Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Multiple-sets split feasibility problem and split equality fixed point problem for firmly quasi-nonexpansive or nonexpansive mappings

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Abstract
In this paper, we propose a new iterative algorithm for solving the multiple-sets split feasibility problem (MSSFP for short) and the split equality fixed point problem (SEFPP for short) with firmly quasi-nonexpansive operators or nonexpansive operators in real Hilbert spaces. Under mild conditions, we prove strong convergence theorems for the algorithm by using the projection method and the properties of projection operators. The result improves and extends the corresponding ones announced by some others in the earlier and recent literature.

Keywords: Mutiple-sets split feasibility problem; Split equality fixed point problem; Strong convergence; Hilbert spaces; Iterative algorithm

1 Introduction and preliminaries

Let H_1, H_2, and H_3 be three real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and induce norm $\|\cdot\|$. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of a mapping T.

The split feasibility problem (SFP) in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [6] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. The SFP can be formulated as finding a point x^* in $\mathbb{R}^n$ with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ matrix. SFP (1.1) has recently been studied in a more general space. For example, Xu [21] studied it in an infinite dimensional Hilbert space.

The SFP has been widely studied in recent years. Recently, it has been found that it can also be used to model the intensity-modulated radiation therapy; see, e.g., [7–11]. One of the well-known methods for solving the SFP is Byrne's CQ algorithm [3, 4], which

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generated a sequence $\{x_n\}$ by the following iterative algorithm:

$$x_{n+1} = P_C\big(x_n - \tau_n A^*(I - P_Q)Ax_n\big), \tag{1.2}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2 and the step size τ_n is located in the interval $(0, 2/\|A\|^2)$, A^* is the adjoint of A, P_C and P_Q are the metric projections onto C and Q.

The multiple-set split feasibility problem (MSSFP), which has functions in the inverse problem of intensity-modulated radiation therapy (see [18]), has recently been presented in [5] and is formulated as follows:

$$\text{find a point } x \in C := \bigcap_{i=1}^{r_1} C_i \text{ such that } Ax \in Q := \bigcap_{j=1}^{r_2} Q_j, \tag{1.3}$$

where $r_1, r_2 \in N, C_1, \dots, C_{r_1}$ are closed convex subsets of H_1, $Q_1, \dots, Q_{r_2}$ are closed convex subsets of H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

Assuming consistency of the MSSFP, Censor et al. [5] introduced the following projection algorithm:

$$x_{n+1} = P_\Omega\left(x_n - \gamma\left(\sum_{i=1}^{r_1} \alpha_i(x_n - P_{C_i}x_n) + \sum_{j=1}^{r_2} \beta_j A^*(I - P_{Q_j})Ax_n\right)\right), \quad n \geq 0, \tag{1.4}$$

where $0 < \gamma < \frac{2}{L}$ with $L = \sum_{i=1}^{r_1} \alpha_i + \rho(A^*A)\sum_{j=1}^{r_2} \beta_j$ and $\rho(A^*A)$ is the spectral radius of A^*A. They proved convergence of algorithm (1.4) in the case where both $H_1 and H_2$ are finite dimensional.

Moudafi [17] came up with the split equality problem (SEP) as follows:

$$\text{find } x \in C, y \in Q, \text{ such that } Ax = By, \tag{1.5}$$

where $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, $C \subset H_1$, $Q \subset H_2$ are two nonempty closed convex sets. Let $B = I$, it is easy to see that the SFP is the special case of the SEP. The SEP has already been applied in game theory (see [1]) and intensity-modulated radiation therapy [5, 12]. Furthermore, the author considered the following scheme for solving the SEP:

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \gamma B^*(Ax_{k+1} - By_k)). \end{cases} \tag{1.6}$$

He obtained a weak convergence of (1.6) under certain appropriate assumptions on the parameters.

Shi [19] proposed a modification of Moudafi's ACQA algorithms to solve the SEP and proved its strong convergence:

$$w_{n+1} = P_S\big\{(1 - \alpha_n)\big[I - \gamma G^*G\big]w_n\big\}, \tag{1.7}$$

i.e.,

$$\begin{cases} x_{k+1} = P_C\{(1-\alpha_k)x_k - \gamma A^*(Ax_k - By_k)\}, & n \geq 0, \\ y_{k+1} = P_Q\{(1-\alpha_k)y_k + \gamma B^*(Ax_k - By_k)\}, & n \geq 0. \end{cases} \tag{1.8}$$

Recently, Moudafi [16] introduced the following split equality fixed point problem (SEFPP):

$$\text{find } x \in C := F(U), y \in Q := F(T) \text{ such that } Ax = By, \tag{1.9}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two firmly quasi-nonexpansive operators. The SEFPP has been proved very useful in decomposition methods for PDEs as well as in game theory and intensity-modulated radiation therapy. For solving SEFPP (1.9), he proposed the following iterative algorithm:

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases} \tag{1.10}$$

Further, he proved a weak convergence theorem for SEFPP (1.9) under some mild restrictions on the parameters.

In this paper, we introduce a multiple-sets split feasibility problem (MSSFP) and a split equality fixed point problem (SEFPP), the MSSFP is to find a pair (x, y) such that

$$(x,y) \in C \times Q := \bigcap_{i=1}^{t_1} C_i \times \bigcap_{j=1}^{r_1} Q_j \quad \text{and} \quad (A_1x, B_1y) \in D \times \Theta := \bigcap_{i=1}^{t_2} D_i \times \bigcap_{j=1}^{r_2} \Theta_j. \tag{1.11}$$

The SEFPP is to find a pair (x, y) such that

$$x \in F(T_1), \qquad y \in F(T_2) \quad \text{and} \quad A_2x = B_2y, \tag{1.12}$$

where T_1, T_2 are two firmly quasi-nonexpansive operators or nonexpansive operators, and $A_1 : H_1 \to H_3, A_2 : H_1 \to H_3, B_1 : H_2 \to H_3, B_2 : H_2 \to H_3$ are four bounded linear operators. $C_i \in H_1, i = 1, 2, \dots, t_1; Q_j \in H_2, j = 1, 2, \dots, r_1; D_i \in H_3, i = 1, 2, \dots, t_2; \Theta_j \in H_3, j = 1, 2, \dots, r_2$, are nonempty closed convex subsets.

Guan [15] proposed a new iterative scheme to solve the above problems:

$$\begin{aligned} x_{k+1} = T_1\Bigg[& x_k - \lambda_k \sum_{i=1}^{t_1} \alpha_i(x_k - P_{C_{i,k}}x_k) - \xi_k \sum_{i=1}^{t_2} \beta_i A_1^*(A_1x_k - P_{D_{i,k}}A_1x_n) \\ & - \tau A_2^*(A_2x_k - B_2y_k)\Bigg] \end{aligned} \tag{1.13}$$

and

$$y_{k+1} = T_2\Bigg[y_k - \sigma_k \sum_{j=1}^{r_1} \gamma_j (y_k - P_{Q_{j,k}} y_k) - \zeta_k \sum_{j=1}^{r_2} \delta_j B_1^* (B_1 y_k - P_{\Theta_{j,k}} B_1 y_k) \\ - \tau B_2^* (B_2 y_n - A_2 x_{k+1}) \Bigg]. \tag{1.14}$$

Further, he proved a weak convergence theorem under some mild restrictions on the parameters.

Inspired by the results, we propose the following questions.

Question 1.1 *Can we modify iterative scheme* (1.8) *to a more general iterative scheme for solving a multiple-sets split feasibility problem and a split equality fixed point problem instead of solving the split equality problem?*

Question 1.2 *Can we obtain a strong convergence by the iterative scheme for MSSFP and SEFPP?*

The purpose of this paper is to construct a new algorithm for MSSFP and SEFPP so that strong convergence is guaranteed. The paper is organized as follows. In Sect. 2, we denote the concept of minimal norm solution of MSSFP and SEFPP. Using Tychonov regularization, we obtain a net of solutions for some minimization problem approximating such minimal norm solutions (see Theorem 2.5). In Sect. 3, we introduce an algorithm and prove the strong convergence of the algorithm, more importantly, its limit is the minimum-norm solution of MSSFP and SEFPP (see Theorem 3.2).

Throughout the rest of this paper, let I denote the identity operator on a Hilbert space H, and let ∇f denote the gradient of the function $f : H \to R$.

