Shanjit and Rohen *Journal of Inequalities and Applications* (2021) 2021:124
https://doi.org/10.1186/s13660-021-02660-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Non-convex proximal pair and relatively nonexpansive maps with respect to orbits

Laishram Shanjit[1*] and Yumnam Rohen[1]

*Correspondence:
shanjit2015@yahoo.co.in
[1]Department of Mathematics, NIT Manipur, Langol, India

Abstract

Every non-convex pair (C,D) may not have proximal normal structure even in a Hilbert space. In this article, we use cyclic relatively nonexpansive maps with respect to orbits to show the presence of best proximity points in $C \cup D$, where $C \cup D$ is a cyclic T-regular set and (C,D) is a non-empty, non-convex proximal pair in a real Hilbert space. Moreover, we show the presence of best proximity points and fixed points for non-cyclic relatively nonexpansive maps with respect to orbits defined on $C \cup D$, where C and D are T-regular sets in a uniformly convex Banach space satisfying $T(C) \subseteq C$, $T(D) \subseteq D$ wherein the convergence of Kranoselskii's iteration process is also discussed.

MSC: 47H09; 47H10; 54H25

Keywords: Proximal parallel pair; Cyclic T-regular set; Kranoselskii's iteration; Fixed point; Best proximity point

1 Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a normed linear space and let C and D be non-empty subsets of X. A map $T: C \cup D \to X$ with $T(C) \subseteq D$, $T(D) \subseteq C$ (or $T(C) \subseteq C$, $T(D) \subseteq D$) and $\|Tu - Tv\| \leq \|u - v\|$, for $u \in C$, $v \in D$ is known as a relatively nonexpansive map (see [1]). A relatively nonexpansive map may not be continuous (see for an example [2]). If $C \cap D \neq \phi$, then $T: C \cap D \to C \cap D$ is a nonexpansive map.

If $Tw \neq w$, then it is endeavoured to get a point $w_0 \in C$ and $d(w_0, Tw_0) = \operatorname{dist}(C,D)$, where $\operatorname{dist}(C,D) := \inf\{d(w,z) : w \in C, z \in D\}$. A point $w_0 \in C \cup D$ is known as a best proximity point for T when $d(w_0, Tw_0) = \operatorname{dist}(C,D)$ holds true.

Eldred et al. [1] introduced proximal normal structure for a non-empty, convex pair (C,D) of X and proved two interesting theorems (See Theorem 2.1 and Theorem 2.2 of [1]). Every non-empty, convex pair of subsets (C,D) in a uniformly convex Banach space has proximal normal structure (see [1, 3]). Every non-empty, non-convex pair of subsets (C,D), even in a Hilbert space, a proximal normal structure may or may not exist (see [4]).

The notion of cyclic T-regular set was introduced by Rajesh et al. [4], which was an extension of T-regular set introduced by Veeramani [5]. The notion of cyclic T-regular set and T-regular set for relatively nonexpansive maps affirms the presence of best proximity points and fixed points on a non-empty non-convex pair (see [4–6]). For any pair of subsets

(C,D) of X, let

$$R(a,D) := \sup\{\|a-b\| : b \in D\}, \quad a \in C \quad \text{and}$$
$$\delta(C,D) := \sup\{R(a,D) : a \in C\}.$$

Definition 1 A non-empty pair (K_1,K_2) of subsets in a normed linear space, X is known as a proximal pair [1] if for every $w_1 \in K_1$, $z_1 \in K_2$, there exist $w_2 \in K_1$, $z_2 \in K_2$ so that $\|w_1 - z_2\| = \operatorname{dist}(K_1,K_2) = \|w_2 - z_1\|$ and proximal parallel pair [7] if

(i) for any $(w_1,z_1) \in K_1 \times K_2$, there is unique $(w_2,z_2) \in K_1 \times K_2$ so that $\|w_1 - z_2\| = \operatorname{dist}(K_1,K_2) = \|w_2 - z_1\|$ and

(ii) $K_2 = K_1 + h$, where $h \in X$.

The proximal parallel pair (K_1,K_2) is said to have the rectangle property [8] if and only if $\|k_1 + h - k_1'\| = \|k_1' + h - k_1\|$, for $k_1, k_1' \in K_1$, where $K_2 = K_1 + h$, $h \in X$.

Proposition 1 ([4, 8]) *Let X be a strictly convex Banach space and (K_1,K_2) be a non-empty, non-convex weakly compact proximal pair with* $\operatorname{dist}(K_1,K_2) = \operatorname{dist}(\overline{\operatorname{conv}}(K_1),\overline{\operatorname{conv}}(K_2))$. *Then the pairs (K_1,K_2) and $(\overline{\operatorname{conv}}(K_1),\overline{\operatorname{conv}}(K_2))$ are proximal parallel pair in X.*

Moreover, if (K_1,K_2) is convex and X is a real Hilbert space, then, for $x,y \in K_1$, $\langle x-y,h\rangle = 0$, where $h \in X$ and $K_2 = K_1 + h$.

