subset of M. The *projection operator* $P_\Gamma(\cdot) : M \to \Gamma$ is defined for any $p \in M$ by $P_\Gamma(p) := \{r : d(p,r) \le d(p,q), \forall q \in \Gamma\}$. The projection operator P_Γ is firmly nonexpansive as described in the following proposition [37].

Proposition 4 ([37]) *Let Γ be a nonempty closed geodesic convex subset of M. Then the following assertions holds*:

(i) P_Γ *is single-valued and firmly nonexpansive*;
(ii) *For all $p \in M$, $r = P_\Gamma(p)$ if and only if $\langle \exp_r^{-1} p, \exp_r^{-1} q\rangle \le 0$, for all $q \in \Gamma$.*

Recently, Ansari et al. [28] obtained the relationship between a fixed point of $T_\lambda^{A,B}$ (see Lemma 5) and a singularity of the inclusion problem (8) as follows.

Proposition 5 ([28]) *For each $p \in \Gamma$, the following assertions are equivalent*:

(i) $p \in (A+B)^{-1}(\mathbf{0})$;
(ii) $p = T_\lambda^{A,B}(p)$, $\forall \lambda \in (0,\infty)$.

Moreover, they [28] also provided the following lemma which is useful in establishing the convergence result of the inclusion problem (8).

Lemma 4 ([29]) *Let Γ be a nonempty closed subset of a Hadamard manifold M and $B \in \mathfrak{X}(\Gamma)$ a maximal monotone. Let $\{\lambda_n\} \subset (0,\infty)$ with $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and a sequence $\{p_n\} \subset \Gamma$ with $\lim_{n\to\infty} p_n = p \in \Gamma$ such that $\lim_{n\to\infty} J_{\lambda_n}^B(p_n) = q$. Then $q = J_\lambda^B(p)$.*

Next, let us introduce the concept of an inverse-strongly-monotone vector field in Hadamard manifolds.

Definition 8 A vector $A \in \Psi(\Gamma)$ is said to be *inverse-strongly-monotone* if there exists $\alpha > 0$ such that

$$\langle A(p), \exp_p^{-1} q\rangle + \langle A(q), \exp_q^{-1} p\rangle \leq -\alpha \|A(p) - \mathrm{P}_{p,q}A(q)\|^2, \quad \forall p, q \in \Gamma;$$

In this case α-inverse-strongly-monotone. The reason for us to provide this definition is that β-strongly monotone and K-Lipschitz continuous vector field must be $\frac{\beta}{K^2}$-inverse-strongly-monotone. (It is seen from the definition.) Moreover, we can see that if A is α-inverse-strongly-monotone, then it is $\frac{1}{\alpha}$-Lipschitz continuous.

Indeed, let A be α-inverse-strongly-monotone, then by the definition we have

$$\begin{aligned}\alpha \|A(p) - \mathrm{P}_{p,q}A(q)\|^2 &\leq -\langle A(p), \exp_p^{-1} q\rangle - \langle A(q), \exp_q^{-1} p\rangle \\ &= \langle \mathrm{P}_{p,q}A(q) - A(p), \exp_p^{-1} q\rangle \\ &\leq \|\mathrm{P}_{p,q}A(q) - A(p)\| \|\exp_p^{-1} q\| \\ &= \|\mathrm{P}_{p,q}A(q) - A(p)\| d(p,q),\end{aligned}$$

this implies that

$$\|\mathrm{P}_{p,q}A(q) - A(p)\| \leq \frac{1}{\alpha} d(p,q),$$

for all $p, q \in \Gamma$, where $\alpha > 0$. Thus, A is $\frac{1}{\alpha}$-Lipschitz continuous.

Conversely, let A be $\frac{1}{\alpha}$-Lipschitz continuous, then by the definition we have

$$\begin{aligned}-\alpha \|\mathrm{P}_{p,q}A(q) - A(p)\|^2 &\geq -d(p,q)\|\mathrm{P}_{p,q}A(q) - A(p)\| \\ &= -\|\exp_p^{-1} q\| \|\mathrm{P}_{p,q}A(q) - A(p)\| \\ &\geq -\langle \mathrm{P}_{p,q}A(q) - A(p), \exp_p^{-1} q\rangle \\ &= \langle A(p), \exp_p^{-1} q\rangle + \langle A(q), \exp_q^{-1} p\rangle,\end{aligned}$$

for all $p, q \in \Gamma$, where $\alpha > 0$. Thus, A is α-inverse-strongly-monotone.

Now, we provide some examples of inverse-strongly-monotone vector fields.

Example 1 Let $\mathbb{R}_n^{++}$ is the product space of $\mathbb{R}^{++}$, that is, $\mathbb{R}_n^{++} = \{p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : p_i > 0,\ i = 1, \ldots, n\}$. Let $M = (\mathbb{R}_n^{++}, \langle\cdot,\cdot\rangle)$ with metric defined by $\langle \upsilon, \nu\rangle := \upsilon^T G(p)\nu$, for $p \in \mathbb{R}_n^{++}$ and $\upsilon, \nu \in T_p\mathbb{R}_n^{++}$ where $G(p)$ is a diagonal metric defined by $G(p) = \mathrm{diag}(p_1^{-2}, p_2^{-2}, \ldots, p_n^{-2})$. Specially, $M = (\mathbb{R}_n^{++}, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold with sectional curvature zero (see [18]).

Let $A : \mathbb{R}_n^{++} \to T\mathbb{R}_n^{++}$ be a single-valued vector field defined by

$$\big(A(p)\big)i := a_i b_i p_i^2 e^{-b_i p_i} + 2c_i p_i \ln(p_i) + d_i p_i, \quad i = 1, 2, \ldots, n,$$

where $a_i, b_i, d_i \in \mathbb{R}^+$ and $c_i \in \mathbb{R}^{++}$ satisfy $c_i > a_i$. Hence, A is a K-Lipschitz continuous with $K < \sum_{i=1}^n (a_i + 2c_i)^2$; for more details see [22]. Thus, we see that A is $\frac{1}{K}$-inverse-strongly-monotone where $K < \sum_{i=1}^n (a_i + 2c_i)^2$.

Example 2 Let $n = 1$ in Example 1 and $A : \mathbb{R}^{++} \to T\mathbb{R}^{++}$ be a single-valued vector field defined by

$$A(p) := \frac{1}{32} p \ln(p) + \ln(p).$$

Hence, A is a $\frac{33}{32}$-Lipschitz continuous; for further details see [38]. Thus, we see that A is $\frac{32}{33}$-inverse-strongly-monotone.

We have the following lemma.

