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Some identities and reciprocity relationsof unipoly-Dedekind type DC sums

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Abstract

Dedekind type DC sums and their generalizations are defined in terms of Euler functions and their generalization. Recently, Ma et al. (Adv. Differ. Equ. 2021:30 2021) introduced the poly-Dedekind type DC sums by replacing the Euler function appearing in Dedekind sums, and they were shown to satisfy a reciprocity relation. In this paper, we consider two kinds of new generalizations of the poly-Dedekind type DC sums. One is a unipoly-Dedekind type DC sum associated with the type 2 unipoly-Euler functions expressed in the type 2 unipoly-Euler polynomials using the modified polyexponential function, and we study some identities and the reciprocity relation for these unipoly-Dedekind type DC sums. The other is a unipoly-Dedekind sums type DC associated with the poly-Euler functions expressed in the unipoly-Euler polynomials using the polylogarithm function, and we derive some identities and the reciprocity relation for those unipoly-Dedekind type DC sums.

MSC: 11F20; 11B68; 11B83

Keywords: Dedekind type DC sums; Type 2 poly-Euler polynomials; Type 2 poly-Genocchi polynomials; Poly-Euler polynomials; Euler function; Polylogarithm function

1 Introduction

Let

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases} \quad \text{(see [1–8, 20–28, 30])},$$

where $[\cdot]$ denotes the greatest integer not exceeding x.

The Dedekind sums are defined by

$$S(h,m) = \sum_{\mu=1}^{m-1} \left(\left(\frac{\mu}{m}\right)\right)\left(\left(\frac{h\mu}{m}\right)\right) \quad \text{(see [1–8, 20–28, 30])}, \tag{1}$$

where h and m are positive integers.

The most fundamental result in the theory of Dedekind sums is the reciprocity theorem. If h and m are relatively prime positive integers, then

$$S(h,m)+S(m,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{m}+\frac{m}{h}+\frac{1}{hm}\right),\quad \text{(see [2])}.$$

It is well known that the classical Dedekind sums $S(h,m)$, initiated by Richard Dedekind [9], first arose in the transformation formula of the logarithm of the Dedekind eta function. Dedekind introduced them to express the functional equation of the Dedekind eta function. These sums have figured prominently in so many different areas such as elliptic modular functions to number theory, analysis, number theory, combinatorics, q-series, Weierstrass elliptic functions, modular forms, and other areas [1–9, 13, 14, 16, 18, 20–30]. In combinatorial number theory, one is interested in partitions of an integer n from a finite set of positive integers. Beck et al. showed that the number of such partitions of n from a finite set is a quasi-polynomial in n, whose coefficients are built up from some generalization of Dedekind sums [3]. Bayad and Simsek [4, 28] studied three new shifted sums of Apostol–Dedekind–Rademacher type. The Dedekind type DC (Daehee and Changhee) sums using Euler functions were first introduced by Kim [13] and have been studied variously by several authors since then [20, 30]. Recently, as a generalization of Dedekind sums, the poly-Dedekind sums associated with the type 2 poly-Bernoulli functions of index k [18] and the unipoly-Dedkind sum [11] were introduced. In addition , Ma et al. introduced the poly-Dedekind sums associated with the poly-Bernoulli functions of index k [21] and the poly-Dedekind type DC sums associated with the type 2 poly-Euler functions of index k [20].

In this paper, we introduce two kinds of new generalizations of the poly Dedekind type DC sums. In Sect. 2, for our goal, we show explicit formulas of type 2 unipoly-Euler polynomials and type 2 unipoly-Genocchi polynomials. In Sect. 3, we introduce a unipoly-Dedekind type DC sum associated with the type 2 unipoly-Euler functions expressed in the type 2 unipoly-Euler polynomials using the modified polyexponential function, and derive the reciprocity relation for these unipoly-Dedekind type DC sums. In Sect. 4, we introduce a unipoly-Dedekind sums type DC associated with the poly-Euler functions expressed in the unipoly-Euler polynomials using the polylogarithm function, and derive the reciprocity relation for those.

The Euler polynomials $E_n(x)$ $(n\in\mathbb{N}\cup\{0\})$ are defined by their generating functions as follows:

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad \text{(see [13, 14, 19, 20])}. \tag{2}$$

When $x=0$, $E_n:=E_n(0)$ are called Euler numbers. The first few of Euler numbers are $E_0=1$, $E_1=-\frac{1}{2}$, $E_2=0$, $E_3=\frac{1}{4}$, $E_4=0,\ldots,$ and $E_{2n}=0$, $(n\in\mathbb{N})$.

The Genocchi polynomials $G_n(x)$, $(n\in\mathbb{N}\cup\{0\})$, are defined by their generating functions as follows:

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\quad \text{(see [9, 20, 29])}. \tag{3}$$

When $x = 0$, $G_n := G_n(0)$ are called Genocchi numbers. The first few of Euler numbers are $G_0 = 0$, $G_1 = 1$, $G_2 = -1$, $G_3 = 0$, $G_4 = 1, \ldots$, and $G_{2n+1} = 0$, $(n \in \mathbb{N})$.

We note that

$$E_n = \frac{G_{n+1}}{n+1} \quad (\text{see [6, 13, 20]}). \tag{4}$$

From (2), we note that

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l} = (E+x)^n, \quad (n \geq 0), (\text{see [13, 19, 20]}) \tag{5}$$

and

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l} = (G+x)^n, \quad (n \geq 0), (\text{see [12, 20]}), \tag{6}$$

with the usual convention about replacing E^n and G^n with E_n and G_n, respectively.

From (2), for $n \equiv 1(\text{mod}2)$, we have

$$(-1)^{n-1}E_l(n) + E_l = 2\sum_{i=0}^{n-1}(-1)^i i^l, \quad (n \in \mathbb{N}), (\text{see [20]}). \tag{7}$$

Let d be an odd positive integer ≥ 3. Then we have the following well-known relation:

$$E_n(x) = d^n \sum_{i=0}^{d-1}(-1)^i E_n\left(\frac{x+i}{d}\right), \quad (\text{see [20]}), \tag{8}$$

where d is an odd positive integer ≥ 3 and $n \geq 0$.

The Euler function is defined by

$$\overline{E}_n(x) = E_n\bigl(x - [x]\bigr), \quad (n \geq 0), (\text{see [13, 20, 30]}), \tag{9}$$

where $[x]$ denotes the greatest integer not exceeding x.

Kim and Kim considered the modified polyexponential function defined by

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!}, \quad (k \in \mathbb{Z}), (\text{see [10]}). \tag{10}$$

Note that $\mathrm{Ei}_1(x) = e^x - 1$.

We introduce the type 2 poly-Euler polynomials, which are given by

$$\frac{\mathrm{Ei}_k(\log(1+2t))}{t(e^t+1)} e^{xt} = \sum_{n=0}^{\infty} E_{k,n}(x)\frac{t^n}{n!} \quad (k \in \mathbb{Z}), (\text{see [19]}). \tag{11}$$

When $x = 0$, $E_{k,n} = E_{k,n}(0)$, $n \geq 0$, are called type 2 poly-Euler numbers.

We also introduce the type 2 poly-Genocchi polynomials, which are given by

$$\frac{\mathrm{Ei}_k(\log(1+2t))}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_{k,n}(x)\frac{t^n}{n!} \quad (k \in \mathbb{Z}), (\text{see [12]}). \tag{12}$$

When $x = 0$, $G_{k,n} = G_{k,n}(0)$, $n \geq 0$, are called type 2 poly-Genocchi numbers.

By (12), we easily get $G_{k,0} = 0$, $G_{k,1} = 1$, $G_{k,2} = -2 + 2^{l-k}, \ldots$. Since $\mathrm{Ei}_1(\log(1+2t)) = 2t$, we see that $E_{1,n}(x) = E_n(x)$ and $G_{1,n}(x) = G_n(x)$ $(n \geq 0)$ are the Euler polynomials and the Genocchi polynomials, respectively.

Kim introduced the Dedekind type DC sums given by

$$T_p(h,m) = 2\sum_{\mu=1}^{m-1}(-1)^{\mu}\frac{\mu}{m}\overline{E_p}\left(\frac{h\mu}{m}\right) \quad (\text{see [13]}), \tag{13}$$

where $\overline{E_p}(x)$ is the pth Euler function.

For $p \in \mathbb{N}$ with $p \equiv 1$ (mod2), the reciprocity law of $T_p(h,m)$ is given by

$$\begin{aligned} &m^p T_p(h,m) + h^p T_p(m,h) \\ &\quad = 2\sum_{\mu=0}^{m-1}\left(mh\left(E+\frac{\mu}{m}\right)+m\left(E+h-\left[\frac{h\mu}{m}\right]\right)\right)^p + (hE+mE)^p + (p+2)E_p, \\ &\mu - \left[\frac{h\mu}{m}\right] \equiv 1(\mathrm{mod}2), \quad (\text{see [13, 30]}), \end{aligned}$$

where h, k are relative prime positive integers and

$$(Eh+Em)^p = \sum_{l=0}^{p}\binom{p}{l}E_l h^l E_{p-l} m^{p-l} (\text{see [13, 30]}).$$

Recently, Ma et al. introduced the poly-Dedekind type DC sums associated with the type 2 poly-Euler functions, which are given by

$$T_p^{(k)}(h,m) = 2\sum_{\mu=1}^{m-1}(-1)^{\mu}\frac{\mu}{m}\overline{E}_p^{(k)}(h\mu/m), \quad (h,m,p \in \mathbb{N}), (\text{see [20]}), \tag{14}$$

where $\overline{E}_p(h\mu/m) = E_p(\langle h\mu/m\rangle)$.