Definition 1.3 ([21]) An operator T on a Hilbert space H is *nonexpansive* if, for each x and y in H,

$$\|Tx - Ty\| \le \|x - y\|.$$

T is said to be *strictly nonexpansive* if, for each x and y in H,

$$\|Tx - Ty\| < \|x - y\|.$$

An operator T on a Hilbert space H is *firmly nonexpansive* if, for each x and y in H,

$$\langle x - y, Tx - Ty\rangle \ge \|Tx - Ty\|^2.$$

T is firmly nonexpansive if $2T - I$ is nonexpansive, Equivalently, $T = (I + S)/2$, where $S : H \to H$ is nonexpansive.

T is said to be *averaged* if there exist $0 < \alpha < 1$ and a nonexpansive operator N such that

$$T = (1 - \alpha)I + \alpha N.$$

T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ for each x in H, q in $F(T)$,

$$\|Tx - q\| \leq \|x - q\|.$$

T is said to be *strictly quasi-nonexpansive* if $F(T) \neq \emptyset$ for each x in H, q in $F(T)$,

$$\|Tx - q\| < \|x - q\|.$$

T is said to be f*irmly quasi-nonexpansive* if $F(T) \neq \emptyset$ for each x in H, q in $F(T)$,

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2.$$

Let P_S denote the projection from H onto a nonempty closed convex subset S of H; that is,

$$P_S(w) = \left\{x \in S, \min_{x \in S} \|x - w\|\right\}.$$

It is well known that $P_S(w)$ is characterized by the inequality

$$\langle w - P_S(w), x - P_S(w)\rangle \leq 0, \quad \forall x \in S,$$

$P_S and (I - P_S)$ are nonexpansive, averaged, and firmly nonexpansive.

Next we should collect some elementary facts which will be used in the proofs of our main results.

Lemma 1.4 ([13, 14]) *Let X be a Banach space, C be a closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 1.5 ([2]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in* $[0,1]$ *with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, and $\{t_n\}$ be a sequence of real numbers with* $\limsup_n t_n \leq 0$. *Suppose that*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n t_n + u_n, \quad \forall n \in N.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 1.6 ([20]) *Let $\{w_n\}, \{z_n\}$ be bounded sequences in a Banach space, and let $\{\beta_n\}$ be a sequence in* $[0,1]$ *which satisfies the following condition*:

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $w_{n+1} = (1 - \beta_n)w_n + \beta_n z_n$ and $\limsup_{n\to\infty} \|z_{n+1} - z_n\| - \|w_{n+1} - w_n\| \leq 0$, *then* $\lim_{n\to\infty} \|z_n - w_n\| = 0$.

Lemma 1.7 ([4]) *Let f be a convex and differentiable function, and let C be a closed convex subset of H. Then* $x \in C$ *is a solution of the problem*

$$\min_{x\in C} f(x)$$

if and only if $x \in C$ *satisfies the following optimality condition*:

$$\langle \nabla f(x), v - x\rangle \geq 0, \quad \forall v \in C.$$

Moreover, if f is, in addition, strictly convex and coercive, then the minimization problem has a unique solution.

Lemma 1.8 ([4]) *Let* A, B *be averaged operators and suppose that* $\mathrm{Fix}(A) \cap \mathrm{Fix}(B)$ *is nonempty. Then* $\mathrm{Fix}(A) \cap \mathrm{Fix}(B) = \mathrm{Fix}(AB) = \mathrm{Fix}(BA)$.

2 Minimum-norm solution of SEFPP and MSSFP

In this section, we define the concept of the minimal norm solution of MSSFP (1.11) and SEFPP (1.12). Using Tychonov regularization, we obtain a net of solutions for some minimization problems approximating such minimal norm solutions.

We use Γ to denote the solution set of SEFPP and MSSFP, i.e.,

$$\Gamma = \left\{ (x,y) \in H_1 \times H_2, x \in \bigcap_{i=1}^{t_1} C_i, y \in \bigcap_{j=1}^{r_1} Q_j, A_1 x \in \bigcap_{i=1}^{t_2} D_i, B_1 y \in \bigcap_{j=1}^{r_2} \Theta_j, A_2 x = B_2 y, \right.$$
$$\left. x \in F(T_1), y \in F(T_2) \right\}$$

and assume the consistency of SEFPP and MSSFP, so that Γ is closed, convex, and nonempty.

We aim to propose a new iterative algorithm for solving MSSFP (1.11) and SEFPP (1.12). Let the sets C_i, Q_i, D_i, Θ_i be defined as

$$C_i = \{x \in H_1 : c_i(x) \leq 0\}, \qquad Q_j = \{y \in H_2 : q_j(y) \leq 0\} \tag{2.1}$$

and

$$D_i = \{u \in H_1 : d_i(u) \leq 0\}, \qquad \Theta_j = \{v \in H_2 : \phi_j(v) \leq 0\}, \tag{2.2}$$

where $c_i : H_1 \to \mathbb{R}, i = 1,2,\dots,t_1; q_i : H_2 \to \mathbb{R}, j = 1,2,\dots,r_1; d_i : H_3 \to \mathbb{R}, i = 1,2,\dots,t_2$; and $\phi_i : H_3 \to \mathbb{R}, j = 1,2,\dots,r_2$, are convex functions.

In order to solve MSSFP (1.11) and SEFPP (1.12), we consider the following minimization problem:

$$\min_{(x,y)\in \mathrm{Fix}(T_1)\times \mathrm{Fix}(T_2)} h(x,y), \tag{2.3}$$

where

$$h(x,y) = f(x) + g(y) + \frac{1}{2}\|A_2 x - B_2 y\|^2,$$

$$f(x) = \frac{1}{2}\sum_{i=1}^{t_1}\alpha_i\left\|(I-P_{C_i})x\right\|^2 + \frac{1}{2}\sum_{i=1}^{t_2}\beta_i\left\|(I-P_{D_i})A_1x\right\|^2,$$

$$g(y) = \frac{1}{2}\sum_{j=1}^{r_1}\gamma_j\left\|(I-P_{Q_j})y\right\|^2 + \frac{1}{2}\sum_{j=1}^{r_2}\delta_j\left\|(I-P_{\Theta_j})B_1y\right\|^2,$$

here $\sum_{i=1}^{t_1}\alpha_i = \sum_{i=1}^{t_2}\beta_i = \sum_{j=1}^{r_1}\gamma_j = \sum_{j=1}^{r_2}\delta_j = 1$. The minimization problem is in general ill-posed. A classical way to deal with such a possibly ill-posed problem is the well-known Tychonov regularization, which approximates a solution of problem (2.3) by the unique minimizer of the regularized problem

$$\min_{(x,y)\in \mathrm{Fix}(T_1)\times\mathrm{Fix}(T_2)} h_\alpha(x,y) = f(x) + g(y) + \frac{1}{2}\|A_2x - B_2y\|^2 + \frac{1}{2}\alpha\left(\|x\|^2 + \|y\|^2\right), \tag{2.4}$$

where $\alpha > 0$ is the regularization parameter. Denote by $w_\alpha = (x_\alpha, y_\alpha)$ the unique solution of (2.4).

Lemma 2.1 *For the sake of convenience, let $H = H_1 \times H_2$, define*:

$$M := \begin{pmatrix} \sum_{i=1}^{t_1}\alpha_i(I-P_{C_i}) & 0 \\ 0 & \sum_{j=1}^{r_1}\gamma_j(I-P_{Q_j}) \end{pmatrix},$$

$$N := \begin{pmatrix} \sum_{i=1}^{t_2}\beta_i A_1^*(I-P_{D_i})A_1 & 0 \\ 0 & \sum_{j=1}^{r_2}\delta_j B_1^*(I-P_{\Theta_j})B_1 \end{pmatrix},$$

and

$$G := (A_2, -B_2), G^*G := \begin{pmatrix} A_2^*A_2 & -A_2^*B_2 \\ -B_2^*A_2 & B_2^*B_2 \end{pmatrix},$$

where $G : H \to H_3$ and $G^*G : H \to H$, then $M, \lambda_1 N$ and $\lambda_2 G^*G$ are firmly nonexpansive operators, where $0 < \lambda_1 < 1/(\max\{\rho(A_1^*A_1), \rho(B_1^*B_1)\})$ and $0 < \lambda_2 < 1/\rho(G^*G)$.