Lemma 5 *Let $A \in \Psi(\Gamma)$ be an α-inverse-strongly-monotone vector field, where $\alpha > 0$, and $B \in \mathfrak{X}(\Gamma)$ a maximal monotone vector field. Then the following properties hold*:

(i) *for each $\lambda \in [0, 2\alpha]$, the mapping $W_\lambda : \Gamma \to \Gamma$ defined by $W_\lambda(p) = \exp_p -\lambda A(p)$ is nonexpansive*;

(ii) *for each $\lambda > 0$, the mapping $T_\lambda^{A,B} : \Gamma \to \Gamma$ defined by $T_\lambda^{A,B}(p) = J_\lambda^B(W_\lambda(p))$ is well defined and $(A + B)^{-1}(\mathbf{0}) = \mathrm{Fix}(T_\lambda^{A,B})$, where $\mathrm{Fix}(T_\lambda^{A,B})$ is the set of fixed points of $T_\lambda^{A,B}$*;

(iii) *for each $\lambda \in (0, 2\alpha]$, $T_\lambda^{A,B}$ is nonexpansive*.

Proof Conclusion (ii) follows from Proposition 5 and the maximal monotonicity of B.

In order to prove (i), let $\triangle(p, W_\lambda(p), W_\lambda(q)) \subseteq M$ be a geodesic triangle with vertices p, $W_\lambda(p)$ and $W_\lambda(q)$, and $\triangle(\overline{p}, \overline{W_\lambda(p)}, \overline{W_\lambda(q)}) \subseteq \mathbb{R}^2$ is the corresponding comparison triangle. Then we have $d(p, W_\lambda(p)) = \|\overline{p} - \overline{W_\lambda(p)}\|$, $d(W_\lambda(p), W_\lambda(q)) = \|\overline{W_\lambda(p)} - \overline{W_\lambda(q)}\|$, and $d(W_\lambda(q), p) = \|\overline{W_\lambda(q)} - \overline{p}\|$. Again, letting $\triangle(p, q, W_\lambda(q)) \subseteq M$ be a geodesic triangle with vertices p, q and $W_\lambda(q)$, and $\triangle(\overline{p}, \overline{q}, \overline{W_\lambda(q)}) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle, one obtains

$$d(p, q) = \|\overline{p} - \overline{q}\|, \qquad d\big(q, W_\lambda(q)\big) = \big\|\overline{q} - \overline{W_\lambda(q)}\big\| \quad \text{and} \quad d\big(W_\lambda(q), p\big) = \big\|\overline{W_\lambda(q)} - \overline{p}\big\|.$$

Now,

$$\begin{aligned}
d^2\big(W_\lambda(p), W_\lambda(q)\big) &= \big\|\overline{W_\lambda(p)} - \overline{W_\lambda(q)}\big\|^2 \\
&= \big\|\big(\overline{W_\lambda(p)} - \overline{p}\big) + (\overline{p} - \overline{q}) - \big(\overline{W_\lambda(q)} - \overline{q}\big)\big\|^2 \\
&= \|\overline{p} - \overline{q}\|^2 + \big\|\big(\overline{W_\lambda(p)} - \overline{p}\big) - \big(\overline{W_\lambda(q)} - \overline{q}\big)\big\|^2 \\
&\quad + 2\big\langle\big(\overline{W_\lambda(p)} - \overline{p}\big) - \big(\overline{W_\lambda(q)} - \overline{q}\big), \overline{p} - \overline{q}\big\rangle \\
&= d^2(p, q) + \big\|\big(\overline{W_\lambda(p)} - \overline{p}\big) - \big(\overline{W_\lambda(q)} - \overline{q}\big)\big\|^2 \\
&\quad + 2\big\langle\overline{W_\lambda(p)} - \overline{p}, \overline{p} - \overline{q}\big\rangle + 2\big\langle\overline{W_\lambda(q)} - \overline{q}, \overline{q} - \overline{p}\big\rangle \\
&\quad + 2\|\overline{p} - \overline{q}\|^2 - 2\|\overline{p} - \overline{q}\|^2 + 2\|\overline{q} - \overline{p}\|^2 - 2\|\overline{q} - \overline{p}\|^2 \\
&= d^2(p, q) + \big\|\big(\overline{W_\lambda(p)} - \overline{p}\big) - \big(\overline{W_\lambda(q)} - \overline{q}\big)\big\|^2 \\
&\quad + 2\big\langle\overline{W_\lambda(p)} - \overline{p}, \overline{p} - \overline{q}\big\rangle + 2\langle\overline{p} - \overline{q}, \overline{p} - \overline{q}\rangle - 2d^2(p, q) \\
&\quad + 2\big\langle\overline{W_\lambda(q)} - \overline{q}, \overline{q} - \overline{p}\big\rangle + 2\langle\overline{q} - \overline{p}, \overline{q} - \overline{p}\rangle - 2d^2(q, p) \\
&= d^2(p, q) + \big\|\big(\overline{W_\lambda(p)} - \overline{p}\big) - \big(\overline{W_\lambda(q)} - \overline{q}\big)\big\|^2
\end{aligned}$$

$$+ 2\langle \overline{W_\lambda(p)} - \overline{q}, \overline{p} - \overline{q} \rangle - 2d^2(p,q)$$
$$+ 2\langle \overline{W_\lambda(q)} - \overline{p}, \overline{q} - \overline{p} \rangle - 2d^2(q,p). \tag{9}$$

Let θ, $\overline{\theta}$ be the angles at the vertices q, $\overline{q}$. By (i) of Lemma 2, we get $\theta \leq \overline{\theta}$. Besides, by Proposition 2, we have

$$\begin{aligned}\langle \overline{W_\lambda(p)} - \overline{q}, \overline{p} - \overline{q} \rangle &= \| \overline{W_\lambda(p)} - \overline{q} \| \| \overline{p} - \overline{q} \| \cos \overline{\theta} \\ &= d\big(W_\lambda(p), q\big) d(q,p) \cos \overline{\theta} \\ &\leq d\big(W_\lambda(p), q\big) d(q,p) \cos \theta \\ &= \langle \exp_q^{-1} W_\lambda(p), \exp_q^{-1} p \rangle. \end{aligned} \tag{10}$$

Repeating the argument above gives

$$\langle \overline{W_\lambda(q)} - \overline{p}, \overline{q} - \overline{p} \rangle \leq \langle \exp_p^{-1} W_\lambda(q), \exp_p^{-1} q \rangle. \tag{11}$$

