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Common fixed point theorems for several multivalued mappings on proximinal sets in regular modular space

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Abstract

In this paper, we found a common fixed point for several multivalued mappings on proximinal sets in regular modular metric space. Also, we introduced the notions of conjoint F-proximinal contraction as well as conjoint F-proximinal contraction of Hardy–Rogers-type for several multivalued mappings. Furthermore, we enhanced our results by giving an application in integral equations.

Keywords: Multivalued mappings; Conjoint F-contraction; Δ_2-condition and Δ_M-condition

1 Introduction

In 2010, the notion of modular metric space was introduced by Chistyakov [3]. In 2012, Wardowski characterized the idea of F-contraction which generalized the Banach contraction principle in various manners and he utilized the new concept of contraction to find the fixed point theorem [15]. Also, Mongkolkeha et al. proved the existence of common fixed points for a generalized weak contractive mapping in modular spaces. Moreover, they proved the existence of some fixed point theorems without the Δ_2-condition [11].

Furthermore, in 2013, Sgroi et al. achieved a multivalued version of Wardowski's result [14].

In 2014, Abdou et al. studied the existence of fixed points for contractive-type multivalued maps in the setting of modular metric spaces [1].

In 2015, Rahimpoor et al. generalized and extended results of Mongkolkeha et al. [11] by proving some coincidence and common fixed point theorems for a contractive mapping in modular metric spaces [13].

Also, in 2016, Dilip Jain et al. presented multivalued F-contraction in the case of modular metric space with specific assumptions [7]. These results were an extension of Nadler, Wardowski, and Sgroi to the case of modular metric spaces [12, 14, 15].

In 2018, Khan et al. presented a common fixed point theorem for a pair of multivalued F-Ψ-proximinal mappings satisfying Ciric–Wardowski-type contraction in partial metric spaces. Also, they introduced an example and application to system of integral equations [10].

Khan et al. [9] introduced the RK-iterative process in the setting of modular function spaces. Also, they studied fixed points of ρ-nonexpansive mappings in modular function spaces using Δ_2 condition.

Moreover, Feng used the concept of w-compatible mappings to establish some new common coupled fixed point theorems for two hybrid pairs of mappings satisfying a symmetric type contractive condition in a partial metric space [6].

Benavides [2] revised some fixed point results for multivalued nonexpansive mappings in Banach and modular spaces. In addition, they found some new results depending either on the Opial modulus or on the Partington modulus in modular spaces.

On the other hand, in 2020, Faried et al., introduced the concepts of conjoint F-contraction and conjoint F-contraction of Hardy–Rogers-type in the case of two multivalued mappings in regular modular metric space [5].

In this work, we generalize these concepts to the case of several multivalued F-proximinal mappings in regular modular metric space. Also, we establish a common fixed point theorems for several multivalued F-proximinal mappings in regular modular metric space. Finally, we give an application from our main results which establish the existence of the solution of integral equations.

2 Preliminaries

Throughout this paper, we use the following results.

Definition 2.1 ([7]) Let X be a nonempty set. A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a *metric modular* on X if it satisfies, for all $x, y, z \in X$, the following conditions (we will write $\omega_\lambda(x,y)$ instead of $\omega(\lambda, x, y)$):

(1) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$,

(2) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$,

(3) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$.

If instead of (3), we have the condition

(4) $\omega_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$,

then ω is called *convex metric modular* on X.

Also, if instead of (1), we have the condition

(5) $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$, then ω is said to be a *metric pseudomodular* on X.

Definition 2.2 ([1]) Let ω be a pseudomodular on X. Fix $x_0 \in X$, the two sets

$X_\omega(x_0) = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x,x_0) = 0\}$ and $X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0$ such that $\omega_\lambda(x,x_0) < \infty\}$ are said to be modular spaces generated by x_0.

