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# The zeros of monotone operators for the variational inclusion problem in Hilbert spaces

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## Abstract

In this paper, we introduce a regularization method for solving the variational inclusion problem of the sum of two monotone operators in real Hilbert spaces. We suggest and analyze this method under some mild appropriate conditions imposed on the parameters, which allow us to obtain a short proof of another strong convergence theorem for this problem. We also apply our main result to the fixed point problem of the nonexpansive variational inequality problem, the common fixed point problem of nonexpansive strict pseudocontractions, the convex minimization problem, and the split feasibility problem. Finally, we provide numerical experiments to illustrate the convergence behavior and to show the effectiveness of the sequences constructed by the inertial technique.

**MSC:** 47H05; 47H09; 47H20

**Keywords:** Common fixed point problem; Convex minimization problem; Split feasibility problem; Variational inclusion problem; Maximal monotone operator; Regularization method

## 1 Introduction

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . The variational inclusion problem is to find  $x^* \in H$  such that

$$0 \in Ax^* + Bx^*, \tag{1.1}$$

where  $A : H \rightarrow H$  is an operator, and  $B : D(B) \subset H \rightarrow 2^H$  is a set-valued operator.

If  $A = \nabla F$  and  $B = \partial G$ , where  $\nabla F$  is the gradient of  $F$ , and  $\partial G$  is the subdifferential of  $G$  defined by

$$\partial G(x) = \{z \in H : \langle y - x, z \rangle + G(x) \leq G(y), \forall y \in H\},$$

then problem (1.1) is reduced to the following convex minimization problem:

$$F(x^*) + G(x^*) = \min_{x \in H} F(x) + G(x) \Leftrightarrow 0 \in \nabla F(x^*) + \partial G(x^*).$$

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If  $A = 0$  and  $B = \partial G$ , then problem (1.1) is reduced to a proximal minimization problem, and if  $A = \nabla F$  and  $B = 0$ , then problem (1.1) is reduced to a constrained convex minimization problem and also to a split feasibility problem. Some typical problems arising in various branches of sciences, applied sciences, economics, and engineering, such as machine learning, image restoration, and signal recovery, can be viewed as problems of the form (1.1).

To solve the variational inclusion problem (1.1) via fixed point theory, for  $r > 0$ , we define the mapping  $T_r : H \rightarrow D(B)$  as

$$T_r = (I + rB)^{-1}(I - rA).$$

For  $x \in H$ , we see that

$$\begin{aligned} T_r x = x &\Leftrightarrow x = (I + rB)^{-1}(x - rAx) \\ &\Leftrightarrow x - rAx \in x + rBx \\ &\Leftrightarrow 0 \in Ax + Bx, \end{aligned}$$

which shows that the fixed point set of  $T_r$  coincides with the solution set of  $(A + B)^{-1}(0)$ . This suggests the following iteration process:  $x_0 \in C$ , and

$$x_{n+1} = (I + r_n B)^{-1}(x_n - r_n A x_n) = T_{r_n} x_n, \quad n = 0, 1, 2, \dots,$$

where  $\{r_n\} \subset (0, \infty)$  and  $D(B) \subset C$ . This method is called the forward–backward splitting algorithm [1, 2]. In the literature, many methods have been suggested to solve the variational inclusion problem (1.1) for maximal monotone operators (see also, e.g., [3–11]).

Very recently, Chulamjiak et al. [12, 13] proved the following theorems in real Hilbert spaces.

**Theorem C1** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $A : C \rightarrow H$  be an  $\alpha$ -inverse strongly monotone mapping, and let  $B$  be a maximal monotone operator on  $H$  such that  $D(B) \subset C$  and  $(A + B)^{-1}(0)$  is nonempty. Let  $f : C \rightarrow C$  be a  $k$ -contraction, and let  $J_{r_n} = (I + r_n B)^{-1}$ . Let  $\{z_n\}$  be a sequence in  $C$  of the following process:  $z_0 \in C$ , and*

$$\begin{cases} w_n = \alpha_n f(z_n) + (1 - \alpha_n)z_n, \\ z_{n+1} = J_{r_n}(w_n - r_n A w_n + e_n), \quad n = 0, 1, 2, \dots, \end{cases}$$

where  $\{\alpha_n\} \subset (0, 1)$ ,  $\{e_n\} \subset H$ , and  $\{r_n\} \subset (0, 2\alpha)$ . Suppose that the control sequences satisfy the following restrictions:

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (C2)  $0 < a \leq r_n \leq b < 2\alpha$  for some  $a, b > 0$ ,
- (C3)  $\sum_{n=0}^{\infty} \|e_n\| < \infty$ , or  $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ .

Then the sequence  $\{z_n\}$  converges strongly to a point  $\bar{x} \in (A + B)^{-1}(0)$ , where  $\bar{x} = P_{(A+B)^{-1}(0)} f(\bar{x})$ .

**Theorem C2** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping of  $H$  into itself, and let  $B$  be a maximal monotone operator on  $H$  such that the domain of  $B$  is included in  $C$ . Let  $J_\lambda = (I + \lambda B)^{-1}$  be the resolvent of  $B$  for  $\lambda > 0$ , let  $S$  be a nonexpansive mapping of  $C$  into itself such that  $\text{Fix}(S) \cap (A + B)^{-1}(0) \neq \emptyset$ , and let  $f : C \rightarrow C$  be a contraction. Let  $x_0, x_1 \in C$ , and let  $\{x_n\} \subset C$  be the sequence generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\alpha_n f(x_n) + (1 - \alpha_n)J_{\lambda_n}(y_n - \lambda_n A y_n)), \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\beta_n\} \subset (0, 1)$ ,  $\{\lambda_n\} \subset (0, 2\alpha)$ , and  $\{\theta_n\} \subset [0, \theta]$  such that  $\theta \in [0, 1)$  satisfy

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^\infty \alpha_n = \infty$ ,
- (C2)  $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$ ,
- (C3)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2\alpha$ ,
- (C4)  $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $\bar{x} \in \text{Fix}(S) \cap (A + B)^{-1}(0)$ , where  $\bar{x} = P_{\text{Fix}(S) \cap (A+B)^{-1}(0)} f(\bar{x})$ .

In this paper, we modify the algorithms in Theorems C1 and C2 under the same assumptions to solve the variational inclusion problem (1.1) as follows: let  $x_0, x_1 \in C$ , and let  $\{x_n\} \subset C$  be the sequence generated by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = S(\alpha_n f(x_n) + (1 - \alpha_n)J_{\lambda_n}(y_n - \lambda_n A y_n + e_n)), \end{cases}$$

for all  $n \in \mathbb{N}$ . We suggest and analyze this method under some mild appropriate conditions imposed on the parameters, which allow us to obtain a short proof of another strong convergence theorem for those problem.

We also apply our main result to the fixed point problem of nonexpansive variational inequality problem, the common fixed point problem of nonexpansive strict pseudocontractions, the convex minimization problem, and the split feasibility problem. Finally, we provide numerical experiments to illustrate the convergence behavior and to show the effectiveness of the sequences constructed by the inertial technique.

## 2 Preliminaries

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . We use the notation:  $\rightarrow$  denotes the strong convergence,  $\rightharpoonup$  denotes the weak convergence,

$$\omega_w(x_n) = \{x : \exists \{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup x\}$$

denotes the weak limit set of  $\{x_n\}$ , and  $\text{Fix}(T) = \{x : x = Tx\}$  is the fixed point set of the mapping  $T$ .

Recall that the metric projection  $P_C : H \rightarrow C$  is defined as follows: for each  $x \in H$ ,  $P_C x$  is the unique point in  $C$  satisfying

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

The operator  $T : H \rightarrow H$  is called:

(i) monotone if

$$\langle x - y, Tx - Ty \rangle \geq 0, \quad \forall x, y \in H,$$

(ii)  $L$ -Lipschitzian with  $L > 0$  if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H,$$

(iii)  $k$ -contraction if it is  $k$ -Lipschitzian with  $k \in (0, 1)$ ,

(iv) nonexpansive if it is 1-Lipschitzian,

(v) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H,$$

(vi)  $\alpha$ -strongly monotone if

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H,$$

(vii)  $\alpha$ -inverse strongly monotone if

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Let  $B$  be a mapping of  $H$  into  $2^H$ . The domain and the range of  $B$  are denoted by  $D(B) = \{x \in H : Bx \neq \emptyset\}$  and  $R(B) = \cup\{Bx : x \in D(B)\}$ , respectively. The inverse of  $B$ , denoted by  $B^{-1}$ , is defined by  $x \in B^{-1}y$  if and only if  $y \in Bx$ . A multivalued mapping  $B$  is said to be a monotone operator on  $H$  if  $\langle x - y, u - v \rangle \geq 0$  for all  $x, y \in D(B), u \in Bx$ , and  $v \in By$ . A monotone operator  $B$  on  $H$  is said to be maximal if its graph is not strictly contained in the graph of any other monotone operator on  $H$ . For a maximal monotone operator  $B$  on  $H$  and  $r > 0$ , we define the single-valued resolvent operator  $J_r : H \rightarrow D(B)$  by  $J_r = (I + rB)^{-1}$ . It is well known that  $J_r$  is firmly nonexpansive and  $\text{Fix}(J_r) = B^{-1}(0)$ .

We collect together some known lemmas, which are the main tools in proving our result.

**Lemma 2.1** *Let  $H$  be a real Hilbert space. Then, for all  $x, y \in H$ ,*

- (i)  $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$ ,
- (ii)  $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$ .

**Lemma 2.2** ([14]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Then*

- (i)  $z = P_Cx \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \forall x \in H, y \in C$ ,
- (ii)  $z = P_Cx \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \forall x \in H, y \in C$ ,
- (iii)  $\|P_Cx - P_Cy\|^2 \leq \langle x - y, P_Cx - P_Cy \rangle, \forall x, y \in H$ .

**Lemma 2.3** ([15]) *Let  $H$  be a real Hilbert space. For any  $x, y \in H$  and  $\lambda \in \mathbb{R}$ , we have*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

**Lemma 2.4** ([16]) *Let  $H$  and  $K$  be two real Hilbert spaces, and let  $T : K \rightarrow K$  be a firmly nonexpansive mapping such that  $\|(I - T)x\|$  is a convex function from  $K$  to  $\mathbb{R} = [-\infty, +\infty]$ . Let  $A : H \rightarrow K$  be a bounded linear operator and  $f(x) = \frac{1}{2}\|(I - T)Ax\|^2$  for all  $x \in H$ . Then*

- (i)  $f$  is convex and differential,
- (ii)  $\nabla f(x) = A^*(I - T)Ax$  for all  $x \in H$  such that  $A^*$  denotes the adjoint of  $A$ ,
- (iii)  $f$  is weakly lower semicontinuous on  $H$ , and
- (iv)  $\nabla f$  is  $\|A\|^2$ -Lipschitzian.