Proof By $(I - P_S)$ and P_S are firmly nonexpansive operators, $x = (x_1, x_2) \in H_1 \times H_2, y = (y_1, y_2) \in H_1 \times H_2$.$\|Mx - My\|^2 = \|\sum_{i=1}^{t_1}\alpha_i(I-P_{C_i})x_1 - \sum_{i=1}^{t_1}\alpha_i(I-P_{C_i})y_1\|^2 + \|\sum_{j=1}^{r_1}\gamma_j(I-P_{Q_j})x_2 - \sum_{j=1}^{r_1}\gamma_j(I-P_{Q_j})y_2\|^2 \le \langle x_1 - y_1, \sum_{i=1}^{t_1}\alpha_i(I-P_{C_i})x_1 - \sum_{i=1}^{t_1}\alpha_i(I-P_{C_i})y_1\rangle + \langle x_2 - y_2, \sum_{j=1}^{r_1}\gamma_j(I-P_{Q_j})x_2 - \sum_{j=1}^{r_1}\gamma_j(I-P_{Q_j})y_2\rangle = \langle x - y, Mx - My\rangle$, so M is a firmly nonexpansive operator. Similarly, we can prove that $\lambda_1 N$ and $\lambda_2 G^*G$ are firmly nonexpansive operators. □

Proposition 2.2 *Let $T = T_1 \times T_2$, which is mentioned in (1.12), $w = (x, y)$. For any $\alpha > 0$, the solution $w_\alpha = (x_\alpha, y_\alpha)$ of (2.4) is uniquely defined. Then $w_\alpha = (x_\alpha, y_\alpha)$ is characterized by the inequality*

$$\left\langle \nabla h(w_\alpha) + \alpha w_\alpha, w - w_\alpha\right\rangle \ge 0, \quad \forall w \in \mathrm{Fix}(T),$$

i.e.,

$$\left\langle \sum_{i=1}^{t_1}\alpha_i(I-P_{C_i})x_\alpha + \sum_{i=1}^{t_2}\beta_i A_1^*(I-P_{D_i})A_1x_\alpha + A_2^*(A_2x_\alpha - B_2y_\alpha) + \alpha x_\alpha, x - x_\alpha\right\rangle \ge 0,$$

$$\forall x \in \mathrm{Fix}(T_1);$$

and

$$\left\langle \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) y_\alpha + \sum_{j=1}^{r_2} \delta_j B_1^* (I - P_{\Theta_j}) B_1 y_\alpha - B_2^* (A_2 x_\alpha - B_2 y_\alpha) + \alpha y_\alpha, y - y_\alpha \right\rangle \geq 0,$$

$$\forall x \in \mathrm{Fix}(T_2).$$

Proof It is well known that $h(x,y) = \frac{1}{2}\sum_{i=1}^{t_1} \alpha_i \|(I - P_{C_i})x\|^2 + \frac{1}{2}\sum_{i=1}^{t_2} \beta_i \|(I - P_{D_i})A_1 x\|^2 + \frac{1}{2}\sum_{j=1}^{r_1} \gamma_j \|(I - P_{Q_j})y\|^2 + \frac{1}{2}\sum_{j=1}^{r_2} \delta_j \|(I - P_{\Theta_j})B_1 y\|^2 + \frac{1}{2}\|A_2 x - B_2 y\|^2$ is convex and differentiable with gradient $\triangledown h(w) = Mw + Nw + G^* G w, h_\alpha(w) = h(w) + \frac{1}{2}\alpha\|w\|^2$. We can get that h_α is strictly convex, coercive, and differentiable with gradient

$$\triangledown h_\alpha(w) = Mw + Nw + G^* G w + \alpha w.$$

It follows from Lemma 1.7 that w_α is characterized by the inequality

$$\langle \triangledown h(w_\alpha) + \alpha w_\alpha, w - w_\alpha \rangle \geq 0, \quad \forall w \in \mathrm{Fix}(T). \tag{2.5}$$

We can get that

$$\left\langle \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) x + \sum_{i=1}^{t_2} \beta_i A_1^* (I - P_{D_i}) A_1 x + A_2^* (A x_\alpha - B y_\alpha) + \alpha x_\alpha, x - x_\alpha \right\rangle \geq 0,$$

$$\forall x \in \mathrm{Fix}(T_1);$$

and

$$\left\langle \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) y + \sum_{j=1}^{r_2} \delta_j B_1^* (I - P_{\Theta_j}) B_1 y - B_2^* (A x_\alpha - B y_\alpha) + \alpha y_\alpha, y - y_\alpha \right\rangle \geq 0,$$

$$\forall x \in \mathrm{Fix}(T_2).$$ □

Definition 2.3 An element $\bar{w} = (\bar{x}, \bar{y}) \in \Gamma$ is said to be the *minimalnormsolution* of MSSFP (1.11) and SEFPP (1.12) if $\|\bar{w}\| = \inf_{w \in \Gamma} \|w\|$.

The next result collects some useful properties of $\{w_\alpha\}$, the unique solution of (2.4).

Proposition 2.4 *Let w_α be given as the unique solution of* (2.4) *for any sequence $\{\alpha_n\}$ such that $\lim_n \alpha_n = 0$, let w_{α_n} be abbreviated as w_n. Then the following assertions hold*:

(i) $\|w_\alpha\|$ *is decreasing for $\alpha \in (0, \infty)$*;

(ii) $\alpha \mapsto w_\alpha$ *defines a continuous curve from $(0, \infty)$ to H.*

Proof Let $\alpha > \beta > 0$; since w_α and w_β are the unique minimizers of h_α and h_β, $w_\alpha = (x_\alpha, y_\alpha), w_\beta = (x_\beta, y_\beta)$, respectively, we can get that

$$h_\alpha(w_\alpha) = h(w_\alpha) + \frac{1}{2}\alpha\|w_\alpha\|^2 \leq h(w_\beta) + \frac{1}{2}\alpha\|w_\beta\|^2 = h_\alpha(w_\beta)$$

and

$$h_\beta(w_\beta) = h(w_\beta) + \frac{1}{2}\beta\|w_\beta\|^2 \le h(w_\alpha) + \frac{1}{2}\beta\|w_\alpha\|^2 = h_\beta(w_\alpha).$$

Hence we can obtain that $\|w_\alpha\| \le \|w_\beta\|$. That is to say, $\|w_\alpha\|$ is decreasing for $\alpha \in (0,\infty)$.

By Proposition 2.2, we have

$$\langle \nabla h(w_\alpha) + \alpha w_\alpha, w_\beta - w_\alpha \rangle \ge 0$$

and

$$\langle \nabla h(w_\beta) + \beta w_\beta, w_\alpha - w_\beta \rangle \ge 0.$$

It follows that

$$\langle w_\alpha - w_\beta, \alpha w_\alpha - \beta w_\beta \rangle \le \langle w_\alpha - w_\beta, \nabla h(w_\beta) - \nabla h(w_\alpha) \rangle.$$

Then

$$\begin{aligned} &\langle w_\alpha - w_\beta, \nabla h(w_\beta) - \nabla h(w_\alpha) \rangle \\ &\quad = \langle w_\alpha - w_\beta, Mw_\beta + Nw_\beta - Mw_\alpha - Nw_\alpha \rangle \\ &\qquad + \langle w_\alpha - w_\beta, G^*G(w_\beta - w_\alpha) \rangle \end{aligned}$$

and

$$\langle w_\alpha - w_\beta, G^*G(w_\beta - w_\alpha) \rangle \le 0. \tag{2.6}$$

Then

$$\begin{aligned} &\left\langle x_\alpha - x_\beta, \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) x_\beta - \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) x_\alpha \right\rangle \\ &\quad \le -\sum_{i=1}^{t_1} \alpha_i \left\| (I - P_{C_i}) x_\alpha - (I - P_{C_i}) x_\beta \right\|^2 \le 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\left\langle x_\alpha - x_\beta, \sum_{i=1}^{t_2} \beta_i A_1^* (I - P_{D_i}) A_1 x_\beta - \sum_{i=1}^{t_2} \beta_i A_1^* (I - P_{D_i}) A_1 x_\alpha \right\rangle \\ &\quad = \sum_{i=1}^{t_2} \beta_i \langle A_1 x_\alpha - A_1 x_\beta, (I - P_{D_i}) A_1 x_\beta - (I - P_{D_i}) A_1 x_\alpha \rangle \\ &\quad \le -\sum_{i=1}^{t_2} \beta_i \left\| (I - P_{D_i}) A_1 x_\alpha - (I - P_{D_i}) A_1 x_\beta \right\|^2 \le 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\left\langle y_\alpha - y_\beta, \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) y_\beta - \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) y_\alpha \right\rangle \\ &\quad \le -\sum_{j=1}^{r_1} \gamma_j \left\| (I - P_{Q_j}) y_\alpha - (I - P_{Q_j}) y_\beta \right\|^2 \le 0, \end{aligned} \tag{2.9}$$

$$\left\langle y_\alpha - y_\beta, \sum_{j=1}^{r_2} \delta_j B_1^*(I - P_{\Theta_j})B_1 y_\beta - \sum_{j=1}^{r_2} \delta_j B_1^*(I - P_{\Theta_j})B_1 y_\alpha \right\rangle$$
$$= \sum_{j=1}^{r_2} \delta_j \langle B_1 y_\alpha - B_1 y_\beta, (I - P_{\Theta_j})B_1 y_\beta - (I - P_{\Theta_j})B_1 y_\alpha \rangle$$
$$\leq -\sum_{j=1}^{r_2} \delta_j \left\| (I - P_{\Theta_j})B_1 y_\alpha - (I - P_{\Theta_j})B_1 y_\beta \right\|^2 \leq 0. \tag{2.10}$$

