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Solutions of nonlinear difference equations in the domain of (ζ_n)-Cesàro matrix in $\ell_{t(\cdot)}$ of nonabsolute type, and its pre-quasi ideal

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Abstract

We have constructed the sequence space $(\Xi(\zeta, t))_{\upsilon}$, where $\zeta = (\zeta_l)$ is a strictly increasing sequence of positive reals tending to infinity and $t = (t_l)$ is a sequence of positive reals with $1 \le t_l < \infty$, by the domain of (ζ_l)-Cesàro matrix in the Nakano sequence space $\ell_{(t_l)}$ equipped with the function $\upsilon(f) = \sum_{l=0}^{\infty} (\frac{|\sum_{z=0}^{l} f_z \Delta\zeta_z|}{\zeta_l})^{t_l}$ for all $f = (f_z) \in \Xi(\zeta, t)$. Some geometric and topological properties of this sequence space, the multiplication mappings defined on it, and the eigenvalues distribution of operator ideal with s-numbers belonging to this sequence space have been investigated. The existence of a fixed point of a Kannan pre-quasi norm contraction mapping on this sequence space and on its pre-quasi operator ideal formed by $(\Xi(\zeta, t))_{\upsilon}$ and s-numbers is presented. Finally, we explain our results by some illustrative examples and applications to the existence of solutions of nonlinear difference equations.

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Keywords: Kannan contraction mapping; Minimum space; Multiplication mapping; Cesàro sequence space of nonabsolute type; s-numbers; Pre-quasi ideal

1 Introduction

Variable exponent Lebesgue spaces go back many years, and in successive centuries, variable Lebesgue and Sobolev spaces have been systematically examined. Many variable exponent real function spaces and complex function spaces have been presented since then, including Hardy spaces, Besov spaces, Bessel potential spaces, Trieble–Lizorkin spaces, Morrey spaces, Herz–Morrey spaces, Herz spaces, Fock spaces, and Bergman spaces. For three centuries, variable exponent function spaces have been widely applied in approximation theory, image processing, and differential equations. Thus far, the theory of variable exponent function spaces has pensively built upon the boundedness of the Hardy–Littlewood maximal operator, and this confines its procedure to differential equations, approximation, and optimization. By $\mathcal{C}^{\mathbb{N}}$, ℓ_∞, ℓ_r, and c_0, we suggest the spaces of each, bounded, r-absolutely summable, and null sequences of complex numbers, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We denote the space of all, finite rank, approximable, and compact bounded linear mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$ by $\mathbb{B}(\mathcal{P},\mathcal{Q})$, $\mathbb{F}(\mathcal{P},\mathcal{Q})$, $\mathcal{A}(\mathcal{P},\mathcal{Q})$, and $\mathcal{K}(\mathcal{P},\mathcal{Q})$, and if $\mathcal{P}=\mathcal{Q}$, we mark $\mathbb{B}(\mathcal{P})$, $\mathbb{F}(\mathcal{P})$, $\mathcal{A}(\mathcal{P})$, and $\mathcal{K}(\mathcal{P})$, respectively. The ideals of all, finite rank, approximable, and compact mappings are denoted by $\mathbb{B}$, $\mathbb{F}$, $\mathcal{A}$, and $\mathcal{K}$. We designate $e_l=(0,0,\dots,1,0,0,\dots)$, as 1 presents at the l^{th} coordinate, with $l\in\mathrm{N}$.

Definition 1.1 ([1]) An s-number function is a mapping defined on $\mathbb{B}(\mathcal{P},\mathcal{Q})$ which maps every mapping $X\in\mathbb{B}(\mathcal{P},\mathcal{Q})$ to a nonnegative scalar sequence $(s_l(X))_{l=0}^{\infty}$ that satisfies the following conditions:

(a) $\|X\|=s_0(X)\geq s_1(X)\geq s_2(X)\geq\cdots\geq 0$ for every $X\in\mathbb{B}(\mathcal{P},\mathcal{Q})$;
(b) $s_{l+a-1}(X_1+X_2)\leq s_l(X_1)+s_a(X_2)$ for each $X_1,X_2\in\mathbb{B}(\mathcal{P},\mathcal{Q})$ and $l,a\in\mathrm{N}$;
(c) Ideal property: $s_a(ZYX)\leq\|Z\|s_a(Y)\|X\|$ for all $X\in\mathbb{B}(\mathcal{P}_0,\mathcal{P})$, $Y\in\mathbb{B}(\mathcal{P},\mathcal{Q})$, and $Z\in\mathbb{B}(\mathcal{Q},\mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are discretionary Banach spaces;
(d) For $G\in\mathbb{B}(\mathcal{P},\mathcal{Q})$ and $\gamma\in\mathcal{C}$, one has $s_a(\gamma G)=|\gamma|s_a(G)$;
(e) Rank property: Assume $\operatorname{rank}(X)\leq a$, then $s_a(X)=0$ for each $X\in\mathbb{B}(\mathcal{P},\mathcal{Q})$;
(f) Norming property: $s_{l\geq a}(I_a)=0$ or $s_{l<a}(I_a)=1$, where I_a mirrors the unit mapping on the a-dimensional Hilbert space ℓ_2^a.

For an assorted illustration of s-numbers, we provide the next setting:

(1) The ath Kolmogorov number, denoted by $d_a(X)$, is defined as

$$d_a(X)=\inf_{\dim J\leq a}\sup_{\|f\|\leq 1}\inf_{g\in J}\|Xf-g\|.$$

(2) The ath approximation number, denoted by $\alpha_a(X)$, is defined as

$$\alpha_a(X)=\inf\bigl\{\|X-Y\|:Y\in\mathbb{B}(\mathcal{P},\mathcal{Q})\text{ and }\operatorname{rank}(Y)\leq a\bigr\}.$$

Notations 1.2 ([2])

$$\begin{aligned}
&\mathbb{B}^s_{\mathcal{V}}:=\bigl\{\mathbb{B}^s_{\mathcal{V}}(\mathcal{P},\mathcal{Q});\mathcal{P}\text{and}\mathcal{Q}\text{are Banach spaces}\bigr\},\quad\text{where}\\
&\mathbb{B}^s_{\mathcal{V}}(\mathcal{P},\mathcal{Q}):=\bigl\{X\in\mathbb{B}(\mathcal{P},\mathcal{Q}):\bigl(\bigl(s_a(X)\bigr)_{a=0}^{\infty}\in\mathcal{V}\bigr\}.\\
&\mathbb{B}^\alpha_{\mathcal{V}}:=\bigl\{\mathbb{B}^\alpha_{\mathcal{V}}(\mathcal{P},\mathcal{Q});\mathcal{P}\text{and}\mathcal{Q}\text{are Banach spaces}\bigr\},\quad\text{where}\\
&\mathbb{B}^\alpha_{\mathcal{V}}(\mathcal{P},\mathcal{Q}):=\bigl\{X\in\mathbb{B}(\mathcal{P},\mathcal{Q}):\bigl(\bigl(\alpha_a(X)\bigr)_{a=0}^{\infty}\in\mathcal{V}\bigr\}.\\
&\mathbb{B}^d_{\mathcal{V}}:=\bigl\{\mathbb{B}^d_{\mathcal{V}}(\mathcal{P},\mathcal{Q})\mathcal{P}\text{and}\mathcal{Q}\text{are Banach spaces}\bigr\},\quad\text{where}\\
&\mathbb{B}^d_{\mathcal{V}}(\mathcal{P},\mathcal{Q}):=\bigl\{X\in\mathbb{B}(\mathcal{P},\mathcal{Q}):\bigl(\bigl(d_a(X)\bigr)_{a=0}^{\infty}\in\mathcal{V}\bigr\}.
\end{aligned}$$

A few of ideals in the class of Banach spaces or Hilbert spaces are evident by inconsistent scalar sequence spaces. For example, the ideal of compact mappings is constructed by the space c_0 and $d_a(X)$, for $X\in\mathbb{B}(\mathcal{P},\mathcal{Q})$. Pietsch [3] approved the quasi-ideals $\mathbb{B}^\alpha_{\ell_b}$ for $0<b<\infty$. He investigated that the ideals of nuclear mappings and of Hilbert–Schmidt mappings between Hilbert spaces are explored by ℓ_1 and ℓ_2, respectively. He examined that $\mathbb{F}(\ell_b)$ are dense in $\mathbb{B}(\ell_b)$, and the algebra $\mathbb{B}(\ell_b)$, where $(1\leq b<\infty)$, constructed a simple Banach space. Pietsch [4] proved that $\mathbb{B}^\alpha_{\ell_b}$ for $0<b<\infty$ is small. Makarov and Faried [5] examined that, for each infinite dimensional Banach space $\mathcal{P}$, $\mathcal{Q}$, and $r>b>0$, then

$\mathbb{B}^{\alpha}_{\ell_b}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}^{\alpha}_{\ell_r}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}(\mathcal{P},\mathcal{Q})$. Yaying et al. [6], introduced the sequence space χ_r^t, the domain of r-Cesàro matrix in ℓ_t, with $r \in (0,1]$ and $1 \leq t \leq \infty$. They investigated the quasi Banach ideal of type χ_r^t for $r \in (0,1]$ and $1 < t < \infty$. They found its Schauder basis, $\alpha-$, $\beta-$, and $\gamma-$ duals, and determined certain matrix classes related to this sequence space. On sequence spaces, Başarir and Kara probed the compact mappings on some Euler $B(m)$-difference sequence spaces [7], some difference sequence spaces of weighted means [8], the Riesz $B(m)$-difference sequence space [9], the B-difference sequence space derived by weighted mean [10], and the mth order difference sequence space of generalized weighted mean [11]. Mursaleen and Noman [12, 13] recognized the compact mappings on some difference sequence spaces. The multiplication mappings on Cesàro sequence spaces with the Luxemburg norm were introduced by Komal et al. [14]. İlkhan et al. [15] analyzed the multiplication mappings on Cesàro second order function spaces. Recently, many authors in the literature have investigated some nonabsolute type sequence spaces and introduced recent high quality papers. For example, Mursaleen and Noman [16] defined the sequence spaces ℓ_p^{λ} and ℓ_∞^{λ} of nonabsolute type and showed that the spaces ℓ_p^{λ} and ℓ_∞^{λ} are linearly isomorphic for $0 < p \leq \infty$, ℓ_p^{λ} is a p-normed space, a BK-space in the cases for $0 < p < 1$ and $1 \leq p \leq \infty$, and formed the basis for the space ℓ_p^{λ} for $1 \leq p < \infty$. In [17], they studied the $\alpha-$, $\beta-$, and $\gamma-$ duals of ℓ_p^{λ} and ℓ_∞^{λ} of nonabsolute type for $1 \leq p < \infty$. They characterized some related matrix classes and derived the characterizations of some other classes by means of a given basic lemma. On Cesàro summable sequences, Mursaleen and Başar [18] defined some spaces of double sequences whose Cesàro transforms are bounded, convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent, and absolutely q-summable, respectively, and examined some topological properties of those sequence spaces. The Banach fixed point theorem [19] opened the door for many mathematicians to investigate many extensions of contraction mappings defined in space or generalize space itself. Kannan [20] examined an instance of a class of operators with the identical fixed point actions as contractions, though it fails to be continuous. Ghoncheh [21] was the only one who described Kannan operators in modular vector spaces. He proved the existence of a fixed point of Kannan mapping in complete modular spaces that have the Fatou property. Bakery and Mohamed [22] introduced the concept of the pre-quasi norm on a Nakano sequence space with its variable exponent in $(0,1]$. They investigated the sufficient conditions on it equipped with the definite pre-quasi norm to form pre-quasi Banach and closed space and examined the Fatou property of different pre-quasi norms on it. Moreover, they proved the existence of a fixed point of Kannan pre-quasi norm contraction mappings on it and on the pre-quasi Banach operator ideal constructed by s-numbers which belong to this sequence space. The given inequality will be used in the sequel [23]: If $t_a \geq 1$ and $x_a, z_a \in \mathcal{C}$, with $a \in \mathrm{N}$, and $\hbar = \sup_a t_a$, then

$$|x_a + z_a|^{t_a} \leq 2^{\hbar-1}\bigl(|x_a|^{t_a} + |z_a|^{t_a}\bigr). \tag{1}$$

The organization of the paper is efficient like so: In Sect. 3, we give the definition and some inclusion relations of the sequence space $(\Xi(\zeta,t))_\upsilon$ under the function υ. In Sect. 4, we explain the sufficient conditions for $\Xi(\zeta,t)$ with definite function υ to become pre-modular private sequence space ($\mathfrak{pss}$). This implies that $(\Xi(\zeta,t))_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$. In Sect. 5, we define a multiplication mapping on $(\Xi(\zeta,t))_\upsilon$ and give the necessary and sufficient conditions on this sequence space such that the multiplication mapping is bounded, approximable, invertible, Fredholm, and closed range. In Sect. 6, firstly, we introduce the sufficient settings (not necessary) on $(\Xi(\zeta,t))_\upsilon$, so that $\mathbb{F}$ is dense in $\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}$. This explains a negative answer of the Rhoades [24] open problem about the linearity of s-type $(\Xi(\zeta,t))_\upsilon$ spaces. Secondly, we introduce the conditions on $(\Xi(\zeta,t))_\upsilon$ so that the components of pre-quasi ideal $\mathbb{B}^s_{\Xi(\zeta,t)}$ are complete and closed. Thirdly, we investigate the sufficient conditions on $(\Xi(\zeta,t))_\upsilon$ for $\mathbb{B}^\alpha_{(\Xi(\zeta,t))_\upsilon}$ to be precisely confined for altered weights and powers. We explain the set-ups for which the pre-quasi ideal $\mathbb{B}^\alpha_{(\Xi(\zeta,t))_\upsilon}$ is minimum. Fourthly, we describe the settings for which the Banach pre-quasi ideal $\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}$ is simple. Fifthly, we expound the sufficient settings on $(\Xi(\zeta,t))_\upsilon$ such that the class of all bounded linear mappings whose sequence of eigenvalues in $(\Xi(\zeta,t))_\upsilon$ equals $\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}$. In Sect. 7, the existence of a fixed point of Kannan pre-quasi norm contraction mapping on this sequence space and on its pre-quasi operator ideal formed by $(\Xi(\zeta,t))_\upsilon$ and s-numbers is given. Finally, in Sect. 8, we explain our results by some illustrative examples and applications to the existence of solutions of nonlinear difference equations.

2 Definitions and preliminaries

Lemma 2.1 ([3]) *If $U \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ and $U \notin \mathcal{A}(\mathcal{P},\mathcal{Q})$, then there are mappings $X \in \mathbb{B}(\mathcal{P})$ and $Y \in \mathbb{B}(\mathcal{Q})$ so that $YUXe_l = e_l$ for every $l \in \mathrm{N}$.*

Definition 2.2 ([3]) A Banach space $\mathcal{V}$ is said to be simple if the algebra $\mathbb{B}(\mathcal{V})$ includes a unique nontrivial closed ideal.