For $n \in \mathbb{N} \cup \{0\}$, as is well known, the Stirling numbers of the first kind are defined by

$$(x)_0 = 1, (x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad (n \geq 1),$$

and

$$\frac{1}{l!}\big(\log(1+t)\big)^l = \sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!} \quad (n,l \geq 0), (\text{see [12, 15, 17]}). \tag{15}$$

For $n \geq 0$, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l$$

and

$$\frac{1}{k!}\left(e^t - 1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \quad \text{(see [12, 15, 17])}, \tag{16}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\ldots(x-n+1)$, $(n \geq 1)$.

2 Type 2 unipoly-Euler numbers and type 2 unipoly-Genocchi numbers

Let τ be any arithmetic function which is real or complex valued and defined on the set of positive integers $\mathbb{N}$. Then Kim and Kim defined the unipoly function attached to polynomials τ by

$$u_k(x|\tau) = \sum_{n=1}^{\infty} \frac{\tau(n)x^n}{n^k}, \quad (k \in \mathbb{Z}), \text{(see [10])}. \tag{17}$$

When $\tau(n) = 1$, $u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = Li_k(x)$ is the ordinary polylogarithm function.

From (17), we have

$$\frac{d}{dx}u_k(x|\tau) = \frac{1}{x}u_{k-1}(x|\tau). \tag{18}$$

Lee et al. introduced the type 2 unipoly-Euler polynomials of index k defined by

$$\frac{u_k(\log(1+2t)|\tau)}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_{k,n,\tau}(x)\frac{t^n}{n!} \quad \text{(see [19])}. \tag{19}$$

When $x = 0$, $\mathcal{E}_{k,n,\tau} := \mathcal{E}_{k,n,\tau}(0)$ are called type 2 unipoly-Euler numbers.

The type 2 unipoly-Genocchi polynomials of index k are defined by

$$\frac{u_k(\log(1+2t)|\tau)}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} \mathbb{G}_{k,n,\tau}(x)\frac{t^n}{n!} \quad \text{(see [12])}. \tag{20}$$

When $x = 0$, $\mathbb{G}_{k,n,\tau} := \mathbb{G}_{k,n,\tau}(0)$ are called type 2 unipoly-Genocchi numbers.

For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, let $\tau(n) = \frac{1}{\Gamma(n)} = \frac{1}{(n-1)!}$. Then, from (12) and (19), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{E}_{k,n,\frac{1}{\Gamma}}(x)\frac{t^n}{n!} &= \frac{u_k(\log(1+2t)|\frac{1}{\Gamma})}{t(e^t+1)}e^{xt} \\ &= \frac{Ei_k(\log(1+2t))}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} E_{k,n}(x)\frac{t^n}{n!}.\end{aligned} \tag{21}$$

Thus, from (21) we have

$$\mathcal{E}_{k,n,\frac{1}{\Gamma}}(x) = E_{k,n}(x). \tag{22}$$

Similarly, we get

$$\mathbb{G}_{k,n,\frac{1}{\Gamma}}(x) = G_{k,n}(x).$$

Moreover, from (21), we note that

$$\begin{aligned}\frac{u_k(\log(1+2t)|\tau)}{t(e^t+1)}e^{xt} &= \left(\sum_{l=0}^{\infty}\mathcal{E}_{k,l,\tau}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{k,l,\tau}x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}\tag{23}$$

Thus, by (23), we get

$$\mathcal{E}_{k,n,\tau}(x) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{k,l,\tau}x^{n-l} \quad (n \geq 0).\tag{24}$$

In the same way as (23), we have

$$\mathbb{G}_{k,n,\tau}(x) = \sum_{l=0}^{n}\binom{n}{l}\mathbb{G}_{k,l,\tau}x^{n-l} \quad (n \geq 0).\tag{25}$$

Furthermore, by (24) and (25), we have

$$\frac{d}{dx}\mathcal{E}_{k,n,\tau}(x) = n\mathcal{E}_{k,n-1,\tau}(x) \quad \text{and} \quad \frac{d}{dx}\mathbb{G}_{k,n,\tau}(x) = n\mathbb{G}_{k,n-1,\tau}(x), \quad (n \geq 1).\tag{26}$$

Theorem 1 *For $n \geq 1$, we have*

$$\mathcal{E}_{k,n,\tau}(x) = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!2^m\tau(l)S_1(m+1,l)}{l^{k-1}(m+1)}E_{n-m}(x), \quad (k \in \mathbb{Z}),\tag{27}$$

and

$$\mathbb{G}_{k,n,\tau}(x) = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!2^m\tau(l)S_1(m+1,l)}{l^{k-1}(m+1)}G_{n-m}(x), \quad (k \in \mathbb{Z}).\tag{28}$$

Proof From (2), (15), and (19), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{k,n,\tau}(x)\frac{t^n}{n!} &= \frac{u_k(\log(1+2t)|\tau)}{t(e^t+1)}e^{xt} = \frac{1}{2}\frac{2e^{xt}}{e^t+1}\frac{1}{t}\sum_{l=1}^{\infty}\frac{\tau(l)(\log(1+2t))^l}{l^k}\\ &= \frac{1}{2}\frac{2e^{xt}}{e^t+1}\frac{1}{t}\sum_{m=1}^{\infty}\sum_{l=1}^{m}\frac{(l-1)!2^m\tau(l)S_1(m,l)}{l^{k-1}}\frac{t^m}{m!}\\ &= \sum_{i=0}^{\infty}E_i(x)\frac{t^i}{i!}\sum_{m=0}^{\infty}\sum_{l=1}^{m+1}\frac{(l-1)!2^m\tau(l)S_1(m+1,l)}{l^{k-1}(m+1)}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!2^m\tau(l)S_1(m+1,l)}{l^{k-1}(m+1)}E_{n-m}\right)\frac{t^n}{n!}.\end{aligned}\tag{29}$$

Therefore, from (29) we get identity (27).

By using (3), (15), and (19) in the same way as (29), we also get identity (28). □

In particular, we get

$$\mathcal{E}_{k,n,\tau} = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!2^m\tau(l)S_1(m+1,l)}{l^{k-1}(m+1)}E_{n-m} \quad (k\in\mathbb{Z}),$$

and the first few of the type 2 unipoly-Euler numbers are $\mathcal{E}_{0,\tau}^{(k)} = \tau(1)$, $\mathcal{E}_{1,\tau}^{(k)} = \frac{1}{2}\tau(1) + \frac{1}{2^{k-1}}\tau(2),\ldots$. In addition,

$$\mathbb{G}_{k,n,\tau} = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!2^m\tau(l)S_1(m+1,l)}{l^{k-1}(m+1)}G_{n-m} \quad (k\in\mathbb{Z}),$$

and the first few of the type 2 unipoly-Genocchi numbers are $\mathbb{G}_{k,0,\tau} = 0$, $\mathbb{G}_{k,1,\tau} = \tau(1),\ldots$. Since $\mathbb{G}_{k,0,\tau} = 0$, we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{k,n,\tau}(x)\frac{t^n}{n!} &= \frac{u_k(\log(1+2t)|\tau)}{t(e^t+1)}e^{xt} \\ &= \sum_{n=0}^{\infty}\mathbb{G}_{k,n,\tau}(x)\frac{t^{n-1}}{n!} = \sum_{n=1}^{\infty}\mathbb{G}_{k,n,\tau}(x)\frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty}\frac{\mathbb{G}_{k,n+1,\tau}(x)}{n+1}\frac{t^n}{n!}.\end{aligned} \tag{30}$$

Thus, from (30), we have

$$\mathcal{E}_{k,n,\tau}(x) = \frac{\mathbb{G}_{k,n+1,\tau}(x)}{n+1}. \tag{31}$$

Lemma 2 *For $n\geq 1$, we have*

$$\mathbb{G}_{k,n,\tau}(1) + \mathbb{G}_{k,n,\tau} = 2^n\sum_{m=1}^{n}\frac{m!}{m^k}\tau(m)S_1(n,m), \quad (k\in\mathbb{Z}), \tag{32}$$

and

$$\mathcal{E}_{k,n-1,\tau}(1) + \mathcal{E}_{k,n-1,\tau} = \frac{2^n}{n}\sum_{m=1}^{n}\frac{m!}{m^k}\tau(m)S_1(n,m), \quad (k\in\mathbb{Z}). \tag{33}$$