The notion of cyclic T-regular set was introduced by Rajesh et al. [4].

Definition 2 ([4]) Let (K_1,K_2) be a non-empty, non-convex proximal pair in a normed linear space X. Let $T : K_1 \cup K_2 \to X$ be a map with $T(K_1) \subseteq K_2$ and $T(K_2) \subseteq K_1$. The set $K_1 \cup K_1$ is known as a cyclic T-regular set if

(i) $\frac{u+Tu'}{2} \in K_1$, for $u \in K_1$, $u' \in K_2$ so that $\|u - u'\| = \operatorname{dist}(K_1,K_2)$ and

(ii) $\frac{v+Tv'}{2} \in K_1$, for $v \in K_2$, $v' \in K_1$ so that $\|v - v'\| = \operatorname{dist}(K_1,K_2)$.

In the above definition, if $K_1 = K_2$, then it reduces to being T-regular as defined by Veeramani [5].

Definition 3 ([5]) Let X be a normed linear space, $K \subseteq X$, and $T : K \to K$. The set K is said to be a T-regular set if $\frac{u+Tu}{2} \in K$, for $u \in K$.

Let L and M be non-empty subsets and let $T : L \cup M \to X$ with $T(L) \subseteq M$, $T(M) \subseteq L$ (or $T(L) \subseteq L$, $T(M) \subseteq M$). Let $a_0 \in L$ (or M). (i) If $T(L) \subseteq M$, $T(M) \subseteq L$, then $O(a_0) := \{a_0, Ta_0, \dots, T^n a_0, \dots\}$, $T^{2n}a_0 \in L$ (or M) and $T^{2n+1}a_0 \in M$ (or L), $n = 0,1,2,\dots$; (ii) If $T(L) \subseteq L$, $T(M) \subseteq M$, then $O(a_0) := \{a_0, Ta_0, \dots, T^n a_0, \dots\}$, $O(a_0) \subseteq L$ (or M), $n = 0,1,2,\dots$.

Definition 4 ([9]) Let X be a Banach space and let L and M be non-empty subsets of X. A map $T : L \cup M \to X$ with $T(L) \subseteq L$, $T(M) \subseteq M$ is said to be a non-cyclic relatively nonexpansive map with respect to orbits provided that for every $a \in L$, $b \in M$ if $\|a - b\| = \mathit{dist}(L,M)$ then $\|Ta - Tb\| = \mathit{dist}(L,M)$, otherwise $\|Ta - Tb\| \le R(a, O(b))$ and $\|Ta - Tb\| \le R(b, O(a))$.

If $L = M$, then it reduces to being nonexpansive with respect to orbits given by Harandi et al. [10]. Motivating by the definitions of Gabeleh et al. [9] and Harandi et al. [10], Shanjit et al. [11] introduced the following definition.

Definition 5 ([11]) Let X be a Banach space and let L and M be non-empty subsets of X. A map $T : L \cup M \to X$ with $T(L) \subseteq M$ and $T(M) \subseteq L$ is said to be a cyclic relatively nonexpansive map with respect to orbits provided that for every $a \in L$, $b \in M$ if $\|a - b\| = \operatorname{dist}(L, M)$, then $\|Ta - Tb\| = \operatorname{dist}(L, M)$, otherwise $\|Ta - Tb\| \le R(a, O(b))$, $\|Tb - Ta\| \le R(b, O(a))$.

Remark 1 Let (L, M) be a non-empty, convex proximal pair in a Banach space X and $T : L \cup M \to L \cup M$ be a relatively nonexpansive map.

(i) If $T(L) \subseteq M$ and $T(M) \subseteq L$, then T is a relatively nonexpansive map with respect to orbits and $L \cup M$ is a cyclic T-regular set.

(ii) If $T(L) \subseteq L$ and $T(M) \subseteq M$, then T is a relatively nonexpansive map with respect to orbits and L and M are T-regular sets.

2 Main results

We prove the following proposition.

Proposition 2 *Let (L, M) be a non-empty, non-convex weakly compact proximal pair in a real Hilbert space satisfying $\operatorname{dist}(\overline{\operatorname{conv}}(L), \overline{\operatorname{conv}}(M)) = \operatorname{dist}(L, M)$. Then (L, M) has the rectangle property.*

Proof From Proposition 1, the pairs $(\overline{\operatorname{conv}}(L), \overline{\operatorname{conv}}(M))$ and (L, M) are proximal parallel pair in X. Let $s_1, s_2 \in L$. Then we have $s_1 + h, s_2 + h \in M$, where $h \in X$. Now,

$$\|s_1 + h - s_2\|^2 = \|s_1 - s_2\|^2 + \|h\|^2 + 2\operatorname{Re}\langle s_1 - s_2, h\rangle.$$

Since $(s_1, s_1 + h), (s_2, s_2 + h) \in (\overline{\operatorname{conv}}(L), \overline{\operatorname{conv}}(M))$, from Proposition 1, $s_1 - s_2$ is orthogonal to h that is, $\langle s_1 - s_2, h\rangle = 0$. Hence, $\|s_1 + h - s_2\| = \|s_2 + h - s_1\|$ for every $s_1, s_2 \in L$. This shows that the pair (L, M) has the rectangle property. □

Lemma 1 *Let X be a strictly convex Banach space and let (L, M) be a non-empty, non-convex weakly compact proximal pair satisfying*

$$\operatorname{dist}\bigl(\overline{\operatorname{conv}}(L), \overline{\operatorname{conv}}(M)\bigr) = \operatorname{dist}(L, M).$$

Let $T : L \cup M \to X$ be a cyclic relatively nonexpansive map with respect to orbits so that $L \cup M$ is a cyclic T-regular set.