Theorem 2.3 ([3]) *Let $\mathcal{V}$ be an infinite dimensional Banach space, then*

$$\mathbb{F}(\mathcal{V}) \subsetneqq \mathcal{A}(\mathcal{V}) \subsetneqq \mathcal{K}(\mathcal{V}) \subsetneqq \mathbb{B}(\mathcal{V}).$$

Definition 2.4 ([25]) A mapping $U \in \mathbb{B}(\mathcal{V})$ is said to be Fredholm if $\dim(\mathrm{Range}(U))^c < \infty$, $\dim(\ker(U)) < \infty$, and $\mathrm{Range}(U)$ is closed, where $(\mathrm{Range}(U))^c$ is the complement of $\mathrm{Range}(U)$.

Definition 2.5 ([26]) A subclass $\mathbb{W}$ of $\mathbb{B}$ is called an operator ideal if every component $\mathbb{W}(\mathcal{P},\mathcal{Q}) = \mathbb{W} \cap \mathbb{B}(\mathcal{P},\mathcal{Q})$ verifies the next set-ups:

(i) $I_\Omega \in \mathbb{W}$ if Ω illustrates a Banach space of one dimension.

(ii) $\mathbb{W}(\mathcal{P},\mathcal{Q})$ is a linear space on $\mathcal{C}$.

(iii) Suppose $X \in \mathbb{B}(\mathcal{P}_0,\mathcal{P})$, $Y \in \mathbb{W}(\mathcal{P},\mathcal{Q})$, and $Z \in \mathbb{B}(\mathcal{Q},\mathcal{Q}_0)$, then $ZYX \in \mathbb{W}(\mathcal{P}_0,\mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are normed spaces.

Faried and Bakery [2] introduced the notion of pre-quasi ideal, which is more general than the quasi ideal.

Definition 2.6 A function $\Psi : \mathbb{W} \to [0,\infty)$ is said to be a pre-quasi norm on the operator ideal $\mathbb{W}$ if the following conditions hold:

(1) For each $X \in \mathbb{W}(\mathcal{P},\mathcal{Q})$, $\Psi(X) \geq 0$ and $\Psi(X) = 0 \Longleftrightarrow X = 0$;

(2) We have $E_0 \geq 1$ such that $\Psi(\kappa X) \leq E_0|\kappa|\Psi(X)$ for all $X \in \mathbb{W}(\mathcal{P},\mathcal{Q})$ and $\kappa \in \mathcal{C}$;

(3) We have $G_0 \geq 1$ for $\Psi(Z_1 + Z_2) \leq G_0[\Psi(Z_1) + \Psi(Z_2)]$ for all $Z_1, Z_2 \in \mathbb{W}(\mathcal{P},\mathcal{Q})$;

(4) We have $D_0 \geq 1$, if $X \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $Y \in \mathbb{W}(\mathcal{P}, \mathcal{Q})$, and $Z \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, then $\Psi(ZYX) \leq D_0\|Z\|\Psi(Y)\|X\|$.

Theorem 2.7 ([2]) *Ψ is a pre-quasi norm on the ideal $\mathbb{W}$, whenever Ψ is a quasi norm on the operator ideal $\mathbb{W}$.*

Definition 2.8 ([27]) The linear space of sequences $\mathcal{V}$ is called a private sequence space ($\mathfrak{pss}$) if it satisfies the following:

(1) $e_b \in \mathcal{V}$ with $b \in \mathrm{N}$;
(2) $\mathcal{V}$ is solid, i.e., for $f = (f_b) \in \mathcal{C}^{\mathrm{N}}$, $|g| = (|g_b|) \in \mathcal{V}$ and $|f_b| \leq |g_b|$ over $b \in \mathrm{N}$, then $|f| \in \mathcal{V}$;
(3) $(|f_{[\frac{b}{2}]}|)_{b=0}^{\infty} \in \mathcal{V}$, while $[\frac{b}{2}]$ illustrates the integral part of $\frac{b}{2}$ if $(|f_b|)_{b=0}^{\infty} \in \mathcal{V}$.

Theorem 2.9 ([27]) *If the linear sequence space $\mathcal{V}$ is a $\mathfrak{pss}$, then $\mathbb{B}^s_{\mathcal{V}}$ is an operator ideal.*

Definition 2.10 ([27]) A subclass of the $\mathfrak{pss}$ is said to be a pre-modular $\mathfrak{pss}$ if there is a mapping $\upsilon : \mathcal{V} \to [0, \infty)$ with the settings:

(i) When $f \in \mathcal{V}, f = \theta \Longleftrightarrow \upsilon(|f|) = 0$, with $\upsilon(f) \geq 0$, where θ is the zero element of $\mathcal{V}$;
(ii) If $f \in \mathcal{V}$ and $\rho \in \mathcal{C}$, we have $E_0 \geq 1$ with $\upsilon(\rho f) \leq |\rho| E_0 \upsilon(f)$;
(iii) $\upsilon(f + g) \leq G_0(\upsilon(f) + \upsilon(g))$ holds for some $G_0 \geq 1$ with $f, g \in \mathcal{V}$;
(iv) For $b \in \mathrm{N}$, $|f_b| \leq |g_b|$, we get $\upsilon((|f_b|)) \leq \upsilon((|g_b|))$;
(v) The inequality $\upsilon((|f_b|)) \leq \upsilon((|f_{[\frac{b}{2}]}|)) \leq D_0 \upsilon((|f_b|))$ holds for $D_0 \geq 1$;
(vi) If $\mathcal{F}$ denotes the space of all sequences with finite nonzero coordinates, then $\overline{\mathcal{F}} = \mathcal{V}_\upsilon$;
(vii) We have $\varpi > 0$ so that $\upsilon(\rho, 0, 0, 0, \ldots) \geq \varpi |\rho| \upsilon(1, 0, 0, 0, \ldots)$ with $\rho \in \mathcal{C}$.

Definition 2.11 ([27]) The $\mathfrak{pss}$ $\mathcal{V}_\upsilon$ is called a pre-quasi normed $\mathfrak{pss}$ if υ supports points (i)–(iii) of Definition 2.10. If $\mathcal{V}$ is complete equipped with υ, then $\mathcal{V}_\upsilon$ is called a pre-quasi Banach $\mathfrak{pss}$.

Theorem 2.12 ([27]) *A pre-quasi normed $\mathfrak{pss}$ $\mathcal{V}_\upsilon$, whenever it is pre-modular $\mathfrak{pss}$.*

Theorem 2.13 ([27]) *The function Ψ is a pre-quasi norm on $\mathbb{B}^s_{(\mathcal{V})_\upsilon}$, where $\Psi(Z) = \upsilon(s_b(Z))_{b=0}^{\infty}$ for all $Z \in \mathbb{B}^s_{(\mathcal{V})_\upsilon}(\mathcal{P}, \mathcal{Q})$ if $(\mathcal{V})_\upsilon$ is a pre-modular $\mathfrak{pss}$.*

Definition 2.14 ([22]) A pre-quasi norm υ on $\mathcal{V}$ verifies the Fatou property if, for every sequence $\{t^a\} \subseteq \mathcal{V}_\upsilon$ with $\lim_{a\to\infty} \upsilon(t^a - t) = 0$ and all $z \in \mathcal{V}_\upsilon$, $\upsilon(z - t) \leq \sup_j \inf_{a \geq j} \upsilon(z - t^a)$.

Definition 2.15 ([22]) A pre-quasi norm Ψ on the ideal $\mathbb{B}^s_{\mathcal{V}}$, where $\Psi(W) = \upsilon((s_a(W))_{a=0}^{\infty})$, verifies the Fatou property if, for all sequence $\{W_a\}_{a\in\mathrm{N}} \subseteq \mathbb{B}^s_{\mathcal{V}}(Z, M)$ with $\lim_{a\to\infty} \Psi(W_a - W) = 0$ and every $V \in \mathbb{B}^s_{\mathcal{V}}(Z, M)$,

$$\Psi(V - W) \leq \sup_a \inf_{i \geq a} \Psi(V - W_i).$$

Definition 2.16 ([22]) An operator $W : \mathcal{V}_\upsilon \to \mathcal{V}_\upsilon$ is said to be a Kannan υ-contraction if there is $\lambda \in [0, \frac{1}{2})$ such that $\upsilon(Wz - Wt) \leq \lambda(\upsilon(Wz - z) + \upsilon(Wt - t))$ for every $z, t \in \mathcal{V}_\upsilon$.

An element $t \in \mathcal{V}_\upsilon$ is called a fixed point of W if $W(t) = t$.

Definition 2.17 ([22]) An operator $W : \mathbb{B}^s_{\mathcal{V}}(Z,M) \to \mathbb{B}^s_{\mathcal{V}}(Z,M)$ is called a Kannan Ψ-contraction if there is $\lambda \in [0, \frac{1}{2})$ such that $\Psi(WV - WT) \leq \lambda(\Psi(WV - V) + \Psi(WT - T))$ for every $V, T \in \mathbb{B}^s_{\mathcal{V}}(Z,M)$.

Definition 2.18 ([22]) Let $\mathcal{V}_\upsilon$ be a pre-quasi normed (sss), $W : \mathcal{V}_\upsilon \to \mathcal{V}_\upsilon$, and $b \in \mathcal{V}_\upsilon$. The operator W is called υ-sequentially continuous at b if and only if, when $\lim_{a\to\infty} \upsilon(t_a - b) = 0$, then $\lim_{a\to\infty} \upsilon(Wt_a - Wb) = 0$.

Definition 2.19 ([22]) For the pre-quasi norm Ψ on the ideal $\mathbb{B}^s_{\mathcal{V}}$, where $\Psi(W) = \upsilon((s_a(W))_{a=0}^\infty)$, $G : \mathbb{B}^s_{\mathcal{V}}(Z,M) \to \mathbb{B}^s_{\mathcal{V}}(Z,M)$, and $B \in \mathbb{B}^s_{\mathcal{V}}(Z,M)$. The operator G is called Ψ-sequentially continuous at B if and only if, when $\lim_{p\to\infty} \Psi(W_p - B) = 0$, then $\lim_{p\to\infty} \Psi(GW_p - GB) = 0$.

Definition 2.20 ([27]) If $\omega = (\omega_k) \in \mathcal{C}^{\mathrm{N}}$ and $\mathcal{V}_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$. The mapping $H_\omega : \mathcal{V}_\upsilon \to \mathcal{V}_\upsilon$ is called a multiplication mapping on $\mathcal{V}_\upsilon$, when $H_\omega f = (\omega_b f_b) \in \mathcal{V}_\upsilon$ with $f \in \mathcal{V}_\upsilon$. The multiplication mapping is called created by ω if $H_\omega \in \mathbb{B}(\mathcal{V}_\upsilon)$.

Theorem 2.21 ([28]) *For s-type $\mathcal{V}_\upsilon := \{f = (s_r(X)) \in \mathrm{R}^{\mathrm{N}} : X \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) and \upsilon(f) < \infty\}$. If $\mathbb{B}^s_{\mathcal{V}_\upsilon}$ is a mapping ideal, then the following conditions are verified*:

1. *$\mathcal{F} \subset$ s-type $\mathcal{V}_\upsilon$.*
2. *Assume $(s_r(X_1))_{r=0}^\infty \in$ s-type $\mathcal{V}_\upsilon$ and $(s_r(X_2))_{r=0}^\infty \in$ s-type $\mathcal{V}_\upsilon$, then $(s_r(X_1 + X_2))_{r=0}^\infty \in$ s-type $\mathcal{V}_\upsilon$.*
3. *If $\lambda \in \mathcal{C}$ and $(s_r(X))_{r=0}^\infty \in$ s-type $\mathcal{V}_\upsilon$, then $|\lambda|(s_r(X))_{r=0}^\infty \in$ s-type $\mathcal{V}_\upsilon$.*
4. *The sequence space $\mathcal{V}_\upsilon$ is solid, i.e., if $(s_r(Y))_{r=0}^\infty \in$ s-type $\mathcal{V}_\upsilon$ and $s_r(X) \leq s_r(Y)$ for all $r \in \mathrm{N}$ and $X, Y \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, then $(s_r(X))_{r=0}^\infty \in$ s-type $\mathcal{V}_\upsilon$.*

3 The sequence space $(\Xi(\zeta, t))_\upsilon$

We introduce in this section the definition and some inclusion relations of the sequence space $(\Xi(\zeta, t))_\upsilon$ under the function υ.

Definition 3.1 For all $(t_l) \in \mathrm{R}^{+\mathrm{N}}$, where $\mathrm{R}^{+\mathrm{N}}$ is the space of all sequences of positive reals and $(\zeta_l) \in \mathrm{R}^{+\mathrm{N}}$ is strictly increasing tending to infinity, the sequence space $(\Xi(\zeta, t))_\upsilon$ under the function υ is defined as follows:

$$\bigl(\Xi(\zeta,t)\bigr)_\upsilon = \bigl\{f = (f_k) \in \mathcal{C}^{\mathrm{N}} : \upsilon(\rho f) < \infty \text{ for some } \rho > 0\bigr\}, \quad \text{where}$$
$$\upsilon(f) = \sum_{l=0}^{\infty} \biggl(\frac{|\sum_{z=0}^{l} f_z \Delta\zeta_z|}{\zeta_l}\biggr)^{t_l} \text{ and } \Delta\zeta_z = \zeta_z - \zeta_{z-1}.$$

Suppose that $\zeta_z = 0$ for $z < 0$.

Theorem 3.2 *If $(t_l) \in \mathrm{R}^{+\mathrm{N}} \cap \ell_\infty$, then*

$$\bigl(\Xi(\zeta,t)\bigr)_\upsilon = \bigl\{f = (f_k) \in \mathcal{C}^{\mathrm{N}} : \upsilon(\rho f) < \infty \textit{ for any } \rho > 0\bigr\}.$$

Proof Assume $(t_l) \in \mathrm{R}^{+\mathrm{N}} \cap \ell_\infty$, one has

$$\begin{aligned}
\big(\Xi(\zeta,t)\big)_\upsilon &= \big\{f=(f_k)\in \mathcal{C}^{\mathrm{N}} : \upsilon(\rho f)<\infty \text{for some} \rho>0\big\} \\
&= \left\{f=(f_k)\in \mathcal{C}^{\mathrm{N}} : \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}\rho f_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}<\infty \text{for some} \rho>0\right\} \\
&= \left\{f=(f_k)\in \mathcal{C}^{\mathrm{N}} : \inf_l \rho^{t_l}\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}f_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}<\infty \text{for some} \rho>0\right\} \\
&= \left\{f=(f_k)\in \mathcal{C}^{\mathrm{N}} : \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}f_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}<\infty\right\} \\
&= \big\{f=(f_k)\in \mathcal{C}^{\mathrm{N}} : \upsilon(\rho f)<\infty \text{ for any } \rho>0\big\}.
\end{aligned}$$

□

Remark 3.3

(1) For $t_z = t$, for all $z \in \mathrm{N}$ and $t \geq 1$, the sequence space $\Xi(\zeta,t) = \ell_t^{\zeta}$ was defined and investigated by Mursaleen and Noman [16].

(2) Assume $t_z = t$, $\Delta\zeta_z = r^z$ for all $z \in \mathrm{N}$, $0 < r \leq 1$, and $t \geq 1$, the sequence space $\Xi(\zeta,t) = \chi_r^t$ was investigated by Yaying et al. [6].

