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A new extragradient algorithm with adaptive step-size for solving split equilibrium problems

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Abstract

He (J. Inequal. Appl. 2012:Article ID 162 2012) introduced the proximal point CQ algorithm (PPCQ) for solving the split equilibrium problem (SEP). However, the PPCQ converges weakly to a solution of the SEP and is restricted to monotone bifunctions. In addition, the step-size used in the PPCQ is a fixed constant μ in the interval $(0, \frac{1}{\|A\|^2})$. This often leads to excessive numerical computation in each iteration, which may affect the applicability of the PPCQ. In order to overcome these intrinsic drawbacks, we propose a robust step-size $\{\mu_n\}_{n=1}^{\infty}$ which does not require computation of $\|A\|$ and apply the adaptive step-size rule on $\{\mu_n\}_{n=1}^{\infty}$ in such a way that it adjusts itself in accordance with the movement of associated components of the algorithm in each iteration. Then, we introduce a self-adaptive extragradient-CQ algorithm (SECQ) for solving the SEP and prove that our proposed SECQ converges strongly to a solution of the SEP with more general pseudomonotone equilibrium bifunctions. Finally, we present a preliminary numerical test to demonstrate that our SECQ outperforms the PPCQ.

MSC: 47H05; 47H09; 49M37; 65K10

Keywords: Split equilibrium problems; Extragradient algorithm; Self-adaptive step-sizes; Pseudomonotone equilibrium problems

1 Introduction

The *equilibrium problem* (EP) associated with a bifunction $f : C \times C \to \mathbb{R}$ and a nonempty subset C of a real Hilbert space $\mathbb{H}$ consists of finding a vector $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1}$$

It is well known that the mathematical basis for the EP pre-dates the works of Ky Fan [10]. However, due to his dedication to the subject, the EP is often called the Ky Fan inequality. The EP is an incredibly important powerful tool that unifies a number of useful and elegant nonlinear problems. In recent days, the EP is one of the major nonlinear methods that has provided significant success in modeling several real world problems (see, e.g., [1, 2, 8, 9, 13, 14, 18, 23, 24]).

The set of solutions of EP is denoted by $\mathrm{SOL}(f, C)$. To solve the EP, it is common to use the proximal point method proposed by Combettes and Hirstoaga [8]: given $x_n \in C$, as a current iterate, the next iterate x_{n+1} solves the following problem:

$$\text{find } x \in C \text{ such that } \quad f(x,y) + \frac{1}{r_n}\langle y - x, x - x_n\rangle \geq 0, \quad \forall y \in C, \{r_n\}_{n=1}^{\infty} \subset (0, \infty). \tag{2}$$

In 2008, Tran et al. [23] (see also Dang [9]) submitted that a large number of important real-world problems can be reformulated as pseudomonotone bifunctions. A well-known example is the Nash–Cournot oligopolistic electricity market model. Unfortunately, the traditional proximal point method (2) does not converge if f is pseudomonotone (see, e.g., [21, Example 2.1]). Hence, Tran et al. [23] introduced the following proximal-extragradient algorithm for solving (1) when f is pseudomonotone.

$$\begin{cases} \text{select arbitrary} \quad x_1 \in C, \\ y_n = \operatorname{argmin}_{y \in C}\{f(x_n, y) + \frac{1}{2\rho_n}\|y - x_n\|^2\}, \\ x_{n+1} = \operatorname{argmin}_{y \in C}\{f(y_n, y) + \frac{1}{2\rho_n}\|y - x_n\|^2\}, \quad n \in \mathbb{N}. \end{cases} \tag{3}$$

It is worthy to mention that, unlike the proximal point method (2), the extragradient algorithm (3) falls within the applicable scope of standard *matlab optimization toolbox.* So, in order to implement algorithm (3), one only needs to solve a pair of strongly convex programs via matlab optimization toolbox.

On the other hand, the study of classical *split feasibility problem* (SFP) is pioneered by Censor and Elfving [5]. It provides effective tools for obtaining the existence of solutions of constrained and inverse problems arising in optimization, engineering, medical sciences, and most notably in image reconstruction, signal processing, and phase retrieval (see, e.g., [3, 4, 6]). The SFP is to find a vector

$$x^* \in C \quad \text{such that} \quad A(x^*) \in Q, \tag{4}$$

where C and Q are given nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : \mathrm{H}_1 \to \mathrm{H}_2$ is a bounded linear operator. Bryne [3] imposed the restrictive condition $\{\gamma_n\}_{n=1}^{\infty} \subset (0, \frac{2}{\|A\|^2})$ on the CQ-method for solving (4):

$$x_1 \in C, \qquad x_{n+1} := \mathrm{P}_C\big(x_n - \gamma_n A^*(I - \mathrm{P}_Q)Ax_n\big), \quad n \in \mathbb{N}, \tag{5}$$

where P_C and P_Q stand for the metric projection onto C and Q, respectively, and A^* denotes the adjoint operator of A from H_2 to H_1. The concept of self-adaptive step-size for solving (4) was introduced by Yang [26] and developed by Lopez et al. [15] in order to dispense with the restrictive condition $\{\gamma_n\}_{n=1}^{\infty} \subset (0, \frac{2}{\|A\|^2})$. In general, self-adaptive algorithms operate under the assumption that future events (inputs) are uncertain so they are characterized by step-sizes that continuously monitor themselves, gather data, analyze data, and adapt when their requirements fail due to unexpected changes in their components. Research found out that an ever-growing number of algorithms with adaptive step-sizes for solving nonlinear problems are faster and robust to failure (see, e.g., [11, 19, 20, 22, 27, 28]).

