Hussain et al. *Journal of Inequalities and Applications* (2021) 2021:123
https://doi.org/10.1186/s13660-021-02650-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

The boundedness of commutators of rough p-adic fractional Hardy type operators on Herz-type spaces

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Abstract
In this paper, we obtain some inequalities about commutators of a rough p-adic fractional Hardy-type operator on Herz-type spaces when the symbol functions belong to two different function spaces.

MSC: 42B35; 26D15; 46B25; 47G10

Keywords: p-adic rough fractional Hardy operator; Commutators; p-adic Herz-type space; p-adic central BMO space; Lipschitz space

1 Introduction

During the last several decades, the p-adic analysis has cemented its role in the field of mathematical physics (see, for example, [1, 22, 32, 33]). That stimulates researchers to pay attention to harmonic analysis on p-adic fields [18–21, 24, 30, 31, 35], which has direct implications in the stochastic process [2, 3], theoretical biology [6], and p-adic pseudo-differential equations [23, 34]. In continuation of the ongoing research, the present paper considers an extension of the investigation of p-adic Hardy-type operators discussed in [19–21, 25, 36, 37].

For every non-zero rational number x there is a unique $\gamma = \gamma(x) \in \mathbb{Z}$ such that $x = p^{\gamma} m/n$, where $p \geq 2$ is a fixed prime number which is coprime to $m, n \in \mathbb{Z}$. We define a mapping $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_+$ as follows:

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.1}$$

The p-adic absolute value $|\cdot|_p$ has many properties of the usual real norm $|\cdot|$ with an additional non-Archimedean property,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

The field of p-adic numbers, denoted by $\mathbb{Q}_p$, is the completion of rational numbers with respect to the p-adic absolute value $|\cdot|_p$. A p-adic number $x \in \mathbb{Q}_p$ can be written in the formal power series as [34]:

$$x = p^{\gamma}\left(\beta_0 + \beta_1 p + \beta_2 p^2 + \cdots\right), \tag{1.2}$$

where $\gamma \in \mathbb{Z}$ and $\beta_i \in \{0, 1, \ldots, p-1\}$, $i = 0, 1, 2, \ldots$. The p-adic absolute value ensures the convergence of series (1.2) in $\mathbb{Q}_p$, because the inequality $|p^{\gamma}\beta_i p^i|_p \le p^{-\gamma-i}$ holds for all $\gamma \in \mathbb{Z}$ and $i \in \mathbb{N}$.

The n-dimensional vector space $\mathbb{Q}_p^n$, $n \ge 1$, consists of the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$ and $j = 1, 2, \ldots, n$, with the following absolute value:

$$|\mathbf{x}|_p = \max_{1\le k\le n} |x_k|_p. \tag{1.3}$$

For $\gamma \in \mathbb{Z}$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Q}_p^n$, we denote by

$$B_{\gamma}(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \le p^{\gamma}\right\}$$

the closed ball with the center $\mathbf{a}$ and radius p^{γ} and by

$$S_{\gamma}(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^{\gamma}\right\}$$

the corresponding sphere. For $\mathbf{a} = \mathbf{0}$, we write $B_{\gamma}(\mathbf{0}) = B_{\gamma}$, and $S_{\gamma}(\mathbf{0}) = S_{\gamma}$. It is easy to see that the equalities

$$\mathbf{a}_0 + B_{\gamma} = B_{\gamma}(\mathbf{a}_0) \quad \text{and} \quad \mathbf{a}_0 + S_{\gamma} = S_{\gamma}(\mathbf{a}_0) = B_{\gamma}(\mathbf{a}_0) \setminus B_{\gamma-1}(\mathbf{a}_0)$$

hold for all $\mathbf{a}_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists a unique Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^n$, such that

$$\int_{B_0} d\mathbf{x} = |B_0|_h = 1,$$

where $|B|_h$ denotes the Haar measure of measurable subset B of $\mathbb{Q}_p^n$. Furthermore, a simple calculation shows that

$$\left|B_{\gamma}(\mathbf{a})\right|_h = p^{n\gamma} \quad \text{and} \quad \left|S_{\gamma}(\mathbf{a})\right|_h = p^{n\gamma}\left(1 - p^{-n}\right)$$

hold for all $\mathbf{a} \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$.

