Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Two-sided fractional quaternion Fourier transform and its application

Zunfeng Li[1*], Haipan Shi[2] and Yuying Qiao[2]

*Correspondence:
zunfeng928@163.com
[1]College of Science, Hebei University of Science and Technology, Shijiazhuang, Hebei Province 050018, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the two-sided fractional quaternion Fourier transform (FrQFT) and give some properties of it. The main results of this paper are divided into three parts. Firstly we give a definition of the FrQFT. Secondly based on properties of the two-sided QFT, we study the relationship between the two-sided QFT and the two-sided FrQFT, and give some differential properties of the two-sided FrQFT and the Parseval identity. Finally, we give an example to illustrate the application of the two-sided FrQFT and its inverse transform in solving partial differential equations.

Keywords: Two-sided fractional quaternion Fourier transform; Differential properties; Parseval identity; Partial differential equations

1 Introduction

There are two aspects of the generalization of the classical Fourier transform: one is to the high dimensional space, the other is to the fractional Fourier transform. The quaternion Fourier transform (QFT) is one of the generalized forms of the classical Fourier transform in high dimensional space and has been proved to be very useful in signal processing, non-marginal color image processing, electromagnetism, multi-channel processing, quantum mechanics, and partial differential systems. Many scholars have done a lot of research on the QFT and got many excellent results. In recent years, some properties of the QFT and the two-sided QFT have been studied [4–11].

In 2007, Hitzer [6] researched the QFT properties useful for applications to differential equations, image processing and optimized numerical implementations and studied the general linear transformation behavior of the QFT with matrices. In 2010, Hitzer [7] derived a directional uncertainty principle for quaternion-valued functions subject to the QFT. In 2016, Hitzer [4] defined the FT on the quaternion domain and analyzed its main properties, including quaternion dilation, modulation, shift properties and Parseval identities. In 2017, Haoui and Fahlaoui [2] presented the Heisenberg inequality and Hardy's theorem for the two-sided QFT. In [11], Yang et al. studied uncertainty principles of the QFT under the polar coordinate form. In [1], Bahri proposed the uncertainty principle for the two-sided QFT. That uncertainty principle described that the spread of a quaternion-valued function and its two-sided QFT was inversely proportional.

On the basis of the above work, we give some properties of the two-sided FrQFT and its application. The structure of the article is as follows: in the second section, we introduce the basic knowledge related to the quaternion Fourier analysis. In the third section, we first give a definition of the two-sided FrQFT. Then based on the nature of the two-sided QFT, we study the relationship between the two-sided QFT and the two-sided FrQFT. We give some differential properties, shift properties of the two-sided FrQFT and Parseval identity. Finally, we give an example to illustrate the application of the two-sided FrQFT and its inverse transform in solving partial differential equations.

2 Preliminaries

Let $\mathbb{R}^2$ be a real linear space with basis $\{e_1, e_2\}$, the quaternion algebra $\mathbb{H}$ which is an associative and noncommutative algebra structure spanned by

$$\{1, e_1, e_2, e_1e_2\}.$$

And basis elements satisfy the following multiplication laws:

$$\begin{cases} e_i^2 = -1, \quad i = 1, 2; \\ e_1e_2 = -e_2e_1 = e_{12}; \\ {e_{12}}^2 = e_{12}e_{12} = -1; \\ e_2e_{12} = -e_{12}e_2 = e_1; \\ e_{12}e_1 = -e_1e_{12} = e_2. \end{cases}$$

Every quaternion

$$q = q_0 + q_1e_1 + q_2e_2 + q_3e_{12} \in \mathbb{H}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

has a quaternion conjugate $\overline{q} = q_0 - q_1e_1 - q_2e_2 - q_3e_{12}$, where $(q)_0 = q_0$.

For arbitrary $p, q \in \mathbb{H}$, $\overline{pq} = \overline{q}\,\overline{p}$.

For quaternion-valued functions $f, g : \mathbb{R}^2 \to \mathbb{H}$, the quaternion-valued inner product is defined by

$$(f, g) = \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}\, d\mathbf{x},$$

and the real scalar part is

$$\langle f, g\rangle = \frac{1}{2}\big[(f, g) + (g, f)\big] = \int_{\mathbb{R}^2} \big(f(\mathbf{x})g(\mathbf{x})\big)_0\, d\mathbf{x},$$

where $d\mathbf{x} = dx_1\, dx_2$.

In particular, when $f = g$, this leads to

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = (f, f) = \langle f, f\rangle = \int_{\mathbb{R}^2} \big|f(\mathbf{x})\big|^2\, d\mathbf{x}.$$

Definition 2.1 ([12]) For any infinitely differentiable function $f(x)$, if

$$\int_{-\infty}^{\infty}\delta(x)f(x)\,dx=\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\delta_\varepsilon(x)f(x)\,dx,$$

we call the weak limit of $\delta_\varepsilon(x)$ a δ function and denote $\lim_{\varepsilon\to 0^+}\delta_\varepsilon(x)=\delta(x)$, where

$$\delta_\varepsilon(x)=\begin{cases}0, & x<0;\\ \frac{1}{\varepsilon}, & 0\le x\le\varepsilon;\\ 0, & x>\varepsilon.\end{cases}$$

Here are some properties of δ function as described below.

Lemma 2.1 ([12]) *Suppose that* $\lim_{\varepsilon\to 0+}\delta_\varepsilon(x)=\delta(x)$, *then*

(1) $\int_{-\infty}^{\infty}\delta(x)\,dx=1$;
(2) $\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{jwx}\,dw=\delta(x)$, $\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{jw(x-x_0)}\,dw=\delta(x-x_0)$;
(3) $\int_{-\infty}^{\infty}\delta(x)f(x)\,dx=f(0)$, $\int_{-\infty}^{\infty}\delta(x-x_0)f(x)\,dx=f(x_0)$.

Definition 2.2 We define

$$L^1(\mathbb{R}^2;\mathbb{H})=\left\{f:\mathbb{R}^2\to\mathbb{H}\,\middle|\,\int_{\mathbb{R}^2}\bigl|f(\mathbf{x})\bigr|d\mathbf{x}<\infty\right\},$$
$$L^2(\mathbb{R}^2;\mathbb{H})=\left\{f:\mathbb{R}^2\to\mathbb{H}\,\middle|\,\int_{\mathbb{R}^2}\bigl|f(\mathbf{x})\bigr|^2\,d\mathbf{x}<\infty\right\}.$$

Next, we define the space

$$S(\mathbb{R}^2;\mathbb{H})=\left\{f\in C^\infty(\mathbb{R}^2;\mathbb{H})\,\middle|\,\sup_{x\in\mathbb{R}^2}\bigl(1+|x|^k\bigr)\bigl|\partial^\alpha f(x)\bigr|<\infty\right\},$$

where $C^\infty(\mathbb{R}^2;\mathbb{H})$ is the set of all infinitely differentiable functions from $\mathbb{R}^2$ to $\mathbb{H}$, and $\alpha=(\alpha_1,\alpha_2)$, $\alpha_1,\alpha_2,k\in\mathbb{Z}_+$.

For convenience, we divide $f\in\mathbb{H}$ into two parts as follows.