**Lemma 2.5** ([16]) *Let  $H$  be a real Hilbert space, and let  $T : H \rightarrow H$  be an operator. The following statements are equivalent:*

- (i)  $T$  is firmly nonexpansive,
- (ii)  $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \forall x, y \in H$ ,
- (iii)  $I - T$  is firmly nonexpansive.

**Lemma 2.6** ([17]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let the mapping  $A : C \rightarrow H$  be an  $\alpha$ -inverse strongly monotone, and let  $r > 0$  be a constant. Then we have*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 - r(2\alpha - r)\|Ax - Ay\|^2$$

for all  $x, y \in C$ . In particular, if  $0 < r \leq 2\alpha$ , then  $I - rA$  is nonexpansive.

**Lemma 2.7** ([18] (Demiclosedness principle)) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ , and let  $S : C \rightarrow C$  be a nonexpansive mapping with  $\text{Fix}(S) \neq \emptyset$ . If the sequence  $\{x_n\} \subset C$  converges weakly to  $x$  and the sequence  $\{(I - S)x_n\}$  converges strongly to  $y$ , then  $(I - S)x = y$ ; in particular, if  $y = 0$ , then  $x \in \text{Fix}(S)$ .*

**Lemma 2.8** ([19, 20]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $\{T_n\}$  and  $\varphi$  be two classes of nonexpansive mappings of  $C$  into itself such that*

$$\emptyset \neq \text{Fix}(\varphi) = \bigcap_{n=0}^{\infty} \text{Fix}(T_n).$$

Then, for any bounded sequence  $\{z_n\} \subset C$ , we have:

- (i) if  $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$ , then  $\lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0$  for all  $T \in \varphi$ , which is called the NST-condition (I),
- (ii) if  $\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0$ , then  $\lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0$  for all  $m \in \mathbb{N} \cup \{0\}$ , which is called the NST-condition (II).

**Lemma 2.9** ([21]) *Let  $\{a_n\}$  and  $\{c_n\}$  be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n = 0, 1, 2, \dots,$$

where  $\{\delta_n\}$  is a sequence in  $(0, 1)$ , and  $\{b_n\}$  is a real sequence. Assume that  $\sum_{n=0}^{\infty} c_n < \infty$ . Then we have:

- (i) if  $b_n \leq \delta_n M$  for some  $M \geq 0$ , then  $\{a_n\}$  is a bounded sequence,
- (ii) if  $\sum_{n=0}^{\infty} \delta_n = \infty$  and  $\limsup_{n \rightarrow \infty} b_n / \delta_n \leq 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 2.10** ([22]) *Let  $\{s_n\}$  be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\delta_n, \quad n = 0, 1, 2, \dots,$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \quad n = 0, 1, 2, \dots,$$

where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$ ,  $\{\eta_n\}$  is a sequence of nonnegative real numbers, and  $\{\delta_n\}, \{\rho_n\}$  are real sequences such that

- (i)  $\sum_{n=0}^{\infty} \gamma_n = \infty$ ,
- (ii)  $\lim_{n \rightarrow \infty} \rho_n = 0$ ,
- (iii) if  $\lim_{k \rightarrow \infty} \eta_{n_k} = 0$ , then  $\limsup_{k \rightarrow \infty} \delta_{n_k} \leq 0$  for any subsequence  $\{n_k\}$  of  $\{n\}$ .

Then  $\lim_{n \rightarrow \infty} s_n = 0$ .

### 3 Main result

**Theorem 3.1** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping of  $H$  into itself, and let  $B$  be a maximal monotone operator on  $H$  such that the domain of  $B$  is included in  $C$ . Let  $J_\lambda = (I + \lambda B)^{-1}$  be the resolvent of  $B$  for  $\lambda > 0$ , let  $S$  be a nonexpansive mapping of  $C$  into itself such that  $\Omega := \text{Fix}(S) \cap (A + B)^{-1}(0) \neq \emptyset$ , and let  $f$  be a  $k$ -contraction mapping of  $C$  into itself. Let  $x_0, x_1 \in C$ , and let  $\{x_n\} \subset C$  be the sequence generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = S(\alpha_n f(x_n) + (1 - \alpha_n)J_{\lambda_n}(y_n - \lambda_n A y_n + e_n)), \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\lambda_n\} \subset (0, 2\alpha)$ ,  $\{e_n\} \subset H$ , and  $\{\theta_n\} \subset [0, \theta]$  such that  $\theta \in [0, 1)$  satisfy the following conditions:

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (C2)  $0 < a \leq \lambda_n \leq b < 2\alpha$  for some  $a, b > 0$ ,
- (C3)  $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ ,
- (C4)  $\sum_{n=1}^{\infty} \|e_n\| < \infty$ , and  $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Omega$ , where  $x^* = P_\Omega f(x^*)$ .

*Proof* Picking  $z \in \text{Fix}(S) \cap (A + B)^{-1}(0)$  and fixing  $n \in \mathbb{N}$ , it follows that  $z = S(z) = J_{\lambda_n}(z - \lambda_n A z)$ . Let

$$w_n = \alpha_n f(x_n) + (1 - \alpha_n)J_{\lambda_n}(y_n - \lambda_n A y_n + e_n).$$

Firstly, we will show that  $\{x_n\}$  and  $\{y_n\}$  are bounded. Since

$$\begin{aligned} \|y_n - z\| &= \|x_n + \theta_n(x_n - x_{n-1}) - z\| \\ &\leq \|x_n - z\| + \theta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{3.1}$$

Therefore by (3.1) and the nonexpansiveness of  $S, J_{\lambda_n}$ , and  $I - \lambda_n A$  in Lemma 2.6 we obtain

$$\|x_{n+1} - z\| = \|S w_n - S z\| \leq \|w_n - z\|$$

$$\begin{aligned}
 &\leq \alpha_n \|f(x_n) - z\| + (1 - \alpha_n) \|J_{\lambda_n}(y_n - \lambda_n A y_n + e_n) - J_{\lambda_n}(z - \lambda_n A z)\| \\
 &\leq \alpha_n (\|f(x_n) - f(z)\| + \|f(z) - z\|) \\
 &\quad + (1 - \alpha_n) \|(y_n - \lambda_n A y_n + e_n) - (z - \lambda_n A z)\| \\
 &\leq \alpha_n (k \|x_n - z\| + \|f(z) - z\|) + (1 - \alpha_n) (\|y_n - z\| + \|e_n\|) \\
 &\leq \alpha_n (k \|x_n - z\| + \|f(z) - z\|) \\
 &\quad + (1 - \alpha_n) (\|x_n - z\| + \theta_n \|x_n - x_{n-1}\| + \|e_n\|) \\
 &\leq (1 - \alpha_n(1 - k)) \|x_n - z\| + \alpha_n \|f(z) - z\| + \theta_n \|x_n - x_{n-1}\| + \|e_n\| \\
 &= (1 - \alpha_n(1 - k)) \|x_n - z\| \\
 &\quad + \alpha_n(1 - k) \left( \frac{\|f(z) - z\|}{1 - k} + \frac{1}{1 - k} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right) + \|e_n\|.
 \end{aligned}$$

So, by condition (C4), putting  $M = \frac{1}{1-k} (\|f(z) - z\| + \sup_{n \in \mathbb{N}} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|) \geq 0$  in Lemma 2.9(i), we conclude that the sequence  $\{\|x_n - z\|\}$  is bounded, that is, the sequence  $\{x_n\}$  is bounded, and so is  $\{y_n\}$ . Moreover, by condition (C4),  $\sum_{n=1}^{\infty} \|e_n\| < \infty$  implies  $\lim_{n \rightarrow \infty} \|e_n\| = 0$ , that is,  $\lim_{n \rightarrow \infty} e_n = 0$ . It follows that the sequence  $\{e_n\}$  is also bounded, and so is  $\{w_n\}$ .

Since  $P_{\text{Fix}(S) \cap (A+B)^{-1}(0)}$  is a  $k$ -contraction on  $C$ , by Banach’s contraction principle there exists a unique element  $x^* \in C$  such that  $x^* = P_{\text{Fix}(S) \cap (A+B)^{-1}(0)} f(x^*)$ , that is,  $x^* \in \text{Fix}(S) \cap (A + B)^{-1}(0)$ . It follows that  $x^* = S(x^*) = J_{\lambda_n}(x^* - \lambda_n A x^*)$ . Now we will show that  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ . On the other hand, we have

$$\begin{aligned}
 \|y_n - x^*\|^2 &= \langle y_n - x^*, y_n - x^* \rangle \\
 &= \langle x_n + \theta_n(x_n - x_{n-1}) - x^*, y_n - x^* \rangle \\
 &= \langle x_n - x^*, y_n - x^* \rangle + \theta_n \langle x_n - x_{n-1}, y_n - x^* \rangle \\
 &\leq \|x_n - x^*\| \|y_n - x^*\| + \theta_n \|x_n - x_{n-1}\| \|y_n - x^*\| \\
 &\leq \frac{1}{2} (\|x_n - x^*\|^2 + \|y_n - x^*\|^2) + \theta_n \|x_n - x_{n-1}\| \|y_n - x^*\|.
 \end{aligned}$$

This implies that

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\| \|y_n - x^*\|. \tag{3.2}$$