The spaces $X_\omega(x_0)$ and $X^*_\omega(x_0)$ are metric spaces with the metrics $d_\omega(x,y) = \inf\{\lambda > 0, \omega_\lambda(x,y) < \lambda\}$ and $d^*_\omega(x,y) = \inf\{\lambda > 0, \omega_\lambda(x,y) < 1\}$, respectively. For each $x, y \in X$ and $\lambda > 0$, [1] defined $\omega_{\lambda^+}(x,y) := \lim_{\epsilon\to 0^+} \omega_{\lambda+\epsilon(x,y)}$ and $\omega_{\lambda^-}(x,y) := \lim_{\epsilon\to 0^+} \omega_{\lambda-\epsilon(x,y)}$.

Remark 2.3 ([3])

(1) A metric modular ω on X is nonincreasing with respect to $\lambda > 0$. In fact, for any $x, y \in X$ and $0 < \mu < \lambda$, we have $\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y)$.

(2) $\omega_{\lambda^+}(x,y) \le \omega_\lambda(x,y) \le \omega_{\lambda^-}(x,y)$.

(3) If a metric modular ω on X possesses a finite value for each $x, y \in X$ and $\omega_\lambda(x,y) = \omega_\mu(x,y)$ for all $\lambda, \omega > 0$, then $d(x,y) = \omega_\lambda(x,y)$ is a metric on X.

The following indexed objects ω are simple examples of (pseudo) modulars on a set X. Let $\lambda > 0$ and $x, y \in X$, we have

Example 2.4 ([3])

$$\begin{aligned} \omega^a_\lambda(x,y) &= \infty \quad \text{if } x \neq y, \\ &= 0 \quad \text{if } x = y, \end{aligned}$$

and if (X,d) is a (pseudo)metric space with (pseudo) metric d, then we also have the following.

Example 2.5 ([3])

$$\omega^b_\lambda(x,y) = \frac{d(x,y)}{\varphi(\lambda)}$$

for all $x, y \in X$, $\lambda > 0$ where $\varphi : (0,\infty) \to (0,\infty)$ is any nondecreasing function.

Example 2.6 ([3])

$$\begin{aligned} \omega^c_\lambda(x,y) &= \infty \quad \text{if } \lambda \leq d(x,y), \\ &= 0 \quad \text{if } \lambda > d(x,y). \end{aligned}$$

Example 2.7 ([3])

$$\begin{aligned} \omega^d_\lambda(x,y) &= \infty \quad \text{if } \lambda < d(x,y), \\ &= 0 \quad \text{if } \lambda \geq d(x,y). \end{aligned}$$

Example 2.8 ([3]) Let (M,d) be a metric space and $X = M^{\mathbb{N}}$ be the set of all sequences $x : \mathbb{N} \to M$. Define $\omega_\lambda(x,y)$ by

$$\omega_\lambda(x,y) = \sup_{n\in\mathbb{N}} \left(\frac{d(x(n), y(n))}{\lambda} \right)^{\frac{1}{n}}, \quad \lambda > 0, x, y \in X.$$

In general, if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$. So, Chistyakov [3, 4] presented the following definition.

Definition 2.9 ([7]; Regular metric modular) A modular metric ω on X is said to be regular if the following condition is satisfied:

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x,y) = 0 \quad \text{for some } \lambda > 0.$$

This condition plays a significant role to ensure the existence of fixed point in modular metric space.

Definition 2.10 ([1]) Let ω be a metric modular on X then

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be ω-convergent if and only if there exists $x \in X$ such that $\omega_1(x_n, x) \to 0$ as $n \to \infty$.
(2) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be ω-Cauchy if $\omega_1(x_m, x_n) \to 0$ as $m, n \to \infty$.
(3) A subset D of X is said to be ω-complete if any ω-Cauchy sequence in D is a convergent sequence and its limit is in D.
(4) A subset D of X is said to be ω-closed if ω-limits of all ω-convergent sequences of D always belong to D.
(5) A subset D of X is said to be ω-bounded if we have $\delta_\omega(D) = \sup\{\omega_1(x, y) : x, y \in D\} < +\infty$.
(6) A subset D of X is said to be ω-compact if for any $\{x_n\}_{n\in\mathbb{N}}$ in D there exists a subsequence $\{x_{n_k}\}$ and $x \in D$ such that $\omega_1(x_{n_k}, x) \to 0$.
(7) ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X ω-convergent to x, we have $\omega_1(x, y) \le \liminf_{n\to\infty} \omega_1(x_n, y)$ for any $y \in X$.