Lemma 2.2 *Let* $e_l\in\mathbb{H}$, $l=1,2$. *For any* $f\in\mathbb{H}$, *we define*

$$f_{+e_l}=\frac{1}{2}(f+e_lfe_l),\qquad f_{-e_l}=\frac{1}{2}(f-e_lfe_l).$$

Then $f=f_{+e_l}+f_{-e_l}$, *and* $\overline{f_{+e_l}}=\overline{f}_{+e_l}$, $\overline{f_{-e_l}}=\overline{f}_{-e_l}$.

Lemma 2.3 ([3]) *Let* $\alpha\in\mathbb{R}$ *and* $e_l\in\mathbb{H}$ *with* $e_l^2=-1$. *We have a natural generalization of Euler's formula in quaternion analysis as follows*:

$$e^{e_l\alpha}=\cos\alpha+e_l\sin\alpha.$$

Theorem 2.1 *Suppose* e_l *is as stated above, for any* $f\in\mathbb{H}$, *the following equation is always true*

(1) $e^{ce_l}f_{+e_l} = f_{+e_l}e^{-ce_l}$, $e^{ce_l}f_{-e_l} = f_{-e_l}e^{ce_l}$.
(2) $\sqrt{\frac{1-e_l\cot\theta_i}{2\pi}}f_{+e_l} = f_{+e_l}\sqrt{\frac{1+e_l\cot\theta_i}{2\pi}}$, $\sqrt{\frac{1-e_l\cot\theta_i}{2\pi}}f_{-e_l} = f_{-e_l}\sqrt{\frac{1-e_l\cot\theta_i}{2\pi}}$,
where $c \in \mathbb{R}$, $i = 1, 2$.

Proof (1) According to Lemma 2.2, we have

$$\begin{aligned} e^{ce_l}f_{+e_l} &= (\cos c + e_l \sin c)\frac{1}{2}(f + e_l f e_l) \\ &= \frac{1}{2}(f + e_l f e_l)\cos c + \frac{1}{2}e_l(f + e_l f e_l)\sin c \\ &= \frac{1}{2}(f + e_l f e_l)\cos c + \frac{1}{2}\big(e_l f e_l(-e_l) - f e_l\big)\sin c \\ &= \frac{1}{2}(f + e_l f e_l)\cos c + \frac{1}{2}(e_l f e_l + f)(-e_l)\sin c \\ &= \frac{f + e_l f e_l}{2}(\cos c - e_l \sin c) = f_{+e_l}e^{-ce_l}. \end{aligned}$$

Similarly, we have $e^{ce_l}f_{-e_l} = f_{-e_l}e^{ce_l}$.
(2) According to Lemma 2.2, we have

$$\begin{aligned} \sqrt{\frac{1 - e_l\cot\theta_i}{2\pi}} &= \sqrt{\frac{\sin\theta_i - e_l\cos\theta_i}{2\pi\,\sin\theta_i}} = \sqrt{\frac{-e_l(e_l\sin\theta_i + \cos\theta_i)}{2\pi\,\sin\theta_i}} \\ &= \sqrt{\frac{-e_l e^{e_l\theta_i}}{2\pi\,\sin\theta_i}} = \sqrt{\frac{e^{-\frac{\pi}{2}e_l}e^{e_l\theta_i}}{2\pi\,\sin\theta_i}} = \sqrt{\frac{e^{(-\frac{\pi}{2}+\theta_i)e_l}}{2\pi\,\sin\theta_i}}. \end{aligned}$$

The above equation as a function of θ_i is periodic with π.
When $\theta_i \in (0, \pi)$, $\sqrt{\frac{e^{(-\frac{\pi}{2}+\theta_i)e_l}}{2\pi\sin\theta_i}} = \frac{e^{\frac{1}{2}(-\frac{\pi}{2}+\theta_i)e_l}}{\sqrt{2\pi\sin\theta_i}}$.
According to Eq. (1), we have

$$\frac{e^{\frac{1}{2}(-\frac{\pi}{2}+\theta_i)e_l}}{\sqrt{2\pi\,\sin\theta_i}}f_{+e_l} = f_{+e_l}\frac{e^{\frac{1}{2}(\frac{\pi}{2}-\theta_i)e_l}}{\sqrt{2\pi\,\sin\theta_i}},$$

that is,

$$\sqrt{\frac{1 - e_l\cot\theta_i}{2\pi}}f_{+e_l} = f_{+e_l}\sqrt{\frac{1 + e_l\cot\theta_i}{2\pi}}.$$

When $\theta_i \in (-\pi, 0)$, $\sqrt{\frac{e^{(-\frac{\pi}{2}+\theta_i)e_l}}{2\pi\sin\theta_i}} = \frac{e^{\frac{1}{2}(\frac{\pi}{2}+\theta_i)e_l}}{\sqrt{-2\pi\sin\theta_i}}$.
So we have $\sqrt{\frac{1-e_l\cot\theta_i}{2\pi}}f_{+e_l} = f_{+e_l}\sqrt{\frac{1+e_l\cot\theta_i}{2\pi}}$.
The other cases are similar. □

Theorem 2.2 *Let* $e_l \in \mathbb{H}$, $l = 1, 2$. *For any* $f_1, f_2 \in \mathbb{H}$, *we have*

$$\begin{aligned} e^{-\alpha e_l}f_{1+}\overline{f_{2+}} &= f_{1+}\overline{f_{2+}}e^{-\alpha e_l}, \qquad e^{-\alpha e_l}f_{1-}\overline{f_{2-}} = f_{1-}\overline{f_{2-}}e^{-\alpha e_l}. \\ e^{-\alpha e_l}f_{1-}\overline{f_{2+}} &= f_{1-}\overline{f_{2+}}e^{\alpha e_l}, \qquad e^{-\alpha e_l}f_{1+}\overline{f_{2-}} = f_{1+}\overline{f_{2-}}e^{\alpha e_l}. \end{aligned}$$

Proof According to Lemma 2.2 and Theorem 2.1, we have

$$e^{-\alpha e_l}f_{1+}\overline{f_{2+}} = f_{1+}e^{\alpha e_l}\overline{f_{2+}} = f_{1+}e^{\alpha e_l}\frac{1}{2}(\overline{f_2} + \overline{e_l f_2 e_l})$$
$$= f_{1+}e^{\alpha e_l}\frac{1}{2}(\overline{f_2} + \overline{e_l}\,\overline{f_2}\,\overline{e_l}) = f_{1+}e^{\alpha e_l}\frac{1}{2}(\overline{f_2} + e_l\overline{f_2}e_l) = f_{1+}\overline{f_{2+}}e^{-\alpha e_l},$$