Therefore by (3.2), Lemma 2.6, and the firm nonexpansiveness of  $J_{\lambda_n}$  we obtain

$$\begin{aligned}
 &\|J_{\lambda_n}(y_n - \lambda_n A y_n + e_n) - x^*\|^2 \\
 &= \|J_{\lambda_n}(y_n - \lambda_n A y_n + e_n) - J_{\lambda_n}(x^* - \lambda_n A x^*)\|^2 \\
 &\leq \|(y_n - \lambda_n A y_n + e_n) - (x^* - \lambda_n A x^*)\|^2 \\
 &\quad - \|(I - J_{\lambda_n})(y_n - \lambda_n A y_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n A x^*)\|^2 \\
 &\leq (\|(y_n - \lambda_n A y_n) - (x^* - \lambda_n A x^*)\| + \|e_n\|)^2 \\
 &\quad - \|(I - J_{\lambda_n})(y_n - \lambda_n A y_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n A x^*)\|^2
 \end{aligned}$$

$$\begin{aligned}
 &= \|(I - \lambda_n A)y_n - (I - \lambda_n A)x^*\|^2 + 2\|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| \|e_n\| \\
 &\quad + \|e_n\|^2 - \|(I - J_{\lambda_n})(y_n - \lambda_n Ay_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n Ax^*)\|^2 \\
 &\leq \|y_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Ax^*\|^2 \\
 &\quad + 2\|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| \|e_n\| + \|e_n\|^2 \\
 &\quad - \|(I - J_{\lambda_n})(y_n - \lambda_n Ay_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n Ax^*)\|^2 \\
 &\leq \|x_n - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\| \|y_n - x^*\| - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Ax^*\|^2 \\
 &\quad + 2\|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| \|e_n\| + \|e_n\|^2 \\
 &\quad - \|(I - J_{\lambda_n})(y_n - \lambda_n Ay_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n Ax^*)\|^2. \tag{3.3}
 \end{aligned}$$

We also have

$$\begin{aligned}
 &\|w_n - x^*\|^2 \\
 &= \langle w_n - x^*, w_n - x^* \rangle \\
 &= \langle \alpha_n f(x_n) + (1 - \alpha_n)J_{\lambda_n}(y_n - \lambda_n Ay_n + e_n) - x^*, w_n - x^* \rangle \\
 &= \langle \alpha_n(f(x_n) - x^*) + (1 - \alpha_n)(J_{\lambda_n}(y_n - \lambda_n Ay_n + e_n) - x^*), w_n - x^* \rangle \\
 &= \alpha_n \langle f(x_n) - f(x^*), w_n - x^* \rangle + \alpha_n \langle f(x^*) - x^*, w_n - x^* \rangle \\
 &\quad + (1 - \alpha_n) \langle J_{\lambda_n}(y_n - \lambda_n Ay_n + e_n) - x^*, w_n - x^* \rangle \\
 &\leq \alpha_n k \|x_n - x^*\| \|w_n - x^*\| + \alpha_n \langle f(x^*) - x^*, w_n - x^* \rangle \\
 &\quad + (1 - \alpha_n) \|J_{\lambda_n}(y_n - \lambda_n Ay_n + e_n) - x^*\| \|w_n - x^*\| \\
 &\leq \frac{1}{2} \alpha_n k (\|x_n - x^*\|^2 + \|w_n - x^*\|^2) + \alpha_n \langle f(x^*) - x^*, w_n - x^* \rangle \\
 &\quad + \frac{1}{2} (1 - \alpha_n) (\|J_{\lambda_n}(y_n - \lambda_n Ay_n + e_n) - x^*\|^2 + \|w_n - x^*\|^2).
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \|w_n - x^*\|^2 &\leq \frac{\alpha_n k}{1 + \alpha_n(1 - k)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1 - k)} \langle f(x^*) - x^*, w_n - x^* \rangle \\
 &\quad + \frac{1 - \alpha_n}{1 + \alpha_n(1 - k)} \|J_{\lambda_n}(y_n - \lambda_n Ay_n + e_n) - x^*\|^2. \tag{3.4}
 \end{aligned}$$

Hence by (3.3), (3.4), and the nonexpansiveness of  $S$  we obtain

$$\begin{aligned}
 &\|x_{n+1} - x^*\|^2 \\
 &= \|Sw_n - Sx^*\|^2 \leq \|w_n - x^*\|^2 \\
 &\leq \frac{\alpha_n k}{1 + \alpha_n(1 - k)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1 - k)} \langle f(x^*) - x^*, w_n - x^* \rangle \\
 &\quad + \frac{1 - \alpha_n}{1 + \alpha_n(1 - k)} (\|x_n - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\| \|y_n - x^*\| \\
 &\quad - \lambda_n(2\alpha - \lambda_n)\|Ay_n - Ax^*\|^2 + 2\|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| \|e_n\|
 \end{aligned}$$



$$+ \|e_n\|^2 - \|(I - J_{\lambda_n})(y_n - \lambda_n Ay_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n Ax^*)\|^2).$$

It follows that

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ & \leq \left(1 - \frac{\alpha_n(1-k)}{1 + \alpha_n(1-k)}\right) \|x_n - x^*\|^2 + \frac{\alpha_n(1-k)}{1 + \alpha_n(1-k)} \left(\frac{2}{1-k} \langle f(x^*) - x^*, w_n - x^* \rangle \right. \\ & \quad + \frac{2(1-\alpha_n)}{1-k} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \|y_n - x^*\| \\ & \quad \left. + \frac{2(1-\alpha_n)}{1-k} \frac{\|e_n\|}{\alpha_n} \|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| + \frac{1-\alpha_n}{1-k} \frac{\|e_n\|}{\alpha_n} \|e_n\| \right) \end{aligned}$$

and

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ & \leq \|x_n - x^*\|^2 - (\lambda_n(2\alpha - \lambda_n) \|Ay_n - Ax^*\|^2 \\ & \quad + \|(I - J_{\lambda_n})(y_n - \lambda_n Ay_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n Ax^*)\|^2) \\ & \quad + \left(\frac{2\alpha_n}{1 + \alpha_n(1-k)} \|f(x^*) - x^*\| \|w_n - x^*\| + 2\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \|y_n - x^*\| \right. \\ & \quad \left. + 2\|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| \|e_n\| + \|e_n\|^2\right), \end{aligned}$$

which are of the forms

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \delta_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n,$$

respectively, where  $s_n = \|x_n - x^*\|^2$ ,  $\gamma_n = \frac{\alpha_n(1-k)}{1 + \alpha_n(1-k)}$ ,  $\delta_n = \frac{2}{1-k} \langle f(x^*) - x^*, w_n - x^* \rangle + \frac{2(1-\alpha_n)}{1-k} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \|y_n - x^*\| + \frac{2(1-\alpha_n)}{1-k} \frac{\|e_n\|}{\alpha_n} \|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| + \frac{1-\alpha_n}{1-k} \frac{\|e_n\|}{\alpha_n} \|e_n\|$ ,  $\eta_n = \lambda_n(2\alpha - \lambda_n) \|Ay_n - Ax^*\|^2 + \|(I - J_{\lambda_n})(y_n - \lambda_n Ay_n + e_n) - (I - J_{\lambda_n})(x^* - \lambda_n Ax^*)\|^2$ , and  $\rho_n = \frac{2\alpha_n}{1 + \alpha_n(1-k)} \|f(x^*) - x^*\| \|w_n - x^*\| + 2\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \|y_n - x^*\| + 2\|(y_n - \lambda_n Ay_n) - (x^* - \lambda_n Ax^*)\| \|e_n\| + \|e_n\|^2$ . Therefore, using conditions (C1) and (C4), we can check that all those sequences satisfy conditions (i) and (ii) in Lemma 2.10. To complete the proof, we verify that condition (iii) in Lemma 2.10 is satisfied. Let  $\lim_{i \rightarrow \infty} \eta_{n_i} = 0$ . Then by condition (C2) we have

$$\lim_{i \rightarrow \infty} \|Ay_{n_i} - Ax^*\| = 0 \tag{3.5}$$

and

$$\lim_{i \rightarrow \infty} \|(I - J_{\lambda_{n_i}})(y_{n_i} - \lambda_{n_i} Ay_{n_i} + e_{n_i}) - (I - J_{\lambda_{n_i}})(x^* - \lambda_{n_i} Ax^*)\| = 0.$$

It follows by conditions (C2) and (C4) and by (3.5) that

$$\begin{aligned} & \lim_{i \rightarrow \infty} \left\| (y_{n_i} - \lambda_{n_i}Ay_{n_i} + e_{n_i}) - J_{\lambda_{n_i}}(y_{n_i} - \lambda_{n_i}Ay_{n_i} + e_{n_i}) \right. \\ & \quad \left. - ((x^* - \lambda_{n_i}Ax^*) - J_{\lambda_{n_i}}(x^* - \lambda_{n_i}Ax^*)) \right\| = 0, \\ & \lim_{i \rightarrow \infty} \|y_{n_i} - J_{\lambda_{n_i}}(y_{n_i} - \lambda_{n_i}Ay_{n_i})\| = 0. \end{aligned} \tag{3.6}$$

Consider a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$ . As  $\{x_n\}$  is bounded, so is  $\{x_{n_i}\}$ , and thus there exists a subsequence  $\{x_{n_{i_j}}\}$  of  $\{x_{n_i}\}$  that weakly converges to  $x \in C$ . Without loss of generality, we can assume that  $x_{n_{i_j}} \rightharpoonup x$  as  $i \rightarrow \infty$ . On the other hand, by conditions (C1) and (C4) we have

$$\lim_{i \rightarrow \infty} \|y_{n_i} - x_{n_i}\| = \lim_{i \rightarrow \infty} \alpha_{n_i} \frac{\theta_{n_i}}{\alpha_{n_i}} \|x_{n_i} - x_{n_{i-1}}\| = 0. \tag{3.7}$$

It follows that  $y_{n_i} \rightharpoonup x$  as  $i \rightarrow \infty$ . Therefore by (3.6) and the demiclosedness at zero in Lemma 2.7 we obtain  $x \in \text{Fix}(J_{\lambda_{n_i}}(I - \lambda_{n_i}A))$ , that is,  $x \in (A + B)^{-1}(0)$ . Next, we will show that  $x \in \text{Fix}(S)$ . By the nonexpansiveness of  $S$  we have

$$\begin{aligned} \|x_{n_{i+1}} - Sx_{n_i}\| &= \|Sw_{n_i} - Sx_{n_i}\| \leq \|w_{n_i} - x_{n_i}\| \\ &\leq \|w_{n_i} - y_{n_i}\| + \|y_{n_i} - x_{n_i}\| \\ &\leq \alpha_{n_i} \|f(x_{n_i}) - y_{n_i}\| \\ &\quad + (1 - \alpha_{n_i}) \|J_{\lambda_{n_i}}(y_{n_i} - \lambda_{n_i}Ay_{n_i} + e_{n_i}) - y_{n_i}\| + \|y_{n_i} - x_{n_i}\|. \end{aligned}$$