The SFP is a particular case of a more general problem, called the *split equilibrium problem* (SEP)

$$\begin{cases} \text{find } x^* \in C \text{ such that } \quad f(x^*,y) \geq 0, \quad \forall y \in C \quad \text{and} \\ A(x^*) = u^* \in Q \quad \text{solves} \quad g(u^*,v) \geq 0, \quad \forall v \in Q. \end{cases} \tag{6}$$

Here, $g : Q \times Q \to \mathbb{R}$ stands for another bifunction on H_2. The SEP was briefly introduced by Moudafi [17] in 2011. Perhaps, due to its relevance, the problem was reintroduced a year after by He [12], and the following *proximal point-CQ algorithm* (PPCQ) for solving SEP (6) was proposed:

$$\begin{cases} \text{Select arbitrary:} \quad x_1 \in C, \qquad \{\rho_n\}_{n=1}^{\infty} \subset (0,\infty), \quad \text{and} \quad \mu \in (0, \frac{1}{\|A\|^2}), \\ f(y_n,y) + \frac{1}{\rho_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ g(v_n,v) + \frac{1}{\rho_n}\langle v - v_n, v_n - Ay_n\rangle \geq 0, \quad \forall v \in Q, \\ x_{n+1} = P_C(y_n - \mu A^*(Ay_n - v_n)), \quad n \in \mathbb{N}. \end{cases} \tag{7}$$

It is worth noting that, as a prototype of the proximal point method, the PPCQ may not converge when f and g are pseudomonotone. In addition, the PPCQ converges weakly to a solution of (6) when it is consistent and the step-size $\mu \in (0, \frac{1}{\|A\|^2})$ depends on $\|A\|$, which, in turn, may lead to excessive numerical computation that may affect the convergence of the PPCQ. The question now becomes: is it possible to develop an extragradient algorithm with an adaptive step-size that converges strongly to a solution of (6) when f and g are pseudomonotone? In answering this question, we present a self-adaptive extragradient-CQ algorithm (SECQ) for solving (6) and prove that a sequence generated by our proposed SECQ converges strongly to solutions of (6) when f and g are pseudomonotone. A numerical example is also given to demonstrate the effectiveness of our iterative scheme.

2 Preliminaries

Let $\mathbb{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let $C \subseteq \mathbb{H}$ be a nonempty, closed, and convex set, and denote by the symbols $\rightharpoonup$ and $\longrightarrow$ weak and strong convergence of a sequence $\{x_n\}_{n=1}^{\infty}$.

Definition 2.1 A bifunction $f : C \times C \to \mathbb{R}$ is said to be

(i) *monotone* on C, if

$$f(x,y) + f(y,x) \leq 0, \quad \forall x,y \in C;$$

(ii) *pseudomonotone* on C with respect to $x \in C$, if

$$f(x,y) \geq 0 \quad \Rightarrow \quad f(y,x) \leq 0, \quad \forall y \in C;$$

(iii) *pseudomonotone* on C with respect to $\emptyset \neq \Omega \subset C$, if $\forall x^* \in \Omega$,

$$f(x^*,y) \geq 0 \quad \Rightarrow \quad f(y,x^*) \leq 0, \quad \forall y \in C;$$

(iv) *Lipschitz-type continuous*, if there are two positive constants L_1, L_2 such that

$$f(x,y)+f(y,z)\geq f(x,z)-L_1\|x-y\|^2-L_2\|y-z\|^2,\quad \forall x,y,z\in C;$$

(v) *jointly weakly continuous* on $C\times C$ in the sense that, given any $x,y\in C$ and $\{x_n\}_{n=1}^{\infty},\{y_n\}_{n=1}^{\infty}\subset C$ converge weakly to x and y, respectively, then

$$\lim_{n\to\infty} f(x_n,y_n)=f(x,y).$$

The following conditions will be used in the sequel.

Assumption A

$(A1)$. f is pseudomonotone with respect to $\mathrm{SOL}(f,C)$;
$(A2)$. f is jointly weakly and Lipschitz-type continuous on C with constants L_1 and L_2;
$(A3)$. $f(x,\cdot)$ is convex and subdifferentiable on C;
$(A4)$. $\Omega=\{x\in\mathrm{SOL}(f,C)$ such that $A(x)\in\mathrm{SOL}(g,Q)\}\neq\emptyset$.