where

$$\begin{aligned}
&e^{\alpha e_l}\frac{1}{2}(\overline{f_2} + e_l\overline{f_2}e_l)\\
&= (\cos\alpha + e_l\sin\alpha)\frac{1}{2}(\overline{f_2} + e_l\overline{f_2}e_l)\\
&= \frac{1}{2}(\overline{f_2} + e_l\overline{f_2}e_l)\cos\alpha + \frac{1}{2}\left(e_l\overline{f_2} + e_l^2\overline{f_2}e_l\right)\sin\alpha\\
&= \frac{1}{2}(\overline{f_2} + e_l\overline{f_2}e_l)\cos\alpha + \frac{1}{2}\left(e_l\overline{f_2}\left(-e_l^2\right) - \overline{f_2}e_l\right)\sin\alpha\\
&= \frac{1}{2}(\overline{f_2} + e_l\overline{f_2}e_l)\cos\alpha + \frac{1}{2}(e_l\overline{f_2}e_l + \overline{f_2})(-e_l)\sin\alpha\\
&= \frac{1}{2}(\overline{f_2} + e_l\overline{f_2}e_l)(\cos\alpha - e_l\sin\alpha) = \overline{f_{2+}}e^{-\alpha e_l}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&e^{-\alpha e_l}f_{1-}\overline{f_{2-}} = f_{1-}\overline{f_{2-}}e^{-\alpha e_l},\\
&e^{-\alpha e_l}f_{1-}\overline{f_{2+}} = f_{1-}\overline{f_{2+}}e^{\alpha e_l},\\
&e^{-\alpha e_l}f_{1+}\overline{f_{2-}} = f_{1+}\overline{f_{2-}}e^{\alpha e_l}.
\end{aligned}$$

□

3 Some properties of the two-sided FrQFT

In this section we state some properties of the two-sided FrQFT. We first give a definition of the two-sided FrQFT and its inverse transformation. Then we get the relationship between two-sided QFT and the two-sided FrQFT. Finally we study the properties of this transformation, such as the shift property, differential properties of functions and their image functions, and differential properties of kernel functions.

Definition 3.1 Suppose that the function $f \in L^1(\mathbb{R}^2;\mathbb{H})$. We define $p = (p_1, p_2)$-order two-sided FrQFT as follows:

$$\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) = \int_{\mathbb{R}^2} K_{\theta_1}(x_1,w_1)f(\mathbf{x})K_{\theta_2}(x_2,w_2)\,d\mathbf{x}, \tag{3.1}$$

where

$$K_{\theta_1}(x_1,w_1) = C_{\theta_1}e^{e_1\frac{x_1^2+w_1^2}{2}\cot\theta_1 - e_1x_1w_1\csc\theta_1}, \qquad C_{\theta_1} = \sqrt{\frac{1-e_1\cot\theta_1}{2\pi}},$$
$$K_{\theta_2}(x_2,w_2) = C_{\theta_2}e^{e_2\frac{x_2^2+w_2^2}{2}\cot\theta_2 - e_2x_2w_2\csc\theta_2}, \qquad C_{\theta_2} = \sqrt{\frac{1-e_2\cot\theta_2}{2\pi}},$$

and $\theta_i \neq n\pi$, $p_i = \frac{2\theta_i}{\pi}$, $i = 1,2$.

Definition 3.2 Suppose that $f \in L^1(\mathbb{R}^2;\mathbb{H})$. We define the inverse transformation of the two-sided FrQFT as follows:

$$\mathcal{H}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) = \int_{\mathbb{R}^2} K_{-\theta_1}(x_1,w_1)f(\mathbf{x})K_{-\theta_2}(x_2,w_2)\,d\mathbf{x}, \tag{3.2}$$

where

$$K_{-\theta_1}(x_1,w_1) = C_{-\theta_1}e^{-e_1\frac{x_1^2+w_1^2}{2}\cot\theta_1+e_1x_1w_1\csc\theta_1}, \qquad C_{-\theta_1} = \sqrt{\frac{1+e_1\cot\theta_1}{2\pi}},$$
$$K_{-\theta_2}(x_2,w_2) = C_{-\theta_2}e^{-e_2\frac{x_2^2+w_2^2}{2}\cot\theta_2+e_2x_2w_2\csc\theta_2}, \qquad C_{-\theta_2} = \sqrt{\frac{1+e_2\cot\theta_2}{2\pi}},$$

where θ_i $(i = 1,2)$ are as mentioned above.

When $p_1 = p_2 = 1$, the two-sided FrQFT becomes the two-sided QFT.
We know that the two-sided QFT is defined as (see [1])

$$\mathcal{F}\{f\}(\mathbf{w}) = \int_{\mathbb{R}^2} e^{-e_1x_1w_1}f(\mathbf{x})e^{-e_2x_2w_2}\,d\mathbf{x}. \tag{3.3}$$

Its inverse transformation is defined as

$$\mathcal{F}^{-1}\{f\}(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{e_1x_1w_1}f(\mathbf{w})e^{e_2x_2w_2}\,d\mathbf{w}. \tag{3.4}$$

The Plancherel identity is given by

$$(f_1,f_2) = \frac{1}{(2\pi)^2}\big(\mathcal{F}\{f_1\}(\mathbf{w}),\mathcal{F}\{f_2\}(\mathbf{w})\big). \tag{3.5}$$

Theorem 3.1 *The two-sided FrQFT and the two-sided QFT have the following relationship*:

$$\begin{aligned}
&\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) \\
&\quad= C_{\theta_1}e^{e_1\frac{w_1^2}{2}\cot\theta_1}\int_{\mathbb{R}^2} e^{e_1\frac{x_1^2}{2}\cot\theta_1-e_1x_1w_1\csc\theta_1}f(\mathbf{x})e^{e_2\frac{x_2^2}{2}\cot\theta_2-e_2x_2w_2\csc\theta_2}\,d\mathbf{x} \\
&\qquad\times\big(C_{\theta_2}e^{e_2\frac{w_2^2}{2}\cot\theta_2}\big) \\
&\quad= C_{\theta_1}e^{e_1\frac{w_1^2}{2}\cot\theta_1}\mathcal{F}\big\{e^{e_1\frac{x_1^2}{2}\cot\theta_1}f(\mathbf{x})e^{e_2\frac{x_2^2}{2}\cot\theta_2}\big\}(\mathbf{w}\csc\theta)\big(C_{\theta_2}e^{e_2\frac{w_2^2}{2}\cot\theta_2}\big),
\end{aligned}$$

where $\mathbf{w}\csc\theta = (w_1\csc\theta_1, w_2\csc\theta_2)$.

Theorem 3.2 *Let* $f, \mathcal{F}_{\theta_1,\theta_2}\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$ *and* $0 < |\theta_i| < \pi$, $i = 1,2$. *Then*

$$(\mathcal{H}_{\theta_1,\theta_2}\circ\mathcal{F}_{\theta_1,\theta_2})\{f\} = (\mathcal{F}_{\theta_1,\theta_2}\circ\mathcal{H}_{\theta_1,\theta_2})\{f\} = f.$$