Moreover, we have

$$\begin{aligned}&\| (\overline{W_\lambda(p)} - \overline{p}) - (\overline{W_\lambda(q)} - \overline{q}) \|^2 \\ &= \| \overline{W_\lambda(p)} - \overline{p} \|^2 - 2\langle \overline{W_\lambda(p)} - \overline{p}, \overline{W_\lambda(q)} - \overline{q} \rangle + \| \overline{W_\lambda(q)} - \overline{q} \|^2 \\ &= d^2\big(W_\lambda(p), p\big) - 2\langle \overline{W_\lambda(p)} - \overline{p}, \overline{W_\lambda(q)} - \overline{q} \rangle + d^2\big(W_\lambda(q), q\big) \\ &= \| \exp_p^{-1} W_\lambda(p) \|^2 - 2\langle \overline{W_\lambda(p)} - \overline{p}, \overline{W_\lambda(q)} - \overline{q} \rangle + \| \exp_q^{-1} W_\lambda(q) \|^2 \\ &= \| -\lambda A(p) \|^2 - 2\langle \overline{W_\lambda(p)} - \overline{p}, \overline{W_\lambda(q)} - \overline{q} \rangle + \| -\lambda A(q) \|^2 \\ &= \lambda^2 \| A(p) \|^2 - 2\langle \overline{W_\lambda(p)} - \overline{p}, \overline{W_\lambda(q)} - \overline{q} \rangle + \lambda^2 \| A(q) \|^2. \end{aligned} \tag{12}$$

Consider

$$\begin{aligned}-2\langle \overline{W_\lambda(p)} - \overline{p}, \overline{W_\lambda(q)} - \overline{q} \rangle &= -2\langle \overline{W_\lambda(p)} - \overline{q} + \overline{q} - \overline{p}, \overline{W_\lambda(q)} - \overline{q} \rangle \\ &= 2\langle \overline{q} - \overline{W_\lambda(p)}, \overline{W_\lambda(q)} - \overline{q} \rangle + 2\langle \overline{p} - \overline{q}, \overline{W_\lambda(q)} - \overline{q} \rangle \\ &= 2\langle \overline{q} - \overline{W_\lambda(p)}, \overline{W_\lambda(q)} - \overline{q} \rangle + 2\| \overline{q} - \overline{W_\lambda(p)} \|^2 \\ &\quad - 2\| \overline{q} - \overline{W_\lambda(p)} \|^2 + 2\langle \overline{p} - \overline{q}, \overline{W_\lambda(q)} - \overline{q} \rangle \\ &= 2\langle \overline{q} - \overline{W_\lambda(p)}, \overline{W_\lambda(q)} - \overline{W_\lambda(p)} \rangle - 2d^2\big(q, W_\lambda(p)\big) \\ &\quad + 2\langle \overline{p} - \overline{q}, \overline{W_\lambda(q)} - \overline{q} \rangle. \end{aligned} \tag{13}$$

Repeating the argument above yields

$$\langle \overline{q} - \overline{W_\lambda(p)}, \overline{W_\lambda(q)} - \overline{W_\lambda(p)} \rangle \leq \langle \exp_{\overline{W_\lambda(p)}}^{-1} q, \exp_{\overline{W_\lambda(p)}}^{-1} W_\lambda(q) \rangle \tag{14}$$

and

$$\langle \overline{p} - \overline{q}, \overline{W_\lambda(q)} - \overline{q} \rangle \leq \langle \exp_q^{-1} p, \exp_q^{-1} W_\lambda(q) \rangle. \tag{15}$$

Substituting (14) and (15) into (13) gives

$$\begin{aligned}-2\langle \overline{W_\lambda(p)}-\overline{p}, \overline{W_\lambda(q)}-\overline{q}\rangle \leq{}& 2\langle \exp^{-1}_{W_\lambda(p)} q, \exp^{-1}_{W_\lambda(p)} W_\lambda(q)\rangle - 2d^2\big(q, W_\lambda(p)\big)\\ &+ 2\langle \exp^{-1}_q p, \exp^{-1}_q W_\lambda(q)\rangle.\end{aligned}$$

Noting Remarks 1 and 2 in the last inequality, we get

$$\begin{aligned}&-2\langle \overline{W_\lambda(p)}-\overline{p}, \overline{W_\lambda(q)}-\overline{q}\rangle\\ &\quad\leq 2\langle \exp^{-1}_{W_\lambda(p)} q, \exp^{-1}_{W_\lambda(p)} q\rangle + 2\langle \exp^{-1}_{W_\lambda(p)} q, \mathrm{P}_{W_\lambda(p),q} \exp^{-1}_q W_\lambda(q)\rangle\\ &\qquad - 2d^2\big(q, W_\lambda(p)\big) + 2\langle \exp^{-1}_q W_\lambda(p), \exp^{-1}_q W_\lambda(q)\rangle\\ &\qquad + 2\langle \mathrm{P}_{q,W_\lambda(p)} \exp^{-1}_{W_\lambda(p)} p, \exp^{-1}_q W_\lambda(q)\rangle\\ &\quad= -2\langle \exp^{-1}_q W_\lambda(p), \exp^{-1}_q W_\lambda(q)\rangle + 2\langle \exp^{-1}_q W_\lambda(p), \exp^{-1}_q W_\lambda(q)\rangle\\ &\qquad - 2\langle \exp^{-1}_p W_\lambda(p), \mathrm{P}_{p,q} \exp^{-1}_q W_\lambda(q)\rangle\\ &\quad= -2\langle -\lambda A(p), \mathrm{P}_{p,q} - \lambda A(q)\rangle\\ &\quad= -2\lambda^2\langle A(p), \mathrm{P}_{p,q} A(q)\rangle. \end{aligned} \tag{16}$$

Substituting (16) into (12) yields

$$\begin{aligned}\big\|\big(\overline{W_\lambda(p)}-\overline{p}\big) - \big(\overline{W_\lambda(q)}-\overline{q}\big)\big\|^2 &\leq \lambda^2\big\|A(p)\big\|^2 - 2\lambda^2\langle A(p), \mathrm{P}_{p,q}A(q)\rangle + \lambda^2\big\|A(q)\big\|^2\\ &= \lambda^2\big(\big\|A(p)\big\|^2 - 2\langle A(p), \mathrm{P}_{p,q}A(q)\rangle + \big\|A(q)\big\|^2\big)\\ &= \lambda^2\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2. \end{aligned} \tag{17}$$

Combining (9), (10), (11) and (17), we obtain

$$\begin{aligned}d^2\big(W_\lambda(p), W_\lambda(q)\big) \leq{}& d^2(p,q) + \lambda^2\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2\\ &+ 2\langle \exp^{-1}_q W_\lambda(p), \exp^{-1}_q p\rangle - 2d^2(p,q)\\ &+ 2\langle \exp^{-1}_p W_\lambda(q), \exp^{-1}_p q\rangle - 2d^2(q,p). \end{aligned} \tag{18}$$