The one-dimensional Hardy operator

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(y)\,dy, \quad x > 0,$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable functions, was introduced by Hardy in [13]. This operator satisfies the inequality:

$$\|\mathcal{H}f\|_{L^q(\mathbb{R}^+)} \le \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty, \tag{1.4}$$

where the constant $q/(q-1)$ is sharp. In [7], Faris proposed an extension of the operator $\mathcal{H}$ on higher dimensional Euclidean space $\mathbb{R}^n$ which is given by

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n} \int_{|\mathbf{y}| \leq |\mathbf{x}|} f(\mathbf{y})\, d\mathbf{y}, \tag{1.5}$$

for $\mathbf{x} = (x_1, \dots, x_n)$. In addition, Christ and Grafakos [4] obtained the exact value of the norm of operator H defined by (1.5). For boundedness results for these operators on function spaces we refer to some recent publications including [8, 10, 16, 17, 28, 29, 38].

On the other hand, the n-dimensional fractional p-adic Hardy operator

$$H_\alpha^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p \leq |\mathbf{x}|_p} f(\mathbf{y})\, d\mathbf{y}$$

was defined and studied for $f \in L_1^{\mathrm{loc}}(\mathbb{Q}_p^n)$ and $0 \leq \alpha < n$ in [36]. When $\alpha = 0$, the operator H_α^p transfers to the p-adic Hardy-type operator (see [10] for more details). Fu et al. in [9], fixed the optimal bounds of p-adic Hardy operator on $L^q(\mathbb{Q}_p^n)$. On the central Morrey space the p-adic Hardy-type operators and their commutators were discussed in [37]. In this connection see also [19, 21, 25].

There is still zero attention towards the rough Hardy operators on the p-adic linear spaces. Motivated by papers cited above and results of Fu et al. in [8], we define the special kind of p-adic rough fractional Hardy operator $H_{\Omega,\alpha}^p$ and its commutators as follows.

Definition 1.1 Let $f: \mathbb{Q}_p^n \to \mathbb{R}$, $b: \mathbb{Q}_p^n \to \mathbb{R}$ be measurable mappings and let $0 < \alpha < n$. Then, for $\mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}$, we define the rough p-adic fractional Hardy operator $H_{\Omega,\alpha}^p$ by

$$H_{\Omega,\alpha}^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p \leq |\mathbf{x}|_p} \Omega\bigl(|\mathbf{y}|_p \mathbf{y}\bigr) f(\mathbf{y})\, d\mathbf{y}, \tag{1.6}$$

and its commutator $H_{\Omega,\alpha}^{p,b}$ by

$$H_{\Omega,\alpha}^{p,b} f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p \leq |\mathbf{x}|_p} \bigl(b(\mathbf{x}) - b(\mathbf{y})\bigr) \Omega\bigl(|\mathbf{y}|_p \mathbf{y}\bigr) f(\mathbf{y})\, d\mathbf{y}, \tag{1.7}$$

whenever

$$\int_{|\mathbf{y}|_p \leq |\mathbf{x}|_p} \bigl|\Omega\bigl(|\mathbf{y}|_p \mathbf{y}\bigr) f(\mathbf{y})\bigr|\, d\mathbf{y} < \infty \tag{1.8}$$

and

$$\int_{|\mathbf{y}|_p \leq |\mathbf{x}|_p} \bigl|b(\mathbf{y}) \Omega\bigl(|\mathbf{y}|_p \mathbf{y}\bigr) f(\mathbf{y})\bigr|\, d\mathbf{y} < \infty, \tag{1.9}$$

where $\Omega \in L^s(S_0(\mathbf{0}))$, $1 \leq s < \infty$.