Proof According to the Fubini replacement theorem and Theorem 2.1, we have

$$
\begin{aligned}
&(\mathcal{H}_{\theta_1,\theta_2} \circ \mathcal{F}_{\theta_1,\theta_2})\{f\}(\mathbf{y}) \\
&\quad = \mathcal{H}_{\theta_1,\theta_2}\bigl\{\mathcal{F}_{\theta_1,\theta_2}\{f\}(w)\bigr\}(\mathbf{y}) \\
&\quad = \int_{\mathbb{R}^2} K_{-\theta_1}(y_1,w_1)\biggl(\int_{\mathbb{R}^2} K_{\theta_1}(x_1,w_1)f(\mathbf{x})K_{\theta_2}(x_2,w_2)\,d\mathbf{x}\biggr)K_{-\theta_2}(w_2,y_2)\,d\mathbf{w} \\
&\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_{-\theta_1}(y_1,w_1)K_{\theta_1}(x_1,w_1)f(\mathbf{x})K_{\theta_2}(x_2,w_2)\,d\mathbf{x}K_{-\theta_2}(w_2,y_2)\,d\mathbf{w} \\
&\quad = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} C_{\theta_1}C_{-\theta_1}e^{e_1\frac{x_1^2-y_1^2}{2}\cot\theta_1-e_1(x_1-y_1)w_1\csc\theta_1}f(\mathbf{x}) \\
&\qquad \times C_{\theta_2}C_{-\theta_2}e^{e_2\frac{x_2^2-y_2^2}{2}\cot\theta_2-e_2(x_2-y_2)w_2\csc\theta_2}\,d\mathbf{x}\,d\mathbf{w} \\
&\quad = \biggl(\frac{1}{2\pi}\biggr)^2\frac{1}{|\sin\theta_1\sin\theta_2|}\int_{\mathbb{R}^2} e^{e_1\frac{-y_1^2}{2}\cot\theta_1}e^{e_1y_1w_1\csc\theta_1}\int_{\mathbb{R}^2} e^{-e_1x_1w_1\csc\theta_1}e^{e_1\frac{x_1^2}{2}\cot\theta_1} \\
&\qquad \times f(\mathbf{x})e^{-e_2x_2w_2\csc\theta_2}e^{e_2\frac{x_2^2}{2}\cot\theta_2}\,d\mathbf{x}e^{e_2\frac{-y_2^2}{2}\cot\theta_2}e^{e_2y_2w_2\csc\theta_2}\,d\mathbf{w}.
\end{aligned}
$$

Let $w_i\csc\theta_i = t_i$, then $w_i = \frac{t_i}{\sin\theta_i}$, $i = 1,2$. According to (3.3) and (3.4), we have

$$
\begin{aligned}
&(\mathcal{H}_{\theta_1,\theta_2} \circ \mathcal{F}_{\theta_1,\theta_2})\{f\}(\mathbf{y}) \\
&\quad = e^{e_1\frac{-y_1^2}{2}\cot\theta_1}\biggl(\frac{1}{2\pi}\biggr)^2\int_{\mathbb{R}^2} e^{e_1y_1t_1}\int_{\mathbb{R}^2} e^{-e_1t_1x_1}e^{e_1\frac{x_1^2}{2}\cot\theta_1}f(\mathbf{x})e^{e_2\frac{x_2^2}{2}\cot\theta_2}e^{-e_2t_2x_2} \\
&\qquad \times d\mathbf{x}e^{e_2y_2t_2}\,d\mathbf{t}e^{e_2\frac{-y_2^2}{2}\cot\theta_2} \\
&\quad = e^{e_1\frac{-y_1^2}{2}\cot\theta_1}\mathcal{F}^{-1}\bigl(\mathcal{F}\bigl\{e^{e_1\frac{y_1^2}{2}\cot\theta_1}f(\mathbf{y})e^{e_2\frac{y_2^2}{2}\cot\theta_2}\bigr\}\bigr)e^{e_2\frac{-y_2^2}{2}\cot\theta_2} \\
&\quad = f(\mathbf{y}).
\end{aligned}
$$

Similarly we have $(\mathcal{F}_{\theta_1,\theta_2} \circ \mathcal{H}_{\theta_1,\theta_2})\{f\} = f$. It means that $\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w})$ and $\mathcal{H}_{\theta_1,\theta_2}\{f\}(\mathbf{w})$ are inverse transformations of each other. □

Next, we give some important properties of the two-sided FrQFT; we begin with the shift property.

Theorem 3.3 *Let* $f \in L^1(\mathbb{R}^2;\mathbb{H})$ *and* $\mathbf{t} = (t_1,t_2)$, *the following property holds*

$$
\begin{aligned}
&\mathcal{F}_{\theta_1,\theta_2}\bigl\{f(\mathbf{x}-\mathbf{t})\bigr\}(\mathbf{w}) \\
&\quad = e^{e_1\frac{t_1^2}{2}\cot\theta_1-e_1t_1w_1\csc\theta_1}\mathcal{F}_{\theta_1,\theta_2}\bigl\{e^{e_1y_1t_1\cot\theta_1}f(\mathbf{y})e^{e_2y_2t_2\cot\theta_2}\bigr\}(\mathbf{w}) \\
&\qquad \times e^{e_2\frac{t_2^2}{2}\cot\theta_2-e_2t_2w_2\csc\theta_2}.
\end{aligned}
$$

Proof

$$
\begin{aligned}
&\mathcal{F}_{\theta_1,\theta_2}\bigl\{f(\mathbf{x}-\mathbf{t})\bigr\}(\mathbf{w}) \\
&\quad = \int_{\mathbb{R}^2} K_{\theta_1}(x_1,w_1)f(\mathbf{x}-\mathbf{t})K_{\theta_2}(x_2,w_2)\,d\mathbf{x}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} C_{\theta_1} e^{e_1 \frac{x_1^2+w_1^2}{2} \cot\theta_1 - e_1 x_1 w_1 \csc\theta_1} f(\mathbf{x}-\mathbf{t}) \\
&\quad \times C_{\theta_2} e^{e_2 \frac{x_2^2+w_2^2}{2} \cot\theta_2 - e_2 x_2 w_2 \csc\theta_2} \, d\mathbf{x}.
\end{aligned}
$$

Let $x_i - t_i = y_i$, $i = 1, 2$, we have

$$
\begin{aligned}
&\mathcal{F}_{\theta_1,\theta_2}\{f(\mathbf{x}-\mathbf{t})\}(\mathbf{w}) \\
&= \int_{\mathbb{R}^2} C_{\theta_1} e^{e_1 \frac{(y_1+t_1)^2+w_1^2}{2} \cot\theta_1 - e_1 (y_1+t_1) w_1 \csc\theta_1} f(\mathbf{y}) \\
&\quad \times C_{\theta_2} e^{e_2 \frac{(y_2+t_2)^2+w_2^2}{2} \cot\theta_2 - e_2 (y_2+t_2) w_2 \csc\theta_2} \, d\mathbf{y} \\
&= e^{e_1 \frac{t_1^2}{2} \cot\theta_1 - e_1 t_1 w_1 \csc\theta_1} \int_{\mathbb{R}^2} C_{\theta_1} e^{e_1 \frac{y_1^2+w_1^2}{2} \cot\theta_1 - e_1 y_1 w_1 \csc\theta_1} e^{e_1 y_1 t_1 \cot\theta_1} f(\mathbf{y}) \\
&\quad \times C_{\theta_2} e^{e_2 \frac{y_2^2+w_2^2}{2} \cot\theta_2 - e_2 y_2 w_2 \csc\theta_2} e^{e_2 y_2 t_2 \cot\theta_2} \, d\mathbf{y} e^{e_2 \frac{t_2^2}{2} \cot\theta_2 - e_2 t_2 w_2 \csc\theta_2} \\
&= e^{e_1 \frac{t_1^2}{2} \cot\theta_1 - e_1 t_1 w_1 \csc\theta_1} \mathcal{F}_{\theta_1,\theta_2}\{e^{e_1 y_1 t_1 \cot\theta_1} f(\mathbf{y}) e^{e_2 y_2 t_2 \cot\theta_2}\}(\mathbf{w}) \\
&\quad \times e^{e_2 \frac{t_2^2}{2} \cot\theta_2 - e_2 t_2 w_2 \csc\theta_2}. \qquad \square
\end{aligned}
$$

In the following theorem we give the differential properties of the two-sided FrQFT. These conclusions are similar in nature to those of the classical FT, although they have different forms.