By (2.6)–(2.10), we can get

$$\langle w_\alpha - w_\beta, \nabla h(w_\beta) - \nabla h(w_\alpha) \rangle \leq 0.$$

Hence

$$\langle w_\alpha - w_\beta, \alpha w_\alpha - \beta w_\beta \rangle \leq 0$$
$$\alpha \|w_\alpha - w_\beta\|^2 \leq \langle w_\alpha - w_\beta, (\beta - \alpha) w_\beta \rangle.$$

It turns out that

$$\|w_\alpha - w_\beta\| \leq |\alpha - \beta|/\alpha \|w_\beta\|.$$

Thus $\alpha \mapsto w_\alpha$ defines a continuous curve from $(0, \infty)$ to H. □

Theorem 2.5 *Let w_α be given as the unique solution of* (2.4). *Then w_α converges strongly as $\alpha \to 0$ to the minimum-norm solution $\bar{w}$ of MSSFP* (1.11) *and SEFPP* (1.12).

Proof For any $0 < \alpha < \infty$, w_α is given as (2.4), it follows that

$$h_\alpha(w_\alpha) = h(w_\alpha) + \frac{1}{2}\alpha \|w_\alpha\|^2 \leq h(\bar{w}) + \frac{1}{2}\alpha \|\bar{w}\|^2 = h_\alpha(\bar{w}).$$

Since $\bar{w} \in \Gamma$ is a solution for MSSFP and SEFPP, we get

$$h(w_\alpha) + \frac{1}{2}\alpha \|w_\alpha\|^2 \leq \frac{1}{2}\alpha \|\bar{w}\|^2.$$

Hence, $\|w_\alpha\| \leq \|\bar{w}\|$ for all $\alpha > 0$. That is to say, $\{w_\alpha\}$ is a bounded net in $H = H_1 \times H_2$.

For any sequence $\{\alpha_n\}$ such that $\lim_n \alpha_n = 0$, let w_{α_n} be abbreviated as w_n. All we need to prove is that $\{w_n\}$ contains a subsequence converging strongly to $\bar{w}$.

Indeed $\{w_n\}$ is bounded and $\mathrm{Fix}(T)$ is bounded convex. By passing to a subsequence if necessary, we may assume that $\{w_n\}$ converges weakly to a point $\hat{w} \in \mathrm{Fix}(T)$. By Proposition 2.2, we get that

$$\langle \nabla h(w_n) + \alpha_n w_n, \bar{w} - w_n \rangle \geq 0$$

and

$$\langle \nabla h(w_n) + \alpha_n w_n, \hat{w} - w_n \rangle \geq 0. \tag{2.11}$$

It follows that

$$\bigl\langle \nabla h(w_n), \bar{w} - w_n \bigr\rangle \geq \alpha_n \langle w_n, w_n - \bar{w} \rangle,$$

i.e.,

$$\left\langle \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) x_n + \sum_{i=1}^{t_2} \beta_i A_1^* (I - P_{D_i}) A_1 x_n + A_2^* (A x_n - B y_n), \bar{x} - x_n \right\rangle \geq \alpha_n \langle x_n, x_n - \bar{x} \rangle,$$

and

$$\left\langle \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) y_n + \sum_{j=1}^{r_2} \delta_j B_1^* (I - P_{\Theta_j}) B_1 y_n - B_2^* (A x_n - B y_n), \bar{y} - y_n \right\rangle \geq \alpha_n \langle y_n, y_n - \bar{y} \rangle$$

i.e.,

$$\bigl\langle M w_n + N w_n + G^* G w_n, \bar{w} - w_n \bigr\rangle \geq \alpha_n \langle w_n, w_n - \bar{w} \rangle.$$

By $\bar{w} \in \Gamma$,

$$\begin{aligned}
&\alpha_n \langle w_n, w_n - \bar{w} \rangle \\
&\quad \leq \langle M w_n - M \bar{w}, \bar{w} - w_n \rangle + \bigl\langle G^* G (w_n - \bar{w}), \bar{w} - w_n \bigr\rangle + \langle N w_n - N \bar{w}, \bar{w} - w_n \rangle \\
&\quad \leq - \left\| \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) x_n - \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) \bar{x} \right\|^2 \\
&\qquad - \left\| \sum_{i=1}^{t_2} \beta_i (I - P_{D_i}) A_1 x_n - \sum_{i=1}^{t_2} \beta_i (I - P_{D_i}) A_1 \bar{x} \right\|^2 \\
&\qquad - \left\| \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) y_n - \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) \bar{y} \right\|^2 \\
&\qquad - \left\| \sum_{j=1}^{r_2} \delta_j (I - P_{\Theta_j}) B_1 y_n - \sum_{j=1}^{r_2} \delta_j (I - P_{\Theta_j}) B_1 \bar{y} \right\|^2 \\
&\qquad - \| G w_n \|^2,
\end{aligned}$$

we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) x_n - \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i}) \bar{x} \right\|^2 \\
&\qquad + \left\| \sum_{i=1}^{t_2} \beta_i (I - P_{D_i}) A_1 x_n - \sum_{i=1}^{t_2} \beta_i (I - P_{D_i}) A_1 \bar{x} \right\|^2 \\
&\qquad + \left\| \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) y_n - \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j}) \bar{y} \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\sum_{j=1}^{r_2}\delta_j(I-P_{\Theta_j})B_1y_n-\sum_{j=1}^{r_2}\delta_j(I-P_{\Theta_j})B_1\bar{y}\right\|^2\\
&+\|Gw_n\|^2\\
\le{}& \alpha_n\langle w_n, w_n-\bar{w}\rangle \le \alpha_n\|w_n\|\|w_n-\bar{w}\| \le 2\alpha_n\|\bar{w}\|^2 \to 0.
\end{aligned}
$$

Furthermore, note that $\{w_n\}$ converges weakly to a point $\hat{w} \in \mathrm{Fix}(T)$, then $\{h(w_n)\}$ converges weakly to $h(\hat{w})$. It follows that $h(\hat{w}) = 0$, i.e., $\hat{w} \in \Gamma$.

By (2.11),

$$
\bigl\langle \nabla h(w_n) + \alpha_n w_n, \hat{w} - w_n\bigr\rangle \ge 0,
$$

i.e.,

$$
\begin{aligned}
&\bigl\langle Mw_n + Nw_n + G^*Gw_n + \alpha_n w_n, \hat{w} - w_n\bigr\rangle \ge 0,\\
&\bigl\langle Mw_n + Nw_n + G^*Gw_n + \alpha_n w_n, \hat{w} - w_n\bigr\rangle\\
&\quad = \langle Mw_n - M\hat{w}, \hat{w} - w_n\rangle + \langle Nw_n - N\hat{w}, \hat{w} - w_n\rangle + \bigl\langle G^*G(w_n - \hat{w}), \hat{w} - w_n\bigr\rangle\\
&\qquad + \langle \alpha_n w_n - \alpha_n\hat{w}, \hat{w} - w_n\rangle + \langle \alpha_n\hat{w}, \hat{w} - w_n\rangle\\
&\quad \le -\|Mw_n - M\hat{w}\|^2 - \|Nw_n - N\hat{w}\|^2 - \bigl\|G(w_n - \hat{w})\bigr\|^2\\
&\qquad + \langle \alpha_n\hat{w}, \hat{w} - w_n\rangle - \alpha_n\|w_n - \hat{w}\|^2\\
&\quad \ge 0.
\end{aligned}
$$

Then

$$
\|Mw_n - M\hat{w}\|^2 + \|Nw_n - N\hat{w}\|^2 + \bigl\|G(w_n - \hat{w})\bigr\|^2 + \alpha_n\|w_n - \hat{w}\|^2 \le \langle \alpha_n\hat{w}, \hat{w} - w_n\rangle,
$$

we have

$$
\|w_n - \hat{w}\|^2 \le \langle \hat{w}, \hat{w} - w_n\rangle.
$$

Consequently, that $\{w_n\}$ converges weakly to $\hat{w}$ actually implies that $\{w_n\}$ converges strongly to $\hat{w}$. At last, we prove that $\hat{w} = \bar{w}$, and this finishes the proof.

