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Regularly ideal invariant convergence of double sequences

Nimet Pancaroğlu Akın[1*]

*Correspondence:
npancaroglu@aku.edu.tr
[1]Department of Mathematics and Science Education, Faculty of Education, Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey

Abstract

In this paper, we introduce the notions of regularly invariant convergence, regularly strongly invariant convergence, regularly p-strongly invariant convergence, regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergence, regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma}{}^*)$-convergence, regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence, regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma}{}^*)$-Cauchy double sequence and investigate the relationship among them.

MSC: 40A05; 40A35

Keywords: Regularly ideal convergence; Regularly ideal Cauchy sequence; Invariant convergence; Double sequence

1 Introduction and background

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of real sequences was extended to statistical convergence independently by Fast [18] and Schoenberg [42]. This concept was extended to the double sequences by Mursaleen and Edely [29].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [19] as a generalization of statistical convergence. Das et al. [5] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Tripathy and Tripathy [45] studied $\mathcal{I}$-convergent and regularly $\mathcal{I}$-convergent double sequences. Dündar and Altay [11] introduced $\mathcal{I}_2$-convergence and regularly $\mathcal{I}$-convergence of double sequences. Also, Dündar [7] introduced regularly $\mathcal{I}$-convergence and regularly $\mathcal{I}$-Cauchy double sequences of functions. Recently, Dündar and Akın [8] introduced the notions of $R(\mathcal{I}_{W_2}, \mathcal{I}_W)$-convergence, $R(\mathcal{I}_{W_2}^*, \mathcal{I}_W^*)$-convergence, $R(\mathcal{I}_{W_2}, \mathcal{I}_W)$-Cauchy, and $R(\mathcal{I}_{W_2}^*, \mathcal{I}_W^*)$-Cauchy double sequence of sets and investigated the relationship among them. A lot of development has been made in this area after the works of [6, 9, 12, 13, 15, 20, 22, 23, 26, 28, 31–34, 36, 43].

Several authors have studied invariant convergent sequences (see, [4, 24, 25, 30, 35, 38–41, 44]). Recently, the concepts of σ-uniform density of the set $A \subseteq \mathbb{N}$, $\mathcal{I}_\sigma$-convergence and $\mathcal{I}_\sigma^*$-convergence of sequences of real numbers were defined by Nuray et al. [35]. The concept of σ-convergence of double sequences was studied by Çakan et al. [4], and the concept of σ-uniform density of $A \subseteq \mathbb{N} \times \mathbb{N}$ was defined by Tortop and Dündar [44]. Dündar et al. [16] studied ideal invariant convergence of double sequences and some properties.

Now, we recall the basic definitions and concepts (see [1–5, 9–15, 17, 19–21, 27, 34, 36, 37, 43–46]).

A double sequence $x = (x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case, we write $P - \lim_{k,j\to\infty} x_{kj} = \lim_{k,j\to\infty} x_{kj} = L$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$, we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$, and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout the paper we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$, we have $B \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(\mathcal{I})$ corresponding to $\mathcal{I}$, given by

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : (\exists A \in \mathcal{I})\ (M = \mathbb{N}\backslash A)\}.$$

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP) if, for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})\ (i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if, for every countable family of mutually disjoint sets $\{E_1, E_2, \ldots\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{F_1, F_2, \ldots\}$ such that $E_j \Delta F_j \in \mathcal{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence $F_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1 $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

2 $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and

3 $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.$$

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \left\{ \sigma(n), \sigma^2(n), \ldots, \sigma^m(n) \right\} \right|$$

and

$$S_m = \max_n \left| A \cap \left\{ \sigma(n), \sigma^2(n), \ldots, \sigma^m(n) \right\} \right|.$$

If the limits $\underline{V}(A) = \lim_{m\to\infty} \frac{s_m}{m}$, $\overline{V}(A) = \lim_{m\to\infty} \frac{S_m}{m}$ exist, then they are called a lower and upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma$-convergent to the number L if, for every $\varepsilon > 0$, $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}_\sigma$, that is, $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim_k = L$.

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma^*$-convergent to the number L if there exists a set $M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that $\lim_{k\to\infty} x_{m_k} = L$. In this case, we write $\mathcal{I}_\sigma^* - \lim_k = L$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mn} = \min_{k,j} \left| A \cap \left\{ \left(\sigma(k), \sigma(j)\right), \left(\sigma^2(k), \sigma^2(j)\right), \ldots, \left(\sigma^m(k), \sigma^n(j)\right) \right\} \right|$$

and

$$S_{mn} = \max_{k,j} \left| A \cap \left\{ \left(\sigma(k), \sigma(j)\right), \left(\sigma^2(k), \sigma^2(j)\right), \ldots, \left(\sigma^m(k), \sigma^n(j)\right) \right\} \right|.$$

If the limits exist, $\underline{V_2}(A) = \lim_{m,n\to\infty} \frac{s_{mn}}{mn}$, $\overline{V_2}(A) = \lim_{m,n\to\infty} \frac{S_{mn}}{mn}$ exist, then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A) = \overline{V_2}(A)$, then $V_2(A) = \underline{V_2}(A) = \overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_2^\sigma$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$.