Theorem 3.4 *Let $f \in S(\mathbb{R}^2; \mathbb{H})$, then, for each component x_s and w_s with $s = 1, 2$, the following relation is fulfilled*:

$$
\begin{aligned}
&\mathcal{F}_{\theta_1,\ldots,\theta_n}\left\{\frac{\partial f}{\partial x_s}\right\}(\mathbf{w}) \\
&= \begin{cases} -e_1 \cot\theta_1 \mathcal{F}_{\theta_1,\theta_2}\{x_1 f\}(\mathbf{w}) + e_1 w_1 \csc\theta_1 \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}), & s = 1; \\ -\cot\theta_2 \mathcal{F}_{\theta_1,\theta_2}\{x_2 f\}(\mathbf{w}) e_2 + w_2 \csc\theta_2 \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) e_2, & s = 2. \end{cases}
\end{aligned}
$$

Proof When $s = 1$, using integration by parts we have

$$
\begin{aligned}
&\mathcal{F}_{\theta_1,\theta_2}\left\{\frac{\partial f}{\partial x_1}\right\}(\mathbf{w}) \\
&= \int_{\mathbb{R}^2} C_{\theta_1} e^{e_1 \frac{x_1^2+w_1^2}{2} \cot\theta_1 - e_1 x_1 w_1 \csc\theta_1} \frac{\partial f(\mathbf{x})}{\partial x_1} C_{\theta_2} e^{e_2 \frac{x_2^2+w_2^2}{2} \cot\theta_2 - e_2 x_2 w_2 \csc\theta_2} \, d\mathbf{x} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(C_{\theta_1} e^{e_1 \frac{x_1^2+w_1^2}{2} \cot\theta_1 - e_1 x_1 w_1 \csc\theta_1} \frac{\partial f(\mathbf{x})}{\partial x_1} \, dx_1 \right) C_{\theta_2} \\
&\quad \times e^{e_2 \frac{x_2^2+w_2^2}{2} \cot\theta_2 - e_2 x_2 w_2 \csc\theta_2} dx_2 \\
&= \int_{\mathbb{R}} \left(C_{\theta_1} e^{e_1 \frac{x_1^2+w_1^2}{2} \cot\theta_1 - e_1 x_1 w_1 \csc\theta_1} f(\mathbf{x})\Big|_{x_1=-\infty}^{x_1=+\infty} \right. \\
&\quad \left. - \int_{\mathbb{R}} \frac{\partial}{\partial x_1} \left(C_{\theta_i} e^{e_1 \frac{x_1^2+w_1^2}{2} \cot\theta_1 - e_1 x_1 w_1 \csc\theta_1} \right) f(\mathbf{x}) \, dx_1 \right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times e^{e_2 \frac{x_2^2+w_2^2}{2} \cot\theta_2 - e_2 x_2 w_2 \csc\theta_2} dx_2 \\
&= -\int_{\mathbb{R}^2} \frac{\partial}{\partial x_1} \bigl(C_{\theta_1} e^{e_1 \frac{x_1^2+w_1^2}{2} \cot\theta_1 - e_1 x_1 w_1 \csc\theta_1}\bigr) f(\mathbf{x}) \\
&\quad \times e^{e_2 \frac{x_2^2+w_2^2}{2} \cot\theta_2 - e_2 x_2 w_2 \csc\theta_2} \, d\mathbf{x} \\
&= -e_1 \cot\theta_1 \mathcal{F}_{\theta_1,\theta_2}\{x_1 f\}(\mathbf{w}) + e_1 w_1 \csc\theta_1 \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}).
\end{aligned}
$$

Using a similar calculation, the conclusion is valid when $s = 2$. □

Theorem 3.4 describes the relationship between the two-sided FrQFT of the derivative of a function f and the two-sided FrQFT of the function f itself.

Theorem 3.5 *Suppose that $f \in S(\mathbb{R}^2;\mathbb{H})$. Then we have*

$$
\frac{\partial \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w})}{\partial w_s} = \begin{cases} e_1[w_s \cot\theta_s \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) - \csc\theta_s \mathcal{F}_{\theta_1,\theta_2}\{x_s f\}(\mathbf{w})], & s = 1; \\ [w_s \cot\theta_s \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) - \csc\theta_s \mathcal{F}_{\theta_1,\theta_2}\{x_s f\}(\mathbf{w})]e_2, & s = 2, \end{cases}
$$

or

$$
\mathcal{F}_{\theta_1,\theta_2}\bigl\{x_s f(\mathbf{x})\bigr\}(\mathbf{w}) = \begin{cases} (-e_1 \frac{\partial}{\partial w_1} \sin\theta_1 + w_1 \cos\theta_1) \mathcal{F}_{\theta_1,\theta_2}\{f(\mathbf{x})\}(\mathbf{w}), & s = 1; \\ (-\frac{\partial}{\partial w_2} \sin\theta_2 e_2 + w_s \cos\theta_2) \mathcal{F}_{\theta_1,\theta_2}\{f(\mathbf{x})\}(\mathbf{w}), & s = 2. \end{cases}
$$

Proof The proof is similar to that of Theorem 3.4, so the proof is not given here. □

Theorem 3.5 describes the relationship between the derivative of the two-sided FrQFT of a function f and the two-sided FrQFT of the function f itself.

Some important differential properties of kernel functions $K_{\theta_1}(x_1, w_1)$ and $K_{\theta_2}(x_2, w_2)$ are stated in the following theorems.

Theorem 3.6 *Suppose that $f \in S(\mathbb{R}^2;\mathbb{H})$,*

$$
\begin{aligned}
\triangle_{x_1} &= \frac{\partial}{\partial x_1} - e_1 x_1 \cot\theta_1, \\
\triangle_{x_2} &= \frac{\partial}{\partial x_2} - x_2 \cot\theta_2 e_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\triangle_{x_1}^m K_{\theta_1}(x_1, w_1) &= e_1^m (-w_1 \csc\theta_1)^m K_{\theta_1}(x_1, w_1), \\
\triangle_{x_2}^m K_{\theta_2}(x_2, w_2) &= (-w_2 \csc\theta_2)^m K_{\theta_2}(x_2, w_2) e_2^m.
\end{aligned}
$$

Proof By simple calculation, we have

$$
\begin{aligned}
&\triangle_{x_1} K_{\theta_1}(x_1, w_1) \\
&\quad = (e_1 x_1 \cot\theta_1 - e_1 w_1 \csc\theta_1 - e_1 x_1 \cot\theta_1) K_{\theta_1}(x_1, w_1) \\
&\quad = -e_1 w_1 \csc\theta_1 K_{\theta_1}(x_1, w_1).
\end{aligned}
$$