Lemma 2.2 ([2]) *If the bifunction f satisfies conditions $(A1)$–$(A4)$, then $\mathrm{SOL}(f,C)$ is weakly closed and convex.*

Recall that a metric projection of $\mathbb{H}$ onto C is the mapping $\mathrm{P}_C:\mathbb{H}\to C$ which assigns to each $x\in\mathbb{H}$ the (nearest) unique point $\mathrm{P}_C(x)$ in C satisfying

$$\|x-\mathrm{P}_C(x)\|=\min\{\|x-y\|:y\in C\}.$$

Lemma 2.3 *Given $u\in\mathbb{H}$ and $z\in C$. Then*

$$z=\mathrm{P}_C(u)\quad\Longleftrightarrow\quad\langle u-z,z-y\rangle\geq 0,\quad\forall y\in C.$$

Lemma 2.4 ([16]) *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i}<a_{n_{i+1}}$ for all $i\geq 0$. Then there exists an increasing sequence $\{m_k\}_{k=1}^{\infty}\subset\mathbb{N}$ such that $m_k\to\infty$ and the following properties are satisfied by all (sufficiently large) numbers $k\in\mathbb{N}$:*

$$a_{m_k}\leq a_{m_{k+1}}\quad \text{and}\quad a_k\leq a_{m_{k+1}}.$$

In fact, m_k is the largest number n in the set $\{1,2,\ldots,k\}$ such that the condition $a_n\leq a_{n+1}$ holds.

Lemma 2.5 ([25]) *Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ and $\{\delta_n\}_{n=1}^{\infty}$ be in $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\to\infty}\delta_n\leq 0$ or $\sum_{n=1}^{\infty}|\gamma_n\delta_n|<\infty$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers such that $a_{n+1}\leq(1-\gamma_n)a_n+\gamma_n\delta_n$, $\forall n\geq 0$, then $\lim_{n\to\infty}a_n=0$.*

3 Main results

The following is our main algorithm for solving (6).

Algorithm 3.1 (Self-adaptive proximal-extragradient algorithm for SEP)

Initialization: Given initial choice x_1 and u in C. Pick parameters $\{\delta_n\}_{n=1}^{\infty} \subset [\underline{\delta}, \bar{\delta}] \subset (0,1)$, $\{\sigma_n\}_{n=1}^{\infty} \subset (0,2)$, $\{\rho_n\}_{n=1}^{\infty} \subset [\underline{\rho}, \bar{\rho}]$, and $\{r_n\}_{n=1}^{\infty} \subset [\underline{r}, \bar{r}]$ such that

$$[\underline{\rho}, \bar{\rho}], [\underline{r}, \bar{r}] \subset \left(0, \min\left\{\frac{1}{2L_1}, \frac{1}{2L_2}\right\}\right), \qquad \lim_{n\to\infty} \delta_n = 0, \quad \text{and} \quad \sum_{i=1}^{N} \delta_n = \infty.$$

Iterative steps: Assume that x_n is known for $n \in \mathbb{N}$, then compute the update x_{n+1} according to the following rule.

Step 1: Compute:

$$y_n = \operatorname*{argmin}_{y\in C}\left\{f(x_n, y) + \frac{1}{2\rho_n}\|y - x_n\|^2\right\} \quad \text{and} \quad z_n = \operatorname*{argmin}_{y\in C}\left\{f(y_n, y) + \frac{1}{2\rho_n}\|y - x_n\|^2\right\}.$$

Step 2: Set $\hat{v_n} := P_Q A(z_n)$.

Step 3: Compute:

$$v_n = \operatorname*{argmin}_{v\in Q}\left\{g(\hat{v_n}, v) + \frac{1}{2r_n}\|v - \hat{v_n}\|^2\right\} \quad \text{and} \quad u_n = \operatorname*{argmin}_{v\in Q}\left\{g(v_n, v) + \frac{1}{2r_n}\|v - \hat{v_n}\|^2\right\}.$$

Step 4: Set $F(z_n) := \frac{1}{2}\|Az_n - u_n\|^2$ and $G(z_n) := A^*(Az_n - u_n)$.

Step 5: Compute

$$x_{n+1} = \delta_n u + (1 - \delta_n) P_C\big(z_n - \mu_n G(z_n)\big), \quad \text{where}$$

$$\mu_n = \begin{cases} \sigma_n \frac{F(z_n)}{\|G(z_n)\|^2}, & \text{if } G(z_n) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Stopping criterion: If $x_{n+1} = x_n$, then x_n is a solution of SEP (6) and the iterative process stops, otherwise, put $n := n + 1$ and go back to *Step 1*.