Throughout the paper we let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal.

A double sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_2^\sigma$-convergent to L if, for every $\varepsilon > 0$, $A(\varepsilon) = \{(k,j) : |x_{kj} - L| \geq \varepsilon\} \in \mathcal{I}_2^\sigma$, that is, $V_2(A(\varepsilon)) = 0$. In this case, we write $\mathcal{I}_2^\sigma - \lim x = L$ or $x_{kj} \to L(\mathcal{I}_2^\sigma)$.

A double sequence (x_{kj}) is $\mathcal{I}_2^*$-invariant convergent or $\mathcal{I}_2^{\sigma*}$-convergent to L if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ $(\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2^\sigma)$ such that, for $(k,j) \in M_2$, $\lim_{k,j\to\infty} x_{kj} = L$. In this case, we write $\mathcal{I}_2^{\sigma*} - \lim_{k,j\to\infty} x_{kj} = L$ or $x_{kj} \to L(\mathcal{I}_2^{\sigma*})$.

A double sequence (x_{kj}) is said to be $\mathcal{I}_2$-invariant Cauchy or $\mathcal{I}_2^\sigma$-Cauchy sequence if, for every $\varepsilon > 0$, there exist numbers $r = r(\varepsilon)$, $s = s(\varepsilon) \in \mathbb{N}$ such that $A(\varepsilon) = \{(k,j) : |x_{kj} - x_{rs}| \geq \varepsilon\} \in \mathcal{I}_2^\sigma$, that is, $V_2(A(\varepsilon)) = 0$.

A double sequence (x_{kj}) is $\mathcal{I}_2^*$-invariant Cauchy or $\mathcal{I}_2^{\sigma*}$-Cauchy sequence if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^\sigma$) such that, for every $(k,j),(r,s) \in M_2$, $\lim_{k,j,r,s\to\infty}|x_{kj}-x_{rs}| = 0$.

A double sequence $x = (x_{kj})$ is said to be regularly $(\mathcal{I}_2,\mathcal{I})$-convergent ($r(\mathcal{I}_2,\mathcal{I})$-convergent) if it is $\mathcal{I}_2$-convergent in Pringsheim's sense and for every $\varepsilon > 0$ the following hold:

$$\bigl\{k\in\mathbb{N}:|x_{kj}-L_j|\geq\varepsilon\bigr\}\in\mathcal{I}\quad\text{for some } L_j\in X \text{ and each } j\in\mathbb{N},$$
$$\bigl\{j\in\mathbb{N}:|x_{kj}-M_k|\geq\varepsilon\bigr\}\in\mathcal{I}\quad\text{for some } M_k\in X \text{ and each } k\in\mathbb{N}.$$

A double sequence $x = (x_{kj})$ is said to be $r(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2)$, $M_1 \in \mathcal{F}(\mathcal{I})$, and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N}\times\mathbb{N}\setminus M\in\mathcal{I}_2$, $\mathbb{N}\setminus M_1\in\mathcal{I}$ and $\mathbb{N}\setminus M_2\in\mathcal{I}$) such that the limits

$$\lim_{\substack{k,j\to\infty\\(k,j)\in M}} x_{kj},\qquad \lim_{\substack{k\to\infty\\k\in M_1}} x_{kj},\quad (j\in\mathbb{N})\quad\text{and}\quad \lim_{\substack{j\to\infty\\j\in M_2}} x_{kj}\quad (k\in\mathbb{N})$$

exist. Note that if $x = (x_{kj})$ is regularly convergent to L, then the limits $\lim_{k\to\infty}\lim_{j\to\infty}x_{kj}$ and $\lim_{j\to\infty}\lim_{k\to\infty}x_{kj}$ exist and are equal to L.

A double sequence $x = (x_{kj})$ is said to be regularly $(\mathcal{I}_2,\mathcal{I})$-Cauchy ($r(\mathcal{I}_2,\mathcal{I})$-Cauchy) if it is $\mathcal{I}_2$-Cauchy in Pringsheim's sense and for every $\varepsilon > 0$ there exist $m_j = m_j(\varepsilon)$, $n_k = n_k(\varepsilon)\in\mathbb{N}$ such that the following hold:

$$A_1(\varepsilon)=\bigl\{k\in\mathbb{N}:|x_{kj}-x_{m_j j}|\geq\varepsilon\bigr\}\in\mathcal{I}\quad(j\in\mathbb{N}),$$
$$A_2(\varepsilon)=\bigl\{j\in\mathbb{N}:|x_{kj}-x_{kn_k}|\geq\varepsilon\bigr\}\in\mathcal{I}\quad(k\in\mathbb{N}).$$