From Remarks 1 and 2, the last inequality becomes

$$\begin{aligned}d^2\big(W_\lambda(p), W_\lambda(q)\big) \leq{}& d^2(p,q) + \lambda^2\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2\\ &+ 2\langle \exp^{-1}_q p, \exp^{-1}_q p\rangle + 2\langle \mathrm{P}_{q,p} \exp^{-1}_p W_\lambda(p), \exp^{-1}_q p\rangle\\ &- 2d^2(p,q) + 2\langle \exp^{-1}_p q, \exp^{-1}_p q\rangle\\ &+ 2\langle \mathrm{P}_{p,q} \exp^{-1}_q W_\lambda(q), \exp^{-1}_p q\rangle - 2d^2(q,p)\\ ={}& d^2(p,q) + \lambda^2\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2\\ &+ 2\langle -\lambda A(p), -\exp^{-1}_p q\rangle + 2\langle -\lambda A(q), -\exp^{-1}_q p\rangle\\ ={}& d^2(p,q) + \lambda^2\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2 + 2\lambda\langle A(p) \exp^{-1}_p q\rangle\\ &+ 2\lambda\langle A(q), \exp^{-1}_q p\rangle. \end{aligned}$$

So,

$$d^2\big(W_\lambda(p), W_\lambda(q)\big) \le d^2(p,q) + \lambda^2\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2 + 2\lambda\big\langle A(p)\exp_p^{-1} q\big\rangle + 2\lambda\big\langle A(q), \exp_q^{-1} p\big\rangle. \tag{19}$$

Since A is an α-inverse-strongly-monotone, we get

$$\big\langle A(p), \exp_p^{-1} q\big\rangle + \big\langle A(q), + \exp_q^{-1} p\big\rangle \le -\alpha\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2. \tag{20}$$

Substituting (20) into (19), we deduce that

$$\begin{aligned} d^2\big(W_\lambda(p), W_\lambda(q)\big) &\le d^2(p,q) + \lambda^2\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2 - 2\alpha\lambda\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2 \\ &= d^2(p,q) - \lambda(2\alpha - \lambda)\big\|A(p) - \mathrm{P}_{p,q}A(q)\big\|^2. \end{aligned}$$

From the fact the $\lambda \in [0, 2\alpha]$

$$d\big(W_\lambda(p), W_\lambda(q)\big) \le d(p,q),$$

hence $W_\lambda(p)$ is nonexpansive. To prove (iii), notice that B is maximal monotone and so the resolvent J_λ^B is firmly nonexpansive. It follows immediately from (i) that $T_\lambda^{A,B}(x) = J_\lambda^B(W_\lambda(x))$ is nonexpansive. □

4 Mann-type splitting method

In this section, we present the conditions that guarantee the convergence of the Mann-type splitting method in Hadamard manifolds and the proof.

Theorem 2 *Let Γ be a nonempty, closed and geodesic convex subset of a Hadamard manifold M. Let $A \in \Psi(\Gamma)$ be an α-inverse-strongly-monotone vector field, where $\alpha > 0$, and $B \in \mathfrak{X}(\Gamma)$ a maximal monotone vector field with $(A+B)^{-1}(\mathbf{0}) \neq \emptyset$. Choose $p_0 \in \Gamma$ and define $\{q_n\}$ and $\{p_n\}$ as follows*:

$$\begin{aligned} q_n &= J_{\lambda_n}^B\big(\exp_{p_n}\big(-\lambda_n A(p_n)\big)\big), \\ p_{n+1} &= \exp_{p_n}(1-\gamma_n)\exp_{p_n}^{-1} q_n, \end{aligned} \tag{21}$$

for all $n \in \mathbb{N}$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(i) $0 < \gamma_1 \le \gamma_n \le \gamma_2 < 1$, $\forall n \in \mathbb{N}$,

(ii) $0 < \hat{\lambda} \le \lambda_n \le 2\alpha < \infty$, $\forall n \in \mathbb{N}$.

Then $\{p_n\}$ is convergent to a solution of the inclusion problem (8).

Proof Let $x \in (A+B)^{-1}(\mathbf{0})$. From (ii) of Lemma 5, we have $x = T_{\lambda_n}^{A,B}(x) = J_\lambda^B(W_\lambda(x))$. By the nonexpansiveness of $J_{\lambda_n}^B$ and W_{λ_n}, gives

$$\begin{aligned} d(q_n, x) &= d\big(J_{\lambda_n}^B\big(\exp_{p_n}\big(-\lambda_n A(p_n)\big)\big), J_{\lambda_n}^B\big(\exp_x\big(-\lambda_n A(x)\big)\big)\big) \\ &\le d\big(\exp_{p_n}\big(-\lambda_n A(p_n)\big), \exp_x\big(-\lambda_n A(x)\big)\big) \end{aligned}$$

$$\leq d(p_n, x), \quad \forall n \in \mathbb{N}. \tag{22}$$

Let $\chi : [0,1] \to M$ be geodesic joining p_n to q_n. Thus, (21) can be written as $p_{n+1} = \chi(1-\gamma_n)$, respectively. By using the geodesic convexity of Riemannian distance, we have

$$\begin{aligned} d(p_{n+1}, x) &= d\big(\chi(1-\gamma_n), x\big) \\ &\leq \gamma_n d(p_n, x) + (1-\gamma_n) d(q_n, x) \\ &\leq \gamma_n d(p_n, x) + (1-\gamma_n) d(p_n, x) \\ &= d(p_n, x). \end{aligned}$$

Hence, $\{p_n\}$ is Fejér monotone with respect to $(A+B)^{-1}(\mathbf{0})$. By (ii) of Lemma 3, $\{p_n\}$ is bounded. Hence, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ which converges to a cluster point y of $\{p_n\}$. Next, we show that

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Fix $n \in \mathbb{N}$ and for $x \in (A+B)^{-1}(\mathbf{0})$. Let $\triangle(p_n, q_n, x) \subseteq M$ be a geodesic triangle with vertices p_n, q_n and x, and $\triangle(\overline{p_n}, \overline{q_n}, \overline{x}) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle, one obtains

$$d(p_n, x) = \|\overline{p_n} - \overline{x}\|, \qquad d(q_n, x) = \|\overline{q_n} - \overline{x}\|, \quad \text{and} \quad d(p_n, q_n) = \|\overline{p_n} - \overline{q_n}\|.$$

Let $\overline{p_{n+1}} = \gamma_n \overline{p_n} + (1-\gamma_n)\overline{q_n}$ be the comparison point of p_{n+1}. Using (ii) of Lemma 2 and (22),

$$\begin{aligned} d^2(p_{n+1}, x) &\leq \|\overline{p_{n+1}} - \overline{x}\|^2 \\ &= \big\|\gamma_n \overline{p_n} + (1-\gamma_n)\overline{q_n} - \overline{x}\big\|^2 \\ &= \gamma_n \|\overline{p_n} - \overline{x}\|^2 + (1-\gamma_n)\|\overline{q_n} - \overline{x}\|^2 - \gamma_n(1-\gamma_n)\|\overline{p_n} - \overline{q_n}\|^2 \\ &= \gamma_n d^2(p_n, x) + (1-\gamma_n) d^2(q_n, x) - \gamma_n(1-\gamma_n) d^2(p_n, q_n) \\ &\leq \gamma_n d^2(p_n, x) + (1-\gamma_n) d^2(p_n, x) - \gamma_n(1-\gamma_n) d^2(p_n, q_n) \\ &= d^2(p_n, x) - \gamma_n(1-\gamma_n) d^2(p_n, q_n), \quad \forall n \in \mathbb{N}, \end{aligned}$$

which implies that

$$\gamma_n(1-\gamma_n) d^2(p_n, q_n) \leq d^2(p_n, x) - d^2(p_{n+1}, x), \quad \forall n \in \mathbb{N}. \tag{23}$$