Proof From (20), we have

$$\begin{aligned}u_k\bigl(\log(1+2t)|\tau\bigr) &= \left(\sum_{l=0}^{\infty}\mathbb{G}_{k,l,\tau}\frac{t^l}{l!}\right)\bigl(e^t+1\bigr) \\ &= \sum_{n=0}^{\infty}\bigl(\mathbb{G}_{k,n,\tau}(1)+\mathbb{G}_{k,n,\tau}\bigr)\frac{t^n}{n!} = \sum_{n=1}^{\infty}\bigl(\mathbb{G}_{k,n,\tau}(1)+\mathbb{G}_{k,n,\tau}\bigr)\frac{t^n}{n!}.\end{aligned} \tag{34}$$

On the other hand, from (17), we have

$$u_k\bigl(\log(1+2t)|\tau\bigr) = \sum_{m=1}^{\infty}\frac{\tau(m)}{m^k}\bigl(\log(1+2t)\bigr)^m \tag{35}$$

$$= \sum_{m=1}^{\infty} \frac{\tau(m)m!}{m^k} \sum_{n=m}^{\infty} S_1(n,m) \frac{2^n t^n}{n!}$$
$$= \sum_{n=1}^{\infty} \left(2^n \sum_{m=1}^{n} \frac{m!}{m^k} \tau(m) S_1(n,m) \right) \frac{t^n}{n!}.$$

Therefore, by (34) and (35), we obtain identity (32). By using (31), we get identity (33). □

Theorem 3 *For an odd positive integer $d \geq 3$ and $n \geq 1$, we have*

$$(-1)^{d-1}\mathbb{G}_{k,n,\tau}(d) + \mathbb{G}_{k,n,\tau} = \sum_{a=1}^{n} \sum_{b=1}^{a} \sum_{i=0}^{d-1} \binom{n}{a} (-1)^i i^{n-a} \frac{(b-1)!2^a}{b^{k-1}} \tau(b) S_1(a,b) \tag{36}$$

and

$$(-1)^{d-1}\mathcal{E}_{k,n-1,\tau}(d) + \mathcal{E}_{k,n-1,\tau}$$
$$= \frac{1}{n} \left(\sum_{a=1}^{n} \sum_{b=1}^{a} \sum_{i=0}^{d-1} \binom{n}{a} (-1)^i i^{n-a} \frac{(b-1)!2^a}{b^{k-1}} \tau(b) S_1(a,b) \right). \tag{37}$$

Proof For an odd positive integer $d \geq 3$, from (15) and (17), we have

$$\sum_{i=0}^{d-1} (-1)^i e^{it} u_k\big(\log(1+2t)|\tau\big) \tag{38}$$
$$= \sum_{i=0}^{d-1} (-1)^i \sum_{j=0}^{\infty} i^j \frac{t^j}{j!} \sum_{b=1}^{\infty} \frac{\tau(b)(\log(1+2t))^b}{b^k}$$
$$= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{d-1} (-1)^i i^j \right) \frac{t^j}{j!} \sum_{a=1}^{\infty} \left(\sum_{b=1}^{a} \frac{b!2^a}{b^k} \tau(b) S_1(a,b) \right) \frac{t^a}{a!}$$
$$= \sum_{n=1}^{\infty} \left(\sum_{a=1}^{n} \sum_{b=1}^{a} \sum_{i=0}^{d-1} \binom{n}{a} (-1)^i i^{n-a} \frac{(b-1)!2^a}{b^{k-1}} \tau(b) S_1(a,b) \right) \frac{t^n}{n!}.$$

On the other hand,

$$\sum_{i=0}^{d-1} (-1)^i e^{it} u_k\big(\log(1+2t)|\tau\big) = \frac{(-1)^{d-1}e^{dt}+1}{e^t+1} u_k\big(\log(1+2t)|\tau\big) \tag{39}$$
$$= \sum_{n=0}^{\infty} \big((-1)^{d-1}\mathbb{G}_{k,n}(d) + \mathbb{G}_{k,n}\big) \frac{t^n}{n!}.$$

Therefore, by (38) and (39), we obtain identity (36).

Moreover, from (31), we get identity (37). □

Theorem 4 *For an odd positive integer $d \geq 1$ and $n \geq 0$, we have*

$$\mathbb{G}_{k,n,\tau}(x) = \sum_{l=0}^{n} \binom{n}{l} \sum_{j=1}^{n-l+1} \sum_{i=0}^{d-1} (-1)^i d^{l-1} G_l\left(\frac{i+x}{d}\right) \frac{(j-1)!2^{n-l}}{j^{k-1}(n-l+1)} \tau(j) S_1(n-l+1,j).$$

Proof From $\sum_{i=0}^{n-1}(-1)^i e^{it} = \frac{(-1)^{n-1}e^{nt}+1}{e^t+1}$, $(n \equiv 1 (\mathrm{mod} 2))$, (2), and (13), we have

$$\begin{aligned}
&\frac{u_k(\log(1+2t)|\tau)}{e^t+1}e^{xt} \\
&= \frac{1}{2d}\sum_{i=0}^{d-1}(-1)^i e^{(\frac{i+x}{d})dt}\frac{2dt}{e^{dt}+1}\frac{1}{t}u_k\big(\log(1+2t)|\tau\big) \\
&= \frac{1}{2d}\sum_{i=0}^{d-1}(-1)^i\sum_{l=0}^{\infty}G_l\left(\frac{i+x}{d}\right)\frac{d^l t^l}{l!}\frac{1}{t}\sum_{j=1}^{\infty}\frac{\tau(j)(\log(1+2t))^j}{j^k} \\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{d-1}(-1)^i d^{l-1}G_l\left(\frac{i+x}{d}\right)\right)\frac{t^l}{l!}\sum_{a=1}^{\infty}\left(\sum_{j=1}^{a}\frac{(j-1)!2^{a-1}}{j^{k-1}}\tau(j)S_1(a,j)\right)\frac{t^{a-1}}{a!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{d-1}d^{l-1}(-1)^i G_l\left(\frac{i+x}{d}\right)\right)\frac{t^l}{l!}\sum_{a=0}^{\infty}\left(\sum_{j=1}^{a+1}\frac{(j-1)!2^{a}}{j^{k-1}(a+1)}\tau(j)S_1(a+1,j)\right)\frac{t^{a}}{a!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{j=1}^{n-l+1}\sum_{i=0}^{d-1}(-1)^i d^{l-1}G_l\left(\frac{i+x}{d}\right)\frac{(j-1)!2^{n-l}}{j^{k-1}(n-l+1)}\tau(j)S_1(n-l+1,j)\right)\frac{t^n}{n!}.
\end{aligned} \tag{40}$$

Therefore, by (40), we get the desired result. □

Corollary 5 *For an odd positive integer $d \geq 1$ and $n \geq 1$, we have*

$$\mathcal{E}_{k,n-1,\tau}(x) = \sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n-1}{l}(-1)^i d^l E_l\left(\frac{i+x}{d}\right)\frac{(j-1)!2^{n-l-1}\tau(j)S_1(n-l,j)}{j^{k-1}(n-l)}.$$

Proof From (31) and Theorem 4, we have

$$\begin{aligned}
\mathcal{E}_{k,n-1,\tau}(x) &= \frac{\mathbb{G}_{n,\tau}(x)}{n} \\
&= \frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}\sum_{j=1}^{n-l+1}\sum_{i=0}^{d-1}d^{l-1}(-1)^i G_l\left(\frac{i+x}{d}\right)\frac{(j-1)!\tau(j)2^{n-l}}{j^{k-1}(n-l+1)}S_1(n-l+1,j) \\
&= \frac{1}{n}\sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n}{l+1}d^l(-1)^i G_{l+1}\left(\frac{i+x}{d}\right)\frac{(j-1)!2^{n-l-1}}{j^{k-1}(n-l)}\tau(j)S_1(n-l,j) \\
&= \frac{n}{n}\sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n-1}{l}d^l(-1)^i\frac{G_{l+1}(\frac{i+x}{d})}{l+1}\frac{(j-1)!2^{n-l-1}\tau(j)S_1(n-l,j)}{j^{k-1}(n-l)} \\
&= \sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n-1}{l}(-1)^i d^l E_l\left(\frac{i+x}{d}\right)\frac{(j-1)!2^{n-l-1}\tau(j)S_1(n-l,j)}{j^{k-1}(n-l)}.
\end{aligned} \tag{41}$$

There, from (41), we arrive at the desired result. □

Lemma 6 *For $l \in \mathbb{N}$ with $l < q$, we have*

$$\binom{q}{l}\mathcal{E}_{k,q-l,\tau}(1) + \binom{q}{l-1}\mathcal{E}_{k,q-l+1,\tau}(1) = \sum_{j=0}^{q}\binom{q-j+1}{l}\binom{q}{j}\mathcal{E}_{k,j,\tau}.$$

Proof For $l \in \mathbb{N}$ with $l < q$, we have

$$\frac{d^l}{dx^l}\big(x\mathcal{E}_{k,q,\tau}(x)\big)|_{x=1} = l!\binom{q}{l}\mathcal{E}_{k,q-l,\tau}(1) + l!\binom{q}{l-1}\mathcal{E}_{k,q-l+1,\tau}(1). \tag{42}$$