Definition 2.11 ([7]; Δ_2-condition) Let (X, ω) be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. The metric modular ω is said to satisfy the Δ_2-condition if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$, then $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$.

Definition 2.12 ([7]; Δ_M-condition) Let (X, ω) be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. The metric modular ω is said to satisfy the Δ_M-condition if $\lim_{n\to\infty} \omega_p(x_{n+p}, x_n) = 0$ or $(n \in \mathbb{N}, p > 0)$ then $\lim_{n\to\infty} \omega_\lambda(x_{n+p}, x_n) = 0$ for some $\lambda > 0$.

3 Multivalued F-contraction on modular metric space

Throughout this paper, let $\mathcal{CB}(D)$ denote the set of all nonempty *closed and bounded* subsets of D, $\mathcal{C}(D)$ denotes the set of all nonempty *closed* subsets of D, and $\mathcal{CP}r(D)$ denotes the set of all *closed proximinal* subsets of D.

Let $A, B \in \mathcal{CP}r(D)$, we define the proximinal Hausdorff metric modular as follows: $H_{\omega_1}(A, B) := \max\{\sup_{a\in A} \omega_1(a, B), \sup_{b\in B} \omega_1(b, A)\}$ where $\omega_1(a, B) := \inf_{b\in B} \omega_1(a, b)$.

Definition 3.1 ([7]) Let $F: \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying the following conditions:

(F1) F is strictly increasing on $\mathbb{R}^+$.

(F2) For every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$.

(F3) There exists a number $k \in (0, 1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

The family of all functions F satisfying the conditions (F1)–(F3) is denoted by $\mathcal{F}$.

Definition 3.2 ([7]; F-contraction) Let D be a nonempty ω-bounded subset of a modular metric space (X, ω). For a fixed $F \in \mathcal{F}$ a multivalued mapping $T: D \to \mathcal{CB}(D)$ is called F-contraction on X if $\exists \tau \in \mathbb{R}^+$ such that for any $x, y \in D$ with $y \in Tx$ there exists $z \in Ty$ such that $\omega_1(y, z) > 0$ and the following inequality holds:

$$\tau + F\big(\omega_1(y, z)\big) \le F\big(M(x, y)\big), \tag{3.1}$$

where $M(x, y) = \max\{\omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty), \omega_1(y, Tx)\}$.

Definition 3.3 ([7]; F-contraction of Hardy–Rogers-type) Let D be a nonempty ω-bounded subset of a modular metric space (X, ω). A multivalued mapping $T : D \to \mathcal{CB}(D)$

is called an F-contraction of Hardy–Rogers-type if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$2\tau + F\big(H_\omega(Tx, Ty)\big) \le F\big(\alpha\omega_1(x,y) + \beta\omega_1(x, Tx) + \gamma\omega_1(y, Ty) + L\omega_1(y, Tx)\big). \tag{3.2}$$

Definition 3.4 ([8]; Proximinal) Let E be a closed bounded subset of a Banach space X. The set E is called proximinal in X if for all $x \in X$ there is some $e \in E$ such that $\|x - e\| = \inf\{\|x - y\| : y \in E\}$.

We will rewrite the following lemmas in the case of $\mathcal{CP}r(X)$.