A double sequence $x = (x_{kj})$ is said to be regularly $(\mathcal{I}_2^*,\mathcal{I}^*)$-Cauchy ($r(\mathcal{I}_2^*,\mathcal{I}^*)$-Cauchy) if there exist the sets $M\in\mathcal{F}(\mathcal{I}_2)$, $M_1\in\mathcal{F}(\mathcal{I})$, and $M_2\in\mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N}\times\mathbb{N}\setminus M\in\mathcal{I}_2$, $\mathbb{N}\setminus M_1\in\mathcal{I}$, and $\mathbb{N}\setminus M_2\in\mathcal{I}$) and for every $\varepsilon>0$ there exist $N=N(\varepsilon)\in\mathbb{N}$, $s=s(\varepsilon)$, $t=t(\varepsilon)$, $m_j=m_j(\varepsilon)$, $n_k=n_k(\varepsilon)\in\mathbb{N}$ such that whenever $k,j,m_j,n_k>N$, we have

$$|x_{kj}-x_{st}|<\varepsilon\quad\bigl(\text{for } (k,j),(s,t)\in M, k,j,s,t>N\bigr),$$
$$|x_{kj},x_{m_j j}|<\varepsilon\quad(\text{for each } k\in M_1 \text{ and each } j\in\mathbb{N}),$$
$$|x_{kj},x_{kn_k}|<\varepsilon\quad(\text{for each } j\in M_2 \text{ and each } k\in\mathbb{N}).$$

Lemma 1 ([16]) *Suppose that $x = (x_{kj})$ is a bounded double sequence. If $x = (x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to L, then $x = (x_{kj})$ is invariant convergent to L.*

Lemma 2 ([16]) *Let $0 < p < \infty$.*

(i) *If $x_{kj}\to L([V_\sigma^2]_p)$, then $x_{kj}\to L(\mathcal{I}_2^\sigma)$.*

(ii) *If $(x_{kj})\in\ell_\infty^2$ and $x_{kj}\to L(\mathcal{I}_2^\sigma)$, then $x_{kj}\to L([V_\sigma^2]_p)$.*

(iii) *If $(x_{kj})\in\ell_\infty^2$, then $x_{kj}\to L(\mathcal{I}_2^\sigma)$ if and only if $x_{kj}\to L([V_\sigma^2]_p)$.*

Lemma 3 ([16]) *If a double sequence (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-convergent to L, then this sequence is $\mathcal{I}_2^\sigma$-convergent to L.*

Lemma 4 ([16]) *Let $\mathcal{I}_2^\sigma$ have the property $(AP2)$. If (x_{kj}) is $\mathcal{I}_2^\sigma$-convergent to L, then (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-convergent to L.*

Lemma 5 ([16]) *If a double sequence (x_{kj}) is $\mathcal{I}_2^\sigma$-convergent, then (x_{kj}) is an $\mathcal{I}_2^\sigma$-Cauchy double sequence.*

Lemma 6 ([16]) *If a double sequence (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-Cauchy, then this sequence is $\mathcal{I}_2^\sigma$-Cauchy.*

2 Main results

Now, we denote the notions of regularly invariant convergence, regularly strongly invariant convergence, regularly p-strongly invariant convergence, regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergence, regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergence, regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence, regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy double sequence and investigate the relationship among them.

Definition 2.1 A double sequence $x = (x_{kj})$ is said to be regularly invariant convergent ($r(\sigma, \sigma_2)$-convergent) if it is invariant convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} x_{\sigma^k(s),\sigma^j(t)} = L_j, \quad \text{uniformly in } s,$$

for some $L_j \in X$ and each $j \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} x_{\sigma^k(s),\sigma^j(t)} = M_k, \quad \text{uniformly in } t,$$

for some $M_k \in X$ and each $k \in \mathbb{N}$. Note that if $x = (x_{kj})$ is $r(\sigma, \sigma_2)$-convergent to L, the following limits hold:

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} x_{\sigma^k(s),\sigma^j(t)} = L, \quad \text{uniformly in } s, t,$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \sum_{j=0}^{n} \sum_{k=0}^{m} x_{\sigma^k(s),\sigma^j(t)} = L, \quad \text{uniformly in } s, t.$$

In this case, we write

$$r(\sigma, \sigma_2) - \lim_{m,n\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n} x_{\sigma^k(s),\sigma^j(t)} = L \quad \text{or} \quad x_{kj} \overset{r(\sigma,\sigma_2)}{\longrightarrow} L, \quad \text{uniformly in } s, t.$$

Definition 2.2 A double sequence $x = (x_{kj})$ is said to be regularly strongly invariant convergent ($r[\sigma, \sigma_2]$-convergent) if it is strongly invariant convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j| = 0, \quad \text{uniformly in } s,$$

for some $L_j \in X$ and each $j \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} |x_{\sigma^k(s),\sigma^j(t)} - M_k| = 0, \quad \text{uniformly in } t,$$

for some $M_k \in X$ and each $k \in \mathbb{N}$.