From the fact that $0 < \gamma_1 \leq \gamma_n \leq \gamma_2 < 1$, we have $\gamma_1(1-\gamma_2) \leq \gamma_n(1-\gamma_n)$ for all $n \in \mathbb{N}$. Summing (23) from $i = 0$ to $i = n$, we obtain

$$\gamma_1(1-\gamma_2) \sum_{i=0}^{n} d^2(p_i, q_i) \leq d^2(p_0, x) - d^2(p_{n+1}, x), \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we have

$$\gamma_1(1-\gamma_2) \sum_{i=0}^{\infty} d^2(p_i, q_i) \leq d^2(p_0, x) < \infty.$$

Hence,

$$\lim_{n\to\infty} d(p_n, q_n) = 0. \tag{24}$$

Next, we prove $y \in (A+B)^{-1}(\mathbf{0})$. Since $\hat{\lambda} \le \lambda_n \le 2\alpha$, we may assume without the loss of generality that $\lim_{k\to\infty} \lambda_{n_k} = \lambda$ for some $\lambda \in [\hat{\lambda}, 2\alpha]$. Recall that $q_n = J^B_{\lambda_n}(\exp_{p_n}(-\lambda_n A(p_n)))$. Then, by (24) and Lemma 4, we obtain $\lim_{k\to\infty} q_{n_k} = y$ and that $y = J^B_{\lambda}(\exp_y(-\lambda A(y)))$. It indicates that $y \in (A+B)^{-1}(\mathbf{0})$ by applying (ii) of Lemma 5. By (iii) of Lemma 3, the sequence $\{p_n\}$ converges to a singularity of the inclusion problem (8). Therefore, the proof is completed. □

In the order to present an example in support of our main theorem, we need the following results.

Let $\mathcal{M}$ be a Riemannian manifold and $\phi : \mathcal{M} \to \mathbb{R}$ a differentiable function. The directional derivative of ϕ at p in direction $v \in T_p\mathcal{M}$ is

$$\phi'(p; v) := \lim_{t\to 0^+} \frac{\phi(\exp_p tv) - \phi(p)}{t}.$$

The gradient of ϕ at $p \in \mathcal{M}$ [39] is given by $\langle \operatorname{grad}\phi(p), v\rangle := \phi'(p; v)$ for all $v \in T_p\mathcal{M}$. If $\phi : \mathcal{M} \to \mathbb{R}$ is a twice differentiable function, then the Hessian of ϕ at $p \in \mathcal{M}$ [40], denoted by Hessϕ, is defined by

$$\operatorname{Hess}\phi(p) := \nabla_v\big(\operatorname{grad}\phi(p)\big), \quad \forall v \in T_p\mathcal{M},$$

where ∇ is the Riemannian connection of $\mathcal{M}$.

The norm of the Hessian, hess ϕ, at $p \in \mathcal{M}$ is given by

$$\big\|\operatorname{hess}\phi(p)\big\| := \sup\big\{\big\|\operatorname{hess}\phi(p)v\big\| : v \in T_p\mathcal{M}, \|v\| = 1\big\}. \tag{25}$$

Proposition 6 ([20]) *Let M be an Hadamard manifold and $\phi : M \to \mathbb{R}$ a twice continuously differentiable function. If* Hess ϕ *is bounded, then the gradient vector field* grad ϕ *is K-Lipschitz continuous.*

Definition 9 ([31]) Let $\varphi : \Gamma \to \mathbb{R}$ be a geodesic convex function. Take $p \in \Gamma$, a vector $v \in T_pM$ is said to be a *subgradient* of φ at p if and only if

$$\varphi(q) \ge \varphi(p) + \big\langle v, \exp_p^{-1} q\big\rangle, \quad \forall q \in \Gamma. \tag{26}$$

The set of all subgradients of φ, denoted by $\partial\varphi(p)$ is said to be the *subdifferential* of φ at p, which is a closed geodesic convex (possibly empty) set.

Lemma 6 ([24]) *Let $\varphi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and geodesic convex function and $D(\varphi) = \Gamma$. Then the subdifferential $\partial\varphi$ of φ is a maximal monotone vector field.*

Next, we present an example in the cone of the positive semidefinite matrices with other metrics.

Example 3 Let $\mathbb{S}^n$ be the set of symmetric matrices, $\mathbb{S}^n_+$ be the cone of the symmetric positive semidefinite matrices and $\mathbb{S}^n_{++}$ be the cone of the symmetric positive-definite matrices both $n \times n$. $X, Y \in \mathbb{S}^n_+$, $X \succeq X$ (or $X \preceq Y$) means that $Y - X \in \mathbb{S}^n_+$ and $X \succ Y$ (or $X \prec Y$) means that $Y - X \in \mathbb{S}^n_{++}$.

Following Rothaus [41], let $M := (\mathbb{S}^n_{++}, \langle \cdot, \cdot \rangle)$ be the Riemannian manifold endowed with the Riemannian metric defined by

$$\langle U, V \rangle = \operatorname{Tr}\big(VX^{-1}UX^{-1}\big), \quad X \in M \text{ and } U, V \in T_X M, \tag{27}$$

where $\operatorname{Tr}(X)$ denotes the trace of matrix $X \in \mathbb{S}^n$ and $T_X M \approx \mathbb{S}^n$, with the corresponding norm denoted by $\| \cdot \|$. The gradient and the Hessian of a twice differentiable function $\phi : \mathbb{S}^n_{++} \to \mathbb{R}$ are given by

$$\operatorname{grad} \phi(X) = X\phi'(X)X, \tag{28}$$

$$\operatorname{hess} \phi(X)V = X\phi''(X)VX + \frac{1}{2}\big[V\phi'(X)X + X\phi'(X)V\big], \tag{29}$$

where $V \in T_X M$, and $\phi'(X)$ and $\phi''(X)$ are the Euclidian gradient and Hessian of ϕ at X, respectively.