It follows by (3.6), (3.7), and conditions (C1) and (C4) that

$$\lim_{i \rightarrow \infty} \|x_{n_{i+1}} - Sx_{n_i}\| = 0.$$

Then by NST-condition (II) in Lemma 2.8(ii) we get

$$\lim_{i \rightarrow \infty} \|x_{n_i} - Sx_{n_i}\| = 0. \tag{3.8}$$

Hence by (3.8) and the demiclosedness at zero in Lemma 2.7 again we obtain  $x \in \text{Fix}(S)$ , that is,  $x \in \text{Fix}(S) \cap (A + B)^{-1}(0)$ . Since

$$\begin{aligned} \|w_{n_i} - x_{n_i}\| &\leq \alpha_{n_i} \|f(x_{n_i}) - x_{n_i}\| + (1 - \alpha_{n_i}) \|J_{\lambda_{n_i}}(y_{n_i} - \lambda_{n_i}Ay_{n_i} + e_{n_i}) - x_{n_i}\| \\ &\leq \alpha_{n_i} \|f(x_{n_i}) - x_{n_i}\| + (1 - \alpha_{n_i}) (\|J_{\lambda_{n_i}}(y_{n_i} - \lambda_{n_i}Ay_{n_i} + e_{n_i}) - y_{n_i}\| \\ &\quad + \|y_{n_i} - x_{n_i}\|), \end{aligned}$$

by (3.6) and (3.7) and conditions (C1) and (C4) we obtain

$$\lim_{i \rightarrow \infty} \|w_{n_i} - x_{n_i}\| = 0.$$

This implies that  $w_{n_i} \rightarrow x$  as  $i \rightarrow \infty$ . Therefore by Lemma 2.2(i) we obtain

$$\limsup_{i \rightarrow \infty} \langle f(x^*) - x^*, w_{n_i} - x^* \rangle = \langle f(x^*) - x^*, x - x^* \rangle \leq 0.$$

It follows by conditions (C1), (C3), and (C4) that  $\limsup_{i \rightarrow \infty} \delta_{n_i} \leq 0$ . So by Lemma 2.10 we conclude that  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ . This completes the proof.  $\square$

*Remark 3.2 ([23])* We remark here that as, by condition (C4),  $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ , the theorem is easily implemented in numerical computation since the values of  $\|x_n - x_{n-1}\|$  are known before choosing  $\theta_n$ . Indeed, the parameter  $\theta_n$  can be chosen as  $0 \leq \theta_n \leq \bar{\theta}_n$  such that

$$\bar{\theta}_n = \begin{cases} \min\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \theta\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise,} \end{cases}$$

where  $\{\omega_n\}$  is a positive sequence such that  $\omega_n = o(\alpha_n)$ .

#### 4 Applications and numerical examples

In this section, we give some applications of our result to the fixed point problem of the nonexpansive variational inequality problem, the common fixed point problem of nonexpansive strict pseudocontractions, the convex minimization problem, and the split feasibility problem.

##### 4.1 Fixed point problem of the nonexpansive variational inequality problem

The variational inequality problem is to find  $x^* \in C$  such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

We denote the solution set of (4.1) by  $VI(C, A)$ . It is well known that  $\text{Fix}(P_C(I - rA)) = VI(C, A)$  for all  $r > 0$ . Define the indicator function of  $C$ , denoted by  $i_C$ , as  $i_C(x) = 0$  if  $x \in C$  and  $i_C(x) = \infty$  if  $x \notin C$ . We see that  $\partial i_C$  is maximal monotone. So, for  $r > 0$ , we can define  $J_r = (I + r\partial i_C)^{-1}$ . Moreover,  $x = J_r y$  if and only if  $x = P_C y$ . Hence by Theorem 3.1 we obtain the following result.

**Theorem 4.1** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping of  $H$  into itself, let  $S$  be a nonexpansive mapping of  $C$  into itself such that  $\Omega := \text{Fix}(S) \cap VI(C, A) \neq \emptyset$ , and let  $f$  be a  $k$ -contraction mapping of  $C$  into itself. Let  $x_0, x_1 \in C$ , and let  $\{x_n\} \subset C$  be a sequence generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = S(\alpha_n f(x_n) + (1 - \alpha_n)P_C(y_n - \lambda_n A y_n + e_n)), \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\lambda_n\} \subset (0, 2\alpha)$ ,  $\{e_n\} \subset H$ , and  $\{\theta_n\} \subset [0, \theta]$  such that  $\theta \in [0, 1)$  satisfy the following conditions:

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (C2)  $0 < a \leq \lambda_n \leq b < 2\alpha$  for some  $a, b > 0$ ,

$$(C3) \lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0,$$

$$(C4) \sum_{n=1}^{\infty} \|e_n\| < \infty, \text{ and } \lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.$$

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Omega$ , where  $x^* = P_{\Omega}f(x^*)$ .

We next provide a formulation that will be used in our example and its numerical results.

**Proposition 4.2** ([24, 25]) For  $\rho > 0$  and  $C = \{x \in \mathbb{R}^N : \|x\|_2 \leq \rho\}$ , we have

$$P_C x = \begin{cases} \frac{\rho x}{\|x\|_2}, & x \notin C, \\ x, & x \in C. \end{cases}$$

*Example 4.3* Let  $C = \{a \in \mathbb{R}^2 : \|a\|_2 \leq 1\}$ . Find a point  $x^* \in C$  that satisfies the following variational inequality:

$$-2\|x\|^2 + 2x^T y + (1, 1)(x - y) \geq 0, \quad \forall y \in C.$$

Let  $H = (\mathbb{R}^2, \|\cdot\|_2)$ . For each  $x = (u_1, u_2)^T, y = (v_1, v_2)^T \in \mathbb{R}^2$ , we have

$$\begin{aligned} & -2\|x\|^2 + 2x^T y + (1, 1)(x - y) \\ &= -2(u_1^2 + u_2^2) + 2(u_1 v_1 + u_2 v_2) + (u_1 - v_1) + (u_2 - v_2) \\ &= (v_1 - u_1)(2u_1 - 1) + (v_2 - u_2)(2u_2 - 1) \\ &= (y - x)^T A x = \langle A x, y - x \rangle, \end{aligned}$$

where  $Ax = (2u_1 - 1, 2u_2 - 1)^T$  for  $x = (u_1, u_2)^T \in \mathbb{R}^2$ . Note that  $A$  is  $\alpha$ -inverse strongly monotone with  $\alpha = \frac{1}{2}$  and  $\frac{1}{L}$ -Lipschitzian with  $L = \frac{1}{2}$ .

We set  $S(x) = P_C(1 - u_1, 1 - u_2)^T$  and  $f(x) = \frac{3x}{5}$  for  $x = (u_1, u_2)^T \in C$ . Then  $S$  is nonexpansive, and  $f$  is  $k$ -contraction such that  $k \in [\frac{3}{5}, 1)$ . For each  $n \in \mathbb{N}$ , we choose  $\alpha_n = \frac{10^{-6}}{n+1}, e_n = \frac{1}{(n+1)^3}(1, 1)^T, \theta = 0.5$ , and  $\omega_n = \frac{1}{(n+1)^3}$ , and we define  $\theta_n = \bar{\theta}_n$  as in Remark 3.2.

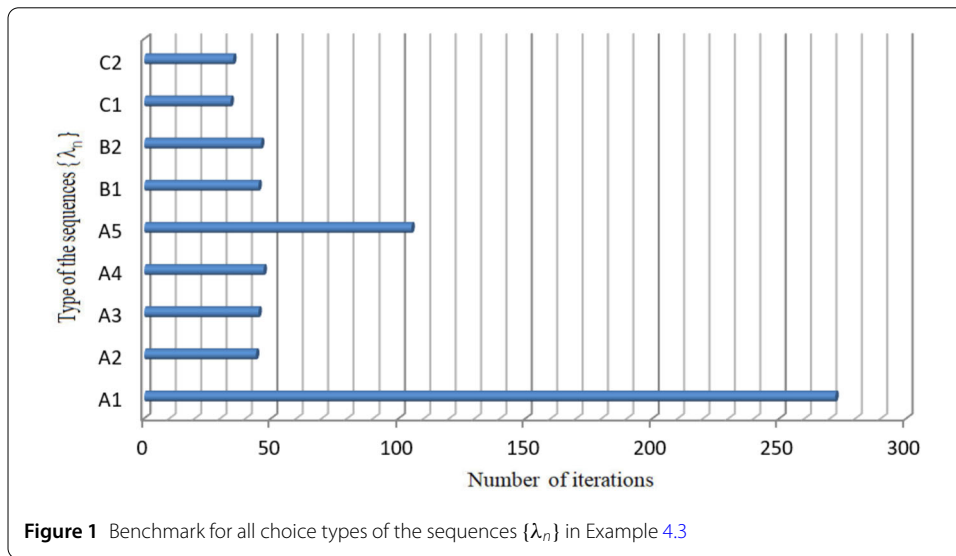
On the best of our result, we consider the type of the sequences  $\{\lambda_n\}$  as in Table 1 so that it converges to  $L$ , and also constant sequences converge to the value near  $L$  and near to its boundary values, to study the best choice types of the sequences  $\{\lambda_n\}$  for amount to the least loops in recursive computing of the sequence  $\{x_n\}$  using the algorithm in Theorem 4.1.

We choose the initial points  $x_0 = (-1, 0)^T, x_1 = (0, 1)^T \in C$  (indeed,  $x_0, x_1$  can be chosen arbitrarily in  $H$ ) for recursive computing of the sequence  $\{x_n\}$  using the algorithm in Theorem 4.1 with an error  $10^{-6}$ . As  $n \rightarrow \infty$ , we obtain  $x_n \rightarrow x^*$  such that an approximate of  $x^*$  is  $(0.5, 0.5)^T$  as in Table 1, and we also show the benchmark for all choice types of the sequences  $\{\lambda_n\}$  in recursive computing of the sequence  $\{x_n\}$  as in Fig. 1, and we also show convergence behavior of the error sequences  $\{\|x_{n+1} - x_n\|_2\}$ , which converge to zero in all the best choice types of the sequences  $\{\lambda_n\}$  as in Fig. 2.

In this example, we found that the sequences  $\{\lambda_n\}$  in the C1 and C2 types are the best choice types in recursive computing of the sequence  $\{x_n\}$ .