Since $\{w_n\}$ converges weakly to $\hat{w}$ and $\|w_n\| \le \|\bar{w}\|$, we can get that

$$
\|\hat{w}\| \le \liminf_n \|w_n\| \le \|\bar{w}\| = \min\bigl\{\|w\| : w \in \Gamma\bigr\}.
$$

This shows that $\hat{w}$ is also a point in Γ which assumes a minimum norm. Due to the uniqueness of a minimum-norm element, we obtain $\hat{w} = \bar{w}$. □

Finally, we introduce another method to get the minimum-norm solution of MSSFP and SEFPP.

Lemma 2.6 *Let* $S = I - \sigma_1 M - \sigma_2\lambda_1 N - \sigma_3\lambda_2 G^*G$, *where* $0 < \lambda_1 < 1/(\max\{\rho(A_1^*A_1), \rho(B_1^*B_1)\}), 0 < \lambda_2 < 1/\rho(G^*G), \sigma_i > 0, i = 1,2,3.$ $\sigma_1 + \sigma_2 + \sigma_3 \le 1$ *with* $\rho(A_1^*A_1), \rho(B_1^*B_1), \rho(G^*G)$ *being the spectral radius of the self-adjoint operator* $A_1^*A_1, B_1^*B_1, G^*G$ *on* H, *then* *we have the following*:

(1) $\|S\| \leq 1$ (*i.e., S is nonexpansive*) *and averaged*;

(2) $\mathrm{Fix}(S) = \{(x,y) \in H_1 \times H_2, x \in \bigcap_{i=1}^{t_1} C_i, y \in \bigcap_{j=1}^{r_1} Q_j, A_1x \in \bigcap_{i=1}^{t_2} D_i, B_1y \in \bigcap_{j=1}^{r_2} \Theta_j, A_2x = B_2y\}$, $\mathrm{Fix}(P_{\mathrm{Fix}(T)}S) = \mathrm{Fix}(P_{\mathrm{Fix}(T)}) \cap \mathrm{Fix}(S) = \Gamma$;

(3) $w \in \mathrm{Fix}(P_{\mathrm{Fix}(T)}S)$ *if and only if w is a solution of the variational inequality* $\langle \nabla h(x,y), v - w\rangle, \forall v \in \mathrm{Fix}(T)$.

Proof (1)

$$\begin{aligned}
&\|Mx - My\|^2 \\
&\quad= \left\| \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i})x_1 - \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i})y_1 \right\|^2 \\
&\qquad+ \left\| \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j})x_2 - \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j})y_2 \right\|^2 \\
&\quad\leq \|x - y\|^2, \\
&\|\lambda_1 Nx - \lambda_1 Ny\|^2 \\
&\quad= \lambda_1^2 \left\| \sum_{i=1}^{t_2} \beta_i A_1^*(I - P_{D_i})A_1x_1 - \sum_{i=1}^{t_2} \beta_i A_1^*(I - P_{D_i})A_1y_1 \right\|^2 \\
&\qquad+ \lambda_1^2 \left\| \sum_{j=1}^{r_2} \delta_j B_1^*(I - P_{\Theta_j})B_1x_2 - \sum_{j=1}^{r_2} \delta_j B_1^*(I - P_{\Theta_j})B_1y_2 \right\|^2 \\
&\quad\leq \lambda_1 \left\| \sum_{i=1}^{t_2} \beta_i (I - P_{D_i})A_1x_1 - \sum_{i=1}^{t_2} \beta_i (I - P_{D_i})A_1y_1 \right\|^2 \\
&\qquad+ \lambda_1 \left\| \sum_{j=1}^{r_2} \delta_j (I - P_{\Theta_j})B_1x_2 - \sum_{j=1}^{r_2} \delta_j (I - P_{\Theta_j})B_1y_2 \right\|^2 \\
&\quad\leq \lambda_1 \|A_1x_1 - A_1y_1\|^2 + \lambda_1 \|B_1x_2 - B_1y_2\|^2 \\
&\quad\leq \|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 = \|x - y\|^2.
\end{aligned}$$

$\lambda_2\|G^*G(x-y)\| \leq \|x-y\|$.

Let $S_1 = \sigma_1 M + \sigma_2\lambda_1 N + \sigma_3\lambda_2 G^*G$, we have

$$\begin{aligned}
&\|S_1x - S_2y\| \\
&\quad= \left\|\sigma_1 Mx - \sigma_1 My + \sigma_2\lambda_1 Nx - \sigma_2\lambda_1 Ny + \sigma_3\lambda_2 G^*Gx - \sigma_3\lambda_2 G^*Gy\right\| \\
&\quad\leq \sigma_1\|Mx - My\| + \sigma_2\|\lambda_1 Nx - \lambda_1 Ny\| + \sigma_3\left\|\lambda_2 G^*Gx - \lambda_2 G^*Gy\right\| \\
&\quad\leq (\sigma_1 + \sigma_2 + \sigma_3)\|x - y\| \leq \|x - y\|.
\end{aligned}$$

We can get that S_1 is a nonexpansive operator.

$$\|Sx - Sy\|^2$$

$$
\begin{aligned}
&= \left\|x-y-(S_1x-S_1y)\right\|^2 \\
&= \|x-y\|^2 + \|S_1x-S_2y\|^2 - 2\langle x-y, S_1x-S_1y\rangle \\
&\le \|x-y\|^2 + 2\|\sigma_1 Mx-\sigma_1 My\|^2 + 2\|\sigma_2\lambda_1 Nx-\sigma_2\lambda_1 Ny\|^2 \\
&\quad + 2\left\|\sigma_3\lambda_2 G^*Gx-\sigma_3\lambda_2 G^*Gy\right\|^2 - 2\langle x-y, S_1x-S_1y\rangle \\
&\le \|x-y\|^2 + 2\sigma_1\langle x-y, \sigma_1 Mx-\sigma_1 My\rangle + 2\sigma_2\langle x-y, \sigma_2\lambda_1 Nx-\sigma_2\lambda_1 Ny\rangle \\
&\quad + 2\sigma_3\left\langle x-y, \sigma_3\lambda_2 G^*Gx-\sigma_3\lambda_2 G^*Gy\right\rangle - 2\langle x-y, S_1x-S_1y\rangle \\
&\le \|x-y\|^2 + 2\left\langle x-y, \sigma_1 Mx-\sigma_1 My+\sigma_2\lambda_1 Nx-\sigma_2\lambda_1 Ny+\sigma_3\lambda_2 G^*G(x-y)\right\rangle \\
&\quad - 2\langle x-y, S_1x-S_1y\rangle \\
&\le \|x-y\|^2
\end{aligned}
$$

so $\|S\| \le 1$, i.e., S is nonexpansive.

Indeed, let $\eta \in (0,1)$ such that $(\sigma_1+\sigma_2+\sigma_3)/(1-\eta) \in (0,1]$, then $S = I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G = \eta I + (1-\eta)V$, where $V = I - \frac{1}{1-\eta}(\sigma_1 M+\sigma_2\lambda_1 N+\sigma_3\lambda_2 G^*G)$ is a nonexpansive mapping. That is to say, S is averaged.