(3) If $t_z = t$, $\Delta\zeta_z = 1$ for all $z \in \mathrm{N}$ and $t \geq 1$, hence $\Xi(\zeta,t) = \mathrm{ces}^t$ was made current and considered by Ng and Lee [29].

Theorem 3.4 *If* $(\Delta\zeta_l),(t_l) \in \mathrm{R}^{+\mathrm{N}}$ *with* $1 \leq t_l < \infty$, *then* $(\Xi(\zeta,t))_\upsilon$ *is of nonabsolute type.*

Proof By taking $f = (1,-1,0,0,0,\dots)$, then $|f| = (1,1,0,0,0,\dots)$. We have

$$\upsilon(f) = 1+\left(\frac{|2\zeta_0-\zeta_1|}{\zeta_1}\right)^{t_1}+\left(\frac{|2\zeta_0-\zeta_1|}{\zeta_2}\right)^{t_2}+\cdots \neq 2+\left(\frac{\zeta_1}{\zeta_2}\right)^{t_2}+\cdots = \upsilon\big(|f|\big).$$

Therefore, the sequence space $(\Xi(\zeta,t))_\upsilon$ is of nonabsolute type. □

Definition 3.5 For all $(\Delta\zeta_l),(t_l) \in \mathrm{R}^{+\mathrm{N}}$. The (ζ_l)-generalized Cesàro sequence space of absolute type $(\mathrm{ces}(\zeta,t))_\varphi$ is defined as follows:

$$\big(\mathrm{ces}(\zeta,t)\big)_\varphi = \big\{f=(f_k)\in \mathcal{C}^{\mathrm{N}} : \varphi(\rho f)<\infty \text{ for some } \rho>0\big\}, \quad \text{where}$$

$$\varphi(f) = \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}|f_z|\Delta\zeta_z}{\zeta_l}\right)^{t_l}.$$

Theorem 3.6 *If* $(\Delta\zeta_l),(t_l) \in \mathrm{R}^{+\mathrm{N}} \cap \ell_\infty$ *with* $\inf_l \Delta\zeta_l > 0$, *then* $(\mathrm{ces}(\zeta,t))_\varphi \subsetneqq (\Xi(\zeta,t))_\upsilon$.

Proof Let $f \in (\mathrm{ces}(\zeta,t))_\varphi$, since

$$\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}f_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l} \leq \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}|f_z|\Delta\zeta_z}{\zeta_l}\right)^{t_l}<\infty.$$

Then $f \in (\Xi(\zeta,t))_\upsilon$. For $(t_l) \in (1,\infty)^{\mathrm{N}} \cap \ell_\infty$, we choose $g = (\frac{(-1)^z}{\Delta\zeta_z})_{z\in\mathrm{N}}$, one has $g \in (\Xi(\zeta,t))_\upsilon$ and $g \notin (\mathrm{ces}(\zeta,t))_\varphi$. For $(t_l) \in (0,1]^{\mathrm{N}}$, we choose $h = (\frac{1}{\zeta_0}, \frac{1}{\zeta_0-\zeta_1}, 0,0,0,\dots)$, one has $h \in (\Xi(\zeta,t))_\upsilon$ and $h \notin (\mathrm{ces}(\zeta,t))_\varphi = \{(0,0,\dots)\}$. □

4 Pre-modular private sequence space

In this section, we offer enough set-ups for $\Xi(\zeta,t)$ with the definite function υ to become pre-modular $\mathfrak{pss}$. This implies that $\Xi(\zeta,t)$ is a pre-quasi normed $\mathfrak{pss}$.

Here and after, we denote the space of all monotonic decreasing and monotonic increasing sequences of positive reals by $\Im_{\searrow}$ and $\Im_{\nearrow}$, respectively.

Theorem 4.1 *$\Xi(\zeta,t)$ is a $\mathfrak{pss}$ if the following conditions hold*:

(f1) $(t_l)\in\Im_{\nearrow}\cap\ell_\infty$ *with* $t_0>1$.

(f2) $(\Delta\zeta_z)_{z=0}^\infty\in\Im_{\searrow}$ *with* $\inf_z\Delta\zeta_z>0$ *or* $(\Delta\zeta_z)_{z=0}^\infty\in\Im_{\nearrow}\cap\ell_\infty$, *and there exists* $C\geq 1$ *such that* $\Delta\zeta_{2z+1}\leq C\Delta\zeta_z$.

Proof (1-i) Assume $f,g\in\Xi(\zeta,t)$. One has

$$\begin{aligned}&\sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l(f_z+g_z)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\\&\quad\leq 2^{\hbar-1}\left(\sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l f_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}+\sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l g_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right)<\infty,\end{aligned}$$

so $f+g\in\Xi(\zeta,t)$.

(1-ii) Suppose $\rho\in\mathcal{C}$, $f\in\Xi(\zeta,t)$ and as $(t_l)\in\Im_{\nearrow}\cap\ell_\infty$, we obtain

$$\sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l\rho f_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\leq\sup_l|\rho|^{t_l}\sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l f_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}<\infty.$$

Hence $\rho f\in\Xi(\zeta,t)$. Relative to (1-i) and (1-ii), we have $\Xi(\zeta,t)$ is a linear space.

Also as $(t_l)\in\Im_{\nearrow}\cap\ell_\infty$ with $t_0>1$, one has

$$\sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l(e_b)_z\Delta\zeta_z|}{\zeta_l}\right)^{t_l}=\sum_{l=b}^\infty\left(\frac{\Delta\zeta_b}{\zeta_l}\right)^{t_l}\leq\sup_l(\Delta\zeta_b)^{t_l}\sum_{l=b}^\infty\left(\frac{1}{\zeta_l}\right)^{t_l}<\infty.$$

Therefore, $e_b\in\Xi(\zeta,t)$ with $b\in\mathrm{N}$.

(2) If $|f_b|\leq|g_b|$ for each $b\in\mathrm{N}$ and $|g|\in\Xi(\zeta,t)$, one can see

$$\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l|f_z|\Delta\zeta_z}{\zeta_l}\right)^{t_l}\leq\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l|g_z|\Delta\zeta_z}{\zeta_l}\right)^{t_l}<\infty,$$

hence $|f|\in\Xi(\zeta,t)$.

(3) Assume $(|f_z|)\in\Xi(\zeta,t)$, where $(t_l),(\Delta\zeta_z)\in\Im_{\nearrow}\cap\ell_\infty$ and there is $C\geq1$ such that $\Delta\zeta_{2z+1}\leq C\Delta\zeta_z$, we get

$$\begin{aligned}&\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l|f_{[\frac{z}{2}]}|\Delta\zeta_z}{\zeta_l}\right)^{t_l}\\&\quad=\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^{2l}|f_{[\frac{z}{2}]}|\Delta\zeta_z}{\zeta_{2l}}\right)^{t_{2l}}+\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^{2l+1}|f_{[\frac{z}{2}]}|\Delta\zeta_z}{\zeta_{2l+1}}\right)^{t_{2l+1}}\\&\quad\leq\sum_{l=0}^\infty\left(\frac{|f_l|\Delta\zeta_{2l}+\sum_{z=0}^l|f_z|(\Delta\zeta_{2z}+\Delta\zeta_{2z+1})}{\zeta_l}\right)^{t_l}+\sum_{l=0}^\infty\left(\frac{\sum_{z=0}^l|f_z|(\Delta\zeta_{2z}+\Delta\zeta_{2z+1})}{\zeta_l}\right)^{t_l}\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}|f_z|\Delta\zeta_{2z}}{\zeta_l}\right)^{t_l}+\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}2C|f_z|\Delta\zeta_z}{\zeta_l}\right)^{t_l}\right)\\
&\quad+\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}2C|f_z|\Delta\zeta_z}{\zeta_l}\right)^{t_l}\\
&\leq \left(2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar}\right)C^{\hbar}\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l}|f_z|\Delta\zeta_z}{\zeta_l}\right)^{t_l}<\infty,
\end{aligned}
$$

so $(|f_{[\frac{z}{2}]}|) \in \Xi(\zeta,t)$. □

By using Theorem 2.9, we can get the next theorem.

Theorem 4.2 *If conditions* (f1) *and* (f2) *are satisfied, then* $\mathbb{B}^{s}_{\Xi(\zeta,t)}$ *is an operator ideal.*

Theorem 4.3 $(\Xi(\zeta,t))_{\upsilon}$ *is a pre-modular* $\mathfrak{pss}$ *if setups* (f1) *and* (f2) *are satisfied.*

Proof

(i) Easily, $\upsilon(f) \geq 0$ and $\upsilon(|f|) = 0 \Leftrightarrow f = \theta$.

(ii) We have $E_0 = \max\{1, \sup_l |\rho|^{t_l-1}\} \geq 1$ with $\upsilon(\rho f) \leq E_0|\rho|\upsilon(f)$ for every $f \in \Xi(\zeta,t)$ and $\rho \in \mathcal{C}$.

(iii) One has $\upsilon(f+g) \leq 2^{\hbar-1}(\upsilon(f)+\upsilon(g))$ for each $f,g \in \Xi(\zeta,t)$.

(iv) Definitely, from the proof part (2) of Theorem 4.1.

(v) Indeed, the proof part (3) of Theorem 4.1 gives that $D_0 \geq (2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar})C^{\hbar} \geq 1$.

(vi) Obviously, $\overline{\mathcal{F}} = \Xi(\zeta,t)$.

(vii) We have $0 < \varpi \leq \sup_l |\rho|^{t_l-1}$ with $\upsilon(\rho,0,0,0,\dots) \geq \varpi|\rho|\upsilon(1,0,0,0,\dots)$ for each $\rho \neq 0$ and $\varpi > 0$, if $\rho = 0$. □

Theorem 4.4 *If settings* (f1) *and* (f2) *are satisfied, then* $(\Xi(\zeta,t))_{\upsilon}$ *is a pre-quasi Banach* $\mathfrak{pss}$.

Proof Let the set-ups be satisfied, then from Theorem 4.3 the space $(\Xi(\zeta,t))_{\upsilon}$ is a pre-modular $\mathfrak{pss}$. By using Theorem 2.12, the space $(\Xi(\zeta,t))_{\upsilon}$ is a pre-quasi normed $\mathfrak{pss}$. To show that $(\Xi(\zeta,t))_{\upsilon}$ is a pre-quasi Banach $\mathfrak{pss}$, assume that $f^a = (f^a_z)_{z=0}^{\infty}$ is a Cauchy sequence in $(\Xi(\zeta,t))_{\upsilon}$, then for all $\varepsilon \in (0,1)$, there is $a_0 \in \mathrm{N}$ so that, for all $a,b \geq a_0$, one has

$$
\upsilon\left(f^a-f^b\right)=\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(f^a_z-f^b_z)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}<\varepsilon^{\hbar}.
$$

Hence, for $a,b \geq a_0$ and $z \in \mathrm{N}$, we have $|f^a_z - f^b_z| < \varepsilon$. So (f^b_z) is a Cauchy sequence in $\mathcal{C}$ for fixed $z \in \mathrm{N}$, this gives $\lim_{b\to\infty} f^b_z = f^0_z$ for fixed $z \in \mathrm{N}$. Hence $\upsilon(f^a - f^0) < \varepsilon^{\hbar}$ for all $a \geq a_0$. Finally, to show that $f^0 \in (\Xi(\zeta,t))_{\upsilon}$, one has $\upsilon(f^0) \leq 2^{\hbar-1}(\upsilon(f^a-f^0)+\upsilon(f^a)) < \infty$, so $f^0 \in (\Xi(\zeta,t))_{\upsilon}$. This means that $(\Xi(\zeta,t))_{\upsilon}$ is a pre-quasi Banach $\mathfrak{pss}$. □

By using Theorem 2.21, we conclude the following properties of the s-type $(\Xi(\zeta,t))_{\upsilon}$.

Theorem 4.5 *For s-type* $(\Xi(\zeta,t))_{\upsilon} := \{f = (s_n(X)) \in \mathrm{R}^{\mathrm{N}} : X \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ *and* $\upsilon(f) < \infty\}$. *The following settings are verified*:

1. *We have s-type* $(\Xi(\zeta,t))_\upsilon \supset \mathcal{F}$.
2. *If* $(s_r(X_1))_{r=0}^\infty \in$ *s-type* $(\Xi(\zeta,t))_\upsilon$ *and* $(s_r(X_2))_{r=0}^\infty \in$ *s-type* $(\Xi(\zeta,t))_\upsilon$, *then* $(s_r(X_1+X_2))_{r=0}^\infty \in$ *s-type* $(\Xi(\zeta,t))_\upsilon$.
3. *For all* $\lambda \in \mathcal{C}$ *and* $(s_r(X))_{r=0}^\infty \in$ *s-type* $(\Xi(\zeta,t))_\upsilon$, *then* $|\lambda|(s_r(X))_{r=0}^\infty \in$ *s-type* $(\Xi(\zeta,t))_\upsilon$.
4. *The s-type* $(\Xi(\zeta,t))_\upsilon$ *is solid.*

5 Multiplication mappings on $(\Xi(\zeta,t))_\upsilon$

In this section, we define a multiplication mapping on the pre-quasi normed $\mathfrak{pss}$ $(\Xi(\zeta,t))_\upsilon$ and investigate the necessary and sufficient conditions on $(\Xi(\zeta,t))_\upsilon$ for the multiplication mapping to be bounded, invertible, approximable, Fredholm, and closed range.

Theorem 5.1 *Suppose* $\omega \in \mathcal{C}^{\mathrm{N}}$, *conditions* (f1) *and* (f2) *are satisfied, then* $\omega \in \ell_\infty$ *if and only if* $H_\omega \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$.

Proof Let $\omega \in \ell_\infty$. Hence there is $\nu > 0$ such that $|\omega_b| \le \nu$ with $b \in \mathrm{N}$. For $f \in (\Xi(\zeta,t))_\upsilon$, one has

$$\begin{aligned}\upsilon(H_\omega f) = \upsilon(\omega f) &= \sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l \omega_z f_z \Delta\zeta_z|}{\zeta_l}\right)^{t_l} \\ &\le \sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l \nu f_z \Delta\zeta_z|}{\zeta_l}\right)^{t_l} \le \sup_l \nu^{t_l}\sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l f_z \Delta\zeta_z|}{\zeta_l}\right)^{t_l} \\ &= \sup_l \nu^{t_l}\upsilon(f).\end{aligned}$$

Therefore, $H_\omega \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$.