In fact, M is a Hadamard manifold with curvature is not identically zero; see [42, Theorem 1.2. p. 325] for further details. The unique geodesic segment connecting any $X, Y \in M$ is given by

$$\chi(t) = X^{1/2}\big(X^{-1/2}YX^{-1/2}\big)^t X^{1/2}, \quad t \in [0, 1];$$

see, for example [43]. From the last equation

$$\chi'(0) = X^{1/2} \ln\big(X^{-1/2}YX^{-1/2}\big)X^{1/2}.$$

Thus, for all $X \in M$, $\exp_X^{-1} : M \to T_X M$ and $\exp_X : T_X M \to M$ are defined, respectively, by

$$\exp_X^{-1} Y = X^{1/2} \ln\big(X^{-1/2}YX^{-1/2}\big)X^{1/2}, \qquad \exp_X V = X^{1/2} e^{(X^{-1/2}VX^{-1/2})} X^{1/2}. \tag{30}$$

Now, since the Riemannian distance d is given by $d(X, Y) = \| \exp_X^{-1} Y \|$, from (27), along with first expression in (30), we have

$$d^2(X, Y) = \operatorname{Tr}\big(\ln^2 X^{-1/2}YX^{-1/2}\big) = \sum_{i=1}^{n} \ln^2 \eta_i\big(X^{-1/2}YX^{-1/2}\big), \tag{31}$$

where $\eta_i(X^{-1/2}YX^{-1/2})$ denotes the ith eigenvalue of the symmetric matrix $X^{-1/2}YX^{-1/2}$.

Following [22], let the function $\phi : \mathbb{S}^n_{++} \to \mathbb{R}$ defined by

$$\phi(X) = a \ln\big(\det(X)\big)^2, \tag{32}$$

where $a > 0$. The Euclidian gradient and Hessian of ϕ are given, respectively, by

$$\phi'(X) = \big[2a \ln\big(\det(X)\big)\big]X^{-1}, \tag{33}$$

$$\phi''(X)V = 2a\,\mathrm{Tr}\big(X^{-1}V\big)X^{-1} - \big[2a\ln\big(\det(X)\big)\big]X^{-1}VX^{-1}, \tag{34}$$

where $X \in \mathbb{S}^n_{++}$ and $V \in \mathbb{S}^n$.

By combining (28), (29), (33) and (34), we obtain, after some calculations,

$$\operatorname{grad}\phi(X) = 2a\ln\big(\det(X)\big)X, \tag{35}$$

$$\operatorname{hess}\phi(X)V = 2a\,\mathrm{Tr}\big(X^{-1}V\big)X, \tag{36}$$

for all $X \in M$ and $V \in T_XM$. We further have $\langle \operatorname{hess}\phi(X)V, V\rangle = 2a\,\mathrm{Tr}(X^{-1}V)^2 \geq 0$. Thus, ϕ is geodesic convex in M. Moreover, (27) together with (36) gives $\|\operatorname{hess}\phi(X)V\| = 2a\,\mathrm{Tr}(X^{-1}V)$ for all $X \in M$ and $V \in T_XM$. If we assume that $\|V\|^2 = \mathrm{Tr}(VX^{-1}VX^{-1}) = 1$, then $\mathrm{Tr}(X^{-1}V) \leq \sqrt{n}$. Hence,

$$\big\|\operatorname{hess}\phi(X)V\big\| \leq 2a\sqrt{n}, \quad X \in M, V \in T_XM, \|V\| = 1.$$

Therefore, (25) and Proposition 6 imply that $\operatorname{grad}\phi$ is Lipschitz with constant $K \leq 2a\sqrt{n}$. We also have $\operatorname{grad}\phi$ is $\frac{1}{K}$-inverse-strongly-monotone vector field with constant $K \leq 2a\sqrt{n}$.

Let $\Gamma = \{X \in \mathbb{S}^n_{++} : \eta_{\min}(X) \geq 1\}$, where $\eta_{\min}(X)$ denotes the minimum eigenvalue of the matrix X, be a nonempty, closed and geodesic convex subset of M and $\varphi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and geodesic convex defined by

$$\varphi(X) = d^2(X, I) = \sum_{i=1}^{n} \ln^2 \eta_i\big(X^{-1}\big),$$

where $\eta_i(X^{-1})$ is the ith eigenvalue of the matrix X^{-1}. One can see that I is a minimizer of φ, where I denotes the identity matrix. By Definition 9, the subdifferential of φ at X is defined by

$$\partial\varphi(X) = \Bigg\{ U \in T_XM \,\Big|\, \sum_{i=1}^{n} \ln^2 \eta_i\big(Y^{-1}\big) - \sum_{i=1}^{n} \ln^2 \eta_i\big(X^{-1}\big) \geq \mathrm{Tr}\big(X^{1/2}\ln\big(X^{-1/2}YX^{-1/2}\big)X^{1/2}X^{-1}UX^{-1}\big)\Bigg\}, \quad \forall Y \in \Gamma.$$

The subdifferential $\partial\varphi$ of φ is a maximal monotone vector field, according to Lemma 6. Moreover, we have

$$J^{\partial\varphi}_{\lambda}(X) = \operatorname*{arg\,min}_{Y\in\Gamma}\Big\{\varphi(Y) + \frac{1}{2\lambda}d^2(Y, X)\Big\}, \quad \forall \lambda > 0.$$

Since the minimizer of φ is I, it is easy to see that $\mathbf{0} \in \partial\varphi(I)$.

Let $A : \Gamma \to \mathbb{S}^n$ is a $\frac{1}{K}$-inverse-strongly-monotone vector field defined by

$$A(X) = 2(\ln\det X)X,$$

where $K \leq 2\sqrt{n}$, and $B : \Gamma \to 2^{\mathbb{S}^n}$ be a maximal monotone multivalued field defined by

$$B(X) = \partial\varphi(X).$$

We see that $(A+B)^{-1}(\mathbf{0}) = \{I\}$. Choose initial point $X_0 \in \Gamma$, then Theorem 2 is applicable leading us to conclude that any sequence generated by Eq. (21) converges to a singularity of the inclusion problem (8).

Remark 3 It is worth noting that ϕ is non-convex with non-Lipschitz continuous gradient on $\mathbb{S}^n_{++}$ endowed with the Euclidean metric. Thus we cannot apply existence results, e.g. [8, 16], to solve the corresponding inclusion problem in the Euclidean setting.

5 Applications

In this section, we shall utilize the Mann-type splitting method presented in the paper to study the convex minimization problems and variational inequality problems.