(2) If $w \in \{(x,y)\in H_1\times H_2, x\in\bigcap_{i=1}^{t_1} C_i, y\in\bigcap_{j=1}^{r_1} Q_j, A_1x\in\bigcap_{i=1}^{t_2} D_i, B_1y\in\bigcap_{j=1}^{r_2}\Theta_j, A_2x = B_2y\}$, it is obvious that $w\in\mathrm{Fix}(S)$. Conversely, assuming that $w\in\mathrm{Fix}(S)$, we have $w = w-\sigma_1 Mw-\sigma_2\lambda_1 Nw-\sigma_3\lambda_2 G^*Gw$, hence $\sigma_1 Mw+\sigma_2\lambda_1 Nw+\sigma_3\lambda_2 G^*Gw = 0, \forall \breve{w}\in\Gamma$,

$$
\begin{aligned}
&\left\langle \sigma_1 Mw+\sigma_2\lambda_1 Nw+\sigma_3\lambda_2 G^*Gw, w-\breve{w}\right\rangle \\
&\quad = \langle \sigma_1 Mw, w-\breve{w}\rangle + \langle \sigma_2\lambda_1 Nw, w-\breve{w}\rangle + \left\langle \sigma_3\lambda_2 G^*Gw, w-\breve{w}\right\rangle \\
&\quad = \langle \sigma_1 Mw-\sigma_1 M\breve{w}, w-\breve{w}\rangle + \langle \sigma_2\lambda_1 Nw-\sigma_2\lambda_1 N\breve{w}, w-\breve{w}\rangle \\
&\qquad + \left\langle \sigma_3\lambda_2 G^*G(w-\breve{w}), w-\breve{w}\right\rangle \\
&\quad \ge \sigma_1\|Mw\|^2 + \sigma_2\|\lambda_1 Nw\|^2 + \sigma_3\left\|\lambda_2 G^*Gw\right\|^2 \\
&\quad \ge \|\sigma_1 Mw\|^2 + \|\sigma_2\lambda_1 Nw\|^2 + \left\|\sigma_3\lambda_2 G^*Gw\right\|^2.
\end{aligned}
$$

This leads to $w \in \{(x,y)\in H_1\times H_2, x\in\bigcap_{i=1}^{t_1} C_i, y\in\bigcap_{j=1}^{r_1} Q_j, A_1x\in\bigcap_{i=1}^{t_2} D_i, B_1y\in\bigcap_{j=1}^{r_2}\Theta_j, A_2x = B_2y\}$, it is obvious that $w\in\mathrm{Fix}(S)$.

(3)

$$
\begin{aligned}
&\left\langle \nabla h(x,y), v-w\right\rangle \ge 0, \quad \forall v\in\mathrm{Fix}(T) \\
&\Leftrightarrow \quad \left\langle w-(w-S_1w), v-w\right\rangle \ge 0, \quad \forall v\in\mathrm{Fix}(T) \\
&\Leftrightarrow \quad w = P_{\mathrm{Fix}(T)}(w-S_1w) \\
&\Leftrightarrow \quad w\in\mathrm{Fix}(P_{\mathrm{Fix}(T)}S).
\end{aligned}
$$

□

Remark 2.7 Take constants $\lambda_1 and \lambda_2$, where $0<\lambda_1<1/(\max\{\rho(A_1^*A_1),\rho(B_1^*B_1)\}), 0<\lambda_2<1/\rho(G^*G)$, with $\rho(A_1^*A_1), \rho(B_1^*B_1), \rho(G^*G)$ being the spectral radius of the self-adjoint operator $A_1^*A_1, B_1^*B_1, G^*G$. For $\tau_1\in(0,(1-\lambda_1(\max\{\|A_1^*A_1\|,\|B_1^*B_1\|\}))/\sigma_2\lambda_1), \tau_2\in(0,(1-\lambda_2\|G^*G\|)/\sigma_3\lambda_2), \tau=\min\{\tau_1,\tau_2\}, (\sigma_1+\sigma_2+\sigma_3)/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)\in(0,1)$, we define a mapping

$$W_\alpha(w) := P_{\mathrm{Fix}(T)}\big[(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w.$$

It is easy to check that W_α is contractive. So, W_α has a unique fixed point denoted by w_α, that is,

$$w_\alpha = P_{\mathrm{Fix}(T)}\big[(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_\alpha. \tag{2.12}$$

Theorem 2.8 *Let w_α be given as (2.12). Then w_α converges strongly as $\alpha \to 0$ to the minimum-norm solution $\bar{w}$ of MSSFP and SEFPP.*

Proof Let $\breve{w}$ be a point in Γ. $I-\sigma_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)M-\sigma_2\lambda_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)N-\sigma_3\lambda_2/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)G^*G$ is nonexpansive. It follows that

$$\begin{aligned}
&\|w_\alpha-\breve{w}\| \\
&\quad= \big\|P_{\mathrm{Fix}(T)}\big[(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_\alpha \\
&\qquad - P_{\mathrm{Fix}(T)}\big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]\breve{w}\big\| \\
&\quad\le \big\|\big[(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_\alpha \\
&\qquad - \big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]\breve{w}\big\| \\
&\quad\le (1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)\big\|\big(w_\alpha-\sigma_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)Mw_\alpha \\
&\qquad -\sigma_2\lambda_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)Nw_\alpha-\sigma_3\lambda_2(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)G^*Gw_\alpha\big) \\
&\qquad -\big(\breve{w}-\sigma_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)M\breve{w} \\
&\qquad -\sigma_2\lambda_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)N\breve{w}-\sigma_3\lambda_2/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)G^*G\breve{w}\big)\big\| \\
&\qquad +\tau(\sigma_2\lambda_1+\sigma_3\lambda_2)\|\breve{w}\| \\
&\quad\le (1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)\|w_\alpha-\breve{w}\|+\tau(\sigma_2\lambda_1+\sigma_3\lambda_2)\|\breve{w}\|.
\end{aligned}$$

Hence,

$$\|w_\alpha-\breve{w}\| \le \|\breve{w}\|.$$

Then $\{w_\alpha\}$ is bounded.

From (2.12), we have

$$\big\|w_\alpha-P_{\mathrm{Fix}(T)}\big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_\alpha\big\| \le \tau\big\|(\sigma_2\lambda_1+\sigma_3\lambda_2)w_\alpha\big\| \to 0.$$

Next we show that w_α is relatively norm compact as $\alpha \to 0^+$. In fact, assuming that $\{\tau_n\} \subseteq (0, \min\{(1-\lambda_1(\max\{\|A_1^*A_1\|, \|B_1^*B_1\|\}))/\sigma_2\lambda_1, (1-\lambda_2\|G^*G\|)/\sigma_3\lambda_2\})$ is such that $\tau_n \to 0^+$ as $n \to \infty$. Put $w_n := w_{\alpha_n}$, we have the following:

$$\big\|w_n-P_{\mathrm{Fix}(T)}\big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_n\big\| \le \tau\big\|(\sigma_2\lambda_1+\sigma_3\lambda_2)w_n\big\| \to 0.$$

We deduce that

$$\begin{aligned}
&\|w_\alpha - \breve{w}\|^2 \\
&\quad = \big\|P_{\mathrm{Fix}(T)}\big[(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_\alpha \\
&\qquad - P_{\mathrm{Fix}(T)}\big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]\breve{w}\big\|^2 \\
&\quad \le \big\langle\big[(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_\alpha \\
&\qquad - \big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]\breve{w}, w_\alpha-\breve{w}\big\rangle \\
&\quad \le (1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)\big\langle\big(w_\alpha-\sigma_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)Mw_\alpha \\
&\qquad - \sigma_2\lambda_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)Nw_\alpha-\sigma_3\lambda_2(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)G^*Gw_\alpha\big) \\
&\qquad - \big(\breve{w}-\sigma_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)M\breve{w}-\sigma_2\lambda_1/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)N\breve{w} \\
&\qquad - \sigma_3\lambda_2/(1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)G^*G\breve{w}\big), w_\alpha-\breve{w}\big\rangle-\tau(\sigma_2\lambda_1+\sigma_3\lambda_2)\langle\breve{w}, w_\alpha-\breve{w}\rangle \\
&\quad \le (1-\sigma_2\lambda_1\tau-\sigma_3\lambda_2\tau)\|w_\alpha-\breve{w}\|^2-\tau(\sigma_2\lambda_1+\sigma_3\lambda_2)\langle\breve{w}, w_\alpha-\breve{w}\rangle.
\end{aligned}$$

Therefore,

$$\|w_\alpha - \breve{w}\|^2 \le \langle -\breve{w}, w_\alpha - \breve{w}\rangle.$$

In particular,

$$\|w_n - \breve{w}\|^2 \le \langle -\breve{w}, w_n - \breve{w}\rangle, \quad \forall \breve{w} \in \Gamma.$$

Since $\{w_n\}$ is bounded, there exists a subsequence of $\{w_n\}$ which converges weakly to a point $\bar{w}$. Without loss of generality, we may assume that $\{w_n\}$ converges weakly to $\bar{w}$. Notice that

$$\big\|w_n - P_{\mathrm{Fix}(T)}\big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]w_n\big\| \le \tau\big\|(\sigma_2\lambda_1+\sigma_3\lambda_2)w_n\big\| \to 0,$$

and by Lemma 1.4, we can get $\bar{w} \in \mathrm{Fix}(TS) = \Gamma$.