On the other hand, by (24), we get

$$\begin{aligned}\frac{d^l}{dx^l}\big(x\mathcal{E}_{k,q,\tau}(x)\big)|_{x=1} &= \sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}\left(\left(\frac{d^l}{dx^l}\right)^l x^{q-j+1}\right)\bigg|_{x=1} \\ &= l!\sum_{j=0}^{q}\binom{q-j+1}{l}\binom{q}{j}\mathcal{E}_{k,j,\tau}.\end{aligned} \tag{43}$$

Therefore, by (42) and (43), we obtain what we want. □

Lemma 7 *For $q \in \mathbb{N}$, we have*

$$\sum_{j=0}^{q}\binom{q}{j}\frac{\mathcal{E}_{k,j,\tau}}{q-j+2} = \frac{1}{q+1}\mathcal{E}_{k,q,\tau}(1) - \frac{1}{(q+1)(q+2)}\mathcal{E}_{k,q+2}(1) + \frac{\mathcal{E}_{k,q+2,\tau}}{(q+1)(q+2)}.$$

Proof By using (26), we observe that

$$\begin{aligned}\int_0^1 x\mathcal{E}_{k,q,\tau}(x)\,dx &= \frac{1}{q+1}\mathcal{E}_{k,q+1}(1) - \frac{1}{q+1}\int_0^1 \mathcal{E}_{k,q+1,\tau}(x)\,dx \\ &= \frac{1}{q+1}\mathcal{E}_{k,q+1,\tau}(1) - \frac{1}{(q+1)(q+2)}\big(\mathcal{E}_{k,q+2,\tau}(1) - \mathcal{E}_{k,q+2,\tau}\big).\end{aligned} \tag{44}$$

On the other hand, by using (24), we have

$$\int_0^1 x\mathcal{E}_{k,q,\tau}(x)\,dx = \sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}\int_0^1 x^{q-j+1}\,dx = \sum_{j=0}^{q}\binom{q}{j}\frac{\mathcal{E}_{k,j,\tau}}{q-j+2}. \tag{45}$$

Therefore, by (44) and (45), we get what we want. □

3 Unipoly-Dedekind type DC sums associated withthe type 2 unipoly-Euler functions of index k

In this section, as a generalization of the poly-Dedekind type DC sums, we consider the unipoly-Dedekind type DC sums associated with the type 2 unipoly-Euler functions of index k and derive several noble identities and the reciprocity relation for these.

Naturally, we consider the unipoly-Dedekind type DC sums associated with the type 2 unipoly-Euler functions of index k as follows:

$$Z_{k,q,\tau}(h,m) = 2\sum_{\mu=1}^{m-1}(-1)^{\mu}(\mu/m)\overline{\mathcal{E}}_{k,q,\tau}(h\mu/m), \quad (h,m,q \in \mathbb{N}, k \in \mathbb{Z}), \tag{46}$$

where $h,m,q \in \mathbb{N}$ with $q \equiv 1 \pmod 2$ and $\overline{\mathcal{E}}_{k,q,\tau}(x) = \mathcal{E}_{k,q,\tau}(x-[x])$ are the type 2 unipoly-Euler functions of index k ($[x]$ is the largest integer less than x).

For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, when $\tau(n) = \frac{1}{\Gamma(n)} = \frac{1}{(n-1)!}$, we note that $Z_{k,q,\frac{1}{\Gamma}}(h,m) = T_q^{(k)}(h,m)$. In addition, we note that

$$Z_{1,q,\frac{1}{\Gamma}}(h,m) = 2\sum_{\mu=1}^{m-1}(-1)^{\mu}(\mu/m)\overline{\mathcal{E}}_{1,q,\frac{1}{\Gamma}}(h\mu/m) = T_q(h,m).$$

Theorem 8 *Let m be an odd positive integer ≥ 1 and $q \in \mathbb{N}$. Then we have*

$$m^q Z_{k,q,\tau}(1,m) = \sum_{j=0}^{q}\sum_{i=0}^{q-j}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau}m^{q-i}E_i + 2\sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{j-1}E_{q-j+1}.$$

Proof From (5), (7), and (46), we have

$$\begin{aligned}
&Z_{k,q,\tau}(1,m) \\
&= 2\sum_{\mu=0}^{m-1}(-1)^{\mu}\left(\frac{\mu}{m}\right)\overline{\mathcal{E}}_{k,q,\tau}\left(\frac{\mu}{m}\right) = 2\sum_{\mu=0}^{m-1}(-1)^{\mu}\left(\frac{\mu}{m}\right)\mathcal{E}_{k,q,\tau}\left(\frac{\mu}{m}\right) \\
&= 2\sum_{\mu=0}^{m-1}(-1)^{\mu}\left(\frac{\mu}{m}\right)\sum_{j=0}^{q}\binom{q}{j}\left(\frac{\mu}{m}\right)^{q-j}\mathcal{E}_{k,j,\tau} \\
&= \sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{-(q-j+1)}\left(2\sum_{\mu=0}^{m-1}\mu^{q-j+1}(-1)^{\mu}\right) \\
&= \sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{-q+j-1}\left((-1)^{m-1}E_{q-j+1}(m) + E_{q-j+1}\right) \\
&= \sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{-q+j-1}\left(\sum_{i=0}^{q-j+1}\binom{q+1-j}{i}m^{q-j+1-i}E_i + E_{q-j+1}\right) \\
&= \sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{-q+j-1}\sum_{i=0}^{q-j}\binom{q-j+1}{i}m^{q-j+1-i}E_i + 2\sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{-q+j-1}E_{q-j+1} \\
&= \sum_{j=0}^{q}\sum_{i=0}^{q-j}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau}m^{-i}E_i + 2\sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{-q+j-1}E_{q-j+1}.
\end{aligned} \tag{47}$$

By multiplying both sides of (47) by m^q, we arrive at the desired result. □

Theorem 9 *Let m, q be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we have*

$$\begin{aligned}
&m^q Z_{k,q,\tau}(1,m) \\
&= \sum_{i=1}^{q-2}\binom{q}{i}\mathcal{E}_{k,q-i,\tau}(1)m^{q-i}E_i + \sum_{i=1}^{q-2}\binom{q}{i-1}\left(\mathcal{E}_{k,q-i+1,\tau}(1) - \mathcal{E}_{k,q-i+1,\tau}\right)m^{q-i}E_i \\
&\quad + 2\sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{j-1}E_{q-j+1} + (q+1)\tau(1)E_q + m^q\mathcal{E}_{k,q,\tau}(1).
\end{aligned}$$

Proof For an odd integer $q \geq 3$, we observe that $E_{q-1} = 0$. Moreover, $E_0 = 1$ and $\mathcal{E}_{k,0,\tau} = \tau(1)$. From (24), we observe that

$$\begin{aligned}
&\sum_{j=0}^{q}\sum_{i=0}^{q-j}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau}m^{q-i}E_i \qquad (48)\\
&= \sum_{i=0}^{q}\sum_{j=0}^{q-i}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau}m^{q-i}E_i\\
&= \sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{q} + \sum_{i=1}^{q-2}\sum_{j=0}^{q-i}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau}m^{q-i}E_i\\
&\quad + \sum_{j=0}^{1}\binom{q}{j}\binom{q-j+1}{q-1}\mathcal{E}_{k,j,\tau}mE_{q-1} + \binom{q+1}{q}\tau(1)E_q\\
&= \sum_{i=1}^{q-2}\sum_{j=0}^{q-i}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau}m^{q-i}E_i + (q+1)\tau(1)E_q + m^q\mathcal{E}_{k,q,\tau}(1).
\end{aligned}$$

In addition, by using Lemma 6, we have

$$\begin{aligned}
\sum_{j=0}^{q-i}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau} &= \sum_{j=0}^{q}\binom{q}{j}\binom{q-j+1}{i}\mathcal{E}_{k,j,\tau} - \binom{q}{i-1}\mathcal{E}_{k,q-i+1,\tau} \qquad (49)\\
&= \binom{q}{i}\mathcal{E}_{k,q-i,\tau}(1) + \binom{q}{i-1}\mathcal{E}_{k,q-i+1,\tau}(1) - \binom{q}{i-1}\mathcal{E}_{k,q-i+1,\tau}\\
&= \binom{q}{i}\mathcal{E}_{k,q-i,\tau}(1) + \binom{q}{i-1}\bigl(\mathcal{E}_{k,q-i+1,\tau}(1) - \mathcal{E}_{k,q-i+1,\tau}\bigr).
\end{aligned}$$

Therefore, from Theorem 8, (48), and (49), we get

$$\begin{aligned}
&m^q Z_{k,q,\tau}(1,m)\\
&= \sum_{i=1}^{q-2}\binom{q}{i}\mathcal{E}_{k,q-i,\tau}(1)m^{q-i}E_i + \sum_{i=1}^{q-2}\binom{q}{i-1}\bigl(\mathcal{E}_{k,q-i+1,\tau}(1) - \mathcal{E}_{k,q-i+1,\tau}\bigr)m^{q-i}E_i\\
&\quad + 2\sum_{j=0}^{q}\binom{q}{j}\mathcal{E}_{k,j,\tau}m^{j-1}E_{q-j+1} + (q+1)\tau(1)E_q + m^q\mathcal{E}_{k,q,\tau}(1).
\end{aligned}$$

□

To prove the next theorem, we employ the symbolic notations as $E_n(x) = (E + x)^n$, $\mathcal{E}_{k,n,\tau}(x) = (\mathcal{E}_{k,\tau} + x)^n$, $(n \geq 0)$, with the usual convention about replacing E^n and $(\mathcal{E}_{k,\tau})^n$ with E_n and $\mathcal{E}_{k,n,\tau}$, respectively.