Additionally, it is assumed that (L, M) is a minimal proximal pair. Then $L \subseteq \overline{\operatorname{conv}}(T(M))$ and $M \subseteq \overline{\operatorname{conv}}(T(L))$.

Proof Let $E = \overline{\operatorname{conv}}(T(M)) \cap L$ and $F = \overline{\operatorname{conv}}(T(L)) \cap M$. Then $E \subseteq L$ and $F \subseteq M$ are non-convex weakly compact subsets of X. Suppose $(u, v) \in (L, M)$ so that $\|u - v\| = \operatorname{dist}(L, M)$. Then $(Tv, Tu) \in T(M) \times T(L)$, which implies $(Tv, Tu) \in (E, F)$. Since $\|u - v\| = \operatorname{dist}(L, M)$, it follows that $\|Tv - Tu\| = \operatorname{dist}(L, M)$. Hence $\operatorname{dist}(E, F) = \operatorname{dist}(L, M)$. To claim that the pair (E, F) is a proximal, it suffices to prove that, for every $u \in E$, we have $v \in F$ so that

$$\operatorname{dist}(L, M) = \|u - v\|.$$

Let $u \in E = \overline{\mathrm{conv}}(T(M)) \cap L$. Then $u = \sum_{i=1}^{\infty} \alpha_i T v_i$, where $v_i \in M$, $\alpha_i \geq 0$ and $\sum_{i=1}^{\infty} \alpha_i = 1$. Since (L, M) is a proximal pair, we have $v_i' \in L$ so that

$$\mathrm{dist}(L, M) = \|v_i' - v_i\|, \quad i = 1, 2, \ldots, n.$$

Then $u' = \sum_{i=1}^{\infty} \alpha_i T v_i' \in \overline{\mathrm{conv}}(T(L))$ so that $\|u - u'\| = \mathrm{dist}(L, M)$ and $u' \in F$. Hence, (E, F) is a proximal parallel pair (and hence proximal parallel pair). Let $(u_1, v_1) \in E \times F$. Then we have $(v_1', u_1') \in E \times F$ so that

$$\|u_1 - u_1'\| = \mathrm{dist}(L, M) = \|v_1' - v_1\|.$$

As $u_1 \in \overline{\mathrm{conv}}(T(M))$ and $Tu_1' \in \overline{\mathrm{conv}}(T(M))$, which implies $\frac{u_1 + Tu_1'}{2} \in \overline{\mathrm{conv}}(T(M))$. Again, $\frac{u_1 + Tu_1'}{2} \in L$. This shows that $\frac{u_1 + Tu_1'}{2} \in E$, where $\|u_1 - u_1'\| = dist(L, M)$. Similarly, $\frac{v_1 + Tv_1'}{2} \in F$, where $\|v_1 - v_1'\| = \mathrm{dist}(L, M)$. This shows that $E \cup F$ is a cyclic T-regular set and $(L, M) := (E, F)$. Hence, $L \subseteq \overline{\mathrm{conv}}(T(M))$ and $M \subseteq \overline{\mathrm{conv}}(T(L))$. □

Lemma 2 *Let X be a strictly convex Banach space and let (L, M) be a non-empty, non-convex weakly compact proximal pair in X with*

$$\mathrm{dist}\big(\overline{\mathrm{conv}}(L), \overline{\mathrm{conv}}(M)\big) = \mathrm{dist}(L, M).$$

Let $T : L \cup M \to X$ be a relatively nonexpansive map with respect to orbits with $T(L) \subseteq L$, $T(M) \subseteq M$ and let L and M be cyclic T-regular sets.

Additionally, it is assumed that (L, M) is a minimal proximal pair. Then $L \subseteq \overline{\mathrm{conv}}(T(L))$ and $M \in \overline{\mathrm{conv}}(T(M))$.