Note that if $x = (x_{kj})$ is $r[\sigma, \sigma_2]$-convergent to L, the following limits hold:

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} |x_{\sigma^k(s),\sigma^j(t)} - L| = 0, \quad \text{uniformly in } s, t,$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \sum_{j=0}^{n} \sum_{k=0}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L| = 0, \quad \text{uniformly in } s, t.$$

In this case, we write

$$r[\sigma, \sigma_2] - \lim_{m,n\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n} |x_{\sigma^k(s),\sigma^j(t)} - L| = 0 \quad \text{or} \quad x_{kj} \xrightarrow{r[\sigma,\sigma_2]} L, \quad \text{uniformly in } s, t.$$

Definition 2.3 Let $0 < p < \infty$. A double sequence $x = (x_{kj})$ is said to be regularly p-strongly invariant convergent ($r[\sigma, \sigma_2]_p$-convergent) if it is p-strongly invariant convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p = 0, \quad \text{uniformly in } s,$$

for some $L_j \in X$ and each $j \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} |x_{\sigma^k(s),\sigma^j(t)} - M_k|^p = 0, \quad \text{uniformly in } t,$$

for some $M_k \in X$ and each $k \in \mathbb{N}$.

Note that if $x = (x_{kj})$ is $r[\sigma, \sigma_2]_p$-convergent to L, the following limits hold:

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p = 0, \quad \text{uniformly in } s, t,$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \sum_{j=0}^{n} \sum_{k=0}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L|^p = 0, \quad \text{uniformly in } s, t.$$

In this case, we write

$$r[\sigma, \sigma_2]_p - \lim_{m,n\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n} |x_{\sigma^k(s),\sigma^j(t)} - L| = 0 \quad \text{or} \quad x_{kj} \xrightarrow{r[\sigma,\sigma_2]_p} L, \quad \text{uniformly in } s, t.$$

Definition 2.4 A double sequence $x = (x_{kj})$ is said to be regularly ideal invariant convergent ($r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent) if it is ideal invariant convergent in Pringsheim's sense and for every $\varepsilon > 0$ the following hold:

$$\left\{k \in \mathbb{N} : |x_{kj} - L_j| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for some $L_j \in X$ and each $j \in \mathbb{N}$, and

$$\left\{j \in \mathbb{N} : |x_{kj} - M_k| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for some $M_k \in X$ and each $k \in \mathbb{N}$.

Note that if $x = (x_{kj})$ is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent to L, then we write

$$r\left(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma\right) - \lim x = L \quad \text{or} \quad x_{kj} \xrightarrow{r\left(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma\right)} L.$$

Theorem 2.1 *Suppose that $x = (x_{kj})$ is a bounded double sequence. If $x = (x_{kj})$ is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent, then $x = (x_{kj})$ is $r(\sigma, \sigma_2)$-convergent.*

Proof Let $x = (x_{kj})$ be a bounded double sequence and $x = (x_{kj})$ be regularly ideal invariant convergent to L. Then $x = (x_{kj})$ is ideal invariant convergent in Pringsheim's sense and for every $\varepsilon > 0$ the following hold:

$$\left\{k \in \mathbb{N} : |x_{kj} - L_j| \geq \varepsilon\right\} \in I_\sigma$$

for some $L_j \in X$ and each $j \in \mathbb{N}$, and

$$\left\{j \in \mathbb{N} : |x_{kj} - M_k| \geq \varepsilon\right\} \in I_\sigma$$

for some $M_k \in X$ and each $k \in \mathbb{N}$. Since $x = (x_{kj})$ is ideal invariant convergent in Pringsheim's sense, then by Lemma 1 $x = (x_{kj})$ is invariant convergent to L.

Now, let $\varepsilon > 0$. We estimate

$$u(m,s) = \left|\frac{1}{m}\sum_{k=0}^{m} x_{\sigma^k(s),\sigma^j(t)} - L_j\right|, \quad \text{uniformly in } s,$$

for some $L_j \in X$ and each $j \in \mathbb{N}$. Then we have

$$u(m,s) \leq u^1(m,s) + u^2(m,s),$$

where

$$u^1(m,s) = \frac{1}{m} \sum_{\substack{k=0 \\ |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon}}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|$$

and

$$u^2(m,s) = \frac{1}{m} \sum_{\substack{k=0 \\ |x_{\sigma^k(s),\sigma^j(t)} - L_j| < \varepsilon}}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|, \quad \text{uniformly in } s,$$

for some $L_j \in X$ and each $j \in \mathbb{N}$. Therefore, we have $u^2(m,s) < \varepsilon$ for every $s = 1, 2, \ldots$. The boundedness of (x_{kj}) implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L_j| \leq K, \quad (k, s = 1, 2, \ldots),$$

then this implies that

$$\begin{aligned} u^1(m,s) &\leq \frac{K}{m} \left| \left\{ 1 \leq k \leq m : |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon \right\} \right| \\ &\leq K \frac{\max_s |\{1 \leq k \leq m : |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon\}|}{m} \\ &= K \frac{S_m}{m}, \end{aligned}$$

and so (x_{kj}) is σ-convergent to L_j.

Similarly, we can show that (x_{kj}) is σ-convergent to M_k. Hence, $x = (x_{kj})$ is $r(\sigma, \sigma_2)$-convergent. □

Theorem 2.2 *Let* $0 < p < \infty$.