5.1 Convex minimization problems

Let $\phi, \varphi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ are proper, lower semicontinuous and geodesic convex functions such that ϕ is differentiable. We consider the problem of finding $p^* \in \Gamma$ such that

$$\phi(p^*) + \varphi(p^*) = \min_{p\in\Gamma}\{\phi(p) + \varphi(p)\}. \tag{37}$$

The problem is said to be a *convex minimization problem*. We denote S by the set of minimizers of the problem (37), that is, $S := \{p \in \Gamma : \phi(p) + \varphi(p) \le \phi(q) + \varphi(q), \forall q \in \Gamma\}$. It is to see that the problem (37) is equivalent to the following inclusion problem: find $p \in \Gamma$ such that $\mathbf{0} \in \operatorname{grad} f(p) + \partial g(p)$, that is,

$$p \in S \quad \Longleftrightarrow \quad \mathbf{0} \in \operatorname{grad}\phi(p) + \partial\varphi(p). \tag{38}$$

For further details see [30].

If $\phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ is a proper, twice continuously differentiable and geodesic convex function such that $\operatorname{Hess}\phi$ is bounded, then, by Proposition 6, $\operatorname{grad}\phi$ is K-Lipschitz continuous vector field. Then $\operatorname{grad}\phi$ is $\frac{1}{K}$-inverse-strongly-monotone vector field. Moreover, from Lemma 6, $\partial\varphi$ is a maximal monotone vector field. By replacing A and B by $\operatorname{grad}\phi$ and $\partial\varphi$, respectively, in Theorem 2, we get the following result for convex minimization problem (37).

Theorem 3 *Suppose that $S \neq \emptyset$. Let Γ be a nonempty, closed and geodesic convex subset of a Hadamard manifold M. Let $\phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ be a proper, differentiable and geodesic convex function such that* $\operatorname{Hess}\phi$ *is bounded, $\varphi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ a proper, lower semicontinuous and geodesic convex function such that $D(\varphi) = \Gamma$. Choose $p_0 \in \Gamma$ and define $\{p_n\}$ as follows*:

$$q_n = J^{\partial\varphi}_{\lambda_n}\big(\exp_{p_n}\big(-\lambda_n \operatorname{grad}\phi(p_n)\big)\big),$$
$$p_{n+1} = \exp_{p_n}(1-\gamma_n)\exp^{-1}_{p_n} q_n,$$

for all $n \in \mathbb{N}$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(i) $0 < \gamma_1 \le \gamma_n \le \gamma_2 < 1$, $\forall n \in \mathbb{N}$,

(ii) $0 < \hat{\lambda} \le \lambda_n \le 2\alpha < \infty$, $\forall n \in \mathbb{N}$.

Then $\{p_n\}$ is convergent to a solution of the convex minimization problem (37).

Proof By replacing A and B by $\operatorname{grad}\phi$ and $\partial\varphi$, respectively, in Theorem 2, we get the required result. □

5.2 Variational inequalities

A monotone variational inequality problem (VIP) on a Hadamard manifold was initially studied by Németh [44]. Then the problem is to find $p^* \in \Gamma$ such that

$$\left\langle V\left(p^*\right), \exp_{p^*}^{-1} q\right\rangle \geq 0, \quad \forall q \in \Gamma, \tag{39}$$

where $V : \Gamma \to TM$ is a single-valued vector field. $VIP(V, \Gamma)$ denotes the set of solutions of the problem (39). Let $N_\Gamma(p)$ denote the normal cone of the set Γ at $p \in \Gamma$:

$$N_\Gamma(p) := \left\{ v \in T_pM : \left\langle v, \exp_p^{-1} q\right\rangle \leq 0, \forall q \in \Gamma \right\}.$$

Let δ_Γ be the indicator function of Γ, that is,

$$\delta_\Gamma(p) = \begin{cases} 0, & \text{if } p \in \Gamma, \\ +\infty, & \text{if } p \notin \Gamma. \end{cases} \tag{40}$$

It is easy to see that δ_Γ is a proper, lower semicontinuous and geodesic convex function on a Hadamard manifold M. By Lemma 6, $\partial\delta_\Gamma$ is a maximal monotone multivalued vector field.

From (40), $\delta_\Gamma(p) = 0$ for all $p \in \Gamma$, and therefore, from (26), we get

$$\begin{aligned} \partial\delta_\Gamma(p) &= \left\{ v \in T_pM : \left\langle v, \exp_p^{-1} q\right\rangle \leq \delta_\Gamma(q) - \delta_\Gamma(p) \right\}, \\ &= \left\{ v \in T_pM : \left\langle v, \exp_p^{-1} q\right\rangle \leq 0 \right\}. \end{aligned} \tag{41}$$

Thus, $\partial\delta_\Gamma(p) = N_\Gamma(p)$. For every $p \in \Gamma$ and $V \in \Psi(\Gamma)$, using (41),

$$\begin{aligned} p \in (V + \partial\delta_\Gamma)^{-1}(\mathbf{0}) &\iff -V(p) \in \partial\delta_\Gamma(p) \\ &\iff \left\langle -V(p), \exp_p^{-1} q\right\rangle \leq 0, \quad \forall q \in \Gamma \\ &\iff p \in VIP(V, \Gamma). \end{aligned}$$

The resolvent operator $J_\lambda^{\partial\delta_\Gamma}$ of $\partial\delta_\Gamma$ for $\lambda > 0$ is given by

$$J_\lambda^{\partial\delta_\Gamma}(p) := \left\{ r \in M : p \in \exp_r \lambda\partial\delta_\Gamma(r) \right\}, \quad \forall p \in M.$$

For every $p \in \Gamma$, we get

$$\begin{aligned} w = J_\lambda^{\partial\delta_\Gamma}(p) &\iff p \in \exp_w \lambda\partial\delta_\Gamma(w) \\ &\iff \frac{1}{\lambda} \exp_w^{-1} p \in \partial\delta_\Gamma(w) = N_\Gamma(w) \\ &\iff \frac{1}{\lambda}\left\langle \exp_w^{-1} p, \exp_w^{-1} q\right\rangle \leq 0, \quad \forall q \in \Gamma \\ &\iff P_\Gamma(p) = w. \end{aligned}$$

We replace B and J_λ by $\partial\delta_\Gamma$ and P_Γ, respectively, in Theorem 2, then we get the following result for variational inequality problem (39).

Theorem 4 *Let Γ be a nonempty, closed and geodesic convex subset of a Hadamard manifold M. Let $V \in \Psi(\Gamma)$ be an α-inverse-strongly-monotone vector field, where $\alpha > 0$ such that $VIP(V,\Gamma) \neq \emptyset$. Choose $p_0 \in \Gamma$ and define $\{p_n\}$ as follows*:

$$q_n = P_\Gamma\big(\exp_{p_n}\big(-\lambda_n A(p_n)\big)\big),$$
$$p_{n+1} = \exp_{p_n}(1-\gamma_n)\exp_{p_n}^{-1} q_n,$$

for all $n \in \mathbb{N}$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(i) $0 < \gamma_1 \leq \gamma_n \leq \gamma_2 < 1$, $\forall n \in \mathbb{N}$,

(ii) $0 < \hat{\lambda} \leq \lambda_n \leq 2\alpha < \infty$, $\forall n \in \mathbb{N}$.