Lemma 3.2 ([1], Lemma 3.1) *Let* H_1 *and* H_2 *be real Hilbert spaces. Let* C *and* Q *be nonempty, closed, and convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator with adjoint* A^**. Assume that* $f : C \times C \to \mathbb{R}$ *satisfies conditions* $(A1)$–$(A4)$*. Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by Algorithm* 3.1*. Then, for all* $x^* \in \mathrm{SOL}(f, C)$*, the following statements hold*:

(a). $\rho_n(f(x_n, y) - f(x_n, y_n)) \geq \langle y_n - x_n, y_n - y\rangle$ *for all* $y \in C$;

(b). $\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\rho_n L_1)\|x_n - y_n\|^2 - (1 - 2\rho_n L_2)\|y_n - z_n\|^2$.

The following theorem gives conditions that guarantee strong convergence of Algorithm 3.1.

Theorem 3.3 *Let* H_1 *and* H_2 *be real Hilbert spaces. Let* C *and* Q *be nonempty, closed, and convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator with adjoint* A^**. Assume that* $f : C \times C \to \mathbb{R}$ *and* $g : Q \times Q \to \mathbb{R}$ *satisfy conditions* $(A1)$–$(A4)$*. Let* $\{x_n\}_{n=1}^{\infty}$ *be any sequence generated by Algorithm* 3.1*. Then* $x_n \to P_\Omega(u)$ *under the following conditions*:

$(C1)$. $0 < t_1 \leq \mu_n \leq t_2$ *for some* $t_1, t_2 \in \mathbb{R}$ *and* $\forall n \in \Gamma = \{n \geq 1 : G(z_n) \neq 0\}$;
$(C2)$. $\liminf_{n\to\infty}(2 - \sigma_n) > 0$;
$(C3)$. $\langle Gz_n, z_n - x^*\rangle \geq F(z_n)$ *for all* $x^* \in \Omega$.

Proof Let $x^* = P_\Omega(u)$ and $\pi_n = \mathrm{P}_C(z_n - \mu_n G(z_n))$. Then, by using $(C3)$, we have

$$\begin{aligned}
\|\pi_n - x^*\|^2 &= \|\mathrm{P}_C(z_n - \mu_n G(z_n)) - \mathrm{P}_C(x^*)\|^2 \\
&\leq \|z_n - \mu_n G(z_n) - x^*\|^2 = \|z_n - x^* - \mu_n G(z_n)\|^2 \\
&\leq \|z_n - x^*\|^2 + \|\mu_n G(z_n)\|^2 - 2\mu_n\langle G(z_n), z_n - x^*\rangle \\
&\leq \|z_n - x^*\|^2 + \mu_n^2\|G(z_n)\|^2 - 2\mu_n F(z_n) \\
&= \|z_n - x^*\|^2 + \sigma_n^2\frac{[F(z_n)]^2}{\|G(z_n)\|^4}\|G(z_n)\|^2 - 2\sigma_n\frac{F(z_n)}{\|G(z_n)\|^2}F(z_n) \\
&= \|z_n - x^*\|^2 + \sigma_n^2\frac{[F(z_n)]^2}{\|G(z_n)\|^2} - 2\sigma_n\frac{[F(z_n)]^2}{\|G(z_n)\|^2}.
\end{aligned}$$

Thus

$$\|\pi_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \sigma_n(2 - \sigma_n)\frac{[F(z_n)]^2}{\|G(z_n)\|^2}. \tag{8}$$

This implies

$$\|\pi_n - x^*\|^2 \leq \|z_n - x^*\|^2. \tag{9}$$

Consequently, by Lemma 3.2 and (9), we get

$$\|\pi_n - x^*\|^2 \leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{10}$$

By Algorithm 3.1 and (10), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\delta_n u + (1 - \delta_n)\pi_n - x^*\| \\
&= \|\delta_n(u - x^*) + (1 - \delta_n)(\pi_n - x^*)\| \\
&\leq \delta_n\|u - x^*\| + (1 - \delta_n)\|\pi_n - x^*\| \\
&\leq \delta_n\|u - x^*\| + (1 - \delta_n)\|x_n - x^*\| \\
&\leq \max\{\|u - x^*\|, \|x_n - x^*\|\} \\
&\ \ \vdots \\
&\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}.
\end{aligned}$$

Hence $\{x_n\}_{n=1}^\infty$ is bounded. Then, from (10), we deduce that $\{\pi_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ are bounded. By using Algorithm 3.1 and *subdifferential inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in \mathbb{H},$$

we obtain

$$\|x_{n+1}-x^*\|^2 \le (1-\delta_n)\|\pi_n - x^*\|^2 + 2\delta_n\langle u-x^*, x_{n+1}-x^*\rangle. \tag{11}$$

Thus, by Lemma 3.2, (10), and (11), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le{}& (1-\delta_n)\|x_n - x^*\|^2 - (1-\delta_n)\big((1-2\rho_n L_1)\|x_n-y_n\|^2 \\ &+ (1-2\rho_n L_2)\|y_n-z_n\|^2\big) + 2\delta_n\langle u-x^*, x_{n+1}-x^*\rangle.\end{aligned} \tag{12}$$