Proof Let $E = \overline{\mathrm{conv}}(T(L)) \cap L$ and $F = \overline{\mathrm{conv}}(T(M)) \cap M$. Then $E \subseteq L$ and $F \subseteq M$ are non-empty, non-convex weakly compact subsets. Suppose $u \in L$ and $v \in M$ so that $\|u - v\| = \mathrm{dist}(L, M)$. Then $(Tu, Tv) \in T(L) \times T(M)$, which implies $(Tu, Tv) \in E \times F$. Since $\|u - v\| = \mathrm{dist}(L, M)$, it follows that $\|Tv - Tu\| = \mathrm{dist}(L, M)$. Hence $\mathrm{dist}(E, F) = \mathrm{dist}(L, M)$. Also, (E, F) is a proximal parallel pair with $T(E) \subseteq E$, $T(F) \subseteq F$ and E and F are T-regular sets. This proves that $(L, M) := (E, F)$. Hence $L \subseteq \overline{\mathrm{conv}}(T(L))$ and $M \in \overline{\mathrm{conv}}(T(M))$. □

Theorem 1 *Let X be a real Hilbert space and let (C, D) be a non-empty, non-convex weakly compact proximal pair of subsets with*

$$\mathrm{dist}\big(\overline{\mathrm{conv}}(C), \overline{\mathrm{conv}}(D)\big) = \mathrm{dist}(C, D).$$

Let $T : C \cup D \to X$ be a cyclic relatively nonexpansive map with respect orbits. Suppose $C \cup D$ is a cyclic T-regular set. Then we have $u \in C \cup D$ so that $\|u - Tu\| = \mathrm{dist}(C, D)$.

Proof Let $\mathcal{F}$ be the collection of all non-empty, non-convex weakly closed proximal pair of subsets (L, M) in (C, D), with $\mathrm{dist}(C, D) = \mathrm{dist}(L, M)$ and $L \cup M$ is a cyclic T-regular set. $\mathcal{F}$ is non-empty as $(C_0, D_0) \in \mathcal{F}$.

By Zorn's lemma, partially ordered set $\mathcal{F}$ has a minimal pair under set inclusion order, say (L, M). Therefore, from Lemma 1, we see that

$$L \subseteq \overline{\mathrm{conv}}(T(M) \text{ and } M \subseteq \overline{\mathrm{conv}}\big(T(L)\big).$$

If $\delta(L,M) = \operatorname{dist}(C,D)$, we get our result and the theorem is complete. Suppose $\delta(L,M) > \operatorname{dist}(C,D)$. Then $Tu \neq u + h$ and $T(u + h) \neq u$ for every $u \in L$. Fix $u_0 \in L$. Since X is a real Hilbert space and $L \cup M$ is a cyclic T-regular set, we have $\beta \in]0,1[$ so that $R(w,M) \leq \beta\delta(L,M)$, $R(w',L) \leq \beta\delta(L,M)$, where $w = \frac{u_0+T(u_0+h)}{2} \in L$ and $w' = \frac{u_0+h+Tu_0}{2} \in M$. Define

$$P = \{u \in L : R(u,M) \leq \beta\delta(L,M)\} \quad \text{and} \quad Q = \{v \in M : R(v,L) \leq \beta\delta(L,M)\}.$$

Then (P,Q) is a non-empty, non-convex weakly compact proximal parallel pair with $\operatorname{dist}(P,Q) = \operatorname{dist}(C,D)$. From Proposition 2, the pair (P,Q) has the rectangle property and, for $u \in P$,

$$\begin{aligned} R(Tu,L) &= \sup\{\|Tu - w\| : w \in L\} \\ &\leq \sup\{\|Tu - w\| : w \in \overline{\operatorname{conv}}(T(M))\} \\ &= \sup\{\|Tu - Tv\| : Tv \in T(M)\} \\ &\leq \sup\{R(u,O(v)) : v \in M\} \leq R(u,M) \leq \beta\delta(L,M). \end{aligned}$$

This shows that $T(P) \subseteq Q$. Similarly, for $v \in Q$,

$$\begin{aligned} R(Tv,M) &= \sup\{\|Tv - z\| : z \in M\} \\ &\leq \sup\{\|Tv - z\| : z \in \overline{\operatorname{conv}}(T(L))\} \\ &= \sup\{\|Tv - Tu\| : Tu \in T(L)\} \\ &\leq \sup\{R(v,O(u)) : u \in L\} \leq R(v,L) \leq \beta\delta(L,M). \end{aligned} \tag{1}$$

This shows that $T(Q) \subseteq P$. Since (P,Q) is a proximal parallel pair, for every $(u,v) \in P \times Q$ we have $(v',u') \in P \times Q$ so that $\|u - u'\| = \|v - v'\| = \operatorname{dist}(C,D)$ and $(Tu',Tv') \in P \times Q$. Clearly, $\frac{u+Tu'}{2} \in L$ and $\frac{v+Tv'}{2} \in M$. Now,

$$\begin{aligned} R\left(\frac{u + Tu'}{2},M\right) &= \sup\left\{\left\|\frac{u + Tu'}{2} - y\right\| : y \in M\right\} \\ &\leq \frac{1}{2}\sup\{\|u - y\| : y \in M\} + \frac{1}{2}\sup\{\|Tu' - y\| : y \in M\} \\ &= \frac{1}{2}R(u,M) + \frac{1}{2}R(Tu',M) \leq \beta\delta(L,M) \quad [\text{by Eq. (1)}], \end{aligned}$$

which means $\frac{u+Tu'}{2} \in P$. Similarly, $\frac{v+Tv'}{2} \in Q$. This shows that $P \cup Q$ is a cyclic T-regular set. Therefore, $(P,Q) \in \mathcal{F}$. But $\delta(L,M) = \sup_{u\in P} R(u,M) \leq \beta\delta(L,M) < \delta(L,M)$, which is a contradiction. Hence, L and M are singleton sets. Therefore, we have $u \in C \cup D$ so that $\|u - Tu\| = \operatorname{dist}(C,D)$. □