On the contrary, let $H_\omega \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$ and $\omega \notin \ell_\infty$. Hence, for all $b \in \mathrm{N}$, there is $x_b \in \mathrm{N}$ such that $\omega_{x_b} > b$. We have

$$\begin{aligned}\upsilon(H_\omega e_{x_b}) = \upsilon(\omega e_{x_b}) &= \sum_{l=0}^\infty\left(\frac{|\sum_{z=0}^l \omega_z (e_{x_b})_z \Delta\zeta_z|}{\zeta_l}\right)^{t_l} \\ &= \sum_{l=x_b}^\infty\left(\frac{|\omega_{x_b}|\Delta\zeta_{x_b}}{\zeta_l}\right)^{t_l} > \sum_{l=x_b}^\infty\left(\frac{b\Delta\zeta_{x_b}}{\zeta_l}\right)^{t_l} > b^{t_{x_b}}\upsilon(e_{x_b}).\end{aligned}$$

Hence $H_\omega \notin \mathbb{B}((\Xi(\zeta,t))_\upsilon)$. So $\omega \in \ell_\infty$. □

Theorem 5.2 *Assume* $\omega \in \mathcal{C}^{\mathrm{N}}$ *and* $(\Xi(\zeta,t))_\upsilon$ *is a pre-quasi normed* $\mathfrak{pss}$, *then* $\omega_b = g$ *for every* $b \in \mathrm{N}$ *and* $g \in \mathcal{C}$ *with* $|g| = 1$ *if and only if* H_ω *is an isometry.*

Proof Let the sufficient condition be verified. One has

$$\upsilon(H_\omega f) = \upsilon(\omega f) = \sum_{l=0}^\infty\left(\frac{|\sum_{k=0}^l \omega_k f_k \Delta\zeta_k|}{\zeta_l}\right)^{t_l} = \sum_{l=0}^\infty\left(\frac{|\sum_{k=0}^l |g| f_k \Delta\zeta_k|}{\zeta_l}\right)^{t_l} = \upsilon(f)$$

with $f \in (\Xi(\zeta,t))_\upsilon$. So H_ω is an isometry.

Let the necessity condition be satisfied and $|\omega_b| < 1$ for some $b = b_0$. We get

$$\begin{aligned}\upsilon(H_\omega e_{b_0}) = \upsilon(\omega e_{b_0}) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l}\omega_k(e_{b_0})_k\Delta\zeta_k|}{\zeta_l}\right)^{t_l} \\ &= \sum_{l=b_0}^{\infty}\left(\frac{|\omega_{b_0}|\Delta\zeta_{b_0}}{\zeta_l}\right)^{t_l} < \sum_{l=b_0}^{\infty}\left(\frac{\Delta\zeta_{b_0}}{\zeta_l}\right)^{t_l} = \upsilon(e_{b_0}).\end{aligned}$$

Also when $|\omega_{b_0}| > 1$, it is easy to show that $\upsilon(H_\omega e_{b_0}) > \upsilon(e_{b_0})$, which is an inconsistency for the two cases. Therefore, $|\omega_b| = 1$ for all $b \in \mathrm{N}$. □

By $\mathfrak{F}$ we denote the space of all sets with a finite number of elements.

Theorem 5.3 *Suppose* $\omega \in \mathcal{C}^{\mathrm{N}}$, *setups* (f1) *and* (f2) *are satisfied, then* $H_\omega \in \mathcal{A}((\Xi(\zeta,t))_\upsilon)$ *if and only if* $(\omega_b)_{b=0}^{\infty} \in c_0$.

Proof Let $H_\omega \in \mathcal{A}((\Xi(\zeta,t))_\upsilon)$, so $H_\omega \in \mathcal{K}((\Xi(\zeta,t))_\upsilon)$. Suppose $\lim_{b\to\infty}\omega_b \neq 0$. Therefore, we have $\varrho > 0$ such that the set $K_\varrho = \{b \in \mathrm{N} : |\omega_b| \geq \varrho\} \nsubseteq \mathfrak{F}$. If $\{\alpha_b\}_{b\in\mathrm{N}} \subset K_\varrho$, hence $\{e_{\alpha_b} : \alpha_b \in K_\varrho\} \in \ell_\infty$ is an infinite set in $(\Xi(\zeta,t))_\upsilon$. Since

$$\begin{aligned}\upsilon(H_\omega e_{\alpha_a} - H_\omega e_{\alpha_b}) = \upsilon(\omega e_{\alpha_a} - \omega e_{\alpha_b}) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l}\omega_k((e_{\alpha_a})_k - (e_{\alpha_b})_k)\Delta\zeta_k|}{\zeta_l}\right)^{t_l} \\ &\geq \sum_{l=0}^{\infty}\left(\frac{|\sum_{k=0}^{l}\varrho((e_{\alpha_a})_k - (e_{\alpha_b})_k)\Delta\zeta_k|}{\zeta_l}\right)^{t_l} \geq \inf_l \varrho^{t_l}\upsilon(e_{\alpha_a} - e_{\alpha_b})\end{aligned}$$

with $\alpha_a, \alpha_b \in K_\varrho$. Therefore, $\{e_{\alpha_b} : \alpha_b \in K_\varrho\} \in \ell_\infty$, which cannot have a convergent subsequence under H_ω. Hence $H_\omega \notin \mathcal{K}((\Xi(\zeta,t))_\upsilon)$. This implies $H_\omega \notin \mathcal{A}((\Xi(\zeta,t))_\upsilon)$, this gives an inconsistency. So, $\lim_{b\to\infty}\omega_b = 0$. On the other hand, let $\lim_{b\to\infty}\omega_b = 0$. Hence, for all $\varrho > 0$, one has $K_\varrho = \{b \in \mathrm{N} : |\omega_b| \geq \varrho\} \subset \mathfrak{F}$. Hence, for each $\varrho > 0$, we have $\dim(((\Xi(\zeta,t))_\upsilon)_{K_\varrho}) = \dim(\mathcal{C}^{K_\varrho}) < \infty$. So $H_\omega \in \mathbb{F}(((\Xi(\zeta,t))_\upsilon)_{K_\varrho})$. Define $\omega_a \in \mathcal{C}^{\mathrm{N}}$ for all $a \in \mathrm{N}$ by

$$(\omega_a)_b = \begin{cases}\omega_b, & b \in K_{\frac{1}{a+1}}, \\ 0, & \text{otherwise.}\end{cases}$$

It is clear that $H_{\omega_a} \in \mathbb{F}(((\Xi(\zeta,t))_\upsilon)_{B_{\frac{1}{a+1}}})$ as $\dim(((\Xi(\zeta,t))_\upsilon)_{B_{\frac{1}{a+1}}}) < \infty$ for all $a \in \mathrm{N}$. From $(t_l) \in \Im_\nearrow \cap \ell_\infty$ with $t_0 > 1$, one can see

$$\begin{aligned}\upsilon\big((H_\omega - H_{\omega_a})f\big) &= \upsilon\big(\big((\omega_b - (\omega_a)_b)f_b\big)_{b=0}^{\infty}\big) \\ &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{b=0}^{l}(\omega_b - (\omega_a)_b)f_b\Delta\zeta_b|}{\zeta_l}\right)^{t_l} \\ &= \sum_{l=0, l\in K_{\frac{1}{a+1}}}^{\infty}\left(\frac{|\sum_{b=0}^{l}(\omega_b - (\omega_a)_b)f_b\Delta\zeta_b|}{\zeta_l}\right)^{t_l} \\ &\quad + \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^{\infty}\left(\frac{|\sum_{b=0}^{l}(\omega_b - (\omega_a)_b)f_b\Delta\zeta_b|}{\zeta_l}\right)^{t_l}\end{aligned}$$

$$= \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^{\infty} \left(\frac{|\sum_{b=0}^{l} \omega_b f_b \Delta\zeta_b|}{\zeta_l} \right)^{t_l}$$
$$\leq \frac{1}{(a+1)^{t_0}} \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^{\infty} \left(\frac{|\sum_{b=0}^{l} \Delta\zeta_b f_b|}{\zeta_l} \right)^{t_l}$$
$$< \frac{1}{(a+1)^{t_0}} \sum_{l=0}^{\infty} \left(\frac{|\sum_{b=0}^{l} f_b \Delta\zeta_b|}{\zeta_l} \right)^{t_l} = \frac{1}{(a+1)^{t_0}} \upsilon(f).$$

Hence $\|H_\omega - H_{\omega_a}\| \leq \frac{1}{(a+1)^{t_0}}$. This gives H_ω is a limit of finite rank mappings. Therefore, $H_\omega \in \mathcal{A}((\Xi(\zeta,t))_\upsilon)$. □

Theorem 5.4 *Assume $\omega \in \mathcal{C}^{\mathrm{N}}$, conditions* (f1) *and* (f2) *are satisfied, then $H_\omega \in \mathcal{K}((\Xi(\zeta,t))_\upsilon)$ if and only if $(\omega_b)_{b=0}^{\infty} \in c_0$.*

Proof Obviously, since $\mathcal{A}((\Xi(\zeta,t))_\upsilon) \subsetneqq \mathcal{K}((\Xi(\zeta,t))_\upsilon)$. □

Corollary 5.5 *If setups* (f1) *and* (f2) *are satisfied, then $\mathcal{K}((\Xi(\zeta,t))_\upsilon) \subsetneqq \mathbb{B}((\Xi(\zeta,t))_\upsilon)$.*

Proof As $\omega = (1,1,\dots)$ creates the multiplication mapping I on $(\Xi(\zeta,t))_\upsilon$. Therefore, $I \notin \mathcal{K}((\Xi(\zeta,t))_\upsilon)$ and $I \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$. □

Theorem 5.6 *If $(\Xi(\zeta,t))_\upsilon$ is a pre-quasi Banach $\mathfrak{pss}$ and $H_\omega \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$, then there are $\alpha > 0$ and $\eta > 0$ such that $\alpha < |\omega_b| < \eta$ with $b \in (\ker(\omega))^c$ if and only if $\mathrm{Range}(H_\omega)$ is closed.*

Proof Assume that the sufficient condition is confirmed. Hence there is $\varrho > 0$ such that $|\omega_b| \geq \varrho$ with $b \in (\ker(\omega))^c$. To show that $\mathrm{Range}(H_\omega)$ is closed, if g is a limit point of $\mathrm{Range}(H_\omega)$, we have $H_\omega f_b \in (\Xi(\zeta,t))_\upsilon$ with $b \in \mathrm{N}$ so that $\lim_{b\to\infty} H_\omega f_b = g$. Obviously, the sequence $H_\omega f_b$ is a Cauchy sequence. As $(t_l) \in \Im_{\nearrow} \cap \ell_\infty$ with $t_0 > 1$, one has

$$\begin{aligned}
\upsilon(H_\omega f_a - H_\omega f_b) &= \sum_{l=0}^{\infty} \left(\frac{|\sum_{k=0}^{l} (\omega_k (f_a)_k - \omega_k (f_b)_k)\Delta\zeta_k|}{\zeta_l} \right)^{t_l} \\
&= \sum_{l=0, l\in(\ker(\omega))^c}^{\infty} \left(\frac{|\sum_{k=0}^{l} (\omega_k (f_a)_k - \omega_k (f_b)_k)\Delta\zeta_k|}{\zeta_l} \right)^{t_l} \\
&\quad + \sum_{l=0, l\notin(\ker(\omega))^c}^{\infty} \left(\frac{|\sum_{k=0}^{l} (\omega_k (f_a)_k - \omega_k (f_b)_k)\Delta\zeta_k|}{\zeta_l} \right)^{t_l} \\
&\geq \sum_{l=0, l\in(\ker(\omega))^c}^{\infty} \left(\frac{|\sum_{k=0}^{l} (\omega_k (f_a)_k - \omega_k (f_b)_k)\Delta\zeta_k|}{\zeta_l} \right)^{t_l} \\
&= \sum_{l=0}^{\infty} \left(\frac{|\sum_{k=0}^{l} (\omega_k (u_a)_k - \omega_k (u_b)_k)\Delta\zeta_k|}{\zeta_l} \right)^{t_l} \\
&> \sum_{l=0}^{\infty} \left(\frac{|\sum_{k=0}^{l} \varrho((u_a)_k - (u_b)_k)\Delta\zeta_k|}{\zeta_l} \right)^{t_l} \geq \inf_l \varrho^{t_l} \upsilon(u_a - u_b),
\end{aligned}$$

where

$$(u_a)_k = \begin{cases} (f_a)_k, & k \in (\ker(\omega))^c, \\ 0, & k \notin (\ker(\omega))^c. \end{cases}$$

This implies that $\{u_a\}$ is a Cauchy sequence in $(\Xi(\zeta,t))_\upsilon$. As $(\Xi(\zeta,t))_\upsilon$ is complete, there is $f \in (\Xi(\zeta,t))_\upsilon$ so that $\lim_{b\to\infty} u_b = f$. Since $H_\omega \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$, we have $\lim_{b\to\infty} H_\omega u_b = H_\omega f$. But $\lim_{b\to\infty} H_\omega u_b = \lim_{b\to\infty} H_\omega f_b = g$. So $H_\omega f = g$. Hence $g \in \mathrm{Range}(H_\omega)$. Therefore, $\mathrm{Range}(H_\omega)$ is closed. Next, suppose that the necessary condition is satisfied. So, there is $\varrho > 0$ such that $\upsilon(H_\omega f) \geq \varrho\upsilon(f)$ with $f \in ((\Xi(\zeta,t))_\upsilon)_{(\ker(\omega))^c}$. If $K = \{b \in (\ker(\omega))^c : |\omega_b| < \varrho\} \neq \emptyset$, then for $a_0 \in K$, one has

$$\begin{aligned} \upsilon(H_\omega e_{a_0}) &= \upsilon\big(\big(\omega_b(e_{a_0})_b\big)\big)_{b=0}^{\infty}\big) \\ &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{b=0}^{l}\omega_b(e_{a_0})_b)\Delta\zeta_b|}{\zeta_l}\right)^{t_l} \\ &< \sum_{l=0}^{\infty}\left(\frac{|\sum_{b=0}^{l}(e_{a_0})_b\varrho\Delta\zeta_b|}{\zeta_l}\right)^{t_l} \leq \sup_l \varrho^{t_l}\upsilon(e_{a_0}), \end{aligned}$$

this gives an inconsistency. Therefore, $K = \phi$, we have $|\omega_b| \geq \varrho$ with $b \in (\ker(\omega))^c$. This proves the theorem. □

Theorem 5.7 *Suppose that $\omega \in \mathcal{C}^{\mathrm{N}}$ and $(\Xi(\zeta,t))_\upsilon$ is a pre-quasi Banach $\mathfrak{pss}$, then there are $\alpha > 0$ and $\eta > 0$ so that $\alpha < |\omega_b| < \eta$ with $b \in \mathrm{N}$ if and only if $H_\omega \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$ is invertible.*

Proof Let the set-up be true. Assume $\kappa \in \mathcal{C}^{\mathrm{N}}$ with $\kappa_b = \frac{1}{\omega_b}$. By using Theorem 5.1, the mappings H_ω and H_κ are bounded linear. We have $H_\omega.H_\kappa = H_\kappa.H_\omega = I$. Therefore, $H_\kappa = H_\omega^{-1}$. Next, let H_ω be invertible. So $\mathrm{Range}(H_\omega) = ((\Xi(\zeta,t))_\upsilon)_{\mathrm{N}}$. Hence $\mathrm{Range}(H_\omega)$ is closed. Therefore, by Theorem 5.6, there is $\alpha > 0$ so that $|\omega_b| \geq \alpha$ for each $b \in (\ker(\omega))^c$. We have $\ker(\omega) = \emptyset$ if $\omega_{b_0} = 0$ with $b_0 \in \mathrm{N}$, this gives $e_{b_0} \in \ker(H_\omega)$ which is an inconsistency as $\ker(H_\omega)$ is trivial. Therefore, $|\omega_b| \geq \alpha$ with $b \in \mathrm{N}$. As $H_\omega \in \ell_\infty$, from Theorem 5.1, there is $\eta > 0$ so that $|\omega_b| \leq \eta$ with $b \in \mathrm{N}$. Hence, one has $\alpha \leq |\omega_b| \leq \eta$ with $b \in \mathrm{N}$. □