Remark 1.2 Obviously

$$\bigl\{|\mathbf{y}|_p : \mathbf{y} \in \mathbb{Q}_p^n\bigr\} = \bigl\{p^\gamma : \gamma \in \mathbb{Z}\bigr\} \cup \{0\}$$

holds for every integer $n \geq 1$ and prime $p \geq 2$. Since the inclusion

$$\{0\} \cup \{p^{\gamma} : \gamma \in \mathbb{Z}\} \subseteq \mathbb{Q}_p$$

holds and $\mathbb{Q}_p^n$ is a linear space over field $\mathbb{Q}_p$, the product $|\mathbf{y}|_p\mathbf{y}$ is well defined. Moreover, if a non-zero $\mathbf{y} \in \mathbb{Q}_p^n$ has the form $\mathbf{y} = (y_1, \ldots, y_n)$ and

$$y_i = p^{\gamma_i}(\beta_{0,i} + \beta_{1,i}p + \beta_{2,i}p^2 + \cdots), \quad i = 1, \ldots, n \tag{1.10}$$

(see (1.2)), then there is $i_0 \in \{1, \ldots, n\}$ such that

$$|y_{i_0}|_p = p^{-\gamma_{i_0}} \geq p^{-\gamma_i} = |y_i|_p \tag{1.11}$$

whenever $y_i \neq 0$. Using (1.3) we obtain $|\mathbf{y}|_p = p^{-\gamma_{i_0}}$. Now from (1.10) and (1.11) it follows that

$$\big||\mathbf{y}|_p\mathbf{y}\big|_p = \max_{\substack{1\leq i\leq n\\ y_i\neq 0}} \big|p^{\gamma_i-\gamma_{i_0}}\big|_p = \max_{\substack{1\leq i\leq n\\ y_i\neq 0}} p^{\gamma_{i_0}-\gamma_i} = p^{\gamma_{i_0}-\gamma_{i_0}} = 1.$$

Thus, for every non-zero $\mathbf{y} \in \mathbb{Q}_p^n$, the vector $|\mathbf{y}|_p\mathbf{y}$ belongs to the sphere

$$S_0(\mathbf{0}) = \{\mathbf{y} \in \mathbb{Q}_p^n : |\mathbf{y}|_p = 1\}.$$

From (1.8) it directly follows that $H^p_{\Omega,\alpha} \in \mathbb{R}$ for every non-zero $\mathbf{x} \in \mathbb{Q}_p^n$ and using (1.8), (1.9) we have

$$\begin{aligned}\big|H^{p,b}_{\Omega,\alpha}f(\mathbf{x})\big| &\leq \frac{|b(\mathbf{x})|}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \big|\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})\big|\,d\mathbf{y} \\ &\quad + \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \big|b(\mathbf{y})\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})\big|\,d\mathbf{y} < \infty\end{aligned}$$

for every $\mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}$. Consequently, the operators $H^p_{\Omega,\alpha}$ and $H^{p,b}_{\Omega,\alpha}$ are well defined.

The aim of the current paper is to study the boundedness of $H^{p,b}_{\Omega,\alpha}$ on p-adic Herz-type spaces by considering the symbol function b belonging to the p-adic CMO and Lipschitz spaces. In Euclidean space $\mathbb{R}^n$, Herz spaces and Morrey–Herz spaces were firstly introduced in [14] and [26], respectively. For more recent developments in the said spaces we mention the articles [15, 27, 39] and the references therein. Also, some operators with rough kernels defined on Euclidian space were recently studied on function spaces; see for example [11, 12]. Before turning to our main results, let us recall the definitions of p-adic function spaces first.

Definition 1.3 ([9]) Suppose $1 < q < \infty$. The p-adic central bounded mean oscillation (CBMO) space $C\dot{M}O^q(\mathbb{Q}_p^n)$ is the set of all measurable functions $f : \mathbb{Q}_p^n \to \mathbb{R}$ which satisfy

$$\|f\|_{CMO^q(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_h} \int_{B_\gamma} \big|f(\mathbf{x}) - f_{B_\gamma}\big|^q\,d\mathbf{x}\right)^{1/q} < \infty, \tag{1.12}$$

where $f_{B_\gamma} = \frac{1}{|B_\gamma|_h}\int_{B_\gamma} f(\mathbf{x})\,d\mathbf{x}$, $|B_\gamma|_h$ is the Haar measure of B_γ.