Example 1 Let $X = (\mathcal{R}^2, \|\cdot\|)$ be a Euclidean space. Let

$$L = \left\{(-1,-c) : c \in \mathcal{Q} \cap \left[-\frac{1}{2},\frac{1}{2}\right]\right\} \quad \text{and} \quad M = \left\{(1,-d) : d \in \mathcal{Q} \cap \left[-\frac{1}{2},\frac{1}{2}\right]\right\},$$

where $\mathcal{Q}$:= the set of rational numbers. Then (L,M) is a non-empty, non-convex proximal parallel pair with $\operatorname{dist}(L,M) = \operatorname{dist}(\overline{\operatorname{conv}}(L),\overline{\operatorname{conv}}(M)) = 2$ and $M = L + h$, $h = (2,0)$. Also, (L,M) has the rectangle property.

Let $T : L \cup M \to L \cup M$ by

$$Tu = T(u_1,u_2) = \left(u_1, -\frac{u_2}{2}\right) + (2,0), \quad u \in L \quad \text{and}$$

$$Tv = T(v_1,v_2) = \left(v_1, -\frac{v_2}{3}\right) - (2,0), \quad v \in M.$$

Clearly, $L \cup M$ is a cyclic T-regular set. The map T is not a relatively nonexpansive map but a relatively nonexpansive map with respect to orbits. Then, from Theorem 1, we have $((-1,0),(1,0)) \in L \times M$ so that $\|(-1,0) - T(-1,0)\| = \operatorname{dist}(L,M) = \|(1,0) - T(1,0)\|$.

If the non-empty pair (C,D) is convex, then from Theorem 1, we obtain the following corollary.

Corollary 1 ([11]) *Let X be a uniformly convex Banach space and let (C,D) be a non-empty, convex weakly compact proximal pair of subsets in X having the rectangle property. Let $T : C \cup D \to X$ be a cyclic relatively nonexpansive map with respect to orbits. Then we have $u \in C \cup D$ so that $\|u - Tu\| = \operatorname{dist}(C,D)$.*

The following theorem proves that a relatively nonexpansive map with respect to orbits T defined on $C \cup D$ has fixed points in C and D.

Theorem 2 *Let X be a uniformly convex Banach space and let (C,D) be a non-empty, non-convex weakly compact proximal pair in X with*

$$\operatorname{dist}\bigl(\overline{\operatorname{conv}}(C),\overline{\operatorname{conv}}(D)\bigr) = \operatorname{dist}(C,D).$$

Let $T : C \cup D \to X$ be a relatively nonexpansive map with respect to orbits with $T(C) \subseteq C$, $T(D) \subseteq D$. Suppose C and D are T-regular sets. Then we have $(Tu,Tv) = (u,v) \in C \times D$ so that $\|u - v\| = \operatorname{dist}(C,D)$.

Proof Let $\mathcal{F}$ be the collection of all non-empty, non-convex weakly closed proximal pair of subsets (L,M) in (C,D), satisfying $\operatorname{dist}(L,M) = \operatorname{dist}(C,D)$, $T(L) \subseteq L$, $T(M) \subseteq M$ and let L and M be T-regular sets. $\mathcal{F}$ is non-empty as $(C_0,D_0) \in \mathcal{F}$. By Zorn's lemma, the partially ordered set $\mathcal{F}$ has the minimal pair under set inclusion order, say (L,M). Therefore, from Lemma 2, we see

$$L \subseteq \overline{\operatorname{conv}}\bigl(T(L)\bigr) \quad \text{and} \quad M \subseteq \overline{\operatorname{conv}}\bigl(T(M)\bigr).$$

If $\delta(L,M) = \operatorname{dist}(C,D)$, we get our result and the theorem is complete. Suppose

$$\delta(L,M) > \operatorname{dist}(C,D).$$

Fix $u_0 \in L$. Since X is a uniformly convex space and L and M are T-regular sets, we have $\beta \in]0,1[$ so that $R(w,M) \leq \beta\delta(L,M)$ and $R(w',L) \leq \beta\delta(L,M)$, where $w = \frac{u_0+Tu_0}{2} \in L$ and

$w' = w + h$. Define

$$P = \{u \in L : R(u, M) \leq \beta\delta(L, M)\} \quad \text{and} \quad Q = \{v \in M : R(v, M) \leq \beta\delta(L, M)\}.$$