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is decreasing for $n \ge n_0$. Then the limit of $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ exists. Consequently,

$$\lim_{n\to\infty}\big(\|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2\big) = 0. \tag{13}$$

Moreover, by using $0 < \underline{\rho} \le \rho_n \le \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$ and (12), we have

$$\begin{aligned} 0 &\le (1-\delta_n)\big((1-2\bar{\rho}L_1)\|x_n-y_n\|^2 + (1-2\bar{\rho}L_2)\|y_n-z_n\|^2\big) \\ &\quad + \delta_n\|x_n-x^*\|^2 - 2\delta_n\langle u-x^*, x_{n+1}-x^*\rangle \\ &\le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2. \end{aligned} \tag{14}$$

Since $\lim_{n\to\infty}\delta_n = 0$, $(1-2\bar{\rho}L_1) > 0$, and $(1-2\bar{\rho}L_2) > 0$, then it follows from (14) and (13) that

$$\lim_{n\to\infty}\|x_n-y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|y_n-z_n\| = 0. \tag{15}$$

This implies that

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \tag{16}$$

By combining (10) and (11), we obtain

$$\begin{aligned} 0 &\le \|x_n-x^*\|^2 - \|\pi_n-x^*\|^2 - \delta_n\|x_n-x^*\|^2 - 2\delta_n\langle u-x^*, x_{n+1}-x^*\rangle \\ &\le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2. \end{aligned} \tag{17}$$

Thus, from (13) and (17), we get

$$\lim_{n\to\infty}\big(\|x_n-x^*\|^2 - \|\pi_n-x^*\|^2\big) = 0. \tag{18}$$

Owing to $(C1)$, (8), and (10), we have

$$0 \le t_1(2-\sigma_n)F(z_n) \le \|x_n-x^*\|^2 - \|\pi_n-x^*\|^2. \tag{19}$$

Clearly, from (19), (18), and $(C2)$, we obtain

$$\lim_{n\to\infty}F(z_n) = 0. \quad \text{Hence} \quad \lim_{n\to\infty}\|Az_n-u_n\| = 0. \tag{20}$$

Since $\|Az_n - Ax^*\| \leq \|Az_n - u_n\| + \|u_n - Ax^*\|$, then it follows from (20) that

$$\lim_{n\to\infty}\left(\left\|Az_n - Ax^*\right\|^2 - \left\|u_n - Ax^*\right\|^2\right) = 0. \tag{21}$$

By Lemma 3.2, we have

$$\left\|u_n - Ax^*\right\|^2 \leq \left\|\hat{v}_n - Ax^*\right\|^2 - (1 - 2r_nL_1)\|\hat{v}_n - v_n\|^2 - (1 - 2r_nL_2)\|v_n - u_n\|^2. \tag{22}$$

Similarly, from $0 < \underline{r} \leq r_n \leq \bar{r} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$ and (22), we get

$$(1 - 2\bar{r}L_1)\|\hat{v}_n - v_n\|^2 + (1 - 2\bar{r}L_2)\|v_n - u_n\|^2 \leq \left\|Az_n - Ax^*\right\|^2 - \left\|u_n - Ax^*\right\|^2. \tag{23}$$

Moreover, since $(1 - 2\bar{r}L_1) > 0$ and $(1 - 2\bar{r}L_2) > 0$, then it follows from (23) and (21) that

$$\lim_{n\to\infty}\|\hat{v}_n - v_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|v_n - u_n\| = 0. \tag{24}$$

Now, let $\zeta_n = z_n - \mu_n G(z_n)$. Then

$$\|\zeta_n - z_n\| = \mu_n\left\|G(z_n)\right\| = \mu_n\left\|A^*\right\|\left\|A(z_n) - u_n\right\|. \tag{25}$$

Again, by using $(C1)$, (20), and (25), we get

$$\lim_{n\to\infty}\|\zeta_n - z_n\| = 0. \tag{26}$$

Since $\|\pi_n - z_n\|^2 \leq \|\zeta_n - z_n\|^2$, then it follows from (26) that

$$\lim_{n\to\infty}\|\pi_n - z_n\| = 0. \tag{27}$$

By the triangle inequality in conjunction with (16) and (27), we obtain

$$\lim_{n\to\infty}\|\pi_n - x_n\| = 0. \tag{28}$$

It is clear that

$$\|x_{n+1} - \pi_n\| \leq \delta_n\|u - \pi_n\|. \tag{29}$$

Since $\delta_n \to 0$ as $n \to \infty$ and $\{\|u - \pi_n\|\}_{n=1}^{\infty}$ is bounded, then

$$\lim_{n\to\infty}\|x_{n+1} - \pi_n\| = 0. \tag{30}$$

Consequently, from (30) and (28), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{31}$$

Furthermore, since H_1 is reflexive and $\{x_n\}_{n=1}^{\infty} \subset H_1$ is bounded, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$x_{n_k} \rightharpoonup e^* \in H_1 \quad \text{as } k \to \infty. \quad \text{Therefore, assume w.o.l.o.g. that} \tag{32}$$