**Table 1** Numerical results of Example 4.3 for the initial points  $x_0 = (-1, 0)^T$  and  $x_1 = (0, 1)^T$  using the algorithm in Theorem 4.1

Type	$\lambda_n$ for $L = \alpha = 1/2$	$n$	CPU (s)	$x^*$	$\ x_{n+1} - x_n\ _2$
A1	$\frac{L}{10}$	272	0.078	$(0.50000, 0.50000)^T$	$9.92819 \times 10^{-7}$
A2	$L - \frac{L}{10}$	44	0.016	$(0.49999, 0.49999)^T$	$9.80129 \times 10^{-7}$
A3	$L$	45	0.015	$(0.49999, 0.49999)^T$	$9.90160 \times 10^{-7}$
A4	$L + \frac{L}{10}$	47	0.031	$(0.49999, 0.49999)^T$	$9.30295 \times 10^{-7}$
A5	$2\alpha - \frac{L}{10}$	105	0.031	$(0.49999, 0.49999)^T$	$9.41142 \times 10^{-7}$
B1	$\frac{Ln}{n+1}$	45	0.015	$(0.49999, 0.49999)^T$	$9.60866 \times 10^{-7}$
B2	$\frac{L(n+2)}{n+1}$	46	0.015	$(0.49999, 0.49999)^T$	$9.35467 \times 10^{-7}$
C1	$L + \frac{(-1)^n L}{n+1}$	34	0.016	$(0.49998, 0.49998)^T$	$9.09566 \times 10^{-7}$
C2	$L + \frac{(-1)^{n+1} L}{n+1}$	35	0.016	$(0.49998, 0.49998)^T$	$8.13627 \times 10^{-7}$



**Figure 1** Benchmark for all choice types of the sequences  $\{\lambda_n\}$  in Example 4.3

### 4.2 Common fixed point problem of nonexpansive strict pseudocontractions

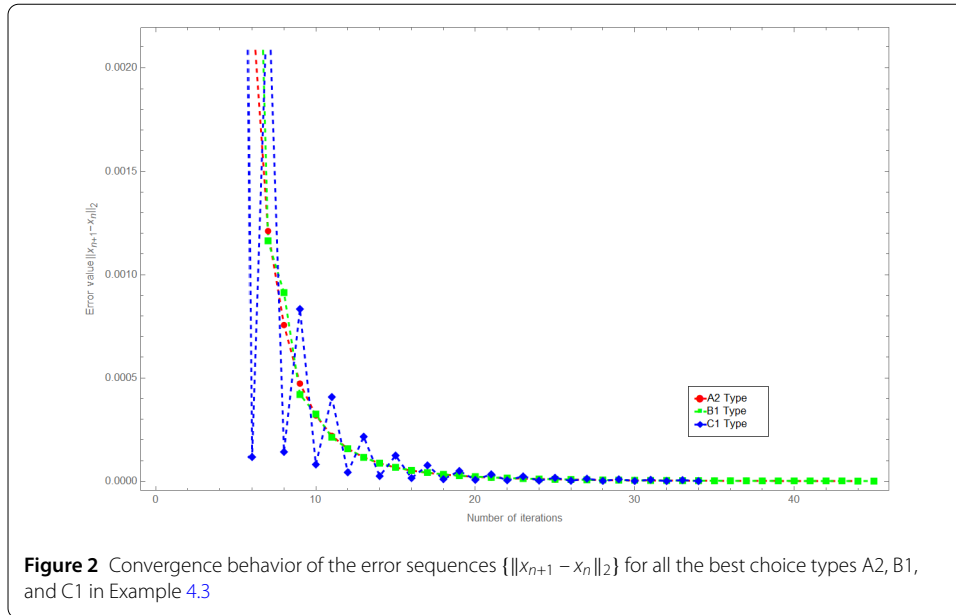
A mapping  $T : C \rightarrow C$  is called  $\beta$ -strictly pseudocontractive if there exists  $\beta \in [0, 1)$  such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta \|(I - T)x - (I - T)y\|^2$$

for all  $x, y \in C$ . It is well known that if  $T$  is  $\beta$ -strictly pseudocontractive, then  $I - T$  is  $\frac{1-\beta}{2}$ -inverse strongly monotone. Moreover, by putting  $A = I - T$  we have  $\text{Fix}(T) = \text{VI}(C, A)$ . So by Theorem 4.1 we obtain the following result.

**Theorem 4.4** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T$  be a  $\beta$ -strict pseudocontraction of  $H$  into itself, let  $S$  be a nonexpansive mapping of  $C$  into itself such that  $\Omega := \text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$ , and let  $f$  be a  $k$ -contraction mapping of  $C$  into itself. Let  $x_0, x_1 \in C$ , and let  $\{x_n\} \subset C$  be a sequence generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = S(\alpha_n f(x_n) + (1 - \alpha_n)P_C((1 - \lambda_n)y_n + \lambda_n T y_n + e_n)), \end{cases}$$



for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\lambda_n\} \subset (0, 1 - \beta)$ ,  $\{e_n\} \subset H$ , and  $\{\theta_n\} \subset [0, \theta]$  such that  $\theta \in [0, 1)$  satisfy the following conditions:

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (C2)  $0 < a \leq \lambda_n \leq b < 1 - \beta$  for some  $a, b > 0$ ,
- (C3)  $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ ,
- (C4)  $\sum_{n=1}^{\infty} \|e_n\| < \infty$ , and  $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Omega$ , where  $x^* = P_{\Omega}f(x^*)$ .

**Example 4.5** Let  $C = \{a \in \mathbb{R}^3 : \|a\|_2 \leq 2\}$ . Find a common fixed point  $x^* \in C$  of the mappings  $S$  and  $T$  defined as follows:

$$S(x) = P_C(2 - u, 2 - v, 2 - w)^T, \quad \forall x = (u, v, w)^T \in C,$$

$$T(x) = (4 - 3u, 4 - 3v, 4 - 3w)^T, \quad \forall x = (u, v, w)^T \in \mathbb{R}^3.$$

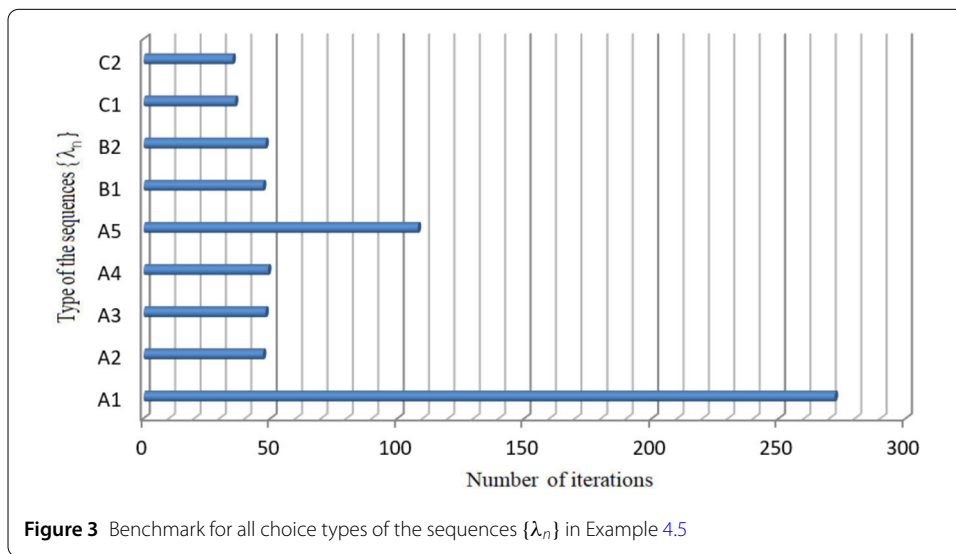
Let  $H = (\mathbb{R}^3, \|\cdot\|_2)$ . Note that  $T$  is  $\beta$ -strictly pseudocontractive with  $\beta = \frac{1}{2}$ ,  $I - T$  is  $\frac{1}{L}$ -Lipschitzian with  $L = \frac{1}{4}$ , and  $S$  is nonexpansive. We set  $f(x) = \frac{3x}{5}$  for  $x \in C$ . For each  $n \in \mathbb{N}$ , we choose  $\alpha_n = \frac{10^{-6}}{n+1}$ ,  $e_n = \frac{1}{(n+1)^3}(1, 1, 1)^T$ ,  $\theta = 0.5$ , and  $\omega_n = \frac{1}{(n+1)^3}$ , and we define  $\theta_n = \bar{\theta}_n$  as in Remark 3.2.

We choose the initial points  $x_0 = (1, -1, -1)^T, x_1 = (-1, 0, 1)^T \in C$  (indeed,  $x_0, x_1$  can be chosen arbitrarily in  $H$ ) for recursive computing of the sequence  $\{x_n\}$  using the algorithm in Theorem 4.4 with an error  $10^{-6}$  in the same of choice types of the sequences  $\{\lambda_n\}$  with  $L = \frac{1}{4}$ . As  $n \rightarrow \infty$ , we obtain  $x_n \rightarrow x^*$  such that an approximate of  $x^*$  is  $(1, 1, 1)^T$  as in Table 2, and we also show the benchmark for all choice types of the sequences  $\{\lambda_n\}$  in recursive computing of the sequence  $\{x_n\}$  as in Fig. 3, and we also show convergence behavior of the error sequences  $\{\|x_{n+1} - x_n\|_2\}$ , which converge to zero in all the best choices types of the sequences  $\{\lambda_n\}$  as in Fig. 4.

In this example, we found that the sequences  $\{\lambda_n\}$  in the C1 and C2 types are the best choice types in recursive computing of the sequence  $\{x_n\}$ .