Theorem 5.8 *Let $(\Xi(\zeta,t))_\upsilon$ be a pre-quasi Banach $\mathfrak{pss}$ and $H_\omega \in \mathbb{B}((\Xi(\zeta,t))_\upsilon)$, then H_ω is a Fredholm mapping if and only if* (i) $\ker(\omega) \subsetneqq \mathrm{N}$ *is finite and* (ii) $|\omega_b| \geq \varrho$ *with $b \in (\ker(\omega))^c$.*

Proof Assume that the sufficient condition is satisfied. Let $\ker(\omega) \subsetneqq \mathrm{N}$ be infinite, hence $e_b \in \ker(H_\omega)$ with $b \in \ker(\omega)$. Since e_bs are linearly independent, this gives that $\dim(\ker(H_\omega)) = \infty$, this implies an inconsistency. Hence, $\ker(\omega) \subsetneqq \mathrm{N}$ must be finite. Condition (ii) comes from Theorem 5.6. Next, let set-ups (i) and (ii) be confirmed. From Theorem 5.6, set-up (ii) implies that $\mathrm{Range}(H_\omega)$ is closed. Setting (i) gives that $\dim(\ker(H_\omega)) < \infty$ and $\dim((\mathrm{Range}(H_\omega))^c) < \infty$. This implies that H_ω is Fredholm. □

6 Pre-quasi ideal

In this section, firstly, we introduce the sufficient settings (not necessary) on $(\Xi(\zeta,t))_\upsilon$ such that $\mathbb{F}$ is dense in $\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}$. This investigates a negative answer of the Rhoades [24] open problem about the linearity of s-type $(\Xi(\zeta,t))_{\upsilon}$ spaces. Secondly, for which conditions on $(\Xi(\zeta,t))_{\upsilon}$ are $\mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}$ complete and closed? Thirdly, we give the sufficient set-ups on $(\Xi(\zeta,t))_{\upsilon}$ such that $\mathbb{B}^{\alpha}_{(\Xi(\zeta,t))_{\upsilon}}$ is strictly contained for different weights and powers. We explain the settings in order that $\mathbb{B}^{\alpha}_{(\Xi(\zeta,t))_{\upsilon}}$ is minimum. Fourthly, we explain the conditions so that the Banach pre-quasi ideal $\mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}$ is simple. Fifthly, we give the sufficient conditions on $(\Xi(\zeta,t))_{\upsilon}$ such that the space of all bounded linear mappings whose sequence of eigenvalues in $(\Xi(\zeta,t))_{\upsilon}$ equals $\mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}$.

6.1 Finite rank pre-quasi ideal

Theorem 6.1 $\mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q}) = \overline{\mathbb{F}(\mathcal{P},\mathcal{Q})}$, *whenever setups* (f1) *and* (f2) *are satisfied. But the converse is not necessarily true.*

Proof To show that $\overline{\mathbb{F}(\mathcal{P},\mathcal{Q})} \subseteq \mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$ as $e_l \in (\Xi(\zeta,t))_{\upsilon}$ with $l \in \mathrm{N}$ and $(\Xi(\zeta,t))_{\upsilon}$ is a linear space. Let $Z \in \mathbb{F}(\mathcal{P},\mathcal{Q})$, one has $(s_l(Z))_{l=0}^{\infty} \in \mathcal{F}$. To show that $\mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q}) \subseteq \overline{\mathbb{F}(\mathcal{P},\mathcal{Q})}$, one can see $\sum_{l=0}^{\infty}(\frac{1}{\zeta_l})^{t_l} < \infty$. For $Z \in \mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$, we have $(s_l(Z))_{l=0}^{\infty} \in (\Xi(\zeta,t))_{\upsilon}$. As $\upsilon(s_l(Z))_{l=0}^{\infty} < \infty$, suppose $\rho \in (0,1)$, then there is $l_0 \in \mathrm{N} - \{0\}$ with $\upsilon((s_l(Z))_{l=l_0}^{\infty}) < \frac{\rho}{2^{\hbar+3}\eta d}$ for some $d \geq 1$, where $\eta = \max\{1, \sum_{l=l_0}^{\infty}(\frac{1}{\zeta_l})^{t_l}\}$. As $s_l(Z)$ is decreasing, one has

$$\begin{aligned}\sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} s_{2l_0}(Z)\Delta\zeta_j}{\zeta_l}\right)^{t_l} &\leq \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l} s_j(Z)\Delta\zeta_j}{\zeta_l}\right)^{t_l}\\ &\leq \sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(Z)\Delta\zeta_j}{\zeta_l}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}.\end{aligned} \tag{2}$$

Therefore, there is $Y \in \mathbb{F}_{2l_0}(\mathcal{P},\mathcal{Q})$ so that $\operatorname{rank}(Y) \leq 2l_0$ and

$$\sum_{l=2l_0+1}^{3l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l} \leq \sum_{l=l_0+1}^{2l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}, \tag{3}$$

since $(t_l) \in \Im_{\nearrow} \cap \ell_{\infty}$, we have

$$\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{l_0}\|Z-Y\|\Delta\zeta_j\right)^{t_l} < \frac{\rho}{2^{2\hbar+2}\eta}. \tag{4}$$

Therefore, one has

$$\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{5}$$

By using inequalities (1)–(5), one has

$$\begin{aligned}d(Z,Y) &= \upsilon\big(s_l(Z-Y)\big)_{l=0}^{\infty}\\ &= \sum_{l=0}^{3l_0-1}\left(\frac{\sum_{j=0}^{l} s_j(Z-Y)\Delta\zeta_j}{\zeta_l}\right)^{t_l} + \sum_{l=3l_0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(Z-Y)\Delta\zeta_j}{\zeta_l}\right)^{t_l}\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l}+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0}s_j(Z-Y)\Delta\zeta_j}{\sum_{j=0}^{l+2l_0}\Delta\zeta_j}\right)^{t_{l+2l_0}}\\
&\leq \sum_{l=0}^{3l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l}+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l+2l_0}s_j(Z-Y)\Delta\zeta_j}{\zeta_l}\right)^{t_l}\\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l}\\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1}s_j(Z-Y)\Delta\zeta_j+\sum_{j=2l_0}^{l+2l_0}s_j(Z-Y)\Delta\zeta_j}{\zeta_l}\right)^{t_l}\\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l}\\
&\quad+2^{\hbar-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1}s_j(Z-Y)\Delta\zeta_j}{\zeta_l}\right)^{t_l}+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=2l_0}^{l+2l_0}s_j(Z-Y)\Delta\zeta_j}{\zeta_l}\right)^{t_l}\right]\\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l}\\
&\quad+2^{\hbar-1}\left[\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{2l_0-1}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l}+\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l}s_{j+2l_0}(Z-Y)\Delta\zeta_{j+2l_0}}{\zeta_l}\right)^{t_l}\right]\\
&\leq 3\sum_{l=0}^{l_0}\left(\frac{\sum_{j=0}^{l}\|Z-Y\|\Delta\zeta_j}{\zeta_l}\right)^{t_l}\\
&\quad+2^{\hbar-1}\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{2l_0-1}\|Z-Y\|\Delta\zeta_j\right)^{t_l}\sum_{l=l_0}^{\infty}(\zeta_l)^{-t_l}+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\sum_{j=0}^{l}s_j(Z)\Delta\zeta_j}{\zeta_l}\right)^{t_l}<\rho.
\end{aligned}$$

Conversely, we give a counterexample as $I_4 \in \mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q})$, where $(\Delta\zeta_j) = (0,0,0,0,1,1,\ldots)$ and $t = (1,1,1,\ldots)$, but $t_0 > 1$ is not verified. This confirms the proof. □

6.2 Pre-quasi Banach and closed ideal

Theorem 6.2 *Suppose that setups* (f1) *and* (f2) *are satisfied, then* $(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon},\Psi)$ *is a pre-quasi Banach ideal, where* $\psi(X) = \upsilon((s_l(X))_{l=0}^{\infty})$.

Proof As $(\Xi(\zeta,t))_\upsilon$ is a pre-modular $\mathfrak{pss}$, hence from Theorem 2.13, Ψ is a pre-quasi norm on $\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}$. Suppose that $(X_b)_{b\in\mathrm{N}}$ is a Cauchy sequence in $\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q})$. As $\mathbb{B}(\mathcal{P},\mathcal{Q}) \supseteq \mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q})$, one has

$$\Psi(X_a - X_b) = \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l}s_j(X_a-X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} \geq \|X_a - X_b\|^{t_0},$$

so $(X_b)_{b\in\mathrm{N}}$ is a Cauchy sequence in $\mathbb{B}(\mathcal{P},\mathcal{Q})$. Since $\mathbb{B}(\mathcal{P},\mathcal{Q})$ is a Banach space, there is $X \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ with $\lim_{b\to\infty}\|X_b - X\| = 0$. Since $(s_l(X_b))_{l=0}^{\infty} \in (\Xi(\zeta,t))_\upsilon$ for all $b \in \mathrm{N}$, therefore, from Definition 2.10 parts (ii), (iii), and (v), one can see

$$\begin{aligned}
\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(X)\Delta\zeta_j}{\zeta_l}\right)^{t_l} \\
&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_{[\frac{j}{2}]}(X-X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} + 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_{[\frac{j}{2}]}(X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} \\
&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\|X-X_b\|^{t_l} + 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} < \infty.
\end{aligned}$$

Hence $(s_l(X))_{l=0}^{\infty} \in (\Xi(\zeta,t))_{\upsilon}$, so $X \in \mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$. □

Theorem 6.3 *Suppose that* $\mathcal{P}$, $\mathcal{Q}$ *are normed spaces, conditions* (f1) *and* (f2) *are satisfied, then* $(\mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}, \Psi)$ *is a pre-quasi closed ideal, where* $\Psi(X) = \upsilon((s_l(X))_{l=0}^{\infty})$.

Proof As $(\Xi(\zeta,t))_{\upsilon}$ is a pre-modular $\mathfrak{pss}$, hence from Theorem 2.13, Ψ is a pre-quasi norm on $\mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}$. Assume $X_b \in \mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$ for each $b \in \mathrm{N}$ and $\lim_{b\to\infty}\Psi(X_b - X) = 0$. As $\mathbb{B}(\mathcal{P},\mathcal{Q}) \supseteq \mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$, we have

$$\Psi(X-X_b) = \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(X-X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} \geq \|X-X_b\|^{t_0},$$

hence $(X_b)_{b\in \mathrm{N}}$ is a convergent sequence in $\mathbb{B}(\mathcal{P},\mathcal{Q})$. Since $(s_l(X_b))_{l=0}^{\infty} \in (\Xi(\zeta,t))_{\upsilon}$ for every $b \in \mathrm{N}$, by using Definition 2.10 parts (ii), (iii), and (v), one can see

$$\begin{aligned}
\Psi(X) &= \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(X)\Delta\zeta_j}{\zeta_l}\right)^{t_l} \\
&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_{[\frac{j}{2}]}(X-X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} + 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_{[\frac{j}{2}]}(X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} \\
&\leq 2^{\hbar-1}\sum_{l=0}^{\infty}\|X-X_b\|^{t_l} + 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(X_b)\Delta\zeta_j}{\zeta_l}\right)^{t_l} < \infty.
\end{aligned}$$

We get $(s_l(X))_{l=0}^{\infty} \in (\Xi(\zeta,t))_{\upsilon}$, so $X \in \mathbb{B}^{s}_{(\Xi(\zeta,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$. □

6.3 Minimum pre-quasi ideal

Theorem 6.4 *For any infinite dimensional Banach spaces* $\mathcal{P}$, $\mathcal{Q}$ *and if conditions* (f1) *and* (f2) *are satisfied with* $1 < t_l^{(1)} < t_l^{(2)}$ *and* $\frac{\Delta\zeta_l^{(2)}}{\zeta_l^{(2)}} \leq \frac{\Delta\zeta_l^{(1)}}{\zeta_l^{(1)}}$ *for all* $l \in \mathrm{N}$, *we have*

$$\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_{\upsilon}}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_{\upsilon}}(\mathcal{P},\mathcal{Q}) \subsetneqq \mathbb{B}(\mathcal{P},\mathcal{Q}).$$

Proof Suppose $Z \in \mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_{\upsilon}}(\mathcal{P},\mathcal{Q})$, then $(s_l(Z)) \in (\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_{\upsilon}$. One has

$$\sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} s_z(Z)\Delta\zeta_z^{(2)}}{\zeta_l^{(2)}}\right)^{t_l^{(2)}} < \sum_{l=0}^{\infty}\left(\frac{\sum_{z=0}^{l} s_z(Z)\Delta\zeta_z^{(1)}}{\zeta_l^{(1)}}\right)^{t_l^{(1)}} < \infty,$$

then $Z \in \mathbb{B}^s_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q})$. Afterwards, if we choose $(s_l(Z))_{l=0}^\infty$ such that $\sum_{z=0}^l s_z(Z)\Delta\zeta_z^{(1)} = \frac{\zeta_l^{(1)}}{t_l^{(1)}\sqrt{l+1}}$, we have $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ such that

$$\sum_{l=0}^\infty \left(\frac{\sum_{z=0}^l s_z(Z)\Delta\zeta_z^{(1)}}{\zeta_l^{(1)}}\right)^{t_l^{(1)}} = \sum_{l=0}^\infty \frac{1}{l+1} = \infty$$

and

$$\sum_{l=0}^\infty \left(\frac{\sum_{z=0}^l s_z(Z)\Delta\zeta_z^{(2)}}{\zeta_l^{(2)}}\right)^{t_l^{(2)}} \le \sum_{l=0}^\infty \left(\frac{\sum_{z=0}^l s_z(Z)\Delta\zeta_z^{(1)}}{\zeta_l^{(1)}}\right)^{t_l^{(2)}} = \sum_{l=0}^\infty \left(\frac{1}{l+1}\right)^{\frac{t_l^{(2)}}{t_l^{(1)}}} < \infty.$$

So $Z \notin \mathbb{B}^s_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q})$ and $Z \in \mathbb{B}^s_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q})$. Clearly, $\mathbb{B}^s_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}) \subset \mathbb{B}(\mathcal{P},\mathcal{Q})$. Next, if we take $(s_l(Z))_{l=0}^\infty$ such that $\sum_{z=0}^l s_z(Z)\Delta\zeta_z^{(2)} = \frac{\zeta_l^{(2)}}{t_l^{(2)}\sqrt{l+1}}$, we have $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ so that $Z \notin \mathbb{B}^s_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q})$. This confirms the proof. $\square$