By

$$\|w_n - \breve{w}\|^2 \le \langle -\breve{w}, w_n - \breve{w}\rangle, \quad \forall \breve{w} \in \Gamma,$$

we have

$$\|w_n - \bar{w}\|^2 \le \langle -\bar{w}, w_n - \bar{w}\rangle.$$

Consequently, that $\{w_n\}$ converges weakly to $\bar{w}$ actually implies that $\{w_n\}$ converges strongly to $\bar{w}$. That is to say, $\{w_\alpha\}$ is relatively norm compact as $\alpha \to 0^+$.

On the other hand, by

$$\|w_n - \breve{w}\|^2 \le \langle -\breve{w}, w_n - \breve{w}\rangle, \quad \forall \breve{w} \in \Gamma,$$

let $n \to \infty$, we have

$$\|\bar{w} - \breve{w}\|^2 \le \langle -\breve{w}, \bar{w} - \breve{w} \rangle, \quad \forall \breve{w} \in \Gamma.$$

This implies that

$$\langle -\breve{w}, \breve{w} - \bar{w} \rangle \le 0, \quad \forall \breve{w} \in \Gamma,$$

which is equivalent to

$$\langle -\bar{w}, \breve{w} - \bar{w} \rangle \le 0, \quad \forall \breve{w} \in \Gamma.$$

It follows that $\bar{w} \in P_{\mathrm{Fix}(T)}(0)$. Therefore, each cluster point of w_α equals $\bar{w}$. So $w_\alpha \to \bar{w}(\alpha \to 0)$ is the minimum-norm solution of SFP and SEFPP. □

3 Main results

In this section, we introduce the following algorithm to solve MSSFP and SEFFP. The purpose for such a modification lies in the hope of strong convergence.

Algorithm 3.1 *For an arbitrary point $w_0 = (x_0, y_0) \in H = H_1 \times H_2$, the sequence $\{w_n\} = \{(x_n, y_n)\}$ is generated by the iterative algorithm*

$$w_{n+1} = P_{\mathrm{Fix}(T)}\bigl\{(1-\tau_n)\bigl[I - \sigma_1 M - \sigma_2\lambda_1 N - \sigma_3\lambda_2 G^* G\bigr]w_n\bigr\}, \tag{3.1}$$

i.e.,

$$\begin{aligned} x_{n+1} = P_{\mathrm{Fix}(T_1)}\Biggl\{(1-\tau_n)\Biggl[x_n - \sigma_1\sum_{i=1}^{t_1}\alpha_i(I - P_{C_i})x_n - \sigma_2\lambda_1\sum_{i=1}^{t_2}\beta_i A_1^*(I - P_{D_i})A_1 x_n \\ - \sigma_3\lambda_2 A_2^*(A_2 x_n - B_2 y_n)\Biggr]\Biggr\}, \quad n \ge 0 \end{aligned}$$

and

$$\begin{aligned} y_{n+1} = P_{\mathrm{Fix}(T_2)}\Biggl\{(1-\tau_n)\Biggl[y_n - \sigma_1\sum_{j=1}^{r_1}\gamma_j(I - P_{Q_j})y_n - \sigma_2\lambda_1\sum_{j=1}^{r_2}\delta_j B_1^*(I - P_{\Theta_j})B_1 y_n \\ + \sigma_3\lambda_2 B_2^*(A_2 x_n - B_2 y_n)\Biggr]\Biggr\}, \quad n \ge 0, \end{aligned}$$

where $\tau_n > 0$ is a sequence in $(0,1)$ *such that*

(i) $\lim_n \tau_n = 0$;
(ii) $\sum_{n=0}^{\infty} \tau_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\tau_{n+1} - \tau_n| < \infty$ *or* $\lim_n |\tau_{n+1} - \tau_n|/\tau_n = 0$.

Now, we prove the strong convergence of the iterative algorithm.

Theorem 3.2 *The sequence $\{w_n\}$ generated by Algorithm 3.1 converges strongly to the minimum-norm solution $\bar{w}$ of MSSFP and SEFPP.*

Proof Let R_n and R be defined by

$$R_n w := P_{\mathrm{Fix}(T)}\big\{(1-\tau_n)\big[I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big]\big\}w = P_{\mathrm{Fix}(T)}\big[(1-\tau_n)Sw\big],$$
$$Rw := P_{\mathrm{Fix}(T)}\big(I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G\big)w = P_{\mathrm{Fix}(T)}(Sw),$$

where $S = I-\sigma_1 M-\sigma_2\lambda_1 N-\sigma_3\lambda_2 G^*G$. By Lemma 2.6 it is easy to see that R_n is a contraction with contractive constant $(1-\tau_n)$; and Algorithm 3.1 can be written as $w_{n+1} = R_n w_n$.

For any $\breve{w}\in\Gamma$, we have

$$\begin{aligned}\|R_n\breve{w}-\breve{w}\| &= \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)S\breve{w}\big]-\breve{w}\big\| \\ &= \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)S\breve{w}\big]-P_{\mathrm{Fix}(T)}(S\breve{w})\big\| \\ &\le \big\|(1-\tau_n)S\breve{w}-S\breve{w}\big\| \\ &= \tau_n\|S\breve{w}\| = \tau_n\|\breve{w}\|.\end{aligned}$$

Hence

$$\begin{aligned}\|w_{n+1}-\breve{w}\| &= \|R_n w_n-\breve{w}\| \le \|R_n w_n - R_n\breve{w}\| + \|R_n\breve{w}-\breve{w}\| \\ &\le \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)Sw_n\big]-P_{\mathrm{Fix}(T)}\big[(1-\tau_n)S\breve{w}\big]\big\| + \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)S\breve{w}\big]-\breve{w}\big\| \\ &\le (1-\tau_n)\|w_n-\breve{w}\| + \tau_n\|\breve{w}\| \\ &\le \max\big\{\|w_n-\breve{w}\|, |\breve{w}\|\big\}, \\ \|Sw_{n+1}-\breve{w}\| &\le \|w_{n+1}-\breve{w}\|.\end{aligned}$$

It follows that $\|w_n-\breve{w}\|\le\max\{\|w_0-\breve{w}\|, |\breve{w}\|\}$. So $\{w_n\}$ and $\{Sw_n\}$ are bounded.

Next we prove that $\lim_n\|w_{n+1}-w_n\| = 0$.

Indeed,

$$\begin{aligned}\|w_{n+1}-w_n\| &= \|R_n w_n - R_{n-1}w_{n-1}\| \\ &\le \|R_n w_n - R_n w_{n-1}\| + \|R_n w_{n-1}-R_{n-1}w_{n-1}\| \\ &\le (1-\tau_n)\|w_n-w_{n-1}\| + \|R_n w_{n-1}-R_{n-1}w_{n-1}\|.\end{aligned}$$

Notice that

$$\begin{aligned}\|R_n w_{n-1}-R_{n-1}w_{n-1}\| &= \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)Sw_{n-1}\big]-P_{\mathrm{Fix}(T)}\big[(1-\tau_{n-1})Sw_{n-1}\big]\big\| \\ &\le \big\|(1-\tau_n)Sw_{n-1}-(1-\tau_{n-1})Sw_{n-1}\big\| \\ &= |\tau_n-\tau_{n-1}|\|Sw_{n-1}\|.\end{aligned}$$

Hence,

$$\|w_{n+1}-w_n\| \le (1-\tau_n)\|w_n-w_{n-1}\| + |\tau_n-\tau_{n-1}|\|Sw_{n-1}\|.$$

By virtue of assumptions (i)–(iii) and Lemma 1.5, we have

$$\lim_n\|w_{n+1}-w_n\| = 0.$$

Therefore,

$$\begin{aligned}\|w_n - Rw_n\| &\leq \|w_{n+1} - w_n\| + \|R_n w_n - Rw_n\| \\ &\leq \|w_{n+1} - w_n\| + \big\|(1-\tau_n)Sw_n - Sw_n\big\| \\ &\leq \|w_{n+1} - w_n\| + \tau_n\|Sw_n\| \to 0.\end{aligned}$$

The demiclosedness principle ensures that each weak limit point of $\{w_n\}$ is a fixed point of the nonexpansive mapping $R = TS$, that is, a point of the solution set Γ of MSSFP and SEFPP.

At last, we will prove that $\lim_n \|w_{n+1} - \bar{w}\| = 0$.