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle u - x^*, x_{n_k} - x^* \rangle. \tag{33}$$

Moreover, since $\{z_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{\pi_n\}_{n=1}^{\infty}$ are bounded, then it follows from (15), (16), (28), and (32) that

$$z_{n_k} \rightharpoonup e^*, \qquad y_{n_k} \rightharpoonup e^*, \quad \pi_{n_k} \rightharpoonup e^* \quad \text{as } k \to \infty. \tag{34}$$

It will now be shown that the weak limit e^* solves SEP (6). That is, $e^* \in \Omega$. Indeed, since C and Q are closed and convex, then

$$C \text{ and } Q \text{ are weakly closed.} \tag{35}$$

Also, since $\{x_n\}_{n=1}^{\infty} \subset C$, then it follows from (32) and (35) that $e^* \in C$. Note that A is linear and bounded. So, from (34), we obtain $Az_{n_k} \rightharpoonup Ae^*$ as $k \to \infty$. In view of (20) and the boundedness of $\{u_n\}_{n=1}^{\infty}$, we see that

$$u_{n_k} \rightharpoonup A(e^*) \quad \text{as } k \to \infty. \tag{36}$$

Likewise, since $\{v_n\}_{n=1}^{\infty}$ and $\{\hat{v_n}\}_{n=1}^{\infty}$ are bounded, then, from (36) and (24), we get

$$v_{n_k} \rightharpoonup A(e^*), \qquad \hat{v_{n_k}} \rightharpoonup A(e^*) \quad \text{as } k \to \infty. \tag{37}$$

Clearly, since $\{\hat{v_{n_k}}\}_{k=1}^{\infty} \subset Q$, then, from (35) and (37), we deduce that $A(e^*) \in Q$. It remains to show that $e^* \in \mathrm{EP}(C,f)$ and $Ae^* \in \mathrm{EP}(Q,g)$. By Lemma 3.2, in particular, for all $k \in \mathbb{N}$, we have

$$\rho_{n_k}\big(f(x_{n_k},y) - f(x_{n_k},y_{n_k})\big) \geq \langle y_{n_k} - x_{n_k}, y_{n_k} - y\rangle, \quad \forall y \in C.$$

This implies

$$\frac{\langle y_{n_k} - x_{n_k}, y_{n_k} - y\rangle}{\rho_{n_k}} \leq f(x_{n_k},y) - f(x_{n_k},y_{n_k}), \quad \forall y \in C. \tag{38}$$

However, since $\rho_{n_k} \geq \underline{\rho} > 0$, then, by applying the Cauchy–Schwartz inequality, we see that

$$\begin{aligned} \left|\frac{\langle y_{n_k} - x_{n_k}, y_{n_k} - y\rangle}{\rho_{n_k}}\right| &= \frac{|\langle y_{n_k} - x_{n_k}, y_{n_k} - y\rangle|}{\rho_{n_k}} \leq \frac{|\langle y_{n_k} - x_{n_k}, y_{n_k} - y\rangle|}{\underline{\rho}} \\ &\leq \frac{\|y_{n_k} - x_{n_k}\|\|y_{n_k} - y\|}{\underline{\rho}}, \quad \forall y \in C. \end{aligned} \tag{39}$$

On the other hand, since $\{y_{n_k}\}_{k=1}^{\infty}$ is bounded and $\lim_{k\to\infty}\|y_{n_k} - x_{n_k}\| = 0$, then $\|y_{n_k} - x_{n_k}\|\|y_{n_k} - y\| \to 0$ as $k \to \infty$. Consequently,

$$\lim_{k\to\infty} \frac{\langle y_{n_k} - x_{n_k}, y_{n_k} - y\rangle}{\rho_{n_k}} = 0. \tag{40}$$

Since f satisfies $(A2)$, then, by passing limit as $k \to \infty$ in (38) in conjunction with (34), (32), and (40), we have

$$\begin{aligned} \lim_{k\to\infty} \frac{\langle y_{n_k} - x_{n_k}, y_{n_k} - y\rangle}{\rho_{n_k}} &\le \lim_{k\to\infty} \bigl(f(x_{n_k}, y) - f(x_{n_k}, y_{n_k})\bigr) \\ &= f\bigl(e^*, y\bigr) - f\bigl(e^*, e^*\bigr), \quad \forall y \in C. \end{aligned} \tag{41}$$

Thus $e^* \in \mathrm{EP}(C,f)$. Again, by Lemma 3.2, we see that

$$r_{n_k}\bigl(g(\hat{v_{n_k}}, y) - g(\hat{v_{n_k}}, v_{n_k})\bigr) \ge \langle v_{n_k} - \hat{v_{n_k}}, v_{n_k} - v\rangle, \quad \forall v \in Q.$$

Therefore

$$\frac{\langle v_{n_k} - \hat{v_{n_k}}, v_{n_k} - v\rangle}{r_{n_k}} \le g(\hat{v_{n_k}}, v) - g(\hat{v_{n_k}}, v_{n_k}), \quad \forall v \in Q. \tag{42}$$