(i) *If* (x_{kj}) *is* $r[\sigma, \sigma_2]_p$*-convergent, then* (x_{kj}) *is* $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$*-convergent.*

(ii) *If* $(x_{kj}) \in \ell_\infty^2$ *and* (x_{kj}) *is* $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$*-convergent, then* (x_{kj}) *is* $r[\sigma, \sigma_2]_p$*-convergent.*

(iii) *If* $(x_{kj}) \in \ell_\infty^2$*, then* (x_{kj}) *is* $r[\sigma, \sigma_2]_p$*-convergent if and only if* (x_{kj}) *is* $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$ *is convergent.*

Proof (*i*) Let $x = (x_{kj})$ be $r[\sigma, \sigma_2]_p$-convergent. Then it is p-strongly invariant convergent in Pringsheim's sense and the following limits hold:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p = 0, \quad \text{uniformly in } s,$$

for some $L_j \in X$ and each $j \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} |x_{\sigma^k(s),\sigma^j(t)} - M_k|^p = 0, \quad \text{uniformly in } t,$$

for some $M_k \in X$ and each $k \in \mathbb{N}$. Since $x = (x_{kj})$ is p-strongly invariant convergent in Pringsheim's sense, then by Lemma 2 $x = (x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent.

Also, for every $\varepsilon > 0$, some $L_j \in X$, and each $j \in \mathbb{N}$, we can write

$$\sum_{k=1}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p \geq \sum_{\substack{k=1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon}}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p$$

$$\geq \varepsilon^p \left|\left\{k \leq m : |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon\right\}\right|$$
$$\geq \varepsilon^p \max_s \left|\left\{k \leq m : |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon\right\}\right|$$

and

$$\begin{aligned}\frac{1}{m}\sum_{k=1}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p &\geq \varepsilon^p \frac{\max_s |\{k \leq m : |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon\}|}{m} \\ &= \varepsilon^p \frac{S_m}{m}\end{aligned}$$

for every $s = 1, 2, \ldots$. This implies $\lim_{m\to\infty} \frac{S_m}{m} = 0$, and so (x_{kj}) is $\mathcal{I}_\sigma$-convergent to L_j.

Similarly, we can show that (x_{kj}) is $\mathcal{I}_\sigma$-convergent to M_k. Hence, $x = (x_{kj})$ is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent.

(ii) Let $(x_{kj}) \in \ell_\infty^2$ and (x_{kj}) be $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent. Then $x = (x_{kj})$ is ideal invariant convergent in Pringsheim's sense and for every $\varepsilon > 0$ the following hold:

$$\left\{k \in \mathbb{N} : |x_{kj} - L_j| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for some $L_j \in X$ and each $j \in \mathbb{N}$, and

$$\left\{j \in \mathbb{N} : |x_{kj} - M_k| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for some $M_k \in X$ and each $k \in \mathbb{N}$. Since $x = (x_{kj})$ is ideal invariant convergent in Pringsheim's sense, then by Lemma 2, $x = (x_{kj})$ is p-strongly σ_2-convergent. Let $0 < p < \infty$ and $\varepsilon > 0$. Since (x_{kj}) is bounded, (x_{kj}) implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L_j| \leq K \quad (j \in \mathbb{N})$$

for all $k, s \in \mathbb{N}$. Then, for every $s = 1, 2, \ldots$, we have

$$\begin{aligned}\frac{1}{m}\sum_{k=1}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p &= \frac{1}{m} \sum_{\substack{k=1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon}}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p \\ &\quad + \frac{1}{m} \sum_{\substack{k=1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L_j| < \varepsilon}}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p \\ &\leq K \frac{\max_s |\{k \leq m : |x_{\sigma^k(s),\sigma^j(t)} - L_j| \geq \varepsilon\}|}{m} + \varepsilon^p \\ &\leq K \frac{S_m}{m} + \varepsilon^p.\end{aligned}$$

Hence, we obtain

$$\lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} |x_{\sigma^k(s),\sigma^j(t)} - L_j|^p = 0$$

uniformly in s, and so $x = (x_{kj})$ is p-strongly σ-convergent to L_j.

Similarly, we show that (x_{kj}) is p-strongly σ-convergent to M_k. Hence, $x = (x_{kj})$ is $r[\sigma, \sigma_2]_p$-convergent.

(*iii*) This is an immediate consequence of (i) and (ii). □

Definition 2.5 A double sequence $x = (x_{kj})$ is said to be regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergent ($r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergent) if and only if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2^\sigma)$, $M_1 \in \mathcal{F}(\mathcal{I}_\sigma)$ and $M_2 \in \mathcal{F}(\mathcal{I}_\sigma)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2^\sigma$, $\mathbb{N} \setminus M_1 \in \mathcal{I}_\sigma$, and $\mathbb{N} \setminus M_2 \in \mathcal{I}_\sigma$) such that the following limits hold:

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M}} x_{kj}, \qquad \lim_{\substack{k\to\infty \\ k\in M_1}} x_{kj} \quad (j \in \mathbb{N}), \quad \text{and} \quad \lim_{\substack{j\to\infty \\ j\in M_2}} x_{kj} \quad (k \in \mathbb{N}).$$