Then (P, Q) is a non-convex weakly compact proximal pair (and hence proximal parallel pair). Since $L \subseteq \overline{\text{conv}}(T(L))$, $M \subseteq \overline{\text{conv}}(T(M))$ and, for $u \in P$,

$$\begin{aligned} R(Tu, M) &= \sup\{\|Tu - w\| : w \in M\} \\ &\leq \sup\{\|Tu - w\| : w \in \overline{\text{conv}}(T(M))\} \\ &= \sup\{\|Tu - Tv\| : Tv \in T(M)\} \\ &\leq \sup\{R(u, O(v)) : v \in M, O(v) \subseteq M\} \\ &\leq R(u, M) \leq \beta\delta(L, M). \end{aligned} \tag{2}$$

This shows that $T(P) \subseteq P$. Similarly, for $v \in Q$,

$$\begin{aligned} R(Tv, L) &= \sup\{\|Tv - z\| : z \in L\} \\ &\leq \sup\{\|Tv - z\| : z \in \overline{\text{conv}}(T(L))\} \\ &= \sup\{\|Tv - Tu\| : Tu \in T(L)\} \\ &\leq \sup\{R(v, O(u)) : u \in L, O(u) \subseteq L\} \\ &\leq R(v, L) \leq \beta\delta(L, M). \end{aligned}$$

This shows that $T(Q) \subseteq Q$. Let $u \in P$, then $Tu \in P$. Since L is a T-regular set, $\frac{u+Tu}{2} \in L$. Now,

$$\begin{aligned} R\left(\frac{u + Tu}{2}, M\right) &= \sup\left\{\left\|\frac{u + Tu}{2} - y\right\| : y \in M\right\} \\ &\leq \frac{1}{2}R(u, M) + \frac{1}{2}R(Tu, M) \leq \beta\delta(L, M) \quad [\text{from Eq. (2)}]. \end{aligned}$$

This shows that $\frac{u+Tu}{2} \in P$. Similarly, $\frac{v+Tv}{2} \in Q$, $v \in Q$. Hence, P and Q are T-regular sets. Therefore, $(P, Q) \in \mathcal{F}$. This forces that $\beta = 1$. Thus, $\delta(L, M) = \text{dist}(L, M)$. Since $M = L + h$, we have $L = \{u\}$ and $M = \{u + h\}$ for some $u \in C$. Therefore, we have $(Tu, Tv) = (u, v) \in C \times D$ so that $\|u - v\| = \text{dist}(C, D)$. □

If the non-empty pair (C, D) is convex, then from Theorem 2, we obtain the following corollary.

Corollary 2 ([9]) *Let X be a uniformly convex Banach space, and let (C, D) be a non-empty, convex weakly compact proximal pair of subsets in X. Let $T : C \cup D \to X$ be a relatively nonexpansive map with respect to orbits with $T(C) \subseteq C$, $T(D) \subseteq D$. Then we have $(Tu, Tv) = (u, v) \in C \times D$ so that $\|u - v\| = \text{dist}(C, D)$.*

In the year 2020, Kim et al. introduced a modified Kranoselskii–Mann interactive method and gave some interesting results (see [12]). Next, we show the convergence of Kranoselskii's iteration process (see [1, 13]) for a non-convex proximal pair.

Theorem 3 *Let (L,M) be a non-empty, non-convex weakly compact proximal pair with* $\operatorname{dist}(\overline{\operatorname{conv}}(L),\overline{\operatorname{conv}}(M)) = \operatorname{dist}(L,M)$ *in a uniformly convex Banach space X. Let $T: L\cup M \to X$ be a relatively nonexpansive map with respect to orbits satisfying $T(L)\subseteq L$, $T(M)\subseteq M$. Further, assume that L and M are T-regular sets. Let an initial point $s_0 \in L$ and define a sequence*

$$s_{n+1} = \frac{s_n + Ts_n}{2}, \quad n = 0,1,2,\dots$$

Then $\lim_{n\to+\infty}\|s_n - Ts_n\| = 0$. *Moreover, if T is continuous and $T(L)$ is contained in a compact set, then* $\lim_{n\to+\infty} s_n = s$ *and* $Ts = s$.

Proof Suppose $\operatorname{dist}(L,M) > 0$. Since $\operatorname{dist}(\overline{\operatorname{conv}}(L),\overline{\operatorname{conv}}(M)) = \operatorname{dist}(L,M)$, by Proposition 1, the pairs (L,M) and $(\overline{\operatorname{conv}}(L),\overline{\operatorname{conv}}(M))$ are proximal parallel pairs in X. From Theorem 2, there exist $s\in L$, $t \in M$ so that $Ts = s$, $Tt = t$ and $\|s-t\| = \operatorname{dist}(L,M)$. L and M being T-regular sets, the sequence $\{s_n\}\subseteq L$. Now,

$$\begin{aligned} \|s_{n+1} - t\| &= \left\| \frac{s_n + Ts_n}{2} - \frac{t+Tt}{2} \right\| \\ &\le \frac{1}{2}\bigl(\|s_n - t\| + \|Ts_n - Tt\|\bigr) \le \frac{1}{2}\bigl(\|s_n - t\| + R\bigl(s_n, O(t)\bigr)\bigr) \\ &= \|s_n - t\| \quad \bigl[\text{since } Tt = t, O(t) = \{t\}, \text{where } t\in M\bigr]. \end{aligned}$$

Hence, $\{\|s_n - t\|\}$ is non-increasing and $\lim_{n\to+\infty}\|s_n - t\| = k$.