Lemma 3.5 ([1]) *Let (X, ω) be a modular metric space and D be a nonempty subset of X_ω. Let $A, B \in \mathcal{CP}r(D)$ then for each $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that*

$$\omega_1(a, b) \le H_1(A, B) + \epsilon.$$

Moreover, if B is ω-compact and ω satisfies the Fatou property, then for any $a \in A$ there exists $b \in B$ such that

$$\omega_1(a, b) \le H_1(A, B).$$

Lemma 3.6 ([1]) *Let D be a nonempty subset of a modular metric space (X, ω). Assume that ω satisfies Δ_2-condition and let A_n be a sequence of sets in $\mathcal{CP}r(D)$ such that* $\lim_{n\to\infty} H_{\omega_1}(A_n, A_0) = 0$ *where $A_0 \in \mathcal{CP}r(D)$. If $x_n \in A_n$ and* $\lim_{n\to\infty} x_n = x_0$ *then $x_0 \in A_0$.*

4 Main results

Now, we are ready to give our main results.

Definition 4.1 Let D be a nonempty ω-bounded subset of a modular metric space (X, ω). For fixed $F \in \mathcal{F}$, we say that multivalued mappings $T_i, T_{i+1} : D \to \mathcal{CP}r(D)$ form a conjoint F-proximinal contraction on X for $i = 1, 2, \dots, k$. If for $0 < q < 1$ and all $x, y \in D$ such that $H_{\omega_1}(T_i x, T_{i+1} y) > 0$ the following inequality holds:

$$0 < \inf\big\{F\big(M_{T_i,T_{i+1}}(x,y)\big) - F\big(H_{\omega_1}(T_i x, T_{i+1} y)\big)\big\} \tag{4.1}$$

and $M_{T_i,T_{i+1}}(x,y) = q(\max\{\omega_1(x,y), \omega_1(y, T_i x), \omega_1(T_i x, x), \omega_1(T_{i+1} y, y)\})$.

Theorem 4.2 *Let D be a nonempty ω-bounded and ω-complete subset of a modular metric space (X, ω). Assume that ω is a regular modular satisfying Δ_M and Δ_2-conditions. If $T_i, T_{i+1} : D \to \mathcal{CP}r(D)$ form continuous conjoint F-proximinal contractions for each $i = 1, 2, \dots, k$ then they have a unique common fixed point for each $i = 1, 2, \dots, k$. In other words, there exists $u \in D$ such that $u \in T_1 u, u \in T_2 u, \dots$ and $u \in T_k u$.*

Proof Since T_1 and T_2 form a continuous conjoint F-proximinal contraction, there exists a unique common fixed point u_1 between T_1 and T_2 or $u_1 \in T_1 u_1$ and $u_1 \in T_2 u_1$. Similarly, T_2 and T_3 form a continuous conjoint F-proximinal contraction; then there exists a unique common fixed point u_2 between T_2 and T_3 or $u_2 \in T_2 u_2$ and $u_2 \in T_3 u_2$.

Now we will show that $u_1 = u_2$.

Assume contrarily that $u_1 \neq u_2$ and we have

$$0 < \inf\{F(M_{T_1,T_2}(u_1,u_2)) - F(H_{\omega_1}(T_1u_1,T_2u_2))\}.$$

Or

$$F(H_{\omega_1}(T_1u_1,T_2u_2)) + \tau \leq F(M_{T_1,T_2}(u_1,u_2)), \quad \text{for some } \tau > 0,$$

i.e.

$$\begin{aligned} H_{\omega_1}(T_1u_1,T_2u_2) &< M_{T_1,T_2}(u_1,u_2) \\ &= q(\max\{\omega_1(u_1,u_2),\omega_1(u_2,T_1u_1),\omega_1(T_1u_1,u_1),\omega_1(T_2u_2,u_2)\}) \\ &= q(\max\{\omega_1(u_1,u_2),\omega_1(u_2,Tu_1)\}). \end{aligned}$$

Since $\omega_1(u_1,u_2) \leq H_{\omega_1}(T_1u_1,T_2u_2)$,

$$\omega_1(u_1,u_2) < q(\max\{\omega_1(u_1,u_2),\omega_1(u_2,T_1u_1)\}). \tag{4.2}$$