**Table 2** Numerical results of Example 4.5 for the initial points  $x_0 = (1, -1, -1)^T$  and  $x_1 = (-1, 0, 1)^T$  using the algorithm in Theorem 4.4

Type	$\lambda_n$ for $L = 1/4, \beta = 1/2$	$n$	CPU (s)	$x^*$	$\ x_{n+1} - x_n\ _2$
A1	$\frac{L}{10}$	272	0.062	$(1.00000, 1.00000, 1.00000)^T$	$9.92684 \times 10^{-7}$
A2	$L - \frac{L}{10}$	47	0.016	$(0.99999, 0.99999, 0.99999)^T$	$9.24684 \times 10^{-7}$
A3	$L$	48	0.016	$(0.99999, 0.99999, 0.99999)^T$	$9.39145 \times 10^{-7}$
A4	$L + \frac{L}{10}$	49	0.015	$(0.99999, 0.99999, 0.99999)^T$	$9.65739 \times 10^{-7}$
A5	$(1 - \beta) - \frac{L}{10}$	108	0.031	$(0.99999, 0.99999, 0.99999)^T$	$9.04352 \times 10^{-7}$
B1	$\frac{Ln}{n+1}$	47	0.016	$(0.99999, 0.99999, 0.99999)^T$	$9.91872 \times 10^{-7}$
B2	$\frac{L(n+2)}{n+1}$	48	0.016	$(0.99999, 0.99999, 0.99999)^T$	$9.66642 \times 10^{-7}$
C1	$L + \frac{(-1)^n L}{n+1}$	36	0.016	$(0.99998, 0.99998, 0.99998)^T$	$8.94335 \times 10^{-7}$
C2	$L + \frac{(-1)^{n+1} L}{n+1}$	35	0.016	$(0.99998, 0.99998, 0.99998)^T$	$9.96769 \times 10^{-7}$



**Figure 3** Benchmark for all choice types of the sequences  $\{\lambda_n\}$  in Example 4.5

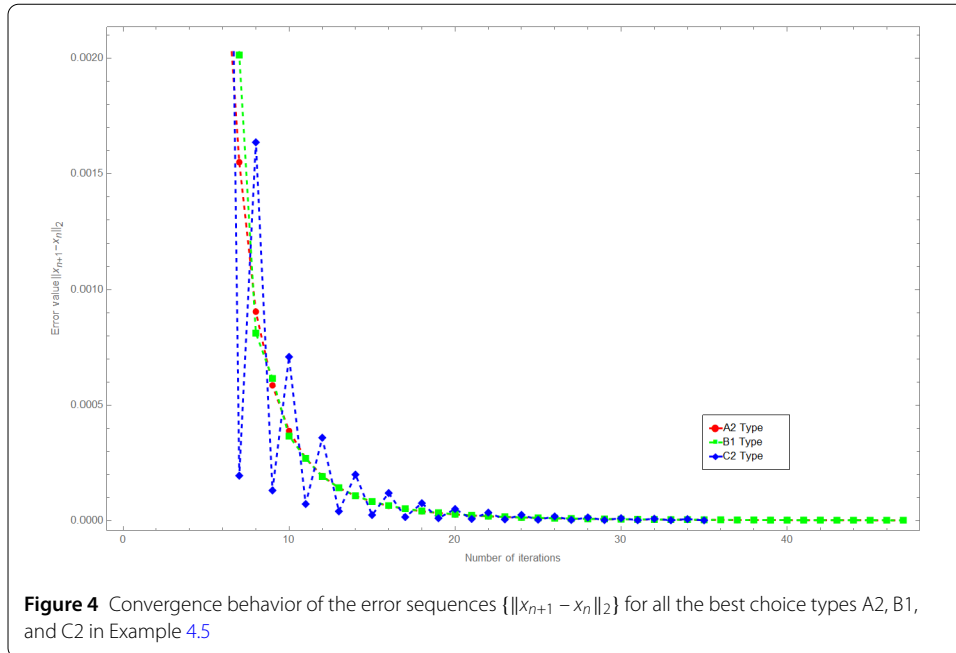
### 4.3 Convex minimization problem

We next consider the following convex minimization problem (CMP): find  $x^* \in H$  such that

$$F(x^*) + G(x^*) = \min_{x \in H} F(x) + G(x) \Leftrightarrow 0 \in \nabla F(x^*) + \partial G(x^*), \tag{4.2}$$

where  $F : H \rightarrow \mathbb{R}$  is a convex differentiable function, and  $G : H \rightarrow \mathbb{R}$  is a convex function. It is well known that if  $\nabla F$  is  $(1/L)$ -Lipschitz continuous, then it is  $L$ -inverse strongly monotone [26]. Moreover,  $\partial G$  is maximal monotone [27]. Putting  $A = \nabla F$  and  $B = \partial G$ , by Theorem 3.1 we obtain the following result.

**Theorem 4.6** *Let  $H$  be a real Hilbert space. Let  $F : H \rightarrow \mathbb{R}$  be a convex differentiable function with  $(1/L)$ -Lipschitz continuous gradient  $\nabla F$ , and let  $G : H \rightarrow \mathbb{R}$  be a convex and lower semicontinuous function. Let  $J_\lambda = (I + \lambda \partial G)^{-1}$  be the resolvent of  $\partial G$  for  $\lambda > 0$ , let  $S$  be a nonexpansive mapping of  $H$  into itself such that  $\Omega := \text{Fix}(S) \cap (\nabla F + \partial G)^{-1}(0) \neq \emptyset$ , and let  $f$  be a  $k$ -contraction mapping of  $H$  into itself. Let  $x_0, x_1 \in H$ , and let  $\{x_n\} \subset H$  be the*



**Figure 4** Convergence behavior of the error sequences  $\{\|x_{n+1} - x_n\|_2\}$  for all the best choice types A2, B1, and C2 in Example 4.5

sequence generated by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = S(\alpha_n f(x_n) + (1 - \alpha_n)J_{\lambda_n}(y_n - \lambda_n \nabla F(y_n) + e_n)), \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\lambda_n\} \subset (0, 2L)$ ,  $\{e_n\} \subset H$ , and  $\{\theta_n\} \subset [0, \theta]$  such that  $\theta \in [0, 1)$  satisfy the following conditions:

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (C2)  $0 < a \leq \lambda_n \leq b < 2L$  for some  $a, b > 0$ ,
- (C3)  $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ ,
- (C4)  $\sum_{n=1}^{\infty} \|e_n\| < \infty$ , and  $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Omega$ , where  $x^* = P_{\Omega}f(x^*)$ .

We next provide the formulation which will be used in our example and its numerical results.

**Proposition 4.7** ([28]) Let  $G : \mathbb{R}^N \rightarrow \mathbb{R}$  be given by  $G(x) = \|x\|_1$  for  $x \in \mathbb{R}^N$ . For  $r > 0$  and  $x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N$ , we have  $(I + r\partial G)^{-1}(x) = y$  such that  $y = (y_1, y_2, \dots, y_N)^T \in \mathbb{R}^N$  where  $y_i = \text{sign}(x_i) \max\{|x_i| - r, 0\}$  for  $i = 1, 2, \dots, N$ .

**Example 4.8** Find a point minimizing the following  $\ell_1$ -least square problem:

$$\min_{x \in \mathbb{R}^3} \|x\|_1 + \frac{1}{2} \|x\|_2^2 + (-2, 1, -3)x + 3,$$

where  $x = (u, v, w)^T \in \mathbb{R}^3$ .

Let  $H = (\mathbb{R}^3, \|\cdot\|_2)$ ,  $F(x) = \frac{1}{2} \|x\|_2^2 + (-2, 1, -3)x + 3$ , and  $G(x) = \|x\|_1$  for all  $x \in \mathbb{R}^3$ . Then  $\nabla F(x) = (u - 2, v + 1, w - 3)^T$  for all  $x \in \mathbb{R}^3$ . It follows that  $F$  is convex and differentiable on



**Table 3** Numerical results of Example 4.8 for the initial points  $x_0 = (1, -2, -1)^T$  and  $x_1 = (-2, -1, 2)^T$  using the algorithm in Theorem 4.6

Type	$\lambda_n$ for $L = 1$	$n$	CPU (s)	$x^*$	$\ x_{n+1} - x_n\ _2$
A1	$\frac{L}{10}$	272	0.109	$(0.99999, -2.48737 \times 10^{-2335}, 2.00000)^T$	$9.92500 \times 10^{-7}$
A2	$L - \frac{L}{10}$	44	0.047	$(0.99999, -1.47065 \times 10^{-351}, 1.99999)^T$	$9.79481 \times 10^{-7}$
A3	$L$	45	0.032	$(0.99999, +6.39411 \times 10^{-360}, 1.99999)^T$	$9.89477 \times 10^{-7}$
A4	$L + \frac{L}{10}$	47	0.031	$(0.99999, +1.13371 \times 10^{-376}, 1.99999)^T$	$9.29597 \times 10^{-7}$
A5	$2L - \frac{L}{10}$	123	0.063	$(0.99999, +7.06031 \times 10^{-1032}, 1.99999)^T$	$8.86229 \times 10^{-7}$
B1	$\frac{Ln}{n+1}$	45	0.016	$(0.99999, +2.30611 \times 10^{-353}, 1.99999)^T$	$9.60213 \times 10^{-7}$
B2	$\frac{L(n+2)}{n+1}$	46	0.046	$(0.99999, -2.72090 \times 10^{-368}, 1.99999)^T$	$9.34783 \times 10^{-7}$
C1	$L + \frac{(-1)^n L}{n+1}$	34	0.015	$(0.99998, +2.17358 \times 10^{-255}, 1.99998)^T$	$9.10791 \times 10^{-7}$
C2	$L + \frac{(-1)^{n+1} L}{n+1}$	35	0.016	$(0.99998, +9.23665 \times 10^{-277}, 1.99998)^T$	$8.14783 \times 10^{-7}$

$\mathbb{R}^3$  with  $L = 1$  of  $\frac{1}{L}$ -Lipschitz continuous gradient  $\nabla F$ . Moreover,  $G$  is convex and lower semicontinuous but not differentiable on  $\mathbb{R}^3$ .

We set  $S(x) = (2 - u, -v, 4 - w)^T$  and  $f(x) = \frac{x}{5}$  for  $x \in \mathbb{R}^3$ . Then  $S$  is nonexpansive, and  $f$  is  $k$ -contraction with  $k \in [\frac{1}{5}, 1)$ . For each  $n \in \mathbb{N}$ , we choose  $\alpha_n = \frac{10^{-6}}{n+1}$ ,  $e_n = \frac{1}{(n+1)^3}(1, 1, 1)^T$ ,  $\theta = 0.5$ , and  $\omega_n = \frac{1}{(n+1)^3}$ , and we define  $\theta_n = \bar{\theta}_n$  as in Remark 3.2.