Note that if $x = (x_{kj})$ is $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergent to L, then we write

$$r\left(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*}\right) - \lim x = L \quad \text{or} \quad x_{kj} \overset{r\left(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*}\right)}{\longrightarrow} L.$$

Theorem 2.3 *If a double sequence $x = (x_{kj})$ is $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergent, then (x_{kj}) is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent.*

Proof Let (x_{kj}) be $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergent. Then (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-convergent, and so by Lemma 3 (x_{kj}) is $\mathcal{I}_2^\sigma$ convergent. Also, there exist the sets $M_1, M_2 \in \mathcal{F}(\mathcal{I}_\sigma)$ such that

$$(\forall \varepsilon > 0)\ (\exists k_0 \in \mathbb{N})\ (\forall k \geq k_0)\ (k \in M_1) \quad |x_{kj} - L_j| < \varepsilon$$

for some L_j and each $j \in \mathbb{N}$, and

$$(\forall \varepsilon > 0)\ (\exists j_0 \in \mathbb{N})\ (\forall j \geq j_0)\ (j \in M_2) \quad |x_{kj} - M_k| < \varepsilon$$

for some M_k and each $k \in \mathbb{N}$. Hence, we have

$$A(\varepsilon) = \left\{k \in \mathbb{N} : |x_{kj} - L_j| \geq \varepsilon\right\} \subset H_1 \cup \left\{1, 2, \ldots, (k_0 - 1)\right\}, \quad (j \in \mathbb{N}),$$
$$B(\varepsilon) = \left\{j \in \mathbb{N} : |x_{kj} - M_k| \geq \varepsilon\right\} \subset H_2 \cup \left\{1, 2, \ldots, (j_0 - 1)\right\}, \quad (k \in \mathbb{N})$$

for $H_1, H_2 \in \mathcal{I}_\sigma$. Since $\mathcal{I}_\sigma$ is an admissible ideal, we get

$$H_1 \cup \left\{1, 2, \ldots, (k_0 - 1)\right\} \in \mathcal{I}_\sigma \quad \text{and} \quad H_2 \cup \left\{1, 2, \ldots, (j_0 - 1)\right\} \in \mathcal{I}_\sigma,$$

and therefore $A(\varepsilon) \in \mathcal{I}_\sigma$ and $B(\varepsilon) \in \mathcal{I}_\sigma$. This shows that the double sequence (x_{kj}) is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent. □

Theorem 2.4 *Let $\mathcal{I}_\sigma$ have the property (AP) and $\mathcal{I}_2^\sigma$ have the property $(AP2)$. If a double sequence (x_{kj}) is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent, then (x_{kj}) is $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergent.*

Proof Let a double sequence (x_{kj}) be $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent. Then (x_{kj}) is $\mathcal{I}_2^\sigma$-convergent, and so by Lemma 4 (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-convergent. Also, for each $\varepsilon > 0$, we have

$$A(\varepsilon) = \left\{k \in \mathbb{N} : |x_{kj} - L_j| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for some L_j and each $j \in \mathbb{N}$, and

$$B(\varepsilon) = \left\{j \in \mathbb{N} : |x_{kj} - M_k| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for some M_k and each $k \in \mathbb{N}$.

Now put

$$A_1 = \left\{k \in \mathbb{N} : |x_{kj} - L_j| \geq 1\right\},$$
$$A_t = \left\{k \in \mathbb{N} : \frac{1}{t} \leq |x_{kj} - L_j| < \frac{1}{t-1}\right\}$$

for $t \geq 2$, some L_j, and each $j \in \mathbb{N}$. It is clear that $A_m \cap A_n = \emptyset$ for $m \neq n$ and $A_m \in \mathcal{I}_\sigma$ for each $m \in \mathbb{N}$. By the property (AP) there is a countable family of sets $\{B_1, B_2, \ldots\}$ in $\mathcal{I}_\sigma$ such that $A_n \triangle B_n$ is a finite set for each $n \in \mathbb{N}$ and $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{I}_\sigma$.

We prove that

$$\lim_{\substack{k\to\infty \\ k\in M}} x_{kj} = L_j, \quad \text{some } L_j \text{ and each } j \in \mathbb{N},$$

for $M = \mathbb{N} \backslash B \in \mathcal{F}(\mathcal{I}_\sigma)$. Let $\delta > 0$ be given. Choose $t \in \mathbb{N}$ such that $1/t < \delta$. Then we have

$$\left\{k \in \mathbb{N} : |x_{kj} - L_j| \geq \delta\right\} \subset \bigcup_{n=1}^{t} x_n \quad \text{for some } L_j \text{ and each } j \in \mathbb{N}.$$

Since $A_n \triangle B_n$ is a finite set for $n \in \{1, 2, \ldots, t\}$, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{n=1}^{t} B_n\right) \cap \{k : k \geq k_0\} = \left(\bigcup_{n=1}^{t} A_n\right) \cap \{k : k \geq k_0\}.$$

If $k \geq k_0$ and $k \notin B$, then

$$k \notin \bigcup_{n=1}^{t} B_n \quad \text{and so} \quad k \notin \bigcup_{n=1}^{t} A_n.$$

Thus, we have $|x_{kj} - L_j| < \frac{1}{t} < \delta$ for some L_j and each $j \in \mathbb{N}$. This implies that

$$\lim_{k\to\infty} x_{kj} = L_j$$

for $k \in M$. Hence, we have

$$\mathcal{I}_\sigma^* - \lim_{k\to\infty} x_{kj} = L_j$$

for some L_j and each $j \in \mathbb{N}$.