Suppose $\lim_{n\to+\infty}\|s_n - Ts_n\| \neq 0$. Then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $\|s_{n_i} - Ts_{n_i}\| \ge \varepsilon > 0$ for $i = 1,2,\dots$. Choose $\theta \in]0,1[$ and ε_1 so that $\frac{\varepsilon}{\theta} > k$ and $0 < \varepsilon_1 < \min\{\frac{k\delta(\theta)}{1-\delta(\theta)}, \frac{\varepsilon}{\theta} - k\}$.

Since X is uniformly convex, $\delta(\varepsilon_1) > 0$ for $\varepsilon_1 > 0$ is a strictly increasing function. Hence, $0 < \delta(\theta) < \frac{\varepsilon}{k+\varepsilon_1}$. So, it is possible to choose $\varepsilon_1 > 0$ so small that

$$\left(1 - \delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\right)(k+\varepsilon_1) < k.$$

As $\lim_{n\to+\infty}\|s_{n_i} - t\| = k$, choose i, so that $\|s_{n_i} - t\| \le k + \varepsilon_1$. Since $Tt = t$, we have $\|Ts_{n_i} - Tt\| \le R(s_{n_i}, O(t)) = \|s_{n_i} - t\| \le k + \varepsilon_1$. Now,

$$\begin{aligned} \|t - s_{n_{i+1}}\| &= \left\| \frac{s_{n_i} + Ts_{n_i}}{2} - \frac{t + Tt}{2} \right\| \\ &\le \left(1 - \delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\right)(k+\varepsilon_1). \end{aligned}$$

By choosing $\varepsilon_1 > 0$ so small, we get

$$\left(1 - \delta\left(\frac{\varepsilon}{k+\varepsilon_1}\right)\right)(k+\varepsilon_1) < k.$$

This shows that $\lim_{n\to+\infty}\|s_n - Ts_n\| = 0$.

Suppose $T(L)$ is contained in a compact set. Then $\{s_n\}$ has a subsequence $\{s_{n_i}\}$ so that $\lim_{i\to+\infty} s_{n_i} = s \in L$. Thus, we have $z \in M$ so that $\|s - z\| = \operatorname{dist}(L,M)$. Now,

$$\begin{aligned}\|s_{n_{i+1}} - Tz\| &= \left\|\frac{s_{n_i} + Ts_{n_i}}{2} - Tz\right\| \\ &\leq \frac{\|s_{n_i} - Tz\|}{2} + \frac{\|Ts_{n_i} - Tz\|}{2}.\end{aligned} \tag{3}$$

Since T is continuous, from Eq. (3), when $i \to +\infty$, we have

$$\|s - Tz\| \leq \frac{\|s - Tz\|}{2} + \frac{\|Ts - Tz\|}{2}.$$

Since $\|s - z\| = \operatorname{dist}(L,M)$, it follows that $\|Ts - Tz\| = \operatorname{dist}(L,M)$. Therefore, $\|s - Tz\| \leq \operatorname{dist}(L,M)$, which implies $\|s - Tz\| = \operatorname{dist}(L,M)$. By strict convexity of the norm, $Tz = z$, which implies $Ts = s$, because s is the unique point of L nearest to z. □

Example 2 Let $X = (\mathcal{R}^2, \|\cdot\|)$ be a Euclidean space. Let

$$L = \{(0,-a) : a \in \mathcal{Q} \cap [-1,1]\} \quad \text{and} \quad M = \{(1,-b) : b \in \mathcal{Q} \cap [-1,1]\},$$

where $\mathcal{Q}$:= the set of rational numbers. Then (L,M) is a non-empty, non-convex proximal parallel pair with $\operatorname{dist}(L,M) = \operatorname{dist}(\overline{\operatorname{conv}}(L), \overline{\operatorname{conv}}(M)) = 1$ and $M = L + h$, $h = (1,0)$.

Let $T : L \to L$ by

$$Ts = T(s_1,s_2) = \left(s_1, -\frac{s_2}{2}\right), \quad s \in L,$$

and $T : M \to M$ by

$$Tt = T(t_1,t_2) = \left(t_1, -\frac{t_2}{3}\right), \quad t \in M.$$

Clearly, $T(L) \subseteq L$, $T(M) \subseteq M$ and L and M are T-regular sets. The map T is not a relatively nonexpansive map but a relatively nonexpansive map with respect to orbits. Then, by Theorem 2, there exist $(0,0) \in L$, $(1,0) \in M$ so that $\|(0,0) - (1,0)\| = \operatorname{dist}(L,M)$.