Theorem 10 *Let h, m be relatively prime positive integers and m, q be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we have*

$$m^q\sum_{\alpha=0}^{m-1}\sum_{j=0}^{q}(-1)^{\alpha}\binom{q}{j}\mathcal{E}_{k,j,\tau}\left(\frac{\alpha}{m}\right)h^jE_{q-j}\left(h - \left[\frac{h\alpha}{m}\right]\right) = \sum_{j=0}^{q}\binom{q}{j}(mh)^{q-j}E_j\mathcal{E}_{k,q-j,\tau}(1).$$

Proof As the index α ranges through the values $\alpha = 0, 1, 2, \dots, m-1$, the product $h\alpha$ ranges over a complete residue system modulo m such that h, m are relatively prime positive integers, and we may replace $< \frac{h\alpha}{m} >= \frac{h\alpha}{m} - [\frac{h\alpha}{m}]$ with $< \frac{h\alpha}{m} >$ without alternating the sum over α.

Therefore, from (8), we observe that

$$\begin{aligned} & m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \sum_{j=0}^{q} \binom{q}{j} \mathcal{E}_{k,j,\tau}\left(\frac{\alpha}{m}\right) h^j E_{q-j}\left(h - \left[\frac{h\alpha}{m}\right]\right) \\ &= m^q \sum_{\alpha=0}^{m-1} (-1)^a \left(h\left(\mathcal{E}_{k,\tau} + \frac{\alpha}{m}\right) + \left(E + h - \left[\frac{h\alpha}{m}\right]\right)\right)^q \\ &= m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \left(h\mathcal{E}_{k,\tau} + E + h + \frac{1}{2} - \frac{1}{2} + \frac{h\alpha}{m} - \left[\frac{h\alpha}{m}\right]\right)^q \\ &= m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \left(h\mathcal{E}_{k,\tau} + E + h + \frac{1}{2} + \overline{E}_1\left(\frac{\alpha}{m}\right)\right)^q \\ &= m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \left(h(\mathcal{E}_{k,\tau} 1) + E + \frac{\alpha}{m}\right)^q \\ &= m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \sum_{j=0}^{q} \binom{q}{j} \left(E + \frac{\alpha}{m}\right)^j h^{q-j} (\mathcal{E}_{k,\tau} + 1)^{q-j} \\ &= m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \sum_{j=0}^{q} \binom{q}{j} E_j\left(\frac{\alpha}{m}\right) h^{q-j} \mathcal{E}_{k,q-j,\tau}(1) \\ &= \sum_{j=0}^{q} \binom{q}{j} m^{q-j} \left(m^j \sum_{\alpha=0}^{m-1} (-1)^\alpha E_s\left(\frac{\alpha}{m}\right)\right) h^{q-j} \mathcal{E}_{k,q-j,\tau}(1) \\ &= \sum_{j=0}^{q} \binom{q}{j} (mh)^{q-j} E_j \mathcal{E}_{k,q-j,\tau}(1). \end{aligned} \tag{50}$$

Therefore, from (50), we obtain what we want. □

Theorem 11 *Let m, q be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we have*

$$\begin{aligned} & m^q Z_{k,q,\tau}(h,m) + h^q Z_{k,q,\tau}(m,h) \\ &= \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{h-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \binom{q}{l} (-1)^{\alpha+\beta-1} \frac{(j-1)!\tau(j) 2^{q-l+1} S_1(q+1-l,j)}{j^{k-1}(q+1-l)} \\ &\quad \times \left((\alpha h) m^{q-l} + (\beta m) h^{q-l}\right) \overline{E_l}\left(\frac{\alpha}{m} + \frac{\beta}{h}\right). \end{aligned}$$

Proof From Corollary 5, we note that

$$\overline{\mathcal{E}}_{k,q,\tau}(x) = \sum_{l=0}^{q} \sum_{j=1}^{q} \sum_{i=0}^{d-1} \binom{q}{l} (-1)^i d^l \overline{E}_l\left(\frac{i+x}{d}\right) \frac{(j-1)! 2^{q-l} \tau(j) S_1(q+1-l,j)}{j^{k-1}(q+1-l)}, \tag{51}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, $k \in \mathbb{Z}$, and $n \geq 0$.

From (46) and (51), we get

$$
\begin{aligned}
&m^q Z_{k,q,\tau}(h,m) + h^q Z_{k,q,\tau}(m,h) \qquad (52)\\
&= 2m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \frac{\alpha}{m} \overline{\mathcal{E}}_{k,q,\tau}\left(\frac{h\alpha}{m}\right) + 2h^q \sum_{\beta=0}^{h-1} (-1)^\beta \frac{\beta}{h} \overline{\mathcal{E}}_{k,q,\tau}\left(\frac{m\beta}{h}\right)\\
&= 2m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \frac{\alpha}{m} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\beta=0}^{h-1} \binom{q}{l} (-1)^\beta h^l \overline{E}_l\left(\frac{\beta}{h} + \frac{\alpha}{m}\right)\\
&\quad \times \frac{(j-1)! 2^{q-l} \tau(j) S_1(q+1-l,j)}{j^{k-1}(q+1-l)}\\
&\quad + 2h^q \sum_{\beta=0}^{h-1} (-1)^\beta \frac{\beta}{h} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\alpha=0}^{m-1} \binom{q}{l} (-1)^\alpha m^l \overline{E}_l\left(\frac{\alpha}{m} + \frac{\beta}{h}\right)\\
&\quad \times \frac{(j-1)! 2^{q-l} \tau(j) S_1(q+1-l,j)}{j^{k-1}(q+1-l)}\\
&= 2 \sum_{\alpha=0}^{m-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\beta=0}^{h-1} \binom{q}{l} (-1)^{\alpha+\beta} \frac{\alpha}{m} m^{q-l} (mh)^l \overline{E}_l\left(\frac{\beta}{h} + \frac{\alpha}{m}\right)\\
&\quad \times \frac{(j-1)! 2^{q-l} \tau(j) S_1(q+1-l,j)}{j^{k-1}(q+1-l)}\\
&\quad + 2 \sum_{\beta=0}^{h-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\alpha=0}^{m-1} \binom{q}{l} (-1)^{\alpha+\beta} \frac{\beta}{h} h^{q-l} (mh)^l \overline{E}_l\left(\frac{\alpha}{m} + \frac{\beta}{h}\right)\\
&\quad \times \frac{(j-1)! 2^{q-l} \tau(j) S_1(q+1-l,j)}{j^{k-1}(q+1-l)}\\
&= \sum_{\alpha=0}^{m-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\beta=0}^{h-1} \binom{q}{l} (-1)^{\alpha+\beta} (\alpha h)(mh)^{-1} m^{q-l} (mh)^l \overline{E}_l\left(\frac{\beta}{h} + \frac{\alpha}{m}\right)\\
&\quad \times \frac{(j-1)! 2^{q-l+1} \tau(j) S_1(q+1-l,j)}{j^{k-1}(q+1-l)}\\
&\quad + \sum_{\beta=0}^{h-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\alpha=0}^{m-1} \binom{q}{l} (-1)^{\alpha+\beta} (\beta m)(mh)^{-1} h^{q-l} (mh)^l \overline{E}_l\left(\frac{\alpha}{m} + \frac{\beta}{h}\right)\\
&\quad \times \frac{(j-1)! 2^{q-l+1} \tau(j) S_1(q+1-l,j)}{j^{k-1}(q+1-l)}\\
&= \sum_{\alpha=0}^{m-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\beta=0}^{h-1} \binom{q}{l} (-1)^{\alpha+\beta} \frac{(j-1)! \tau(j) 2^{q-l+1} S_1(q+1-l,j)}{j^{k-1}(q+1-l)}\\
&\quad \times \left((\alpha h) m^{q-l} + (\beta m) h^{q-l}\right) \overline{E}_l\left(\frac{\alpha}{m} + \frac{\beta}{h}\right).
\end{aligned}
$$

Therefore, from (52), we obtain the reciprocity relation for the type 2 unipoly-Dedekind type DC sums. □

Corollary 12 *Let m, q, be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we get*

$$m^q Z_{1,q,\frac{1}{\Gamma}}(h,m) + h^q Z_{1,q,\frac{1}{\Gamma}}(m,h) = m^q T_q(h,m) + h^q T_q(m,h)$$
$$= 2\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{h-1}(mh)^{q-1}(-1)^{\alpha+\beta}(\alpha h + m\beta)\overline{E}_q\left(\frac{\beta}{h} + \frac{\alpha}{m}\right),$$

where $\Gamma(n) = (n-1)!$.