Since $r_{n_k} \ge \underline{r} > 0$, applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left|\frac{\langle v_{n_k} - \hat{v_{n_k}}, v_{n_k} - v\rangle}{r_{n_k}}\right| &= \frac{|\langle v_{n_k} - \hat{v_{n_k}}, v_{n_k} - v\rangle|}{r_{n_k}} \le \frac{|\langle v_{n_k} - \hat{v_{n_k}}, v_{n_k} - v\rangle|}{\underline{r}} \\ &\le \frac{\|v_{n_k} - \hat{v_{n_k}}\|\|v_{n_k} - v\|}{\underline{r}}, \quad \forall v \in Q. \end{aligned} \tag{43}$$

Now, since $\{v_{n_k}\}_{k=1}^{\infty}$ is bounded and $\lim_{k\to\infty}\|v_{n_k} - \hat{v_{n_k}}\| = 0$, then $\|v_{n_k} - \hat{v_{n_k}}\|\|v_{n_k} - v\| \to 0$ as $k \to \infty$. Hence,

$$\lim_{k\to\infty} \frac{\langle v_{n_k} - \hat{v_{n_k}}, v_{n_k} - v\rangle}{r_{n_k}} = 0. \tag{44}$$

Again since g satisfies $(A2)$, then by passing limit as $k \to \infty$ in (42) using (36), (37), and (44), we get

$$0 \le \lim_{k\to\infty} \bigl(g(\hat{v_{n_k}}, v) - g(\hat{v_{n_k}}, v_{n_k})\bigr) = g\bigl(Ae^*, v\bigr) - g\bigl(Ae^*, Ae^*\bigr), \quad \forall v \in Q. \tag{45}$$

Hence $e^* \in \Omega$. Since Ω is closed, convex, and $x^* = \mathrm{P}_{\Omega}(u)$, then it follows from (31), (32), (33), and Lemma 2.3 that

$$\begin{aligned} \limsup_{n\to\infty}\bigl\langle u - x^*, x_{n+1} - x^*\bigr\rangle &\le \limsup_{n\to\infty}\bigl\langle u - x^*, x_n - x^*\bigr\rangle = \lim_{k\to\infty}\bigl\langle u - x^*, x_{n_k} - x^*\bigr\rangle \\ &= \bigl\langle u - x^*, e^* - x^*\bigr\rangle \le 0. \end{aligned} \tag{46}$$

Consequently, by Lemma 2.5, (11), and (46), we conclude that

$$\lim_{n\to\infty}\bigl\|x_n - x^*\bigr\|^2 = 0.$$

Case 2. Suppose that $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is not decreasing such that there exists a subsequence $\{\|x_{n_l} - x^*\|\}_{l=1}^{\infty}$ of $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ satisfying

$$\bigl\|x_{n_l} - x^*\bigr\| < \bigl\|x_{n_{l+1}} - x^*\bigr\| \quad \text{for all } l \in \mathbb{N}.$$

In view of Lemma 2.4, there exists a nondecreasing sequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - x^*\| \le \|x_{m_{k+1}} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \le \|x_{m_{k+1}} - x^*\|. \tag{47}$$

Note that by discarding the repeated terms of $\{m_k\}_{k=1}^{\infty}$, but still denoted by $\{m_k\}_{k=1}^{\infty}$, we can have $\{x_{m_k}\}_{k=1}^{\infty}$, as a subsequence of $\{x_n\}_{n=1}^{\infty}$. Since $\{x_{m_k}\}_{k=1}^{\infty}$ is bounded, then $\lim_{n\to\infty}(\|x_{m_k} - x^*\| - \|x_{m_{k+1}} - x^*\|) = 0$. Then, following arguments similar to those in Case 1, we deduce that

$$\lim_{k\to\infty} \|x_{m_{k+1}} - x_{m_k}\| = 0 \quad \text{and} \tag{48}$$

$$\limsup_{k\to\infty} \langle u - x^*, x_{m_{k+1}} - x^*\rangle \le 0, \quad \text{where } x^* = P_\Omega(u). \tag{49}$$

It follows from (47) and (11) that

$$\begin{aligned} \|x_{m_{k+1}} - x^*\|^2 &\le (1-\delta_{m_k})\|x_{m_k} - x^*\|^2 + 2\delta_{m_k}\langle u - x^*, x_{m_{k+1}} - x^*\rangle \\ &\le (1-\delta_{m_k})\|x_{m_{k+1}} - x^*\|^2 + 2\delta_{m_k}\langle u - x^*, x_{m_{k+1}} - x^*\rangle. \end{aligned} \tag{50}$$

Clearly, by dividing through by δ_{m_k}, we get

$$\|x_{m_{k+1}} - x^*\|^2 \le 2\langle u - x^*, x_{m_{k+1}} - x^*\rangle. \tag{51}$$