For each  $n \in \mathbb{N}$ , by Proposition 4.7 we have

$$(I + \lambda_n \partial G)^{-1}(x) = (\text{sign}(u) \max\{|u| - \lambda_n, 0\}, \text{sign}(v) \max\{|v| - \lambda_n, 0\}, \text{sign}(w) \max\{|w| - \lambda_n, 0\})^T.$$

We choose the initial points  $x_0 = (1, -2, -1)^T$  and  $x_1 = (-2, -1, 2)^T$  for recursive computing of the sequence  $\{x_n\}$  using the algorithm in Theorem 4.6 with an error  $10^{-6}$  in the same of choice types of the sequences  $\{\lambda_n\}$  with  $L = 1$ . As  $n \rightarrow \infty$ , we obtain  $x_n \rightarrow x^*$  such that an approximate minimum of  $F + G$  is  $(1, 0, 2)^T$  and its approximate minimum value is 0.5 as in Table 3, and we also show the benchmark for all choice types of the sequences  $\{\lambda_n\}$  in recursive computing of the sequence  $\{x_n\}$  as in Fig. 5, and we also show convergence behavior of the error sequences  $\{\|x_{n+1} - x_n\|_2\}$ , which converge to zero in all the best choice types of the sequences  $\{\lambda_n\}$  as in Fig. 6.

In this example, we found that the sequences  $\{\lambda_n\}$  in the C1 and C2 types are the best choice types in recursive computing of the sequence  $\{x_n\}$ .

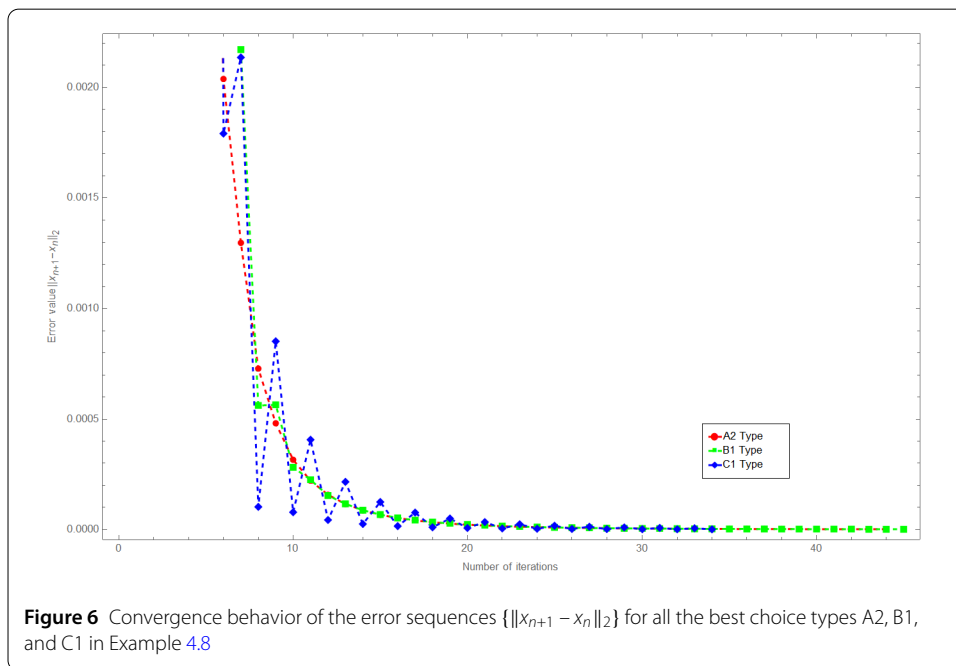
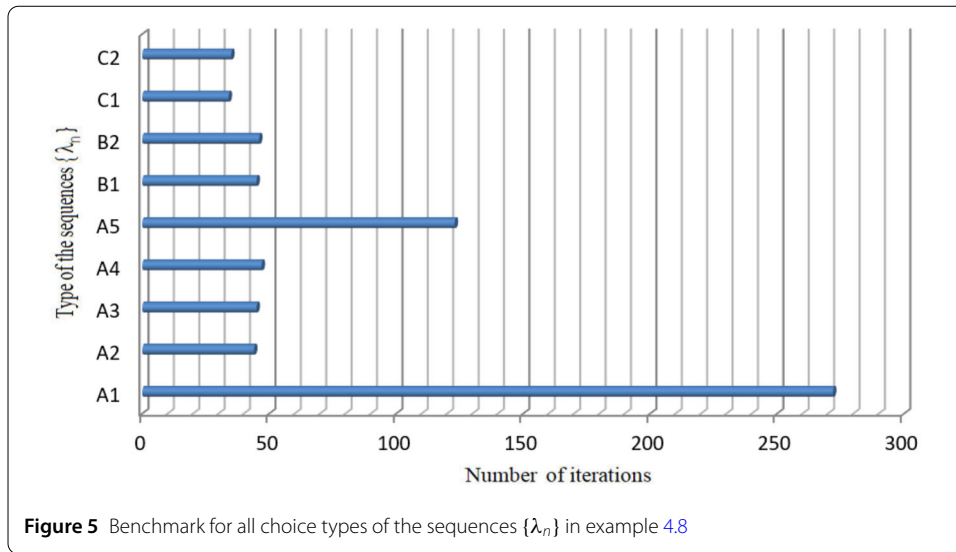
### 4.4 Split feasibility problem

We next consider the following split feasibility problem (SFP), which was first introduced by Censor and Elfving [29]: find

$$x^* \in C \quad \text{such that } Ax^* \in Q, \tag{4.3}$$

where  $C$  and  $Q$  are two nonempty closed convex subsets of two real Hilbert spaces  $H$  and  $K$ , respectively, and  $A : H \rightarrow K$  is a bounded linear operator. Then problem (4.3) becomes to find  $x^* \in C$  in the following minimization problem:

$$F(x^*) = \min_{x \in C} F(x) := \frac{1}{2} \|Ax - P_Q Ax\|^2 \Leftrightarrow 0 \in \nabla F(x^*), \tag{4.4}$$



which is a particular case of the convex minimization problem (4.2) when  $G = 0$ . It is well known from Lemma 2.4 that  $F$  is a convex differentiable function with  $\|A\|^2$ -Lipschitz continuous gradient  $\nabla F$  and weakly lower semicontinuous function on  $H$ , and  $\nabla F(x) = A^*(I - P_Q)Ax$  for all  $x \in H$ , where  $A^*$  denotes the adjoint of  $A$ . Putting  $F(x) = \frac{1}{2}\|Ax - P_QAx\|^2$  for  $x \in H$ ,  $\partial G = 0$ , and  $S = P_C$ , by Theorem 4.6 we obtain the following result.

**Theorem 4.9** *Let  $C$  and  $Q$  are two nonempty closed convex subsets of two real Hilbert spaces  $H$  and  $K$ , respectively. Let  $A : H \rightarrow K$  be a bounded linear operator, and let  $f$  be a  $k$ -contraction mapping of  $H$  into itself. Assume that the SFP (4.3) has a nonempty solution*

set  $\Gamma$ . Let  $x_0, x_1 \in H$ , and let  $\{x_n\} \subset H$  be a sequence generated by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = P_C(\alpha_n f(x_n) + (1 - \alpha_n)(y_n - \lambda_n A^*(I - P_Q)Ay_n + e_n)), \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\lambda_n\} \subset (0, \frac{2}{\|A\|^2})$ ,  $\{e_n\} \subset H$ , and  $\{\theta_n\} \subset [0, \theta]$  such that  $\theta \in [0, 1)$  satisfy the following conditions:

- (C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (C2)  $0 < a \leq \lambda_n \leq b < \frac{2}{\|A\|^2}$  for some  $a, b > 0$ ,
- (C3)  $\lim_{n \rightarrow \infty} \frac{\|e_n\|}{\alpha_n} = 0$ ,
- (C4)  $\sum_{n=1}^{\infty} \|e_n\| < \infty$ , and  $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $x^* \in \Gamma$ , where  $x^* = P_{\Gamma}f(x^*)$ , which is a minimizer of the minimum-norm solution of the SFP (4.3).

**Example 4.10** Let  $C = \{a \in \mathbb{R}^4 : \|a\|_2 \leq 2\}$ . Find some point  $x^* \in C$  that satisfies the following system of linear equations:

$$\begin{aligned} x + y - 2z + w &= 1, \\ x - y + 3z + w &= 2, \\ x + y + z - 3w &= 3, \end{aligned}$$

where  $x, y, z, w \in \mathbb{R}$ .

Let  $H = (\mathbb{R}^4, \|\cdot\|_2)$  and  $K = (\mathbb{R}^3, \|\cdot\|_2)$ . We set

$$A = \begin{pmatrix} 1 & 1 & -2 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix},$$

$Q = \{b : b = (1, 2, 3)^T\}$  and  $f(u) = \frac{u}{5}$  for  $u \in \mathbb{R}^4$ . For each  $n \in \mathbb{N}$ , we choose  $\alpha_n = \frac{10^{-6}}{n+1}$ ,  $e_n = \frac{1}{(n+1)^3}(1, 1, 1, 1)^T$ ,  $\theta = 0.5$ , and  $\omega_n = \frac{1}{(n+1)^3}$ , and we define  $\theta_n = \bar{\theta}_n$  as in Remark 3.2.