4 Unipoly-Dedekind type DC sums associated withunipoly-Euler functions of index *k*

In this section, as another generalization of the poly Dedekind type DC sums, we consider unipoly-Dedekind type DC sums associated with the unipoly-Euler functions of index k and derive the reciprocity relation for these. For the purposes of this section, we first introduce two new polynomials, the poly-Euler polynomials and poly-Genocchi polynomials, using the polylogarithm function of arbitrary index k.

It is well known that the polylogarithm function of index k is defined by

$$Li_k(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^k}, \quad (k \in \mathbb{Z}), (\text{see } [8, 10, 12]).$$

Note that $Li_1(x) = -\log(1-x)$.

We consider the poly-Euler polynomials given by

$$\frac{Li_k(1-e^{-2t})}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty}E^*_{k,n}(x)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{53}$$

When $x = 0$, $E^*_{k,n} = E^*_{k,n}(0)$, $n \geq 0$, are called poly-Euler numbers.

We also introduce the poly-Genocchi polynomials, which are given by

$$\frac{Li_k(1-e^{-2t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty}G^*_{k,n}(x)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{54}$$

When $x = 0$, $G^*_{k,n} = G^*_{k,n}(0)$, $n \geq 0$, are called poly-Genocchi numbers.

Since $Li_1(1-e^{-2t}) = 2t$, we see that $E^{*(1)}_n(x) = E_n(x)$ and $G^{*(1)}_n(x) = G_n(x)$ $(n \geq 0)$ are the Euler polynomials and the Genocchi polynomials, respectively.

In addition, we define the unipoly-Euler polynomials of arbitrary index k defined by

$$\frac{u_k(1-e^{-2t})|\tau)}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty}E^*_{k,n,\tau}(x)\frac{t^n}{n!}. \tag{55}$$

When $x = 0$, $E^*_{k,n,\tau} = E^*_{k,n,\tau}(0)$ are called unipoly-Euler numbers.

When $\tau(n) = 1$ for all n, $E^*_{k,n,1}(x) = E^*_{k,n}(x)$ is the poly-Euler polynomials.

The unipoly-Genocchi polynomials of arbitrary index k are defined by

$$\frac{u_k(1-e^{-2t})|\tau)}{e^t+1}e^{xt} = \sum_{n=0}^{\infty}G^*_{k,n,\tau}(x)\frac{t^n}{n!}. \tag{56}$$

When $x = 0$, $\mathbb{G}^*_{k,n,\tau} = \mathbb{G}^*_{k,n,\tau}(0)$ are called unipoly-Genocchi numbers.

When $\tau(n) = 1$ for all n, $G^*_{k,n,1}(x) = G^*_{k,n}(x)$ is the poly-Genocchi polynomials.

Now, we consider a new type of unipoly-Dedekind type DC sums associated with the unipoly-Euler function of index k as follows:

$$Y_{k,q,\tau}(h,m) = 2\sum_{\mu=1}^{m-1}(-1)^{\mu}(\mu/m)\overline{E}^*_{k,q,\tau}(h\mu/m), \quad (h,m,q \in \mathbb{N}, k \in \mathbb{Z}), \tag{57}$$

where $h,m,q \in \mathbb{N}$ with $q \equiv 1 \pmod 2$ and $\overline{E}^*_{k,q,\tau}(x) = E^*_{k,q,\tau}(x-[x])$ are the unipoly-Euler functions of index k ($[x]$ is the largest integer less than x).

For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, let $\tau(n) = 1$. From $\overline{E}^*_{k,q,1} = \overline{E}^*_{k,q}$ and (13), we note that $Y_{k,q,1}(h,m) = T_{k,q}(h,m)$.

$$Y_{1,q,1}(h,m) = 2\sum_{\mu=1}^{m-1}(-1)^{\mu}(\mu/m)\overline{E}_{1,q,1}(h\mu/m) = T_q(h,m).$$

We note that

$$\frac{u_k(1-e^{-2t})|\tau)}{t(e^t+1)}e^{xt} = \left(\sum_{l=0}^{\infty}E^*_{k,l,\tau}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E^*_{k,l,\tau}x^{n-l}\right)\frac{t^n}{n!}. \tag{58}$$

Thus, by (58), we get

$$E^*_{k,n,\tau}(x) = \sum_{l=0}^{n}\binom{n}{l}E^*_{k,l,\tau}x^{n-l} \quad (n \geq 0). \tag{59}$$

In the same way as (59), we have

$$G^*_{k,n,\tau}(x) = \sum_{l=0}^{n}\binom{n}{l}G^*_{k,l,\tau}x^{n-l} \quad (n \geq 0). \tag{60}$$

Furthermore, by (59) and (60), we have

$$\frac{d}{dx}E^*_{k,n,\tau}(x) = nE^*_{k,n-1,\tau}(x) \quad \text{and} \quad \frac{d}{dx}G^*_{k,n,\tau}(x) = nG^*_{k,n-1,\tau}(x), \quad (n \geq 1). \tag{61}$$

Theorem 13 *For $n \geq 1$, we have*

$$E^*_{k,n,\tau}(x) = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!(-1)^{l+m+1}2^m\tau(l)S_2(m+1,l)}{l^{k-1}(m+1)}E_{n-m}(x), \quad (k \in \mathbb{Z}) \tag{62}$$

and

$$G^*_{k,n,\tau}(x) = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!(-1)^{l+m+1}2^m\tau(l)S_2(m+1,l)}{l^{k-1}(m+1)}G_{n-m}(x), \quad (k \in \mathbb{Z}). \tag{63}$$

Proof From (2), (16), and (55), we have

$$\sum_{n=0}^{\infty}E^*_{k,n,\tau}(x)\frac{t^n}{n!} = \frac{u_k(1-e^{-2t})|\tau)}{t(e^t+1)}e^{xt} = \frac{1}{2}\frac{2e^{xt}}{e^t+1}\frac{1}{t}\sum_{l=1}^{\infty}\frac{\tau(l)(1-e^{-2t})^l}{l^k} \tag{64}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{2e^{xt}}{e^t+1}\frac{1}{t}\sum_{m=1}^{\infty}\sum_{l=1}^{m}\frac{(l-1)!(-1)^l(-2)^m\tau(l)S_2(m,l)}{l^{k-1}}\frac{t^m}{m!} \\
&= \sum_{i=0}^{\infty}E_i(x)\frac{t^i}{i!}\sum_{m=0}^{\infty}\sum_{l=1}^{m+1}\frac{(l-1)!(-1)^{l+m+1}2^m\tau(l)S_2(m+1,l)}{l^{k-1}(m+1)}\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!(-1)^{l+m+1}2^m\tau(l)S_2(m+1,l)}{l^{k-1}(m+1)}E_{n-m}(x)\right)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, from (64), we get identity (62).

By using (3), (16), and (56), in the same way as (64), we also get identity (63). □

In (62), when $x = 1$, we get

$$
E^*_{k,n,\tau} = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!(-1)^{l+m+1}2^m\tau(l)S_2(m+1,l)}{l^{k-1}(m+1)}E_{n-m} \quad (k\in\mathbb{Z}), \tag{65}
$$

and the first few of the unipoly-Euler numbers are $E^*_{k,0,\tau} = \tau(1), E^*_{k,1,\tau} = -\tau(1) + \frac{1}{2^{k-1}}\tau(2), \dots$.

In (63), when $x = 1$, we get

$$
G^*_{k,n,\tau} = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{(l-1)!(-1)^{l+m+1}2^m\tau(l)S_2(m+1,l)}{l^{k-1}(m+1)}G_{n-m} \quad (k\in\mathbb{Z}), \tag{66}
$$

and the first few of the unipoly-Genocchi numbers are $G^*_{k,0,\tau} = 0$, $G^*_{k,1,\tau} = \tau(1), \dots$.