Theorem 6.5 *For any infinite dimensional Banach spaces* $\mathcal{P}$, $\mathcal{Q}$, *if setups* (f1) *and* (f2) *are satisfied, then* $\mathbb{B}^\alpha_{(\Xi(\zeta,t))_\upsilon}$ *is minimum.*

Proof Assume that the set-ups are confirmed. So $(\mathbb{B}^\alpha_{\Xi(\zeta,t)},\Psi)$, where $\Psi(Z) = \sum_{l=0}^\infty (\frac{\sum_{j=0}^l \alpha_j(X)\Delta\zeta_j}{\zeta_l})^{t_l}$ is a pre-quasi Banach ideal. Let $\mathbb{B}^\alpha_{\Xi(\zeta,t)}(\mathcal{P},\mathcal{Q}) = \mathbb{B}(\mathcal{P},\mathcal{Q})$, hence there is $\eta > 0$ such that $\Psi(Z) \le \eta\|Z\|$ for each $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$. Then, by Dvoretzky's theorem [30] with $b \in \mathrm{N}$, one has quotient spaces $\mathcal{P}/Y_b$ and subspaces M_b of $\mathcal{Q}$ which can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \le 2$ and $\|X_b\|\|X_b^{-1}\| \le 2$. If I_b is the identity mapping on ℓ_2^b, T_b is the quotient mapping from $\mathcal{P}$ onto $\mathcal{P}/Y_b$, and J_b is the natural embedding mapping from M_b into $\mathcal{Q}$. Assume m_z to be the Bernstein numbers [31], hence

$$\begin{aligned} 1 &= m_z(I_b) = m_z\big(X_bX_b^{-1}I_bV_bV_b^{-1}\big) \\ &\le \|X_b\|m_z\big(X_b^{-1}I_bV_b\big)\big\|V_b^{-1}\big\| = \|X_b\|m_z\big(J_bX_b^{-1}I_bV_b\big)\big\|V_b^{-1}\big\| \\ &\le \|X_b\|d_z\big(J_bX_b^{-1}I_bV_b\big)\big\|V_b^{-1}\big\| = \|X_b\|d_z\big(J_bX_b^{-1}I_bV_bT_b\big)\big\|V_b^{-1}\big\| \\ &\le \|X_b\|\alpha_z\big(J_bX_b^{-1}I_bV_bT_b\big)\big\|V_b^{-1}\big\| \end{aligned}$$

for $0 \le l \le b$. We have

$$\begin{aligned} &\zeta_l \le \sum_{z=0}^l \|X_b\|\big\|V_b^{-1}\big\|\alpha_z\big(J_bX_b^{-1}I_bV_bT_b\big)\Delta\zeta_z \quad \Rightarrow \\ &1 \le \big(\|X_b\|\big\|V_b^{-1}\big\|\big)^{t_l}\left(\frac{\sum_{z=0}^l \alpha_z(J_bX_b^{-1}I_bV_bT_b)\Delta\zeta_z}{\zeta_l}\right)^{t_l}. \end{aligned}$$

Hence, for some $\varrho \ge 1$, one has

$$b+1 \le \varrho\|X_b\|\big\|V_b^{-1}\big\|\sum_{l=0}^b \left(\frac{\sum_{z=0}^l \alpha_z(J_bX_b^{-1}I_bV_bT_b)\Delta\zeta_z}{\zeta_l}\right)^{t_l} \quad \Rightarrow$$

$$\begin{aligned} b+1 &\le \varrho\|X_b\|\,\|V_b^{-1}\|\,\Psi\big(J_bX_b^{-1}I_bV_bT_b\big) \le \varrho\eta\|X_b\|\,\|V_b^{-1}\|\,\|J_bX_b^{-1}I_bV_bT_b\| \\ &\le \varrho\eta\|X_b\|\,\|V_b^{-1}\|\,\|J_bX_b^{-1}\|\,\|I_b\|\,\|V_bT_b\| \\ &= \varrho\eta\|X_b\|\,\|V_b^{-1}\|\,\|X_b^{-1}\|\,\|I_b\|\,\|V_b\| \le 4\varrho\eta. \end{aligned}$$

We have an inconsistency as b is arbitrary. Then $\mathcal{P}$ and $\mathcal{Q}$ both cannot be infinite dimensional when $\mathbb{B}^{\alpha}_{\Xi(\zeta,t)}(\mathcal{P},\mathcal{Q}) = \mathbb{B}(\mathcal{P},\mathcal{Q})$. This completes the proof. □

Theorem 6.6 *For any infinite dimensional Banach spaces $\mathcal{P}$, $\mathcal{Q}$ and if setups* (f1) *and* (f2) *are satisfied, then $\mathbb{B}^{d}_{\Xi(\zeta,t)}$ is minimum.*

6.4 Simple Banach pre-quasi ideal

Theorem 6.7 *Presume that $\mathcal{P}$ and $\mathcal{Q}$ are infinite dimensional Banach spaces. Let setups* (f1) *and* (f2) *be satisfied with $1 < t_l^{(1)} < t_l^{(2)}$ and $\frac{\Delta\zeta_l^{(2)}}{\zeta_l^{(2)}} \le \frac{\Delta\zeta_l^{(1)}}{\zeta_l^{(1)}}$ for all $l \in \mathrm{N}$, then*

$$\begin{aligned} &\mathbb{B}\big(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q})\big) \\ &\quad= \mathcal{A}\big(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q})\big). \end{aligned}$$

Proof For $X \in \mathbb{B}(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q}))$ and $X \notin \mathcal{A}(\mathbb{B}^{s}_{(\Xi((r_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q}))$. From Lemma 2.1, one has $Y \in \mathbb{B}(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}))$ and $Z \in \mathbb{B}(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q}))$ with $ZXYI_b = I_b$. Therefore, for each $b \in \mathrm{N}$, we have

$$\begin{aligned} \|I_b\|_{\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q})} &= \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(I_b)\Delta\zeta_j^{(1)}}{\zeta_l^{(1)}}\right)^{t_l^{(1)}} \\ &\le \|ZXY\|\,\|I_b\|_{\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q})} \\ &\le \sum_{l=0}^{\infty}\left(\frac{\sum_{j=0}^{l} s_j(I_b)\Delta\zeta_j^{(2)}}{\zeta_l^{(2)}}\right)^{t_l^{(2)}}. \end{aligned}$$

This defies Theorem 6.4. Then $X \in \mathcal{A}(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q}))$, which confirms the proof. □

Corollary 6.8 *For any infinite dimensional Banach spaces $\mathcal{P}$ and $\mathcal{Q}$, if setups* (f1) *and* (f2) *are satisfied with $1 < t_l^{(1)} < t_l^{(2)}$ and $\frac{\Delta\zeta_l^{(2)}}{\zeta_l^{(2)}} \le \frac{\Delta\zeta_l^{(1)}}{\zeta_l^{(1)}}$ for all $l \in \mathrm{N}$, then*

$$\begin{aligned} &\mathbb{B}\big(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q})\big) \\ &\quad= \mathcal{K}\big(\mathbb{B}^{s}_{(\Xi((\zeta_l^{(2)}),(t_l^{(2)})))_\upsilon}(\mathcal{P},\mathcal{Q}),\mathbb{B}^{s}_{(\Xi((\zeta_l^{(1)}),(t_l^{(1)})))_\upsilon}(\mathcal{P},\mathcal{Q})\big). \end{aligned}$$

Proof Clearly, as $\mathcal{A} \subset \mathcal{K}$. □

Theorem 6.9 *Assume that setups* (f1) *and* (f2) *are satisfied, then $\mathbb{B}^{s}_{(\Xi(\zeta,t))_\upsilon}$ is simple.*

Proof Let the closed ideal $\mathcal{K}(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$ include a mapping $X \notin \mathcal{A}(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$. From Lemma 2.1, one has $Y, Z \in \mathbb{B}(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$ with $ZXYI_b = I_b$. This gives that $I_{\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q})} \in \mathcal{K}(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$. Accordingly, $\mathbb{B}(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q})) = \mathcal{K}(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$. Hence $\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}$ is a simple Banach space. □

6.5 Eigenvalues of s-type mappings

Notation 6.10

$$\begin{aligned}
&\left(\mathbb{B}^s_{\mathcal{V}}\right)^\rho := \left\{\left(\mathbb{B}^s_{\mathcal{V}}\right)^\rho(\mathcal{P},\mathcal{Q}); \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach spaces}\right\}, \quad \text{where} \\
&\left(\mathbb{B}^s_{\mathcal{V}}\right)^\rho(\mathcal{P},\mathcal{Q}) \\
&\quad := \left\{X \in \mathbb{B}(\mathcal{P},\mathcal{Q}) : (\left(\rho_l(X)\right)_{n=0}^\infty \in \mathcal{V} \text{ and } \left\|X - \rho_l(X)I\right\| \text{ is not invertible for all } l \in \mathrm{N}\right\}.
\end{aligned}$$

Theorem 6.11 *For any infinite dimensional Banach spaces $\mathcal{P}$ and $\mathcal{Q}$, suppose that setups* (f1) *and* (f2) *are satisfied, then*

$$\left(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}\right)^\rho(\mathcal{P},\mathcal{Q}) = \mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q}).$$

Proof Let $X \in (\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon})^\rho(\mathcal{P},\mathcal{Q})$, hence $(\rho_l(X))_{l=0}^\infty \in (\Xi(\zeta,t))_\upsilon$ and $\|X - \rho_l(X)I\| = 0$ for all $l \in \mathrm{N}$. We have $X = \rho_l(X)I$ with $l \in \mathrm{N}$, so $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$ with $l \in \mathrm{N}$. Therefore, $(s_l(X))_{l=0}^\infty \in (\Xi(\zeta,t))_\upsilon$, so $X \in \mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q})$.

Secondly, let $X \in \mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q})$. Therefore, $(s_l(X))_{l=0}^\infty \in (\Xi(\zeta,t))_\upsilon$. Hence, we have

$$\sum_{l=0}^\infty \left(\frac{\sum_{z=0}^l s_z(X)\Delta\zeta_z}{\zeta_l}\right)^{t_l} \geq \sum_{l=0}^\infty \left[s_l(X)\right]^{t_l}.$$

So $\lim_{l\to\infty} s_l(X) = 0$. Assume that $\|X - s_l(X)I\|^{-1}$ exists for every $l \in \mathrm{N}$. Therefore, $\|X - s_l(X)I\|^{-1}$ exists and is bounded for every $l \in \mathrm{N}$. So, $\lim_{l\to\infty} \|X - s_l(X)I\|^{-1} = \|X\|^{-1}$ exists and is bounded. From the pre-quasi operator ideal of $(\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}, \Psi)$, we obtain

$$I = XX^{-1} \in \mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon}(\mathcal{P},\mathcal{Q}) \quad \Rightarrow \quad \left(s_l(I)\right)_{l=0}^\infty \in \Xi(\zeta,t) \quad \Rightarrow \quad \lim_{l\to\infty} s_l(I) = 0.$$

We have a contradiction since $\lim_{l\to\infty} s_l(I) = 1$. Therefore, $\|X - s_l(X)I\| = 0$ for every $l \in \mathrm{N}$. This gives $X \in (\mathbb{B}^s_{(\Xi(\zeta,t))_\upsilon})^\rho(\mathcal{P},\mathcal{Q})$. This provides the proof. □

7 Kannan contraction mapping

Theorem 7.1 *The function $\upsilon(f) = [\sum_{l=0}^\infty (\frac{|\sum_{z=0}^l f_z\Delta\zeta_z|}{\zeta_l})^{t_l}]^{\frac{1}{\hbar}}$ for every $f \in \Xi(\zeta,t)$ satisfies the Fatou property if setups* (f1) *and* (f2) *are satisfied.*

Proof Assume that the set-ups are verified and $\{g^b\} \subseteq (\Xi(\zeta,t))_\upsilon$ with $\lim_{b\to\infty} \upsilon(g^b - g) = 0$. As the space $(\Xi(\zeta,t))_\upsilon$ is a pre-quasi closed space, then $g \in (\Xi(\zeta,t))_\upsilon$. Hence, for all $f \in (\Xi(\zeta,t))_\upsilon$, we have

$$\upsilon(f-g) = \left[\sum_{l=0}^\infty \left(\frac{|\sum_{z=0}^l (f_z - g_z)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}}$$

$$
\begin{aligned}
&\leq \left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(f_z - g_z^b)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}} + \left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(g_z^b - g_z)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}} \\
&\leq \sup_j \inf_{b\geq j} \upsilon\bigl(f - g^b\bigr).
\end{aligned}
$$

□

Theorem 7.2 *The function* $\upsilon(f) = \sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l} f_z\Delta\zeta_z|}{\zeta_l})^{t_l}$ *does not verify the Fatou property for every* $f \in \Xi(\zeta,t)$ *if setups* (f1) *and* (f2) *are satisfied.*

Proof Assume that set-ups are verified and $\{g^b\} \subseteq (\Xi(\zeta,t))_\upsilon$ with $\lim_{b\to\infty} \upsilon(g^b - g) = 0$. As the space $(\Xi(\zeta,t))_\upsilon$ is a pre-quasi closed space, then $g \in (\Xi(\zeta,t))_\upsilon$. Hence, for all $f \in (\Xi(\zeta,t))_\upsilon$, we have

$$
\begin{aligned}
\upsilon(f - g) &= \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(f_z - g_z)\Delta\zeta_z|}{\zeta_l}\right)^{t_l} \\
&\leq 2^{\hbar-1}\left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(f_z - g_z^b)\Delta\zeta_z|}{\zeta_l}\right)^{t_l} + \sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(g_z^b - g_z)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right] \\
&\leq 2^{\hbar-1}\sup_j \inf_{b\geq j} \upsilon\bigl(f - g^b\bigr).
\end{aligned}
$$

Therefore, υ does not verify the Fatou property. □

Now, we study the sufficient settings on $(\Xi(\zeta,t))_\upsilon$ constructed with definite pre-quasi norm so that there is one and only one fixed point of the Kannan pre-quasi norm contraction mapping.