Since $u_1 \in T_1u_1$ then $\omega_1(u_2,T_1u_1) \leq \omega_1(u_1,u_2)$, i.e.

$$\max\{\omega_1(u_1,u_2),\omega_1(u_2,T_1u_1)\} = \omega_1(u_1,u_2). \tag{4.3}$$

Then from Eqs. (4.2) and (4.3), we get

$$\omega_1(u_1,u_2) < q\omega_1(u_1,u_2).$$

Then $u_1 = u_2$, which gives a contradiction with $u_1 \neq u_2$ or $\omega_1(u_1,u_2) \neq 0$. So $u_1 = u_2 = u$ such that u is a unique common fixed point for T_1, T_2 and T_3 or $u \in T_1u$, $u \in T_2u$ and $u \in T_3u$.

By repeating this procedure for T_2, T_3 and T_4 we can deduce that there exists a unique common fixed point $v \in D$ for T_2, T_3 and T_4 such that $v \in T_2v$, $v \in T_3v$ and $v \in T_4v$. u is unique for T_1, T_2 and T_3 and v is unique for T_2, T_3 and T_4. Now

$$\begin{aligned} \omega_1(u,v) &\leq H_{\omega_1}(T_2u,T_3v) \\ &< M_{T_1,T_2}(u,v) \\ &= q(\max\{\omega_1(u,v),\omega_1(v,T_2u),\omega_1(T_2u,u),\omega_1(T_3v,v)\}) \\ &= q(\max\{\omega_1(u,v),\omega_1(v,T_2u)\}). \end{aligned}$$

Since $u \in T_2u$ we have $\omega_1(v,T_2u) \leq \omega_1(u,v)$.

Then

$$\omega_1(u,v) < q\omega_1(u,v),$$

which gives a contradiction. Therefore, $\omega_1(u,v) = 0$ and $u = v$.

We conclude that there exists a unique common fixed point for T_1, T_2, T_3 and T_4. Finally, we see that u is a common fixed point for $T_1, T_2, \dots,$ and T_k. □

Definition 4.3 Let D be a nonempty bounded subset of a modular metric space (X,ω). The multivalued mappings T_i, T_{i+1}: $D \to \mathcal{CP}r(D)$ are called conjoint F-proximinal contraction of Hardy–Rogers-type on X if there exists $F \in F$, and

$$\begin{aligned} 0 < \inf\bigl\{&F\alpha\omega_1(x,y) + \beta\omega_1(x,T_ix) + \gamma\omega_1(y,T_{i+1}y) \\ &+ L\omega_1(y,T_ix)) - F\bigl(H_{\omega_1}(T_ix,T_{i+1}y)\bigr)\bigr\} \end{aligned} \tag{4.4}$$

for all $x,y \in D$ with $H_{\omega_1}(T_ix,T_{i+1}y) > 0$, where $\alpha,\beta,\gamma,L \geq 0$, $\alpha+\beta+\gamma = 1$, $\gamma < 1$, $\beta + L < 1$ and $\alpha + L < 1$ for each $i = 1,2,\dots,k$.

Theorem 4.4 *Let D be a nonempty ω-bounded and ω-complete subset of a modular metric space (X,ω). Assume that ω is a regular modular satisfying Δ_M- and Δ_2-conditions and $T_i, T_{i+1} : D \to \mathcal{CP}r(D)$ are continuous conjoint F-proximinal contractions of Hardy–Rogers-type on X for each $i = 1,2,\dots,k$. Consequently, they have a common fixed point $u \to D$ such that $u \to T_1u, u \to T_2u,\dots$ and $u \to T_ku$.*

Proof Since T_1 and T_2 be continuous conjoint F-proximinal contraction of Hardy–Rogers-type on X, there exists $u_1 \to D$ such that $u_1 \to T_1u_1$, $u_1 \to T_2u_1$. Also, let T_2 and T_3 be continuous conjoint F-proximinal contraction of Hardy–Rogers-type on X; then there exists $u_2 \to D$ such that $u_2 \to T_2u_2$, $u_2 \to T_3u_2$.