Passing limit as $k \to \infty$ in (51) using (49), we obtain

$$\lim_{k\to\infty} \|x_{m_{k+1}} - x^*\|^2 = 0.$$

Consequently, from (47), we see that

$$\lim_{k\to\infty} \|x_k - x^*\|^2 = 0.$$

Hence $x_n \to x^* \in \Omega$ in both cases and this ends the proof. □

Remark 1

(1) Theorem 3.3 will take the form of the extragradient method studied by Tran et al. [23] if we set $g \equiv 0$ and $H_1 \equiv H_2$.
(2) Theorem 3.3 coincides with the work of Lopez et al. [15], whenever $g = f \equiv 0$.
(3) The restriction on the step-size $\mu \in (0, \frac{1}{\|A\|^2})$ is imposed by He [12], while the step-size $\mu \in (0, \frac{1}{\|A\|^2})$ is relaxed with adaptive step-size that uses a simpler initialization sequence $\{\sigma_n\}_{n=1}^{\infty} \subset (0, 2)$ in our work.
(4) Moreover, in the work of He [12], a weak convergence result was obtained for solving (6) and the strong convergence follows only through the hybrid proximal point algorithm, which is not easy to implement. In this paper, we obtain a strong convergence result for solving (6) without using the hybrid scheme.
(5) The conclusion of our Theorem 3.3 holds for pseudomonotone bifunctions, while the corresponding result by He [12] is restricted to monotone bifunctions.

Table 1 Example 4.1: Comparison of Algorithm 3.1 with He's algorithm (7)

m	Number of iterations (n)		CPU time (s)	
	Algorithm 3.1	He's algorithm (7)	Algorithm 3.1	He's algorithm (7)
5	13	111	0.0942812	0.8982632
10	23	197	0.8638912	2.9385731
20	33	201	1.2636421	4.3829392
50	53	403	1.5485732	8.0197482
70	63	445	1.8735482	8.6749302
100	57	501	2.1294532	9.1485839
150	51	499	2.5394313	12.9829482

4 Numerical results

In this section, a preliminary numerical test is presented to compare the convergence behavior of proposed Algorithm 3.1 with algorithm (7).

Example 4.1 Consider the Nash–Cournot equilibrium problem studied in [20, 23], where $f : C \times C \to \mathbb{R}$ is defined by

$$f(x,y) = \langle Ux + Vy + c, y - x\rangle,$$

where $c \in \mathbb{R}^m$ and U, V are two matrices of order m such that V is symmetric positive semidefinite and $V - U$ is negative semidefinite with Lipschitz constants $L_1 = L_2 = \frac{1}{2}\|U - V\|$. The matrices U, V are randomly generated [1] and the entries of c randomly belong to $[-1, 1]$. The constraint set $C \subset \mathbb{R}^m$ is taken as follows:

$$C := \left\{ x \in \mathbb{R}^m : \sum_{i=1}^{m} x_i \geq -1, -10 \leq x_i \leq 10 \right\}.$$

Assume that $g : Q \times Q \to \mathbb{R}$ is defined by $g(x,y) = x(y - x)$, $\forall x, y \in Q = [-1, \infty)$. Suppose that $A : \mathbb{R}^m \to Q$ is a linear operator defined by $Ax = \langle a, x\rangle$, $\forall x \in \mathbb{R}^m$, where a is a vector in $\mathbb{R}^m$ whose elements are randomly generated from $[1; m]$. Thus, $A^* : [-1, \infty) \to \mathbb{R}^m$ is of the form $A^*y = y.a$ for all $y \in \mathbb{R}$ and $\|A\| = \|a\|$. The starting points $x_1 \in C$ are randomly generated in the interval $[-10, 10]$, and we choose $\mu = \frac{1}{2\|a\|^2}$, $\rho_n = r_n = \frac{1}{4L_1}$, $\delta_n = \frac{1}{n+2}$, and $\sigma_n = 2 - \frac{1}{n+1}$. We define the function TOL_n by $TOL_n := \|x_{n+1} - x_n\|$ and use the stopping rule $TOL_n < \epsilon$ for the iterative process, where ϵ is the predetermined error. The equivalent convex quadratic problems are solved using the function *fmincon* and implemented in MATLAB 7.0 running on an HP Compaq510, Core(TM)2 Duo Processor T5870 with 2.00 GHz and 2 GB RAM. Table 1 shows that Algorithm 3.1 outperforms He's algorithm (7) in running time and in the number of iterations for different cases of m.

5 Conclusion

In this paper, we have proposed a self-adaptive extragradient iterative process for solving split pseudomonotone equilibrium problems. We established strong convergence of our proposed algorithm, and the performance of the algorithm such as CPU time and the

[1] Two matrices are randomly generated E and F with entries from $[-1, 1]$. The matrix $V = E^T E$, $S = F^T F$ and $U = S + V$.

number of iterations required for convergence is highlighted through preliminary numerical tests that show that our proposed algorithm is faster than the corresponding algorithm by He [12].

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