Choose $0 < \delta < 1$ such that $(\sigma_1 + \sigma_2 + \sigma_3)/(1-\delta) \in (0,1)$, then $S = I - \sigma_1 M - \sigma_2\lambda_1 N - \sigma_3\lambda_2 G^*G = \delta I + (1-\delta)V$, where $V = I - \sigma_1/(1-\delta)M - \sigma_2\lambda_1/(1-\delta)N - \sigma_3\lambda_2/(1-\delta)G^*G$ is a nonexpansive mapping. Taking $z \in \Gamma$, we deduce that

$$\begin{aligned}\|w_{n+1} - z\|^2 &= \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)Sw_n\big] - z\big\|^2 \\ &\leq \big\|[(1-\tau_n)Sw_n - z\big\|^2 \\ &\leq (1-\tau_n)\|Sw_n - z\|^2 + \tau_n\|z\|^2 \\ &\leq \big\|\delta(w_n - z) + (1-\delta)(Vw_n - z)\big\|^2 + \tau_n\|z\|^2 \\ &\leq \delta\big\|(w_n - z)\big\|^2 + (1-\delta)\big\|(Vw_n - z)\big\|^2 - \delta(1-\delta)\|w_n - Vw_n\|^2 + \tau_n\|z\|^2 \\ &\leq \big\|(w_n - z)\big\|^2 - \delta(1-\delta)\|w_n - Vw_n\|^2 + \tau_n\|z\|^2.\end{aligned}$$

Then

$$\begin{aligned}\delta(1-\delta)\|w_n - Vw_n\|^2 &\leq \big\|(w_n - z)\big\|^2 - \|w_{n+1} - z\|^2 + \tau_n\|z\|^2 \\ &= \big(\big\|(w_n - z)\big\| + \|w_{n+1} - z\|\big)\big(\big\|(w_n - z)\big\| - \|w_{n+1} - z\|\big) + \tau_n\|z\|^2 \\ &\leq \big(\big\|(w_n - z)\big\| + \|w_{n+1} - z\|\big)\big(\|w_n - w_{n+1}\|\big) + \tau_n\|z\|^2 \to 0.\end{aligned}$$

Note that $S = I - \sigma_1 M - \sigma_2\lambda_1 N - \sigma_3\lambda_2 G^*G = \delta I + (1-\delta)V$, it follows that $\lim_n \|Sw_n - w_n\| = 0$.

Take a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\limsup_n \langle w_n - \bar{w}, -\bar{w}\rangle = \lim_k \langle w_{n_k} - \bar{w}, -\bar{w}\rangle$.

By virtue of the boundedness of $\{w_n\}$, we may further assume, with no loss of generality, that w_{n_k} converges weakly to a point $\breve{w}$. Since $\|Rw_n - w_n\| \to 0$, using the demiclosedness principle, we know that $\breve{w} \in \mathrm{Fix}(R) = \mathrm{Fix}(P_{\mathrm{Fix}(T)}S) = \Gamma$. Noticing that $\bar{w}$ is the projection of the origin onto Γ, we get that

$$\limsup_n \langle w_n - \bar{w}, -\bar{w}\rangle = \lim_k \langle w_{n_k} - \bar{w}, -\bar{w}\rangle = \langle \breve{w} - \bar{w}, -\bar{w}\rangle \leq 0.$$

Finally, we compute

$$\begin{aligned}\|w_{n+1} - \bar{w}\|^2 &= \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)Sw_n\big] - \bar{w}\big\|^2 \\ &= \big\|P_{\mathrm{Fix}(T)}\big[(1-\tau_n)Sw_n\big] - TS\bar{w}\big\|^2 \\ &\leq \big\|(1-\tau_n)Sw_n - S\bar{w}\big\|^2\end{aligned}$$

Table 1 Effectiveness of Iterative method

Iterative method	Error	Number of iteration	Time/s
iterative method (3.1)	10^{-5}	18	0.0000
iterative method (1.13) (1.14)	10^{-5}	107	0.078125
iterative method (3.1)	10^{-7}	24	0.015625
iterative method (1.13) (1.14)	10^{-7}	142	0.09375
iterative method (3.1)	10^{-10}	32	0.015625
iterative method (1.13) (1.14)	10^{-10}	194	0.125
iterative method (3.1)	10^{-12}	38	0.03125
iterative method (1.13) (1.14)	10^{-12}	228	0.1875
iterative method (3.1)	10^{-15}	47	0.03125
iterative method (1.13) (1.14)	10^{-15}	280	0.203125

$$\begin{aligned}
&\le \big\|(1-\tau_n)Sw_n-\bar{w}\big\|^2\\
&= \big\|(1-\tau_n)(Sw_n-\bar{w})+\tau_n(-\bar{w})\big\|^2\\
&= (1-\tau_n)^2\big\|(Sw_n-\bar{w})\big\|^2+\tau_n^2\|\bar{w}\|^2+2\tau_n(1-\tau_n)\langle Sw_n-\bar{w},-\bar{w}\rangle\\
&= (1-\tau_n)^2\big\|(Sw_n-\bar{w})\big\|^2+\tau_n\big[\tau_n\|\bar{w}\|^2+2(1-\tau_n)\langle Sw_n-\bar{w},-\bar{w}\rangle\big].
\end{aligned}$$

Since $\limsup_n\langle w_n-\bar{w},-\bar{w}\rangle\le 0$, $\|Sw_n-w_n\|\to 0$, we know that $\limsup_n(\tau_n\|\bar{w}\|^2+2(1-\tau_n)\langle Sw_n-\bar{w},-\bar{w}\rangle)\le 0$. By Lemma 1.5, we conclude that $\lim_n\|w_{n+1}-\bar{w}\|=0$. This completes the proof. □

4 Numerical experiments

We provide a numerical example to illustrate the effectiveness of our algorithm. The program was written in Mathematica. All results are carried out on a personal DELL computer with Intel(R) Core(TM)i5-5200 CPU @ 2.20 GHz and RAM 4.00 GB.

In this algorithm, we take error = $10^{-5}, 10^{-7}, 10^{-10}, 10^{-12}, 10^{-15}$, respectively. We consider the split feasibility problem (1.1) with $H_1=\mathbb{R}, H_2=\mathbb{R}$, $C=(-\infty,0]$, $Q=(-\infty,0]$, $D=(-\infty,0]$, $\Theta=(-\infty,0]$. $T_1x=x, T_2y=y$, $A_1=B_1=A_2=1, B_2=-1$, $\sigma_1=\sigma_2=\sigma_3=\frac{1}{3}$, $\lambda_1=\frac{1}{\|A_1\|^2}, \lambda_2=1$. Take $\tau_n=\frac{2}{3}$, an initial point $x_1=-20, y_1=-10$. Obviously, $x^*=0, y^*=0$ is a solution of this problem. In consideration of Algorithm 3.1, we have

$$\begin{cases} x_{n+1}=P_{\mathrm{Fix}(T_1)}[\frac{1}{3}(x_n-\frac{1}{3}(x_n+y_n))]; & n\ge 0;\\ y_{n+1}=P_{\mathrm{Fix}(T_2)}[\frac{1}{3}(y_n-\frac{1}{3}(x_n+y_n))]; & n\ge 0.\end{cases}$$

As for iterative method (1.13) and (1.14), we take $H_1=\mathbb{R}, H_2=\mathbb{R}$, $C=(-\infty,0]$, $Q=(-\infty,0]$, $D=(-\infty,0]$, $\Theta=(-\infty,0]$. $T_1x=x, T_2y=y$, $A_1=B_1=A_2=1, B_2=-1$, $\lambda=\xi=\sigma=\zeta=\frac{1}{3}$. Take $\tau=\frac{1}{8}$, an initial point $x_1=-20, y_1=-10$.

In consideration of algorithms (1.13) and (1.14), we have

$$\begin{cases} x_{n+1}=T_1(x_n-\frac{1}{8}(x_n+y_n)); & n\ge 0;\\ y_{n+1}=T_2(y_n-\frac{1}{8}(x_{n+1}+y_n))]; & n\ge 0.\end{cases}$$

From Table 1, it is easy to see that our iterative method converges faster in less time.

5 Conclusions

The paper proposed a new iterative method to solve the split equality fixed point problem of firmly quasi-nonexpansive or nonexpansive operators and multiple-sets split feasibility problem and obtained a strong convergence result without any semi-compact assumption imposed on operators. The results improved and unified many recent results.

Acknowledgements
The authors would like to express their sincere thanks to the editors and reviewers for their noteworthy comments, suggestions, and ideas, which helped to improve this paper.

Funding
This research was supported by NSFC Grants No: 11301379; No: 11226125; No: 11671167.

Availability of data and materials
All data generated or analysed during this study are included in this published article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by TX, and LS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 17 January 2021 Accepted: 23 June 2021 Published online: 13 July 2021

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