Theorem 7.3 *If setups* (f1) *and* (f2) *are satisfied and* $W : (\Xi(\zeta,t))_\upsilon \to (\Xi(\zeta,t))_\upsilon$ *is a Kannan* υ*-contraction mapping, where* $\upsilon(f) = [\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l} f_z\Delta\zeta_z|}{\zeta_l})^{t_l}]^{\frac{1}{\hbar}}$*, for every* $f \in \Xi(\zeta,t)$*, so* W *has a unique fixed point.*

Proof Assume that the conditions are verified. For all $f \in \Xi(\zeta,t)$, then $W^p f \in \Xi(\zeta,t)$. Since W is a Kannan υ-contraction mapping, we have

$$
\begin{aligned}
&\upsilon\bigl(W^{p+1}f - W^p f\bigr) \leq \lambda\bigl(\upsilon\bigl(W^{p+1}f - W^p f\bigr) + \upsilon\bigl(W^p f - W^{p-1}f\bigr)\bigr) \quad \Rightarrow \\
&\upsilon\bigl(W^{p+1}f - W^p f\bigr) \leq \frac{\lambda}{1-\lambda}\upsilon\bigl(W^p f - W^{p-1}f\bigr) \\
&\qquad \leq \left(\frac{\lambda}{1-\lambda}\right)^2 \upsilon\bigl(W^{p-1}f - W^{p-2}f\bigr) \leq \cdots \leq \left(\frac{\lambda}{1-\lambda}\right)^p \upsilon(Wf - f).
\end{aligned}
$$

Therefore, for every $p, q \in N$ with $q > p$, we have

$$
\begin{aligned}
\upsilon\bigl(W^p f - W^q f\bigr) &\leq \lambda\bigl(\upsilon\bigl(W^p f - W^{p-1}f\bigr) + \upsilon\bigl(W^q f - W^{q-1}f\bigr)\bigr) \\
&\leq \lambda\left(\left(\frac{\lambda}{1-\lambda}\right)^{p-1} + \left(\frac{\lambda}{1-\lambda}\right)^{q-1}\right)\upsilon(Wf - f).
\end{aligned}
$$

Hence, $\{W^p f\}$ is a Cauchy sequence in $(\Xi(\zeta,t))_\upsilon$. Since the space $(\Xi(\zeta,t))_\upsilon$ is pre-quasi Banach space, there is $g \in (\Xi(\zeta,t))_\upsilon$ so that $\lim_{p\to\infty} W^p f = g$. To show that $Wg = g$, as υ

has the Fatou property, we get

$$\upsilon(Wg-g)\leq \sup_i \inf_{p\geq i} \upsilon\bigl(W^{p+1}f-W^pf\bigr)\leq \sup_i \inf_{p\geq i}\left(\frac{\lambda}{1-\lambda}\right)^p \upsilon(Wf-f)=0,$$

so $Wg=g$. Then g is a fixed point of W. To prove that the fixed point is unique, assume that we have two different fixed points $b,g\in(\Xi(\zeta,t))_\upsilon$ of W. Therefore, one can see

$$\upsilon(b-g)\leq\upsilon(Wb-Wg)\leq\xi\bigl(\upsilon(Wb-b)+\upsilon(Wg-g)\bigr)=0.$$

Hence, $b=g$. □

Corollary 7.4 *Suppose that setups* (f1) *and* (f2) *are satisfied and* $W:(\Xi(\zeta,t))_\upsilon\to(\Xi(\zeta,t))_\upsilon$ *is a Kannan* υ*-contraction mapping, where* $\upsilon(f)=[\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}f_z\Delta\zeta_z|}{\zeta_l})^{t_l}]^{\frac{1}{\hbar}}$, *for every* $f\in\Xi(\zeta,t)$, *then* W *has a unique fixed point* b *with* $\upsilon(W^pf-b)\leq\lambda(\frac{\lambda}{1-\lambda})^{p-1}\upsilon(Wf-f)$.

Proof Assume that the set-ups are verified. By Theorem 7.3, there is a unique fixed point b of W. Therefore, one can see

$$\begin{aligned}\upsilon\bigl(W^pf-b\bigr)&=\upsilon\bigl(W^pf-Wb\bigr)\\&\leq\lambda\bigl(\upsilon\bigl(W^pf-W^{p-1}f\bigr)+\upsilon(Wb-b)\bigr)=\lambda\left(\frac{\lambda}{1-\lambda}\right)^{p-1}\upsilon(Wf-f).\end{aligned}$$ □

Theorem 7.5 *If setups* (f1) *and* (f2) *are satisfied and* $W:(\Xi(\zeta,t))_\upsilon\to(\Xi(\zeta,t))_\upsilon$, *where* $\upsilon(f)=\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}f_z\Delta\zeta_z|}{\zeta_l})^{t_l}$, *for every* $f\in\Xi(\zeta,t)$. *The point* $g\in(\Xi(\zeta,t))_\upsilon$ *is the only fixed point of* W *if the following conditions are verified*:

(a) *W is a Kannan υ-contraction mapping*;
(b) *W is υ-sequentially continuous at $g\in(\Xi(\zeta,t))_\upsilon$*;
(c) *We have $v\in(\Xi(\zeta,t))_\upsilon$ such that the sequence of iterates $\{W^pv\}$ has a subsequence $\{W^{p_i}v\}$ converging to g.*

Proof If the settings are satisfied, let g be not a fixed point of W, then $Wg\neq g$. By set-ups (b) and (c), one can see

$$\lim_{p_i\to\infty}\upsilon\bigl(W^{p_i}f-g\bigr)=0\quad\text{and}\quad\lim_{p_i\to\infty}\upsilon\bigl(W^{p_i+1}f-Wg\bigr)=0.$$

Since the operator W is a Kannan υ-contraction, we have

$$\begin{aligned}0&<\upsilon(Wg-g)\\&=\upsilon\bigl(\bigl(Wg-W^{p_i+1}f\bigr)+\bigl(W^{p_i}f-g\bigr)+\bigl(W^{p_i+1}f-W^{p_i}f\bigr)\bigr)\\&\leq 2^{2\hbar-2}\upsilon\bigl(W^{p_i+1}v-Wg\bigr)+2^{2\hbar-2}\upsilon\bigl(W^{p_i}v-g\bigr)+2^{\hbar-1}\lambda\left(\frac{\lambda}{1-\lambda}\right)^{p_i-1}\upsilon(Wf-f).\end{aligned}$$

Since $p_i\to\infty$, we get a contradiction. Hence, g is a fixed point of W. To show that the fixed point g is unique, suppose that we have two different fixed points $g,b\in(\Xi(\zeta,t))_\upsilon$

of W. Therefore, we have

$$\upsilon(g-b) \le \upsilon(Wg-Wb) \le \lambda\big(\upsilon(Wg-g)+\upsilon(Wb-b)\big)=0.$$

So, $g=b$. $\square$

Example 7.6 Let $W:(\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon} \to (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$, where $\upsilon(f)=\sqrt{\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}\frac{z+2}{z+1}f_z|}{\sum_{z=0}^{l}\frac{z+2}{z+1}})^{\frac{2l+3}{l+2}}}$, for all $f\in\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty})$ and

$$W(f)=\begin{cases}\frac{f}{4}, & \upsilon(f)\in[0,1),\\ \frac{f}{5}, & \upsilon(f)\in[1,\infty).\end{cases}$$

Since for all $f,g\in(\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $\upsilon(f),\upsilon(g)\in[0,1)$, we have

$$\begin{aligned}\upsilon(Wf-Wg)&=\upsilon\left(\frac{f}{4}-\frac{g}{4}\right)\\&\le\frac{1}{\sqrt[4]{27}}\left(\upsilon\left(\frac{3f}{4}\right)+\upsilon\left(\frac{3g}{4}\right)\right)=\frac{1}{\sqrt[4]{27}}\big(\upsilon(Wf-f)+\upsilon(Wg-g)\big).\end{aligned}$$

For all $f,g\in(\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $\upsilon(f),\upsilon(g)\in[1,\infty)$, we have

$$\begin{aligned}\upsilon(Wf-Wg)&=\upsilon\left(\frac{f}{5}-\frac{g}{5}\right)\\&\le\frac{1}{\sqrt[4]{64}}\left(\upsilon\left(\frac{4f}{5}\right)+\upsilon\left(\frac{4g}{5}\right)\right)=\frac{1}{\sqrt[4]{64}}\big(\upsilon(Wf-f)+\upsilon(Wg-g)\big).\end{aligned}$$

For all $f,g\in(\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $\upsilon(f)\in[0,1)$ and $\upsilon(g)\in[1,\infty)$, we have

$$\begin{aligned}\upsilon(Wf-Wg)&=\upsilon\left(\frac{f}{4}-\frac{g}{5}\right)\\&\le\frac{1}{\sqrt[4]{27}}\upsilon\left(\frac{3f}{4}\right)+\frac{1}{\sqrt[4]{64}}\upsilon\left(\frac{4g}{5}\right)\le\frac{1}{\sqrt[4]{27}}\left(\upsilon\left(\frac{3f}{4}\right)+\upsilon\left(\frac{4g}{5}\right)\right)\\&=\frac{1}{\sqrt[4]{27}}\big(\upsilon(Wf-f)+\upsilon(Wg-g)\big).\end{aligned}$$

Therefore, the mapping W is a Kannan υ-contraction mapping. Since υ satisfies the Fatou property, by Theorem 7.3, the mapping W has a unique fixed point $\theta\in(\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$.

Let $\{f^{(n)}\}\subseteq(\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ be such that $\lim_{n\to\infty}\upsilon(f^{(n)}-f^{(0)})=0$, where $f^{(0)}\in(\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $\upsilon(f^{(0)})=1$. Since the pre-quasi norm υ is continuous, we have

$$\lim_{n\to\infty}\upsilon\big(Wf^{(n)}-Wf^{(0)}\big)=\lim_{n\to\infty}\upsilon\left(\frac{f^{(n)}}{4}-\frac{f^{(0)}}{5}\right)=\upsilon\left(\frac{f^{(0)}}{20}\right)>0.$$

Hence W is not υ-sequentially continuous at $f^{(0)}$. So, the mapping W is not continuous at $f^{(0)}$.

If $\upsilon(f) = \sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}\frac{z+2}{z+1}f_z|}{\sum_{z=0}^{l}\frac{z+2}{z+1}})^{\frac{2l+3}{l+2}}$ for all $f \in \Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty})$. Since for all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $\upsilon(f),\upsilon(g) \in [0,1)$, we have

$$\begin{aligned}\upsilon(Wf-Wg) &= \upsilon\left(\frac{f}{4}-\frac{g}{4}\right)\\ &\le \frac{2}{\sqrt{27}}\left(\upsilon\left(\frac{3f}{4}\right)+\upsilon\left(\frac{3g}{4}\right)\right) = \frac{2}{\sqrt{27}}\big(\upsilon(Wf-f)+\upsilon(Wg-g)\big).\end{aligned}$$

For all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $\upsilon(f),\upsilon(g) \in [1,\infty)$, we have

$$\upsilon(Wf-Wg) = \upsilon\left(\frac{f}{5}-\frac{g}{5}\right) \le \frac{1}{4}\left(\upsilon\left(\frac{4f}{5}\right)+\upsilon\left(\frac{4g}{5}\right)\right) = \frac{1}{4}\big(\upsilon(Wf-f)+\upsilon(Wg-g)\big).$$

For all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $\upsilon(f) \in [0,1)$ and $\upsilon(g) \in [1,\infty)$, we have

$$\begin{aligned}\upsilon(Wf-Wg) &= \upsilon\left(\frac{f}{4}-\frac{g}{5}\right)\\ &\le \frac{2}{\sqrt{27}}\upsilon\left(\frac{3f}{4}\right)+\frac{1}{4}\upsilon\left(\frac{4g}{5}\right) \le \frac{2}{\sqrt{27}}\left(\upsilon\left(\frac{3f}{4}\right)+\upsilon\left(\frac{4g}{5}\right)\right)\\ &= \frac{2}{\sqrt{27}}\big(\upsilon(Wf-f)+\upsilon(Wg-g)\big).\end{aligned}$$

Therefore, the mapping W is a Kannan υ-contraction mapping and

$$W^p(f) = \begin{cases}\frac{f}{4^p}, & \upsilon(f) \in [0,1),\\ \frac{f}{5^p}, & \upsilon(f) \in [1,\infty).\end{cases}$$

It is clear that W is υ-sequentially continuous at $\theta \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ and $\{W^pf\}$ has a subsequence $\{W^{p_i}f\}$ converging to θ. By Theorem 7.5, the point $\theta \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ is the only fixed point of W.

Example 7.7 Let $W : (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon} \to (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$, where $\upsilon(f) = \sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}\frac{z+2}{z+1}f_z|}{\sum_{z=0}^{l}\frac{z+2}{z+1}})^{\frac{2l+3}{l+2}}$, for all $f \in \Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty})$ and

$$W(f) = \begin{cases}\frac{1}{4}(e_1+f), & f_0 \in (-\infty,\frac{1}{3}),\\ \frac{1}{3}e_1, & f_0 = \frac{1}{3},\\ \frac{1}{4}e_1, & f_0 \in (\frac{1}{3},\infty).\end{cases}$$

Since for all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $f_0,g_0 \in (-\infty,\frac{1}{3})$, we have

$$\begin{aligned}\upsilon(Wf-Wg) &= \upsilon\left(\frac{1}{4}(f_0-g_0,f_1-g_1,f_2-g_2,\dots)\right)\\ &\le \frac{2}{\sqrt{27}}\left(\upsilon\left(\frac{3f}{4}\right)+\upsilon\left(\frac{3g}{4}\right)\right)\\ &\le \frac{2}{\sqrt{27}}\big(\upsilon(Wf-f)+\upsilon(Wg-g)\big).\end{aligned}$$

For all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $f_0, g_0 \in (\frac{1}{3},\infty)$, then for any $\varepsilon > 0$ we have

$$\upsilon(Wf - Wg) = 0 \leq \varepsilon\big(\upsilon(Wf - f) + \upsilon(Wg - g)\big).$$

For all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $f_0 \in (-\infty,\frac{1}{3})$ and $g_0 \in (\frac{1}{3},\infty)$, we have

$$\begin{aligned}\upsilon(Wf - Wg) &= \upsilon\left(\frac{f}{4}\right) \leq \frac{1}{\sqrt{27}}\upsilon\left(\frac{3f}{4}\right)\\ &= \frac{1}{\sqrt{27}}\upsilon(Wf - f) \leq \frac{1}{\sqrt{27}}\big(\upsilon(Wf - f) + \upsilon(Wg - g)\big).\end{aligned}$$

Therefore, the mapping W is a Kannan υ-contraction mapping. It is clear that W is υ-sequentially continuous at $\frac{1}{3}e_1 \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ and there is $f \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $f_0 \in (-\infty,\frac{1}{3})$ such that the sequence of iterates $\{W^p f\} = \{\sum_{n=1}^{p}\frac{1}{4^n}e_1 + \frac{1}{4^p}f\}$ has a subsequence $\{W^{p_i}f\} = \{\sum_{n=1}^{p_i}\frac{1}{4^n}e_1 + \frac{1}{4^{p_i}}f\}$ converging to $\frac{1}{3}e_1$. By Theorem 7.5, the mapping W has one fixed point $\frac{1}{3}e_1 \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$. Note that W is not continuous at $\frac{1}{3}e_1 \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$.