Then, by induction, assuming

$$\triangle_{x_1}^{m-1} K_{\theta_1}(x_1,w_1) = (-e_1 w_1 \csc\theta_1)^{m-1} K_{\theta_1}(x_1,w_1)$$

is true, then

$$\begin{aligned}
&\triangle_{x_1}^{m} K_{\theta_1}(x_1,w_1) \\
&\quad = \triangle_{x_1}\big(\triangle_{x_1}^{m-1} K_{\theta_1}(x_1,w_1)\big) \\
&\quad = \triangle_{x_1}\big((-e_1 w_1 \csc\theta_1)^{m-1} K_{\theta_1}(x_1,w_1)\big) \\
&\quad = (-e_1 w_1 \csc\theta_1)^{m-1} \triangle_{x_1} K_{\theta_1}(x_1,w_1) \\
&\quad = (-e_1 w_1 \csc\theta_1)^{m} K_{\theta_1}(x_1,w_1) \\
&\quad = e_1^m (-w_1 \csc\theta_1)^{m} K_{\theta_1}(x_1,w_1).
\end{aligned}$$

Using a similar calculation, we can prove that the other equation is valid. □

Theorem 3.7 *Let* $f \in L^1(\mathbb{R}^2;\mathbb{H})$, *for any* $m_1, m_2 \in \mathbb{Z}_+$, *we have*

$$\begin{aligned}
&\triangle_{x_1}^{m_1} \triangle_{x_2}^{m_2} \big(K_{\theta_1}(x_1,w_1) K_{\theta_2}(x_2,w_2)\big) \\
&\quad = (-w_1 \csc\theta_1)^{m_1} e_1^{m_1} K_{\theta_1}(x_1,w_1) K_{\theta_2}(x_2,w_2) (-w_2 \csc\theta_2)^{m_2} e_2^{m_2}.
\end{aligned}$$

Theorem 3.8 *Let* $f \in L^1(\mathbb{R}^2;\mathbb{H})$,

$$\begin{aligned}
\overline{\triangle}_{x_1} &= \frac{\partial}{\partial x_1} + e_1 x_1 \cot\theta_1, \\
\overline{\triangle}_{x_2} &= \frac{\partial}{\partial x_2} + x_2 \cot\theta_2 e_2.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{F}_{\theta_1,\theta_2}\big\{\overline{\triangle}_{x_1}^{m} f\big\}(\mathbf{w}) &= e_1^m (w_1 \csc\theta_1)^m \mathcal{F}_{\theta_1,\theta_1}\{f\}(\mathbf{w}), \\
\mathcal{F}_{\theta_1,\theta_2}\big\{\overline{\triangle}_{x_2}^{m} f\big\}(\mathbf{w}) &= (w_2 \csc\theta_2)^m \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) e_2^m.
\end{aligned}$$

Proof Using integration by parts we get

$$\begin{aligned}
&\mathcal{F}_{\theta_1,\theta_2}\big\{\overline{\triangle}_{x_1} f(\mathbf{x})\big\}(\mathbf{w}) \\
&\quad = \int_{\mathbb{R}^2} K_{\theta_1}(x_1,w_1) \frac{\partial f(\mathbf{x})}{\partial x_1} K_{\theta_2}(x_2,w_2)\, d\mathbf{x} \\
&\qquad + \int_{\mathbb{R}^2} e_1 x_1 \cot\theta_1 K_{\theta_1}(x_1,w_1) f(\mathbf{x}) K_{\theta_2}(x_2,w_2)\, d\mathbf{x} \\
&\quad = \int_{\mathbb{R}} \bigg(K_{\theta_1}(x_1,w_1) f(\mathbf{x})\big|_{x_1\to-\infty}^{x_1\to\infty} - \frac{\partial K_{\theta_1}(x_1,w_1)}{\partial x_1} f(\mathbf{x})\, dx_1 \bigg) K_{\theta_2}(x_2,w_2)\, dx_2 \\
&\qquad + e_1 x_1 \cot\theta_1 \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w})
\end{aligned}$$

$$
\begin{aligned}
&= (e_1 w_1 \csc\theta_1 - e_1 x_1 \cot\theta_1)\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) + e_1 x_1 \cot\theta_1 \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) \\
&= e_1 w_1 \csc\theta_1 \mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}).
\end{aligned}
$$

Then, by induction, assuming

$$\mathcal{F}_{\theta_1,\theta_2}\left\{\overline{\triangle}_{x_1}^{(m-1)} f\right\}(\mathbf{w}) = (e_1 w_1 \csc\theta_1)^{m-1}\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w})$$

is true, then

$$
\begin{aligned}
\mathcal{F}_{\theta_1,\theta_2}\left\{\overline{\triangle}_{x_1}^{m} f\right\}(\mathbf{w}) &= \mathcal{F}_{\theta_1,\theta_2}\left\{\overline{\triangle}_{x_1}^{(m-1)}(\overline{\triangle}_{x_1}) f\right\}(\mathbf{w}) \\
&= (e_1 w_1 \csc\theta_1)^{m-1}\mathcal{F}_{\theta_1,\theta_2}\{\overline{\triangle}_{x_1} f\}(\mathbf{w}) \\
&= (e_1 w_1 \csc\theta_1)^{m}\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}).
\end{aligned}
$$

Using a similar calculation, we can prove that the other equation is valid. □

Lemma 3.1 *Suppose that $f \in S(\mathbb{R}^2;\mathbb{H})$, $\overline{\triangle}_{x}^{m_1,m_2} = \overline{\triangle}_{x_1}^{m_1}\overline{\triangle}_{x_2}^{m_2}$. Then, for any positive integers m_1, m_2, we have*

$$\mathcal{F}_{\theta_1,\theta_2}\left\{\overline{\triangle}_{x}^{m_1,m_2} f\right\}(\mathbf{w}) = (w_1 \csc\theta_1)^{m_1}(w_2 \csc\theta_2)^{m_2} e_1^{m_1}\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) e_2^{m_2}. \tag{3.6}$$

Lemma 3.1 describes the relationship between the two-sided FrQFT of the derivative of a function f and the two-sided FrQFT of the function itself.

Lemma 3.2 describes the relationship between the derivative of the two-sided FrQFT of a function f and the two-sided FrQFT of $(-e_1 x_1 \csc\theta_1)^{m_1} f$ and $(-x_1 \csc\theta_1)^{m_2} f e_2^{m_2}$.