Similarly, for the set

$$B(\varepsilon) = \left\{j \in \mathbb{N} : |x_{kj} - M_k| \geq \varepsilon\right\} \in \mathcal{I}_\sigma,$$

we have

$$\mathcal{I}_\sigma^* - \lim_{j\to\infty} x_{kj} = M_k$$

for some M_k and each $k \in \mathbb{N}$. Hence, a double sequence (x_{kj}) is $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-convergent. □

Definition 2.6 A double sequence (x_{kj}) is said to be regularly $(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence ($r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence) if it is $\mathcal{I}_2^\sigma$-Cauchy in Pringsheim's sense and for every $\varepsilon > 0$ there exist numbers $m_j = m_j(\varepsilon)$, $n_k = n_k(\varepsilon) \in \mathbb{N}$ such that

$$A_1(\varepsilon) = \{k \in \mathbb{N} : |x_{kj} - x_{m_j j}| \ge \varepsilon\} \in \mathcal{I}_2^\sigma \quad (j \in \mathbb{N}),$$
$$A_2(\varepsilon) = \{j \in \mathbb{N} : |x_{kj} - x_{kn_k}| \ge \varepsilon\} \in \mathcal{I}_2^\sigma \quad (k \in \mathbb{N}).$$

Theorem 2.5 *If a double sequence (x_{kj}) is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent, then (x_{kj}) is an $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence.*

Proof Let (x_{kj}) be $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-convergent. Then (x_{kj}) is $\mathcal{I}_2^\sigma$-convergent, and by Lemma 5, it is $\mathcal{I}_2^\sigma$-Cauchy double sequence. Also, for every $\varepsilon > 0$, we have

$$A_1\left(\frac{\varepsilon}{2}\right) = \left\{k \in \mathbb{N} : |x_{kj} - L_j| \ge \frac{\varepsilon}{2}\right\} \in \mathcal{I}_\sigma$$

for some L_j and each $j \in \mathbb{N}$, and

$$A_2\left(\frac{\varepsilon}{2}\right) = \left\{j \in \mathbb{N} : |x_{kj} - M_k| \ge \frac{\varepsilon}{2}\right\} \in \mathcal{I}_\sigma$$

for some M_k and each $k \in \mathbb{N}$. Since $\mathcal{I}_\sigma$ is an admissible ideal, the sets

$$A_1^c\left(\frac{\varepsilon}{2}\right) = \left\{k \in \mathbb{N} : |x_{kj} - L_j| < \frac{\varepsilon}{2}\right\}$$

for some L_j and each $j \in \mathbb{N}$, and

$$A_2^c\left(\frac{\varepsilon}{2}\right) = \left\{j \in \mathbb{N} : |x_{kj} - M_k| < \frac{\varepsilon}{2}\right\}$$

for some M_k and each $k \in \mathbb{N}$ are nonempty and belong to $\mathcal{F}(\mathcal{I}_\sigma)$. For $m_j \in A_1^c(\frac{\varepsilon}{2})$, $(j \in \mathbb{N}$ and $m_j > 0)$ we have

$$|x_{m_j j} - L_j| < \frac{\varepsilon}{2}$$

for some L_j and each $j \in \mathbb{N}$. Now, for each $\varepsilon > 0$, we define the set

$$B_1(\varepsilon) = \{k \in \mathbb{N} : |x_{kj} - x_{m_j j}| \ge \varepsilon\}, \quad (j \in \mathbb{N}),$$

where $m_j = m_j(\varepsilon) \in \mathbb{N}$. We must prove $B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2})$. Let $k \in B_1(\varepsilon)$. Then, for $m_j \in A_1^c(\frac{\varepsilon}{2})$, ($j \in \mathbb{N}$ and $m_j > 0$) we have

$$\begin{aligned} \varepsilon \leq |x_{kj} - x_{m_j j}| &\leq |x_{kj} - L_j| + |x_{m_j j} - L_j| \\ &< |x_{kj} - L_j| + \frac{\varepsilon}{2} \end{aligned}$$

for some L_j and each $j \in \mathbb{N}$. This shows that $\frac{\varepsilon}{2} < |x_{kj} - L_j|$, and so $k \in A_1(\frac{\varepsilon}{2})$. Hence, we have $B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2})$.