Now we will show that $u_1 = u_2$.

Since T_1 and T_2 are continuous conjoint F-proximinal contraction of Hardy–Rogers-type on X,

$$\begin{aligned} 0 < \inf\bigl\{&F\bigl(\alpha\omega_1(u_1,u_2) + \beta\omega_1(u_1,T_1u_1) + \gamma\omega_1(u_2,T_2u_2) \\ &+ L\omega_1(u_2,T_1u_1)\bigr) - F\bigl(H_{\omega_1}(T_1u_1,T_2u_2)\bigr)\bigr\}. \end{aligned}$$

Or

$$\begin{aligned} F\bigl(H_{\omega_1}(T_1u_1,T_2u_2)\bigr) + \tau \leq{}& F\bigl(\alpha\omega_1(u_1,u_2) + \beta\omega_1(u_1,T_1u_1) + \gamma\omega_1(u_2,T_2u_2) \\ &+ L\omega_1(u_2,T_1u_1)\bigr) \end{aligned}$$

for some $\tau > 0$, i.e.,

$$H_{\omega_1}(T_1u_1,T_2u_2) < \alpha\omega_1(u_1,u_2) + \beta\omega_1(u_1,T_1u_1) + \gamma\omega_1(u_2,T_2u_2) + L\omega_1(u_2,T_1u_1).$$

Or

$$\omega_1(u_1,u_2) < \alpha\omega_1(u_1,u_2) + \beta\omega_1(u_1,T_1u_1) + \gamma\omega_1(u_2,T_2u_2) + L\omega_1(u_2,T_1u_1).$$

Since $u_1 \in T_1u_1$, $u_1 \in T_2u_1$, $u_2 \in T_2u_2$ and $u_2 \in T_3u_2$

$$\begin{aligned} \omega_1(u_1,u_2) &< \alpha\omega_1(u_1,u_2) + L\omega_1(u_2,T_1u_1) \\ &\leq \alpha\omega_1(u_1,u_2) + L\omega_1(u_2,u_1) \\ &= (\alpha + L)\omega_1(u_2,u_1), \end{aligned}$$

which gives a contradiction since $\alpha + L < 1$.

Hence, $u_1 = u_2 = u$.

Therefore, there exists $u \in D$ such that u is a common fixed point for T_1, T_2 and T_3.

By repeating this procedure for T_2, T_3 and T_4 we deduce that v is a common fixed point for T_2, T_3 and T_4.

Now we will show that $u = v$. Assume contrarily that $u \neq v$; hence for $u \in T_2u$ and $v \in T_3v$ we have

$$\begin{aligned}\omega_1(u,v) &< H\omega_1(T_2u, T_3v)\\ &< \alpha\omega_1(u,v) + \beta\omega_1(u,T_2u) + \gamma\omega_1(v,T_3v) + L\omega_1(v,T_2u)\\ &\leq \alpha\omega_1(u,v) + L\omega_1(u,v)\\ &\leq (\alpha + L)\omega_1(u,v).\end{aligned}$$

Since $\alpha + L < 1$ we have $\omega_1(u,v) = 0$ and $u = v$. Therefore, u is a common fixed point for T_1, T_2, T_3 and T_4. Finally, we get a common fixed point for $T_1, T_2, \dots,$ and T_k. □