Since $G^*_{k,0,\tau} = 0$, we note that

$$
E^*_{k,n,\tau}(x) = \frac{G^*_{k,n+1,\tau}(x)}{n+1}. \tag{67}
$$

Lemma 14 *For $n \geq 1$, we have*

$$
G^*_{k,n,\tau}(1) + G^*_{k,n,\tau} = 2^n\sum_{m=1}^{n}\frac{(-1)^{n+m}m!}{m^k}\tau(m)S_2(n,m), \quad (k\in\mathbb{Z}) \tag{68}
$$

and

$$
E^*_{k,n-1,\tau}(1) + E^*_{k,n-1,\tau} = \frac{2^n}{n}\sum_{m=1}^{n}\frac{(-1)^{n+m}m!}{m^k}\tau(m)S_2(n,m), \quad (k\in\mathbb{Z}). \tag{69}
$$

Proof From (5) and (56), we have

$$
\begin{aligned}
u_k\left(1-e^{-2t}|\tau\right) &= \left(\sum_{l=0}^{\infty}G^*_{k,l,\tau}\frac{t^l}{l!}\right)\left(e^t+1\right) \\
&= \sum_{n=0}^{\infty}\left(G^*_{k,n,\tau}(1) + G^*_{k,n,\tau}\right)\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\left(G^*_{k,n,\tau}(1) + G^*_{k,n,\tau}\right)\frac{t^n}{n!}.
\end{aligned} \tag{70}
$$

On the other hand, from (17), we have

$$\begin{aligned} u_k\left(1-e^{-2t}|\tau\right) &= \sum_{m=1}^{\infty} \frac{\tau(m)}{m^k}(-1)^m\left(e^{-2t}-1\right)^m \\ &= \sum_{m=1}^{\infty} \frac{\tau(m)m!}{m^k}(-1)^m \sum_{n=m}^{\infty} S_2(n,m)\frac{(-2)^n t^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(2^n \sum_{m=1}^{n} \frac{(-1)^{n+m}m!}{m^k}\tau(m)S_2(n,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{71}$$

Therefore, by (70) and (71), we obtain identity (68). By using (67), we get identity (69). □

Theorem 15 *For an odd positive integer $d \geq 3$ and $n \geq 1$, we have*

$$\begin{aligned} &(-1)^{d-1}G_{k,n,\tau}(d) + G^*_{k,n,\tau} \\ &= \sum_{a=1}^{n}\sum_{b=0}^{a}\sum_{i=0}^{d-1}\binom{n}{a}(-1)^i i^{n-a}\frac{(b-1)!(-1)^{a+b}2^a}{b^{k-1}}\tau(b)S_2(a,b) \end{aligned} \tag{72}$$

and

$$\begin{aligned} &(-1)^{d-1}E^*_{k,n-1,\tau}(d) + E^*_{k,n-1,\tau} \\ &= \frac{1}{n}\left(\sum_{a=1}^{n}\sum_{b=0}^{a}\sum_{i=0}^{d-1}\binom{n}{a}(-1)^i i^{n-a}\frac{(b-1)!(-1)^{a+b}2^a}{b^{k-1}}\tau(b)S_2(a,b)\right). \end{aligned} \tag{73}$$

Proof For an odd positive integer $d \geq 3$, from (15) and (17), we have

$$\begin{aligned} &\sum_{i=0}^{d-1}(-1)^i e^{it}u_k\left(1-e^{-2t}|\tau\right) \\ &= \sum_{i=0}^{d-1}(-1)^i\sum_{j=0}^{\infty} i^j\frac{t^j}{j!}\sum_{b=1}^{\infty}\frac{\tau(b)(-1)^b(e^{-2t}-1)^b}{b^k} \\ &= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{d-1}(-1)^i i^j\right)\frac{t^j}{j!}\sum_{b=1}^{\infty}\frac{\tau(b)(-1)^b b!}{b^k}\sum_{a=b}^{\infty}S_2(a,b)\frac{(-2t)^a}{a!} \\ &= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{d-1}(-1)^i i^j\right)\frac{t^j}{j!}\sum_{a=1}^{\infty}\left(\sum_{b=1}^{a}\frac{(b-1)!(-1)^b(-2)^a}{b^{k-1}}\tau(b)S_2(a,b)\right)\frac{t^a}{a!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{a=1}^{n}\sum_{b=0}^{a}\sum_{i=0}^{d-1}\binom{n}{a}(-1)^i i^{n-a}\frac{(b-1)!(-1)^{a+b}2^a}{b^{k-1}}\tau(b)S_2(a,b)\right)\frac{t^n}{n!}. \end{aligned} \tag{74}$$

On the other hand, from $\sum_{i=0}^{n-1}(-1)^i e^{it} = \frac{(-1)^{n-1}e^{nt}+1}{e^t+1}$, $(n \equiv 1(\mathrm{mod}\,2))$, we have

$$\sum_{i=0}^{d-1}(-1)^i e^{it}u_k\left(1-e^{-2t}|\tau\right) = \frac{(-1)^{d-1}e^{dt}+1}{e^t+1}u_k\left(1-e^{-2t}|\tau\right) \tag{75}$$

$$= \sum_{n=0}^{\infty}\left((-1)^{d-1}G^{*}_{k,n}(d) + G^{*}_{k,n}\right)\frac{t^n}{n!}.$$

Therefore, by (74) and (75), we obtain identity (72).

Moreover, from (67), we get identity (73). □

Theorem 16 *For an odd positive integer $d \geq 1$ and $n \geq 0$, we have*

$$G^{*}_{k,n,\tau}(x) = \sum_{l=0}^{n}\binom{n}{l}\sum_{j=1}^{n-l+1}\sum_{i=0}^{d-1}(-1)^i d^{l-1} \times G_l\left(\frac{i+x}{d}\right)\frac{(j-1)!(-1)^{j+n-l+1}2^{n-l}}{j^{k-1}(n-l+1)}\tau(j)S_2(n-l+1,j).$$

Proof From (2) and (14), for an odd positive integer $d \geq 1$, we have

$$\begin{aligned}
&\frac{u_k(1-e^{-2t}|\tau)}{e^t+1}e^{xt} \qquad (76)\\
&= \frac{1}{2d}\sum_{i=0}^{d-1}(-1)^i e^{(\frac{i+x}{d})dt}\frac{2dt}{e^{dt}+1}\frac{1}{t}u_k\left(1-e^{-2t}|\tau\right)\\
&= \frac{1}{2d}\sum_{i=0}^{d-1}(-1)^i\sum_{l=0}^{\infty}G_l\left(\frac{i+x}{d}\right)\frac{d^l t^l}{l!}\frac{1}{t}\sum_{j=1}^{\infty}\frac{\tau(j)(-1)^j(e^{-2t}-1)^j}{j^k}\\
&= \frac{1}{2d}\sum_{i=0}^{d-1}(-1)^i\sum_{l=0}^{\infty}G_l\left(\frac{i+x}{d}\right)\frac{d^l t^l}{l!}\frac{1}{t}\sum_{j=1}^{\infty}\frac{\tau(j)(-1)^j j!}{j^k}\sum_{a=j}^{\infty}S_2(a,j)\frac{(-2t)^a}{a!}\\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{d-1}(-1)^i d^{l-1}G_l\left(\frac{i+x}{d}\right)\right)\frac{t^l}{l!}\sum_{a=1}^{\infty}\left(\sum_{j=1}^{a}\frac{(j-1)!(-1)^{j+a}2^{a-1}}{j^{k-1}}\tau(j)S_2(a,j)\right)\frac{t^{a-1}}{a!}\\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{d-1}d^{l-1}(-1)^i G_l\left(\frac{i+x}{d}\right)\right)\frac{t^l}{l!}\sum_{a=0}^{\infty}\left(\sum_{j=1}^{a+1}\frac{(j-1)!(-1)^{j+a+1}2^{a}}{j^{k-1}(a+1)}\tau(j)S_2(a+1,j)\right)\frac{t^{a}}{a!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{j=1}^{n-l+1}\sum_{i=0}^{d-1}(-1)^i d^{l-1}\right.\\
&\qquad\left.\times G_l\left(\frac{i+x}{d}\right)\frac{(j-1)!(-1)^{j+n-l+1}2^{n-l}}{j^{k-1}(n-l+1)}\tau(j)S_2(n-l+1,j)\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (76), we get the desired result. □

Corollary 17 *For an odd positive integer $d \geq 1$ and $n \geq 1$, we have*

$$E^{*}_{k,n-1,\tau}(x) = \sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n-1}{l}(-1)^i d^l E_l\left(\frac{i+x}{d}\right)\frac{(j-1)!(-1)^{n+j-l}2^{n-l-1}\tau(j)S_2(n-l,j)}{j^{k-1}(n-l)}.$$

Proof From (67) and Theorem 16, we have

$$
\begin{aligned}
&E^{*}_{k,n-1,\tau}(x) \\
&= \frac{G^{*}_{k,n,\tau}(x)}{n} \\
&= \frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}\sum_{j=1}^{n-l+1}\sum_{i=0}^{d-1} d^{l-1}(-1)^{i}G_{l}\left(\frac{i+x}{d}\right)\frac{(j-1)!\tau(j)(-1)^{j+n-l+1}2^{n-l}}{j^{k-1}(n-l+1)}S_{2}(n-l+1,j) \\
&= \frac{1}{n}\sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n}{l+1} d^{l}(-1)^{i}G_{l+1}\left(\frac{i+x}{d}\right)\frac{(j-1)!(-1)^{j+n-l}2^{n-l-1}}{j^{k-1}(n-l)}\tau(j)S_{2}(n-l,j) \\
&= \frac{n}{n}\sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n-1}{l} d^{l}(-1)^{i}\frac{G_{l+1}(\frac{i+x}{d})}{l+1}\frac{(j-1)!(-1)^{j+n-l}2^{n-l-1}\tau(j)S_{2}(n-l,j)}{j^{k-1}(n-l)} \\
&= \sum_{l=0}^{n-1}\sum_{j=1}^{n-l}\sum_{i=0}^{d-1}\binom{n-1}{l}(-1)^{i}d^{l}E_{l}\left(\frac{i+x}{d}\right)\frac{(j-1)!(-1)^{n+j-l}2^{n-l-1}\tau(j)S_{2}(n-l,j)}{j^{k-1}(n-l)}.
\end{aligned} \tag{77}
$$

There, from (77), we arrive at the desired result. □

We can obtain the following lemmas in the same way as Lemma 6 and Lemma 7, respectively, in Sect. 2.