Lemma 3.2 *Suppose that $f \in S(\mathbb{R}^2;\mathbb{H})$,*

$$
\begin{aligned}
\triangle_{w_1} &= \frac{\partial}{\partial w_1} - e_1 w_1 \cot\theta_1, \\
\triangle_{w_2} &= \frac{\partial}{\partial w_2} - w_2 \cot\theta_2 e_2.
\end{aligned}
$$

Then, for any positive integers m_1, m_2, we have

$$\triangle_{w_i}\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) = \begin{cases} \mathcal{F}_{\theta_1,\theta_2}\{(-e_1 x_1 \csc\theta_1)^{m_1} f\}(\mathbf{w}), & i = 1; \\ \mathcal{F}_{\theta_1,\theta_2}\{(-x_2 \csc\theta_2)^{m_2} f e_2^{m_2}\}(\mathbf{w}), & i = 2. \end{cases}$$

Lemma 3.3 *Suppose that $f \in S(\mathbb{R}^2;\mathbb{H})$ and $\triangle_{w}^{m_1,m_2} = \triangle_{w_1}^{m_1}\triangle_{w_2}^{m_2}$. Then, for any positive integers m_1, m_2, we have*

$$\triangle_{w}^{m_1,m_2}\mathcal{F}_{\theta_1,\theta_2}\{f\}(\mathbf{w}) = \mathcal{F}_{\theta_1,\theta_2}\left\{(-x_1 \csc\theta_1)^{m_1}(-x_2 \csc\theta_2)^{m_2} e_1^{m_1} f e_2^{m_2}\right\}(\mathbf{w}).$$

We will give the properties of inner and scalar products.

Theorem 3.9 *Suppose that $f_1, f_2 \in L^2(\mathbb{R}^2;\mathbb{H})$, $f_{1+}\overline{f_{2-}} + f_{1-}\overline{f_{2+}} = 0$. Then we have*

$$\left(\mathcal{F}_{\theta_1,\theta_2}\{f_1\}, \mathcal{F}_{\theta_1,\theta_2}\{f_2\}\right) = (f_1, f_2).$$

Proof By the definition of the inner product and the definition of the two-sided FrQFT, we get

$$
\begin{aligned}
&\big(\mathcal{F}_{\theta_1,\theta_2}\{f_1\}(\mathbf{w}),\mathcal{F}_{\theta_1,\theta_2}\{f_2\}(\mathbf{w})\big)\\
&\quad=\int_{\mathbb{R}^2}\big(\mathcal{F}_{\theta_1,\theta_2}\{f_1\}(\mathbf{w})\overline{\mathcal{F}_{\theta_1,\theta_2}\{f_2\}(\mathbf{w})}\big)\,d\mathbf{w}\\
&\quad=\int_{\mathbb{R}^2}\bigg(\int_{\mathbb{R}^2}K_{\theta_1}(x_1,w_1)f(\mathbf{x})K_{\theta_2}(x_2,w_2)\,d\mathbf{x}\\
&\qquad\times\overline{\int_{\mathbb{R}^2}K_{\theta_1}(x_1,w_1)f(\mathbf{y})K_{\theta_2}(x_2,w_2)\,d\mathbf{y}}\bigg)\,d\mathbf{w}\\
&\quad=\int_{\mathbb{R}^2}\bigg(\int_{\mathbb{R}^2}C_{\theta_1}e^{e_1\frac{x_1^2+w_1^2}{2}\cot\theta_1-e_1x_1w_1\csc\theta_1}f_1(\mathbf{x})e^{e_2\frac{x_2^2+w_2^2}{2}\cot\theta_2}C_{\theta_2}e^{e_2\frac{w_2^2}{2}\cot\theta_2}\,d\mathbf{x}\\
&\qquad\times\overline{\int_{\mathbb{R}^2}C_{\theta_1}e^{e_1\frac{x_1^2+w_1^2}{2}\cot\theta_1-e_1x_1w_1\csc\theta_1}f_1(\mathbf{x})e^{e_2\frac{x_2^2+w_2^2}{2}\cot\theta_2}C_{\theta_2}e^{e_2\frac{w_2^2}{2}\cot\theta_2}\,d\mathbf{y}}\bigg)\,d\mathbf{w}\\
&\quad=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}C_{\theta_1}e^{e_1\frac{x_1^2+w_1^2}{2}\cot\theta_1-e_1x_1w_1\csc\theta_1}f_1(\mathbf{x})\\
&\qquad\times C_{\theta_2}C_{-\theta_2}e^{e_2\frac{x_2^2+w_2^2}{2}\cot\theta_2-e_2x_2w_2\csc\theta_2}e^{-e_2\frac{y_2^2+w_2^2}{2}\cot\theta_2+e_2y_2w_2\csc\theta_2}\\
&\qquad\times\overline{f_2(\mathbf{y})}C_{-\theta_1}e^{-e_1\frac{y_1^2+w_1^2}{2}\cot\theta_1+e_1y_1w_1\csc\theta_1}\,d\mathbf{x}\,d\mathbf{y}\,d\mathbf{w};
\end{aligned}
$$

note that $C_{\theta_2}C_{-\theta_2}=\frac{\csc\theta_2}{2\pi}$,

$$
\int_{\mathbb{R}}C_{\theta_2}C_{-\theta_2}e^{e_2(y_2-x_2)w_2\csc\theta_2}\,dw_2=\delta(y_2-x_2),
$$

then using the properties of δ, we have

$$
\int_{\mathbb{R}}C_{\theta_2}C_{-\theta_2}\delta(y_2-x_2)e^{e_2\frac{x_2^2-y_2^2}{2}}\overline{f_2(\mathbf{y})}\,dy_2=\overline{f_2(y_1,x_2)}.
$$

So

$$
\begin{aligned}
&\big(\mathcal{F}_{\theta_1,\theta_2}\{f_1\}(\mathbf{w}),\mathcal{F}_{\theta_1,\theta_2}\{f_2\}(\mathbf{w})\big)\\
&\quad=\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}C_{\theta_1}e^{e_1\frac{x_1^2+w_1^2}{2}\cot\theta_1-e_1x_1w_1\csc\theta_1}f_1(\mathbf{x})\overline{f_2(y_1,x_2)}\\
&\qquad\times C_{-\theta_1}e^{-e_1\frac{y_1^2+w_1^2}{2}\cot\theta_1+e_1y_1w_1\csc\theta_1}\,d\mathbf{x}\,dy_1dw_1.
\end{aligned}
$$

Now, let us write $f_1\overline{f_2}=f_{1+}\overline{f_{2+}}+f_{1-}\overline{f_{2-}}+f_{1+}\overline{f_{2-}}+f_{1-}\overline{f_{2+}}$.

Using the definition and properties of the function δ and the well-known conditions $f_{1+}\overline{f_{2-}}+f_{1-}\overline{f_{2+}}=0$, then, by Theorem 2.2, we get

$$
\begin{aligned}
\big(\mathcal{F}_{\theta_1,\theta_2}f_1(\mathbf{w}),\mathcal{F}_{\theta_1,\theta_2}\{f_2\}(\mathbf{w})\big)&=\int_{\mathbb{R}^2}f_1(\mathbf{y})\overline{f_2(\mathbf{y})}\,d\mathbf{y}\\
&=(f_1,f_2).
\end{aligned}
$$

□

In particular, when $f_1 = f_2 = f$, by Theorem 3.9, we immediately arrive at the following conclusion. Of course, the following equality can also be proved to be true by the definition of the norm.