Then $\{p_n\}$ is convergent to a solution of the variational inequality problem (39).

Proof By replacing B by $\partial\delta_\Gamma$, in Theorem 2, we get the required result. □

6 Numerical example

We present an illustrative example in Hadamard manifolds to show the applicability of the Mann-type splitting method in this section. All the program are written in Matlab R2016b and computed on PC Intel(R) Core(TM) i5 @1.80 GHz, Ram 8.00 GB.

Let $M := \mathbb{R}^{++} = \{p \in \mathbb{R} : p > 0\}$ and $\mathbb{R}^{+} = \{p \in \mathbb{R} : p \geq 0\}$. Following [20, Example 1], let $(\mathbb{R}^{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold, and $\langle\cdot,\cdot\rangle$ the Riemannian metric defined by

$$\langle \upsilon, \nu\rangle := \frac{1}{p^2}\upsilon\nu, \quad \forall \upsilon, \nu \in T_pM.$$

The Riemannian distance $d : M \times M \to \mathbb{R}^{+}$ is defined by

$$d(p,q) := \left|\ln\frac{p}{q}\right|, \quad \forall p, q \in M;$$

for further details see [39]. Then $(\mathbb{R}^{++}, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold and the unique geodesic $\chi : \mathbb{R} \to M$ starting from $\chi(0) = p$ with $\nu = \chi'(0) \in T_pM$ is defined by $\chi(t) := pe^{(\nu t/p)}$. Therefore,

$$\exp_p t\nu = pe^{(\nu t/p)}.$$

The inverse of exponential map is defined by

$$\exp_p^{-1} q = p\ln\frac{q}{p}.$$

Example 4 Let $\Gamma = [1, +\infty)$ be a closed and geodesic convex subset of $\mathbb{R}^{++}$ and $A : \Gamma \to \mathbb{R}$ a single-valued vector field defined by

$$A(p) := p\ln(p), \quad \forall p \in \Gamma.$$

This is 1-Lipschitz continuous vector field, that is,

$$\left\| \mathrm{P}_{p,q} A(q) - A(p) \right\| \leq d(p,q),$$

for all $p, q \in \Gamma$; see [30] for more details. Hence, A is a 1-inverse strongly monotone vector field. Indeed, let $x, y \in \Gamma$, we obtain

$$\begin{aligned}
\left\langle A(p), \exp_p^{-1} q \right\rangle + \left\langle A(q), \exp_q^{-1} p \right\rangle &= \frac{1}{p^2}(p \ln p)\left(p \ln\left(\frac{q}{p}\right)\right) + \frac{1}{q^2}(q \ln q)\left(q \ln\left(\frac{p}{q}\right)\right) \\
&= \ln p \ln\left(\frac{q}{p}\right) + \ln q \ln\left(\frac{p}{q}\right) \\
&= -\ln^2\left(\frac{p}{q}\right) \\
&= -d^2(p,q) \\
&\leq -\left\| \mathrm{P}_{p,q} A(q) - A(p) \right\|^2.
\end{aligned}$$

Let $\varphi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and geodesic convex function defined by

$$\varphi(p) := d(1,p), \quad \forall p \in \Gamma.$$

From Definition 9, we have

$$\partial\varphi(p) = \left\{ \upsilon \in T_p M \;\middle|\; d(1,q) - d(1,p) \geq \frac{\upsilon}{p} \ln\left(\frac{p}{q}\right) \right\}, \quad \text{for all } q \in \Gamma.$$

Lemma 6 shows that the subdifferential $\partial\varphi$ of φ is a maximal monotone vector field. Now, consider the maximal monotone vector field B by $\partial\varphi$. Moreover, we have

$$J_\lambda^{\partial\varphi}(p) = \arg\min_{q \in \Gamma} \left\{ \varphi(q) + \frac{1}{2\lambda} d^2(q,p) \right\}, \quad \forall \lambda > 0.$$

It is clear that $\{1\} = (A + B)^{-1}(\mathbf{0})$.

Setting $\gamma_n = \lambda_n = \frac{1}{2} + \frac{1}{n+4}$, so the sequences $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfied the conditions (i) and (ii) of Theorem 2. With initial point $x_0 = 2$ and $x_0 = 3$, we show the numerical behaviour of Mann-type splitting method in Table 1 and Fig. 1.

7 Concluding remarks

The problem of finding singularities of inclusion problems, which is defined by the sum of an inverse-strongly-monotone vector field and a multivalued maximal monotone vector field on a Hadamard manifold, is examined in this paper. A Mann-type splitting method is proposed for solving this problem. Under suitable conditions, the convergence theorem for sequences generated by the presented methods was established. We gave an example that demonstrates the usefulness of the Mann splitting method's generalization from Hilbert spaces to Hadamard manifolds, in the sense that it must help in the explanation of some previously unconsidered situations. Furthermore, our algorithm was used to solve convex minimization problems and variational inequalities.

Table 1 Numerical results of Mann-type splitting method

Iteration No.	p_n with initial point $p_0 = 2$	p_n with initial point $p_0 = 3$
0	2	3
1	1.6245	2.1577
2	1.3819	1.6698
3	1.2311	1.3904
4	1.1388	1.2287
5	1.0827	1.1341
6	1.0488	1.0785
7	1.0286	1.0456
8	1.0166	1.0264
9	1.0095	1.0151
10	1.0054	1.0086
11	1.0031	1.0049
12	1.0017	1.0027
13	1.0010	1.0015
14	1.0005	1.0008
15	1.0003	1.0005

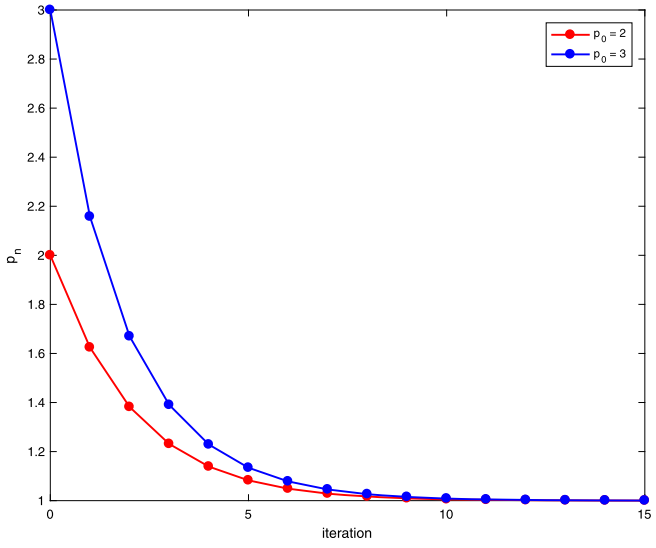

Figure 1 Iterative process of Mann-type splitting method

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