Similarly, for each $\varepsilon > 0$ and for $n_k \in A_2^c(\frac{\varepsilon}{2})$ ($k \in \mathbb{N}$ and $n_k > 0$), we have

$$|x_{kn_k} - M_k| < \frac{\varepsilon}{2}$$

for some M_k and each $k \in \mathbb{N}$. Therefore, it can be seen that

$$B_2(\varepsilon) \subset A_2\left(\frac{\varepsilon}{2}\right),$$

where

$$B_2(\varepsilon) = \left\{j \in \mathbb{N} : |x_{kj} - x_{kn_k}| \geq \varepsilon\right\},$$

where $n_k = n_k(\varepsilon) \in \mathbb{N}$ and each $k \in \mathbb{N}$.

Hence, we have $B_1(\varepsilon) \in \mathcal{I}_\sigma$ and $B_2(\varepsilon) \in \mathcal{I}_\sigma$. This shows that $\{x_{kj}\}$ is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence. □

Definition 2.7 A double sequence (x_{kj}) is regularly $(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy double sequence ($r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy double sequence) if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2^\sigma)$, $M_1 \in \mathcal{F}(\mathcal{I}^\sigma)$ and $M_2 \in \mathcal{F}(\mathcal{I}^\sigma)$ (that is, $\mathbb{N} \times \mathbb{N} \backslash M = H \in \mathcal{I}_2^\sigma$, $\mathbb{N} \backslash M_1 \in \mathcal{I}_\sigma$, and $\mathbb{N} \backslash M_2 \in \mathcal{I}_\sigma$) and for every $\varepsilon > 0$, there exist $N = N(\varepsilon)$, $s = s(\varepsilon)$, $t = t(\varepsilon)$, $m_j = m_j(\varepsilon)$, $n_k = n_k(\varepsilon) \in \mathbb{N}$ such that whenever $k, j, s, t, m_j, n_k \geq N$, we have

$$\begin{aligned} &|x_{kj} - x_{st}| < \varepsilon \quad \bigl(\text{for } (k,j),(s,t) \in M, k,j,s,t \geq N\bigr), \\ &|x_{kj} - x_{m_j j}| < \varepsilon \quad (\text{for each } k \in M_1 \text{ and each } j \in \mathbb{N}), \\ &|x_{kj} - x_{kn_k}| < \varepsilon \quad (\text{for each } j \in M_2 \text{ and each } k \in \mathbb{N}). \end{aligned}$$

Theorem 2.6 *If a double sequence (x_{kj}) is $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy double sequence, then (x_{kj}) is $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence.*

Proof Since a double sequence (x_{kj}) is $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy, then (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-Cauchy implies $\mathcal{I}_2^\sigma$-Cauchy by Lemma 6. Also, since (x_{kj}) is $r(\mathcal{I}_\sigma^*, \mathcal{I}_2^{\sigma*})$-Cauchy, there exist the sets $M_1 \in \mathcal{F}(\mathcal{I}^\sigma)$ and $M_2 \in \mathcal{F}(\mathcal{I}^\sigma)$ (that is, $\mathbb{N} \backslash M_1 \in \mathcal{I}_\sigma$ and $\mathbb{N} \backslash M_2 \in \mathcal{I}_\sigma$), and for every $\varepsilon > 0$, there exist $N = N(\varepsilon)$, $m_j = m_j(\varepsilon)$, $n_k = n_k(\varepsilon) \in \mathbb{N}$ such that we have

$$\begin{aligned} &|x_{kj} - x_{m_j j}| < \varepsilon \quad (\text{for each } k \in M_1 \text{ and each } j \in \mathbb{N}), \\ &|x_{kj} - x_{kn_k}| < \varepsilon \quad (\text{for each } j \in M_1 \text{ and each } k \in \mathbb{N}), \end{aligned}$$

whenever $k, j, m_j, n_k \geq N$. Therefore, $H_1 = \mathbb{N}\backslash M_1 \in \mathcal{I}_\sigma$ and $H_2 = \mathbb{N}\backslash M_2 \in \mathcal{I}_\sigma$ we have

$$A_1(\varepsilon) = \{k \in \mathbb{N} : |x_{kj} - x_{m_j j}| \geq \varepsilon\} \subset H_1 \cup \{1, 2, \dots, (N-1)\}, \quad (j \in \mathbb{N})$$

for each $k \in M_1$ and

$$A_2(\varepsilon) = \{j \in \mathbb{N} : |x_{kj} - x_{kn_k}| \geq \varepsilon\} \subset H_2 \cup \{1, 2, \dots, (N-1)\}, \quad (k \in \mathbb{N})$$

for each $j \in M_2$. Since $\mathcal{I}_\sigma$ is an admissible ideal, $H_1 \cup \{1, 2, 3, \dots, (N-1)\} \in \mathcal{I}_\sigma$ and $H_2 \cup \{1, 2, 3, \dots, (N-1)\} \in \mathcal{I}_\sigma$. Hence, we have $A_1(\varepsilon) \in \mathcal{I}_\sigma$ and $A_2(\varepsilon) \in \mathcal{I}_\sigma$, and so (x_{kj}) is an $r(\mathcal{I}_\sigma, \mathcal{I}_2^\sigma)$-Cauchy double sequence. □

3 Conclusions

We investigated the concepts of regularly invariant convergence types and regularly ideal invariant convergence and Cauchy sequence types. These concepts can also be studied for the lacunary sequence in the future.

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