If $\upsilon(f) = \sqrt{\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}\frac{z+2}{z+1}f_z|}{\sum_{z=0}^{l}\frac{z+2}{z+1}})^{\frac{2l+3}{l+2}}}$ for all $f \in \Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty})$. Since for all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $f_0, g_0 \in (-\infty,\frac{1}{3})$, we have

$$\begin{aligned}\upsilon(Wf - Wg) &= \upsilon\left(\frac{1}{4}(f_0 - g_0, f_1 - g_1, f_2 - g_2, \dots)\right)\\ &\leq \frac{1}{\sqrt[4]{27}}\left(\upsilon\left(\frac{3f}{4}\right) + \upsilon\left(\frac{3g}{4}\right)\right)\\ &\leq \frac{1}{\sqrt[4]{27}}\big(\upsilon(Wf - f) + \upsilon(Wg - g)\big).\end{aligned}$$

For all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $f_0, g_0 \in (\frac{1}{3},\infty)$, then for any $\varepsilon > 0$ we have

$$\upsilon(Wf - Wg) = 0 \leq \varepsilon\big(\upsilon(Wf - f) + \upsilon(Wg - g)\big).$$

For all $f,g \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$ with $f_0 \in (-\infty,\frac{1}{3})$ and $g_0 \in (\frac{1}{3},\infty)$, we have

$$\begin{aligned}\upsilon(Wf - Wg) &= \upsilon\left(\frac{f}{4}\right) \leq \frac{1}{\sqrt[4]{27}}\upsilon\left(\frac{3f}{4}\right) = \frac{1}{\sqrt[4]{27}}\upsilon(Wf - f)\\ &\leq \frac{1}{\sqrt[4]{27}}\big(\upsilon(Wf - f) + \upsilon(Wg - g)\big).\end{aligned}$$

Therefore, the mapping W is a Kannan υ-contraction mapping. Since υ satisfies the Fatou property, by Theorem 7.3, the mapping W has a unique fixed point $\frac{1}{3}e_1 \in (\Xi((\sum_{z=0}^{l}\frac{z+2}{z+1})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}$.

We study the existence of a fixed point of the Kannan pre-quasi norm contraction mapping in the pre-quasi Banach operator ideal constructed by $(\Xi(\zeta,t))_{\upsilon}$ and s-numbers.

Theorem 7.8 *The pre-quasi norm* $\Psi(W) = [\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}s_z(W)\Delta\zeta_z|}{\zeta_l})^{t_l}]^{\frac{1}{\hbar}}$ *for each* $W \in S_{(\Xi(\zeta,t))_{\upsilon}}(Z,M)$ *does not verify the Fatou property if setups* (f1) *and* (f2) *are satisfied.*

Proof Suppose that the conditions are verified and $\{W_p\}_{p\in\mathbb{N}} \subseteq S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$ with $\lim_{p\to\infty}\Psi(W_p - W) = 0$. As the space $S_{(\Xi(\zeta,t))_\upsilon}$ is a pre-quasi closed ideal, hence $W \in S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$. Then, for all $V \in S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$, one has

$$\begin{aligned}\Psi(V-W) &= \left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} s_z(V-W)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}} \\ &\leq \left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} s_{[\frac{z}{2}]}(V-W_i)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}} \\ &\quad + \left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} s_{[\frac{z}{2}]}(W-W_i)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}} \\ &\leq 2^{\frac{1}{\hbar}}\sup_p\inf_{i\geq p}\left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l} s_z(V-W_i)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}}.\end{aligned}$$

Therefore, Ψ does not verify the Fatou property. □

Theorem 7.9 *If setups* (f1) *and* (f2) *are satisfied and* $G : S_{(\Xi(\zeta,t))_\upsilon}(Z,M) \to S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$, *where* $\Psi(W) = [\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l} s_z(W)\Delta\zeta_z|}{\zeta_l})^{t_l}]^{\frac{1}{\hbar}}$ *for all* $W \in S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$. *The point* $A \in S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$ *is the unique fixed point of* G *if the following settings are verified*:

(a) *G is a Kannan Ψ-contraction mapping*;
(b) *G is Ψ-sequentially continuous at a point $A \in S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$*;
(c) *We have $B \in S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$ such that the sequence of iterates $\{G^p B\}$ has a subsequence $\{G^{p_i}B\}$ converging to A.*

Proof Suppose that the settings are satisfied. If A is not a fixed point of G, then $GA \neq A$. From conditions (b) and (c), one has

$$\lim_{p_i\to\infty}\Psi\big(G^{p_i}B - A\big) = 0 \quad\text{and}\quad \lim_{p_i\to\infty}\Psi\big(G^{p_i+1}B - GA\big) = 0.$$

As G is a Kannan Ψ-contraction mapping, we have

$$\begin{aligned}0 < \Psi(GA-A) &= \Psi\big(\big(GA - G^{p_i+1}B\big) + \big(G^{p_i}B - A\big) + \big(G^{p_i+1}B - G^{p_i}B\big)\big) \\ &\leq 2^{\frac{1}{\hbar}}\Psi\big(G^{p_i+1}B - GA\big) + 2^{\frac{2}{\hbar}}\Psi\big(G^{p_i}B - A\big) + 2^{\frac{2}{\hbar}}\lambda\left(\frac{\lambda}{1-\lambda}\right)^{p_i-1}\Psi(GB-B).\end{aligned}$$

Since $p_i \to \infty$, one has a contradiction. Hence, A is a fixed point of G. To prove that the fixed point A is unique, assume that we have two different fixed points $A, D \in S_{(\Xi(\zeta,t))_\upsilon}(Z,M)$ of G. Therefore, we have

$$\Psi(A-D) \leq \Psi(GA-GD) \leq \lambda\big(\Psi(GA-A) + \Psi(GD-D)\big) = 0.$$

So, $A = D$. □

Example 7.10 Let Z and M be Banach spaces,

$$G : S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_\upsilon}(Z,M) \to S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_\upsilon}(Z,M),$$

where

$$\Psi(W)=\sqrt{\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}\frac{z+1}{z+2}s_z|}{\sum_{z=0}^{l}\frac{z+1}{z+2}}\right)^{\frac{2l+3}{l+2}}},$$

for every $W\in S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}(Z,M)$ and

$$G(W)=\begin{cases}\frac{W}{6}, & \Psi(W)\in[0,1),\\ \frac{W}{7}, & \Psi(W)\in[1,\infty).\end{cases}$$

Since for all $W_1,W_2\in S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}$ with $\Psi(W_1),\Psi(W_2)\in[0,1)$, we have

$$\begin{aligned}\Psi(GW_1-GW_2)&=\Psi\left(\frac{W_1}{6}-\frac{W_2}{6}\right)\\&\le\frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Psi\left(\frac{5W_1}{6}\right)+\Psi\left(\frac{5W_2}{6}\right)\right)\\&=\frac{\sqrt{2}}{\sqrt[4]{125}}\big(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\big).\end{aligned}$$

For all $W_1,W_2\in S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}$ with $\Psi(W_1),\Psi(W_2)\in[1,\infty)$, we have

$$\begin{aligned}\Psi(GW_1-GW_2)&=\Psi\left(\frac{W_1}{7}-\frac{W_2}{7}\right)\\&\le\frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Psi\left(\frac{6W_1}{7}\right)+\Psi\left(\frac{6W_2}{7}\right)\right)\\&=\frac{\sqrt{2}}{\sqrt[4]{216}}\big(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\big).\end{aligned}$$

For all $W_1,W_2\in S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}$ with $\Psi(W_1)\in[0,1)$ and $\Psi(W_2)\in[1,\infty)$, we have

$$\begin{aligned}\Psi(GW_1-GW_2)&=\Psi\left(\frac{W_1}{6}-\frac{W_2}{7}\right)\\&\le\frac{\sqrt{2}}{\sqrt[4]{125}}\Psi\left(\frac{5W_1}{6}\right)+\frac{\sqrt{2}}{\sqrt[4]{216}}\Psi\left(\frac{6W_2}{7}\right)\\&\le\frac{\sqrt{2}}{\sqrt[4]{125}}\big(\Psi(GW_1-W_1)+\Psi(GW_2-W_2)\big).\end{aligned}$$

Therefore, the mapping W is a Kannan Ψ-contraction mapping and

$$G^p(W)=\begin{cases}\frac{W}{6^p}, & \Psi(W)\in[0,1),\\ \frac{W}{7^p}, & \Psi(W)\in[1,\infty).\end{cases}$$

It is clear that G is Ψ-sequentially continuous at the zero operator $\Theta\in S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}$ and $\{G^pW\}$ has a subsequence $\{G^{p_i}W\}$ converging to Θ. By Theorem 7.9, the zero operator $\Theta\in S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}$ is the only fixed point of G.

Let $\{W^{(n)}\} \subseteq S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}$ be such that $\lim_{n\to\infty}\Psi(W^{(n)}-W^{(0)})=0$, where $W^{(0)} \in S_{(\Xi((\sum_{z=0}^{l}\frac{z+1}{z+2})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty}))_{\upsilon}}$ with $\Psi(W^{(0)})=1$. Since the pre-quasi norm Ψ is continuous, we have

$$\lim_{n\to\infty}\Psi\left(GW^{(n)}-GW^{(0)}\right)=\lim_{n\to\infty}\Psi\left(\frac{W^{(0)}}{6}-\frac{W^{(0)}}{7}\right)=\Psi\left(\frac{W^{(0)}}{42}\right)>0.$$

Hence G is not Ψ-sequentially continuous at $W^{(0)}$. So, the mapping G is not continuous at $W^{(0)}$.

8 Application to the existence of solutions of nonlinear difference equations

Summable equations like (6) were studied by Salimi et al. [32], Agarwal et al. [33], and Hussain et al. [34]. In this section, we search for a solution to (6) in $(\Xi(\zeta,t))_{\upsilon}$, where setups (f1) and (f2) are satisfied and $\upsilon(f)=[\sum_{l=0}^{\infty}(\frac{|\sum_{z=0}^{l}f_z\Delta\zeta_z|}{\zeta_l})^{t_l}]^{\frac{1}{\hbar}}$ for all $f\in\Xi(\zeta,t)$. Consider the summable equations

$$f_z=p_z+\sum_{m=0}^{\infty}A(z,m)g(m,f_m), \tag{6}$$

and let $W:(\Xi(\zeta,t))_{\upsilon}\to(\Xi(\zeta,t))_{\upsilon}$ defined by

$$W(f_z)_{z\in\mathbb{N}}=\left(p_z+\sum_{m=0}^{\infty}A(z,m)g(m,f_m)\right)_{z\in\mathbb{N}}. \tag{7}$$

Theorem 8.1 *Summable equation (6) has a solution in $(\Xi(\zeta,t))_{\upsilon}$, if $A:\mathbb{N}^2\to\mathbb{R}$, $g:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$, $p:\mathbb{N}\to\mathbb{R}$, suppose that there is a number λ such that $\sup_l\lambda^{\frac{t_l}{\hbar}}\in[0,\frac{1}{2})$, and for all $l\in\mathbb{N}$, we have*

$$\begin{aligned}&\left|\sum_{z=0}^{l}\left(\sum_{m\in\mathbb{N}}A(z,m)\big[g(m,f_m)-g(m,r_m)\big]\right)\Delta\zeta_z\right|\\&\quad\le\lambda\left[\left|\sum_{z=0}^{l}\left(p_z-f_z+\sum_{m=0}^{\infty}A(z,m)g(m,f_m)\right)\Delta\zeta_z\right|\right.\\&\qquad\left.+\left|\sum_{z=0}^{l}\left(p_z-r_z+\sum_{m=0}^{\infty}A(z,m)g(m,r_m)\right)\Delta\zeta_z\right|\right].\end{aligned}$$

Proof Let the conditions be verified. Consider the mapping $W:(\Xi(\zeta,t))_{\upsilon}\to(\Xi(\zeta,t))_{\upsilon}$ defined by equation (7). We have

$$\begin{aligned}\upsilon(Wf-Wr)&=\left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(Wf_z-Wr_z)\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}}\\&=\left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(\sum_{m\in\mathbb{N}}A(z,m)[g(m,f_m)-g(m,r_m)])\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}}\\&\le\sup_l\lambda^{\frac{t_l}{\hbar}}\left[\sum_{l=0}^{\infty}\left(\frac{|\sum_{z=0}^{l}(p_z-f_z+\sum_{m=0}^{\infty}A(z,m)g(m,f_m))\Delta\zeta_z|}{\zeta_l}\right)^{t_l}\right]^{\frac{1}{\hbar}}\end{aligned}$$

$$+ \sup_l \lambda^{\frac{t_l}{\hbar}} \left[\sum_{l=0}^{\infty} \left(\frac{|\sum_{z=0}^{l} (p_z - r_z + \sum_{m=0}^{\infty} A(z,m) g(m, r_m)) \Delta \zeta_z|}{\zeta_l} \right)^{t_l} \right]^{\frac{1}{\hbar}}$$

$$= \sup_l \lambda^{\frac{t_l}{\hbar}} \big(\upsilon(Wf - f) + \upsilon(Wr - r) \big).$$

Then, from Theorem 7.3, we have a solution of equation (6) in $(\Xi(\zeta, t))_\upsilon$. □

Example 8.2 Given the sequence space $(\Xi((\sum_{z=0}^{l} \frac{z+2}{z+1})_{l=0}^{\infty}, (\frac{2l+3}{l+2})_{l=0}^{\infty}))_\phi$, where

$$\upsilon(f) = \sqrt{\sum_{l=0}^{\infty} \left(\frac{|\sum_{z=0}^{l} \frac{z+2}{z+1} f_z|}{\sum_{z=0}^{l} \frac{z+2}{z+1}} \right)^{\frac{2l+3}{l+2}}},$$

for all $f \in \Xi((\sum_{z=0}^{l} \frac{z+2}{z+1})_{l=0}^{\infty}, (\frac{2l+3}{l+2})_{l=0}^{\infty})$. Consider the nonlinear difference equations:

$$f_z = e^{-(3z+6)} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{f_{z-2}^p}{f_{z-1}^q + m^2 + 1} \tag{8}$$

with $p, q, f_{-2}, f_{-1} > 0$, and let $W : \Xi((\sum_{z=0}^{l} \frac{z+2}{z+1})_{l=0}^{\infty}, (\frac{2l+3}{l+2})_{l=0}^{\infty}) \to \Xi((\sum_{z=0}^{l} \frac{z+2}{z+1})_{l=0}^{\infty}, (\frac{2l+3}{l+2})_{l=0}^{\infty})$ defined by

$$W(f_z)_{z=0}^{\infty} = \left(e^{-(3z+6)} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{f_{z-2}^p}{f_{z-1}^q + m^2 + 1} \right)_{z=0}^{\infty}. \tag{9}$$

Clearly, there is a number λ such that $\sup_l \lambda^{\frac{2l+3}{2l+4}} \in [0, \frac{1}{2})$, and for all $l \in \mathbb{N}$, we have

$$\begin{aligned}
&\left| \sum_{z=0}^{l} \left(\sum_{m=0}^{\infty} (-1)^z \frac{f_{z-2}^p}{f_{z-1}^q + m^2 + 1} \big((-1)^m - (-1)^m \big) \right) \frac{z+2}{z+1} \right| \\
&\quad \le \lambda \left| \sum_{z=0}^{l} \left(e^{-(3z+6)} - f_z + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{f_{z-2}^p}{f_{z-1}^q + m^2 + 1} \right) \frac{z+2}{z+1} \right| \\
&\qquad + \lambda \left| \sum_{z=0}^{l} \left(e^{-(3z+6)} - r_z + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{r_{z-2}^p}{r_{z-1}^q + m^2 + 1} \right) \frac{z+2}{z+1} \right|.
\end{aligned}$$

By Theorem 8.1, the nonlinear difference equation (8) has a solution in $\Xi((\sum_{z=0}^{l} \frac{z+2}{z+1})_{l=0}^{\infty}, (\frac{2l+3}{l+2})_{l=0}^{\infty})$.

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