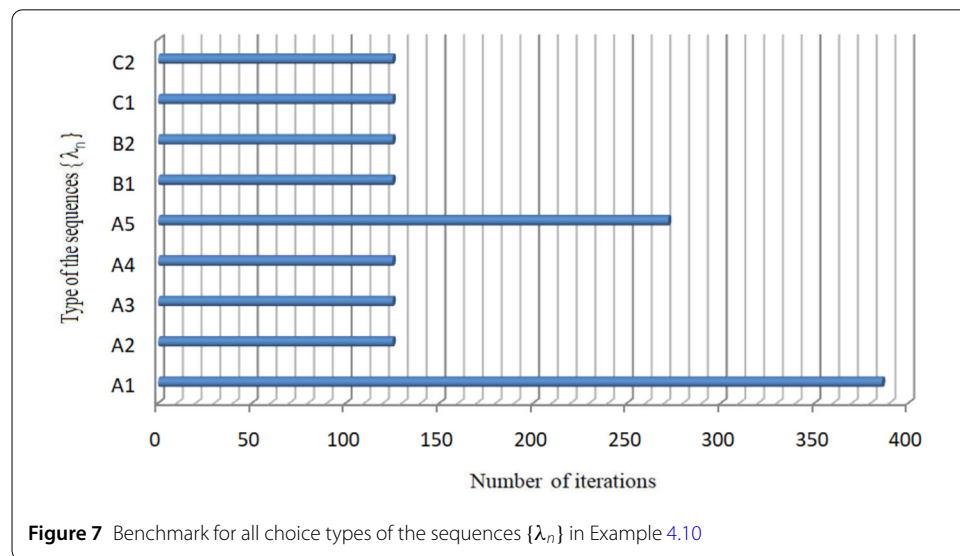
We choose the initial points  $x_0 = (1, 1, 0, 2)^T$  and  $x_1 = (2, 1, 3, 0)^T$  for recursive computing of the sequence  $\{x_n\}$  using the algorithm in Theorem 4.9 with an error  $10^{-6}$  in the same of choice types of the sequences  $\{\lambda_n\}$  with  $L = \frac{1}{\|A\|^2}$  of  $\frac{1}{L}$ -Lipschitz continuous gradient  $\nabla F$  defined by (4.4) such that  $\|A\|$  is the square root of the maximum eigenvalue of  $A^T A$ . By Proposition 4.2, as  $n \rightarrow \infty$ , we obtain  $x_n \rightarrow x^*$  such that  $x^*$  is our solution, and the numerical results are listed in Table 4. We also show the benchmark for all choice types of the sequences  $\{\lambda_n\}$  in recursive computing of the sequence  $\{x_n\}$  as in Fig. 7, and we also show convergence behavior of the error sequences  $\{\|x_{n+1} - x_n\|_2\}$ , which converge to zero in all the best choice types of the sequences  $\{\lambda_n\}$  as in Fig. 8.

In this example, we found that the sequences  $\{\lambda_n\}$  in the A2, A3, A4, B1, B2, C1, and C2 types are the best choice types in recursive computing of the sequence  $\{x_n\}$ .

**Remark 4.11** A new iterative shrinkage thresholding algorithm (NISTA) with an error is obtained in our main result, based on the forward-backward splitting method with an

**Table 4** Numerical results of Example 4.10 for the initial points  $x_0 = (1, 1, 0, 2)^T$  and  $x_1 = (2, 1, 3, 0)^T$  using the algorithm in Theorem 4.9

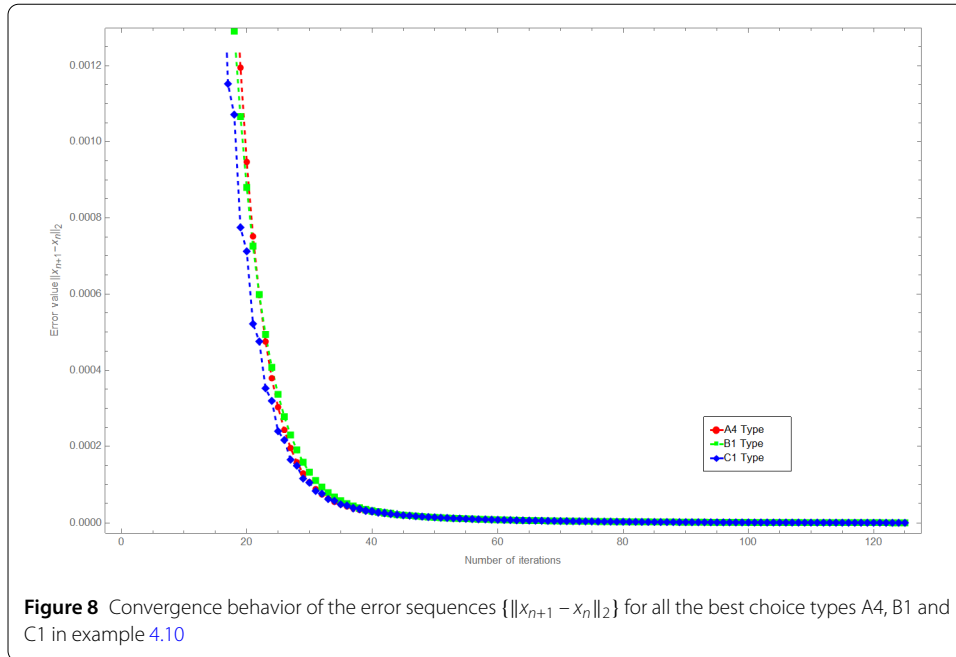
Type	$\lambda_n$ for $L = 1/\ A\ ^2$	$n$	CPU (s)	$x^*$	$\ x_{n+1} - x_n\ _2$
A1	$\frac{L}{10}$	386	0.062	$(1.18574, 1.12842, 0.651375, -0.114802)^T$ $Ax^* = (0.99993, 1.99996, 2.99998)^T$	$9.98894 \times 10^{-7}$ $\ x^*\ _2 = 1.76174$
A2	$L - \frac{L}{10}$	125	0.031	$(1.08648, 1.32706, 0.730823, 0.048118)^T$ $Ax^* = (1.00001, 2.00001, 3)^T$	$9.87153 \times 10^{-7}$ $\ x^*\ _2 = 1.86492$
A3	$L$	125	0.031	$(1.08274, 1.33452, 0.733809, 0.050358)^T$ $Ax^* = (1.00001, 2, 3)^T$	$9.86456 \times 10^{-7}$ $\ x^*\ _2 = 1.8693$
A4	$L + \frac{L}{10}$	125	0.032	$(1.08078, 1.33845, 0.735378, 0.0515346)^T$ $Ax^* = (1, 2, 3)^T$	$9.85966 \times 10^{-7}$ $\ x^*\ _2 = 1.87162$
A5	$2L - \frac{L}{10}$	272	0.063	$(1.20335, 1.09329, 0.637318, -0.0220119)^T$ $Ax^* = (0.999999, 2, 3)^T$	$9.92596 \times 10^{-7}$ $\ x^*\ _2 = 1.74643$
B1	$\frac{L_n}{n+1}$	125	0.047	$(1.13489, 1.23023, 0.692093, 0.0190705)^T$ $Ax^* = (1.00001, 2, 3)^T$	$9.87325 \times 10^{-7}$ $\ x^*\ _2 = 1.8113$
B2	$\frac{L(n+2)}{n+1}$	125	0.046	$(1.09528, 1.30945, 0.723779, 0.0428354)^T$ $Ax^* = (1.00001, 2, 3)^T$	$9.86595 \times 10^{-7}$ $\ x^*\ _2 = 1.85472$
C1	$L + \frac{(-1)^n L}{n+1}$	125	0.031	$(1.13411, 1.23179, 0.692716, 0.0195383)^T$ $Ax^* = (1.00001, 2, 3)^T$	$9.86427 \times 10^{-7}$ $\ x^*\ _2 = 1.81211$
C2	$L + \frac{(-1)^{n+1} L}{n+1}$	125	0.031	$(1.07889, 1.34224, 0.736895, 0.0526719)^T$ $Ax^* = (1.00001, 2, 3)^T$	$9.87253 \times 10^{-7}$ $\ x^*\ _2 = 1.87387$



error as follows:  $x_1 \in C$ , and

$$x_{n+1} = \underbrace{(I + \lambda_n B)^{-1}}_{\text{backward step}} \underbrace{((I - \lambda_n A)x_n + e_n)}_{\text{forward step with an error}} = J_{\lambda_n}^B ((I - \lambda_n A)x_n + e_n), \quad \forall n \in \mathbb{N},$$

where  $\{\lambda_n\} \subset (0, \infty)$ ,  $\{e_n\} \subset H$ ,  $D(B) \subset C$ , and  $J_{\lambda_n}^B = J_{\lambda_n} = (I + \lambda_n B)^{-1}$ . It can be applied to solve many kinds of problems in optimization. For the fast convergence of the sequence  $\{x_n\}$  to its solution using this method with an  $\alpha$ -inverse  $A$  strongly monotone (or  $\frac{1}{L}$ -Lipschitzian with  $L = \alpha$ ), we choose the inertial parameter  $\lambda_n$ , which depends on  $L$  as a momentum to controlled by the operator  $A$  in the forward step of algorithm using an alternating sequence  $\{\lambda_n\} \subset (0, 2L)$  such that  $\lambda_n \rightarrow L$  as  $n \rightarrow \infty$ , which guarantees the fast



convergence of the sequence  $\{x_n\}$  to its solution. For instance,

$$\lambda_n = \begin{cases} L + \frac{(-1)^n L}{n+1} & \text{or} \\ L + \frac{(-1)^{n+1} L}{n+1}, & n \in \mathbb{N}. \end{cases}$$

Furthermore, we can choose the parameter  $\theta_n$  that controls the momentum of  $x_n - x_{n-1}$  for the fast convergence of the sequence  $\{x_n\}$  to its solution as follows:

$$\theta_n = \begin{cases} \sigma_n \in [0, 1) \text{ such that } \sigma_n \rightarrow 0 \text{ as } n \rightarrow \infty & \text{if } n \leq N, \\ \begin{cases} \min\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \theta\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise,} \end{cases} & \forall n \in \mathbb{N}, \end{cases} \text{ otherwise,}$$

where  $N \in \mathbb{N}, \theta \in [0, 1)$ , and  $\{\omega_n\}$  is a positive sequence such that  $\omega_n = o(\alpha_n)$ . For instance,  $\sigma_n = \frac{1}{2^n}$  for all  $n \in \mathbb{N}$ , which guarantees the fast convergence of the sequence  $\{x_n\}$  to its solution, except for complex problems (e.g., the image/signal recovery problems). In this case, the parameter  $\theta_n$  can be chosen as follows:

$$\theta_n = \begin{cases} \sigma_n = \frac{t_n - 1}{t_{n+1}} \text{ such that } t_1 = 1 \text{ and } t_{n+1} = \frac{1 + \sqrt{1 + 4t_n^2}}{2} & \text{if } n \leq N, \\ \begin{cases} \min\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \theta\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise,} \end{cases} & \forall n \in \mathbb{N}, \end{cases} \text{ otherwise,}$$

where  $N \in \mathbb{N}, \theta \in [0, 1), \sigma_n \in [0, 1)$  are such that  $\sigma_n \rightarrow 1$  as  $n \rightarrow \infty$ , and  $\{\omega_n\}$  is a positive sequence such that  $\omega_n = o(\alpha_n)$ .

### 5 Conclusion

We obtain the regularization method for solving the variational inclusion problem of the sum of two monotone operators in real Hilbert spaces. Under some mild appropriate con-

ditions on the parameters, we obtain a short proof of another strong convergence theorem for this problem.

#### Acknowledgements

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