5 Application to integral equations

In this section, we give an application of Theorem 4.2 to Volterra-type integral equations. Let $(C[0,a], \|\cdot\|_\tau)$ be a Banach space where $C[0,a]$ is the set of all continuous functions on $[0,a]$. Consider the integrals

$$u_i(t) = \int_0^t K_i\bigl(t,s,u(s)\bigr)\,ds + f_i(t) \tag{5.1}$$

for all $t \in [0,a]$ and $i = 1,2,\dots,k$. We take $u_i \in C[0,a]$ with the norm

$$\|u_i\|_\tau = \max_{t\in[0,a]}\bigl|u_i(t)e^{-\tau t}\bigr|$$

for arbitrary $\tau > 0$ and the metric

$$\omega_\lambda(u_i,u_{i+1}) = \frac{1}{\lambda}\|u_i - u_{i+1}\|_\tau = \frac{1}{\lambda}\max_{t\in[0,a]}\bigl|\bigl(u_i(t) - u_{i+1}(t)\bigr)e^{-\tau t}\bigr|$$

for all $u_i, u_{i+1} \in C[0,a]$.

Now we will prove the following theorem to ensure the existence of the solution of the system of integral equations.

Theorem 5.1 *Consider $K_i : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}, f_i : [0,a] \to \mathbb{R}$ to be continuous and $T_i : C[0,a] \to \mathcal{CP}r(C[0,a])$ as*

$$T_iu_i(t) = \biggl(\int_0^t K_i\bigl(t,s,u_i(s)\bigr)\,ds + f_i(t)\biggr)e^{-\frac{n}{i}t} \tag{5.2}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $i = 1,2,\dots,k$. If there exists $\tau > 1$, such that

$$\begin{aligned}&\sup_{n,m\in\mathbb{N}\cup\{0\}}\Bigl\{\bigl|K_i\bigl(t,s,u_i(s)\bigr)e^{-\frac{n}{i}t} - K_{i+1}\bigl(t,s,u_{i+1}(s)\bigr)e^{-\frac{m}{i+1}t}\bigr| + \bigl|f_i(t)e^{-\frac{n}{i}t} - f_{i+1}(t)e^{-\frac{m}{i+1}t}\bigr|\Bigr\}e^{-\tau t}\\ &\quad\leq \tau e^{-\tau}\bigl|M_{T_i,T_{i+1}}\bigl(u_i(t),u_{i+1}(t)\bigr)\bigr|\end{aligned} \tag{5.3}$$

for some $t \in [0,a]$, *for every* $n, m \in \mathbb{N} \cup \{0\}$, $u_i \in C[0,a]$ *and* $i = 1, 2, \dots, k$, *then the system of integral equations* (5.2) *has a solution.*

Proof Choosing x^* and y^* to be among the best approximations of $T_{i+1}u_{i+1}(t)$ and $T_i u_i(t)$, we have

$$\begin{aligned} H\bigl(T_i u_i(t), T_{i+1}u_{i+1}(t)\bigr) &= \max\Bigl\{\sup_{x\in T_i u_i(t)} \omega_1\bigl(x, T_{i+1}u_{i+1}(t)\bigr), \sup_{y\in T_{i+1}u_{i+1}(t)} \omega_1\bigl(y, T_i u_i(t)\bigr)\Bigr\} \\ &= \max\Bigl\{\sup_{x\in T_i u_i(t)} \omega_1\bigl(x, x^*\bigr), \sup_{y\in T_{i+1}u_{i+1}(t)} \omega_1\bigl(y, y^*\bigr)\Bigr\} \\ &\le \sup_{x\in T_i u_i(t), y\in T_{i+1}u_{i+1}(t)} \omega_1(x,y) \end{aligned}$$

but $\sup_{x\in T_i u_i(t), y\in T_{i+1}u_{i+1}(t)} \omega_1(x,y) \le \sup_{n,m\in\mathbb{N}\cup\{0\}}\{\int_0^t |K_i(t,s,u_i(s))e^{-\frac{n}{i}t} - K_{i+1}(t,s,u_{i+1}(s)) \times e^{-\frac{m}{i+1}t}|\,ds + |f_i(t)e^{-\frac{n}{i}t} - f_{i+1}(t)e^{-\frac{m}{i+1}t}|\,ds\}e^{-\tau t}$ so