Lemma 18 *For $l \in \mathbb{N}$ with $l < q$, we have*

$$
\binom{q}{l}E^{*}_{k,q-l,\tau}(1) + \binom{q}{l-1}E^{*}_{k,q-l+1,\tau}(1) = \sum_{j=0}^{q}\binom{q-j+1}{l}\binom{q}{j}E^{*}_{k,j,\tau}.
$$

Lemma 19 *For $q \in \mathbb{N}$, we have*

$$
\sum_{j=0}^{q}\binom{q}{j}\frac{E^{*}_{k,j,\tau}}{q-j+2} = \frac{1}{q+1}E^{*}_{k,q,\tau}(1) - \frac{1}{(q+1)(q+2)}E^{*}_{k,q+2,\tau}(1) + \frac{E^{*}_{k,q+2,\tau}}{(q+1)(q+2)}.
$$

In addition, we can obtain the following theorems in the same way as Theorem 9 and Theorem 10, respectively, in Sect. 3.

Theorem 20 *Let m, q be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we have*

$$
\begin{aligned}
&m^{q}Y_{k,q,\tau}(1,m) \\
&= \sum_{i=1}^{q-2}\binom{q}{i}E^{*}_{k,q-i,\tau}(1)m^{q-i}E_{i} + \sum_{i=1}^{q-2}\binom{q}{i-1}\left(E^{*}_{k,q-i+1,\tau}(1) - E^{*}_{k,q-i+1,\tau}\right)m^{q-i}E_{i} \\
&\quad + 2\sum_{j=0}^{q}\binom{q}{j}E^{*}_{k,j,\tau}m^{j-1}E_{q-j+1} + (q+1)\tau(1)E_{q} + m^{q}E^{*}_{k,q,\tau}(1).
\end{aligned}
$$

Theorem 21 *Let h, m be relatively prime positive integers and m, q be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we have*

$$m^q \sum_{\alpha=0}^{m-1} \sum_{j=0}^{q} (-1)^\alpha \binom{q}{j} E^*_{k,j,\tau}\left(\frac{\alpha}{m}\right) h^j E_{q-j}\left(h - \left[\frac{h\alpha}{m}\right]\right) = \sum_{j=0}^{q} \binom{q}{j} (mh)^{q-j} E_j E^*_{k,q-j,\tau}(1).$$

Now, we obtain the following reciprocity theorem for the unipoly-Dedekind type DC sums associated with the unipoly-Euler function with index k.

Theorem 22 *Let m, q be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we have*

$$\begin{aligned} & m^q Y_{k,q,\tau}(h,m) + h^q Y_{k,q,\tau}(m,h) \\ &= \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{h-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \binom{q}{l} (-1)^{\alpha+\beta+q+1+j-l} \frac{(j-1)!\tau(j)2^{q-l+1} S_2(q+1-l,j)}{j^{k-1}(q+1-l)} \\ &\quad \times \left((\alpha h) m^{q-l} + (\beta m) h^{q-l}\right) \overline{E}_l\left(\frac{\alpha}{m} + \frac{\beta}{h}\right). \end{aligned}$$

Proof From Corollary 17, we note that

$$\begin{aligned} \overline{E}^*_{k,q,\tau}(x) &= \sum_{l=0}^{q} \sum_{j=1}^{q} \sum_{i=0}^{d-1} \binom{q}{l} (-1)^i d^l \overline{E}_l\left(\frac{i+x}{d}\right) \\ &\quad \times \frac{(j-1)!(-1)^{q+1+j-l} 2^{q-l} \tau(j) S_2(q+1-l,j)}{j^{k-1}(q+1-l)}, \end{aligned} \tag{78}$$

where $d \in \mathbb{N}$ with $d \equiv 1$ (mod2), $k \in \mathbb{Z}$, and $n \geq 0$.

From (78), in the same way as (53),

$$\begin{aligned} & m^q Y_{k,q,\tau}(h,m) + h^q Y_{k,q,\tau}(m,h) \\ &= 2m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \frac{\alpha}{m} \overline{E}_{k,q,\tau}\left(\frac{h\alpha}{m}\right) + 2h^q \sum_{\beta=0}^{h-1} (-1)^\beta \frac{\beta}{h} \overline{E}_{k,q,\tau}\left(\frac{m\beta}{h}\right) \\ &= 2m^q \sum_{\alpha=0}^{m-1} (-1)^\alpha \frac{\alpha}{m} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\beta=0}^{h-1} \binom{q}{l} (-1)^\beta h^l \overline{E}_l\left(\frac{\beta}{h} + \frac{\alpha}{m}\right) \\ &\quad \times \frac{(j-1)!(-1)^{q+1+j-l} 2^{q-l} \tau(j) S_2(q+1-l,j)}{j^{k-1}(q+1-l)} \\ &\quad + 2h^q \sum_{\beta=0}^{h-1} (-1)^\beta \frac{\beta}{h} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\alpha=0}^{m-1} \binom{q}{l} (-1)^\alpha m^l \overline{E}_l\left(\frac{\alpha}{m} + \frac{\beta}{h}\right) \\ &\quad \times \frac{(j-1)!(-1)^{q+1+j-l} 2^{q-l} \tau(j) S_2(q+1-l,j)}{j^{k-1}(q+1-l)} \\ &= 2 \sum_{\alpha=0}^{m-1} \sum_{l=0}^{q} \sum_{j=1}^{q+1-l} \sum_{\beta=0}^{h-1} \binom{q}{l} (-1)^{\alpha+\beta} \frac{\alpha}{m} m^{q-l} (mh)^l \overline{E}_l\left(\frac{\beta}{h} + \frac{\alpha}{m}\right) \\ &\quad \times \frac{(j-1)!(-1)^{q+1+j-l} 2^{q-l} \tau(j) S_2(q+1-l,j)}{j^{k-1}(q+1-l)} \end{aligned} \tag{79}$$

$$+2\sum_{\beta=0}^{h-1}\sum_{l=0}^{q}\sum_{j=1}^{q+1-l}\sum_{\alpha=0}^{m-1}\binom{q}{l}(-1)^{\alpha+\beta}\frac{\beta}{h}h^{q-l}(mh)^{l}\overline{E_l}\left(\frac{\alpha}{m}+\frac{\beta}{h}\right)$$
$$\times\frac{(j-1)!(-1)^{q+1+j-l}2^{q-l}\tau(j)S_2(q+1-l,j)}{j^{k-1}(q+1-l)}$$
$$=\sum_{\alpha=0}^{m-1}\sum_{l=0}^{q}\sum_{j=1}^{q+1-l}\sum_{\beta=0}^{h-1}\binom{q}{l}(-1)^{\alpha+\beta+q+1+j-l}\frac{(j-1)!\tau(j)2^{q-l+1}S_2(q+1-l,j)}{j^{k-1}(q+1-l)}$$
$$\times\big((\alpha h)m^{q-l}+(\beta m)h^{q-l}\big)\overline{E_l}\left(\frac{\alpha}{m}+\frac{\beta}{h}\right).$$

Therefore, from (79), we obtain the reciprocity relation for the unipoly-Dedekind type DC sums. □

Corollary 23 *Let m, q, be odd positive integers $m \geq 1$ and $q \geq 3$, respectively. Then we get*

$$m^q Y_{1,q,1}(h,m)+h^q Y_{1,q,1}(m,h)=m^q T_q(h,m)+h^q T_q(m,h)$$
$$=2\sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{h-1}(mh)^{q-1}(-1)^{\alpha+\beta}(\alpha h+m\beta)\overline{E}_q\left(\frac{\beta}{h}+\frac{\alpha}{m}\right),$$

where $\tau(n) = 1$ for all n.

5 Conclusion

In this paper, as further generalizations of the poly-Dedekind type DC sums, we introduced two kinds of unipoly-Dedekind type DC sums. In Sect. 3, the type 2 unipoly-Dedekind type DC sums associated with the type 2 unipoly-Euler functions of index k were introduced, and some interesting identities and the reciprocity relation were shown. In Sect. 4, the unipoly-Dedekind type DC sums associated with the unipoly-Euler functions of index k were introduced, and some interesting identities and the reciprocity relation were shown. We would like to further study another Dedekind type DC sums.

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Authors' contributions
HKK structured and wrote the whole paper. DSL and HKK checked the results of the paper and completed the revision of the article. All authors read and approved the final manuscript.

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