Theorem 3.10 (Parseval identity) *Suppose that* $f \in L^2(\mathbb{R}^2;\mathbb{H})$. *Then we have*

$$\|f\|^2_{L^2(R^2;\mathbb{H})} = \left\|\mathcal{F}_{\theta_1,\theta_2}\{f\}\right\|^2_{L^2(R^2;\mathbb{H})}. \tag{3.7}$$

Proof Using the definition of the norm

$$\begin{aligned}
&\left\|\mathcal{F}_{\theta_1,\theta_2}\{f\}\right\|^2_{L^2(R^2;\mathbb{H})} \\
&\quad = \int_{\mathbb{R}^2} \left|K_{\theta_1}(x_1,w_1)f(\mathbf{x})K_{\theta_2}(x_2,w_2)\right|^2 d\mathbf{x} \\
&\quad = \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x} \\
&\quad = \|f\|^2_{L^2(R^2;\mathbb{H})}.
\end{aligned}$$

□

From this conclusion, we can see that the two-sided FrQFT has norm-preserving properties.

4 The application of the two-sided fractional QFT

Next, we give an application of differential properties of the two-sided FrQFT in solving partial differential equations.

Example. Find solutions to the following partial differential equations.

$$\left(\frac{\partial}{\partial x_1} + e_1 x_1 \cot\theta_1\right)^4 y(\mathbf{x}) \left(\frac{\partial}{\partial x_2} + e_2 x_2 \cot\theta_2\right)^5 \overline{e_2}^5 - y(\mathbf{x}) = f(\mathbf{x}), \tag{4.1}$$

where $f(x)$ is a known quaternion-valued function and $y(x)$ is an unknown quaternion-valued function.

Solution. Using differential properties of the two-sided FrQFT, we take the two-sided FrQFT at both sides of differential equation (4.1). Then, by Theorem 3.4, we have

$$\begin{aligned}
&(e_1 w_1 \csc\theta_1)^4 \mathcal{F}_{\theta_1,\theta_2}\big\{y(\mathbf{x})\big\}(\mathbf{w})(e_2 w_2 \csc\theta_2)^5 \overline{e_2}^5 - \mathcal{F}_{\theta_1,\theta_2}\big\{y(\mathbf{x})\big\}(\mathbf{w}) \\
&\quad = \mathcal{F}_{\theta_1,\theta_2}\big\{f(\mathbf{x})\big\}(\mathbf{w}).
\end{aligned}$$

Then

$$\big[(w_1 \csc\theta_1)^4 (w_2 \csc\theta_2)^5 - 1\big]\mathcal{F}_{\theta_1,\theta_2}\big\{y(\mathbf{x})\big\}(\mathbf{w}) = \mathcal{F}_{\theta_1,\theta_2}\big\{f(\mathbf{x})\big\}(\mathbf{w}).$$

That is,

$$\mathcal{F}_{\theta_1,\theta_2}\big\{y(\mathbf{x})\big\}(\mathbf{w}) = \frac{\mathcal{F}_{\theta_1\theta_2}\{f(\mathbf{x})\}(\mathbf{w})}{(w_1 \csc\theta_1)^4 (w_2 \csc\theta_2)^5 - 1}.$$

According to the Fourier inverse transform of the two-sided FrQFT, we can get

$$\begin{aligned} y(\mathbf{x}) &= \mathcal{F}^{-1}_{\theta_1,\theta_2}\left\{\frac{\mathcal{F}_{\theta_1,\theta_2}\{y(\mathbf{x})\}(\mathbf{w})}{(w_1\csc\theta_1)^4(w_2\csc\theta_2)^5-1}\right\} \\ &= \int_{\mathbb{R}^2} K_{-\theta_1}(x_1,w_1)\frac{\mathcal{F}_{\theta_1,\theta_2}\{y(\mathbf{x})\}(\mathbf{w})}{(w_1\csc\theta_1)^4(w_2\csc\theta_2)^5-1}K_{-\theta_2}(x_2,w_2)\,d\mathbf{x} \\ &= \int_{\mathbb{R}^2} A(\mathbf{w})\mathcal{F}_{\theta_1,\theta_2}\big\{y(\mathbf{x})\big\}(\mathbf{w})K_{-\theta_2}(x_2,w_2)\,d\mathbf{x}, \end{aligned}$$

where $A(\mathbf{w}) = \frac{K_{-\theta_1}(x_1,w_1)}{(w_1\csc\theta_1)^4(w_2\csc\theta_2)^5-1}$.

5 Conclusion

Using the basic concepts of quaternion algebra we introduced a two-sided FrQFT. Important properties of the two-sided FrQFT such as shift, differential properties, Parseval identities were demonstrated.

But so far there are still some problems to be studied. Firstly, we mention the relationship between the integral expression of the two-sided FrQFT of f when $\theta_i = n\pi$ and that when $\theta_i \neq n\pi$. Secondly, we mention that applications of the two-sided FrQFT in signal processing, non-marginal color image processing and electromagnetism etc. are not given.

Acknowledgements
We thank the editors and all reviewers for taking time out of their busy schedules to read our papers and thank them for valuable suggestions.

Funding
This work was supported by the Doctoral Foundation of Heibei University of Science and Technology (No. 1181348), the National Natural Science Foundation of China (No. 11871191), the Innovative Foundation Project of Hebei Province (No. CXZZBS2021066), the Natural Science Foundation of Hebei Province (No. A2020205008), and the Key Foundation of Hebei Normal University (No. L2018Z01).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZFL has presented the main purpose of the article. All authors read and approved the final manuscript.

Author details
[1]College of Science, Hebei University of Science and Technology, Shijiazhuang, Hebei Province 050018, P.R. China.
[2]College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei Province 050024, P.R. China.

Received: 8 November 2020 Accepted: 26 April 2021 Published online: 14 July 2021

References

1. Bahri, M.: A modified uncertainty principle for two-sided quaternion Fourier transform. Adv. Appl. Clifford Algebras **26**, 513–527 (2016)
2. El Haoui, Y., Fahlaoui, S.: The uncertainty principle for the two-sided quaternion Fourier transform. Mediterr. J. Math. **14**, 221–232 (2017)
3. El Haoui, Y., Fahlaoui, S.: Donoho–Stark's uncertainty principles in real Clifford algebras. Adv. Appl. Clifford Algebras **29**, 94 (2019)

4. Fei, M., Xu, Y., Yan, J.: Real Paley–Wiener theorem for the quaternion Fourier transform. Complex Var. Elliptic Equ. **62**, 1072–1080 (2017)
5. Hitzer, E.: Quaternion Fourier transform on quaternion fields and generalizations. Adv. Appl. Clifford Algebras **17**, 497–517 (2007)
6. Hitzer, E.: Directional uncertainty principle for quaternion Fourier transform. Adv. Appl. Clifford Algebras **20**, 271–284 (2010)
7. Hitzer, E.: The quaternion domain Fourier transform and its properties. Adv. Appl. Clifford Algebras **26**, 969–984 (2016)
8. Mustard, D.: Fractional convolution. J. Aust. Math. Soc. Ser. B **40**, 257–265 (1998)
9. Sangwine, S.J.: Fourier transforms of colour images using quaternion, or hypercomplex numbers. Electron. Lett. **32**, 1979–1980 (1996)
10. Sangwine, S.J.: Color image edge detector based on quaternion convolution. Electron. Lett. **34**, 969–971 (1998)
11. Yang, Y., Dang, P., Qian, T.: Tighter uncertainty principles based on quaternion Fourier transform. Adv. Appl. Clifford Algebras **26**, 479–497 (2016)
12. Zhang, Y.L.: Engineering Mathematics. Higher Education Press, Beijing (2012)