$$\begin{aligned} H\bigl(T_i u_i(t), T_{i+1}u_{i+1}(t)\bigr) &\le \tau e^{-\tau} \int_0^t \bigl|M_{T_i,T_{i+1}}\bigl(u_i(t), u_{i+1}(t)\bigr)\bigr| e^{-\tau s} e^{\tau s}\,ds \\ &\le \bigl\|M_{T_i,T_{i+1}}\bigl(u_i(t), u_{i+1}(t)\bigr)\bigr\|_\tau \tau e^{-\tau} \int_0^t e^{\tau s}\,ds \\ &= \bigl\|M_{T_i,T_{i+1}}\bigl(u_i(t), u_{i+1}(t)\bigr)\bigr\|_\tau \tau e^{-\tau} \frac{e^{\tau t}}{\tau} \\ &= \bigl\|M_{T_i,T_{i+1}}\bigl(u_i(t), u_{i+1}(t)\bigr)\bigr\|_\tau e^{-\tau} e^{\tau t} \end{aligned}$$

for any $t \in [0,a]$, for every $n, m \in \mathbb{N} \cup \{0\}$, $u_i \in C[0,a]$ and $i = 1, 2, \dots, k$. Dividing by $e^{\tau t}$, we get

$$H\bigl(T_i u_i(t), T_{i+1}u_{i+1}(t)\bigr)e^{-\tau t} \le e^{-\tau} \bigl\|M_{T_i,T_{i+1}}\bigl(u_i(t), u_{i+1}(t)\bigr)\bigr\|_\tau.$$

So,

$$\bigl\|H\bigl(T_i u_i(t), T_{i+1}u_{i+1}(t)\bigr)\bigr\|_\tau \le e^{-\tau} \bigl\|M_{T_i,T_{i+1}}\bigl(u_i(t), u_{i+1}(t)\bigr)\bigr\|_\tau.$$

This implies that

$$\tau + \ln \bigl\|H\bigl(T_i u_i(t), T_{i+1}u_{i+1}(t)\bigr)\bigr\|_\tau \le \bigl\|M_{T_i,T_{i+1}}\bigl(u_i(t), u_{i+1}(t)\bigr)\bigr\|_\tau.$$

So, all the conditions of Theorem 4.2 are satisfied if $F(\alpha) = \ln\alpha$. Hence there exists $r \in C[0,a]$ such that

$$\begin{aligned} r(t) \in T_1 r(t) &= \biggl\{\biggl(\int_0^t K_1\bigl(t,s,r(s)\bigr)\,ds + f_1(t)\biggr)e^{-nt}\biggr\}, \\ r(t) \in T_2 r(t) &= \biggl\{\biggl(\int_0^t K_2\bigl(t,s,r(s)\bigr)\,ds + f_2(t)\biggr)e^{-\frac{n}{2}t}\biggr\}, \dots, \quad \text{and} \\ r(t) \in T_k r(t) &= \biggl\{\biggl(\int_0^t K_k\bigl(t,s,r(s)\bigr)\,ds + f_k(t)\biggr)e^{-\frac{n}{k}t}\biggr\}. \end{aligned}$$

Finally, there exist $K_i, f_i, n_1, n_2, \dots, n_k \in \mathbb{N} \cup \{0\}$ such that $r(t)$ is a solution of the system of integral equations given in (5.2) for $i = 1, 2, \dots, k$. □

6 Conclusion

In this paper we presented the new concepts of *conjoint F-contraction and conjoint F-contraction of Hardy–Rogers-type to the case of several multivalued F-proximinal mappings in regular modular metric space*. Also, we used these concepts to found a common fixed point theorems for several multivalued F-proximinal mappings in regular modular metric space. The solution of integral equations was obtained by employing the condition of conjoint F-contraction for several multivalued F-proximinal mappings in regular modular metric space.

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Abbreviations
MVM, multivalued mappings.

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