Let $s_0 = (u_0,v_0) \in L$ be an initial point. Then $Ts_0 = T(u_0,v_0) = (0, -\frac{v_0}{2})$. Now,

$$s_1 = (u_1,v_1) = \frac{(u_0,v_0) + T(u_0,v_0)}{2} = \frac{(0,v_0) + (0,-\frac{v_0}{2})}{2} = \left(0, \frac{v_0}{2^2}\right).$$

Similarly, $s_2 = (u_2,v_2) = (0, \frac{v_0}{2^4})$, $s_3 = (u_3,v_3) = (0, \frac{v_0}{2^6})$ and so on. In general, $s_n = (u_n,v_n) = (0, \frac{v_0}{2^{2n}})$ and $\lim_{n\to+\infty}(u_n,v_n) = (0,0)$ and $T(0,0) = (0,0)$. In a similar way, if $s'_0 = (u'_0,v'_0) \in M$ be an initial point, then $\lim_{n\to+\infty}(u'_n,v'_n) = (1,0)$ and $T(1,0) = (1,0)$.

From Theorem 3, if $\operatorname{dist}(L,M) = 0$, $L \cap M$ is convex and T is a nonexpansive map, then we have the next result.

Corollary 3 ([13]) *Let L be a non-empty, bounded closed convex subset in a uniformly convex Banach space X and let $T : L \to L$ be a nonexpansive map. Let an initial point $s_0 \in L$ and define a sequence*

$$s_{n+1} = \frac{s_n + Ts_n}{2}, n = 1, 2, \ldots .$$

Then $\lim_{n\to+\infty} \|s_n - Ts_n\| = 0$. Moreover, if $T(L)$ is contained in a compact set, then $\lim_{n\to+\infty} s_n = s$ and $Ts = s$.

3 Conclusion

Relatively nonexpansive maps with respect to orbits, cyclic T-regular sets and T-regular sets are used to obtain our main results. The results, Theorem 1, Theorem 2 and Theorem 3, that are obtained in this article are more generalized than the results obtained in the literature. To converge Kranoselskii's iteration process to a fixed point, the map T in Theorem 3 should be continuous.

Acknowledgements
The authors are thankful to the anonymous reviewer for their valuable comments and suggestions.

Funding
The first author, Laishram Shanjit, is financially supported by the University Grant Commission, India, fellowship granting no. 420004.

Availability of data and materials
Not applicable

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LS and YR contributed equally to the preparation of this manuscript. All authors read and approved the final manuscript.

Publisher's Note

Received: 26 November 2020 Accepted: 8 July 2021 Published online: 23 July 2021

References

1. Eldred, A.A., Kirk, W.A., Veeramani, P.: Proximal normal structure and relatively nonexpansive mappings. Stud. Math. **171**, 283–293 (2005)
2. Gabeleh, M., Shahzad, N.: Seminormal structure and fixed points of cyclic relatively nonexpansive mappings. Abstr. Appl. Anal. **2014**, 123613 (2014)
3. Gao, J.: Normal structure, fixed points and modulus of *n*-dimensional *U*-convexity in Banach spaces *X* and *X**. Nonlinear Funct. Anal. Appl. **26**, 433–442 (2021)
4. Rajesh, S., Veeramani, P.: Non-convex proximal pairs on Hilbert spaces and best proximity points. J. Nonlinear Convex Anal. **15**, 515–524 (2014)
5. Veeramani, P.: On some fixed point theorems on uniformly convex Banach spaces. J. Math. Anal. Appl. **167**, 160–166 (1992)
6. Rajesh, S.: Chebyshev center, fixed point theorems and best proximity point theorems. PhD Thesis, Indian Institute of Technology Madras, India (2014)
7. Espinola, R.: A new approach to relatively nonexpansive mappings. Proc. Am. Math. Soc. **136**, 1987–1995 (2008)
8. Rajesh, S., Veeramani, P.: Best proximity point theorems for asymptotically relatively nonexpansive mappings. Numer. Funct. Anal. Optim. **37**, 80–91 (2016)
9. Gabeleh, M., Moshokoa, S.P.: Study of minimal invariant pairs for relatively nonexpansive mappings with respect to orbits. Optimization **70**, 1029–1045 (2020)
10. Harandi, A.A., Fakhar, M., Hajisharifi, H.R.: Weak fixed point property for nonexpansive mappings with respect to orbits in Banach spaces. J. Fixed Point Theory Appl. **18**, 601–607 (2016)
11. Shanjit, L., Rohen, Y.: Cyclic relatively nonexpansive mappings with respect to orbits and best proximity point theorems. J. Math. **2021**, 1–7 (2021)
12. Kim, J.K., Majee, P.: Modified Krasnoselskii–Mann iterative method for hierarchical fixed point problem and split mixed equilibrium problem. J. Inequal. Appl. **2020**, 1 (2020)
13. Kranoselskii, M.A.: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123–127 (1955)