Definition 1.4 ([9]) Suppose $0 < r < \infty$, $0 < q < \infty$ and $\beta \in \mathbb{R}$. The homogeneous p-adic Herz space $\dot{K}_q^{\beta,r}(\mathbb{Q}_p^n)$ is defined by

$$\dot{K}_q^{\beta,r}(\mathbb{Q}_p^n) = \{f \in L^q(\mathbb{Q}_p^n) : \|f\|_{\dot{K}_q^{\beta,r}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\beta,r}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{\infty} p^{k\beta r}\|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^r\right)^{1/r},$$

and χ_k is the characteristic function of S_k.

Obviously, the equalities $\dot{K}_q^{0,q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$ and $\dot{K}_q^{\beta/q,q}(\mathbb{Q}_p^n) = L^q(|\mathbf{x}|_p^\beta)$ hold.

Definition 1.5 ([5]) Suppose $0 < r < \infty$, $0 < q < \infty$, $\beta \in \mathbb{R}$ and $\lambda \geq 0$. The homogeneous p-adic Morrey–Herz space is defined by

$$M\dot{K}_{q,r}^{\beta,\lambda}(\mathbb{Q}_p^n) = \{f \in L_{\mathrm{loc}}^q(\mathbb{Q}_p^n \setminus \{0\}) : \|f\|_{M\dot{K}_{r,q}^{\beta,\lambda}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{M\dot{K}_{r,q}^{\beta,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\beta r}\|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^r\right)^{1/r}.$$

It is evident that $M\dot{K}_{r,q}^{\beta,0}(\mathbb{Q}_p^n) = \dot{K}_q^{\beta,r}(\mathbb{Q}_p^n)$ and $M\dot{K}_{q,q}^{\beta/q,0}(\mathbb{Q}_p^n) = L^q(|\mathbf{x}|_p^\alpha)$.

Definition 1.6 ([5]) Suppose δ is a positive real number. The Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$ is defined to be the space of all measurable function f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{\Lambda_\delta(\mathbb{Q}_p^n)} = \sup_{\mathbf{x},\mathbf{h}\in\mathbb{Q}_p^n,\mathbf{h}\neq 0} \frac{|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|}{|\mathbf{h}|_p^\delta} < \infty.$$

2 *CBMO* estimates for commutators of p-adic rough fractional Hardy operator

The present section discusses the boundedness of p-adic rough fractional Hardy operator on p-adic Herz-type spaces. We begin this section with the following useful lemma.

Lemma 2.1 ([36]) *Suppose b is a $CMO^1(\mathbb{Q}_p^n)$ function and suppose $i,j \in \mathbb{Z}$. Then the inequality*

$$|b(\mathbf{y}) - b_{B_j}| \leq |b(\mathbf{y}) - b_{B_i}| + p^n|i-j|\|b\|_{CMO^1(\mathbb{Q}_p^n)},$$

holds.

Remark 2.2 From now on the letter C indicates a positive constant which may vary from line to line.

Theorem 2.3 *Let $0 < r_1 \le r_2 < \infty$, $1 \le q_1, q_2 < \infty$. Also, let $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\alpha}{n}$, $q_1' < s < \infty$, $\frac{1}{q_1'} - \frac{1}{t} = \frac{1}{s}$. If $\beta < \frac{n}{t}$, then the inequality*

$$\|H^{p,b}_{\Omega,\alpha}f\|_{\dot{K}^{\beta,r_2}_{q_2}(\mathbb{Q}_p^n)} \le C\|f\|_{\dot{K}^{\beta,r_1}_{q_1}(\mathbb{Q}_p^n)},$$

holds for all $\Omega \in L^s(S_0(\mathbf{0}))$, $b \in CMO^{\max\{q_2,t\}}(\mathbb{Q}_p^n)$, and $f \in L^{q_1}_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

Proof of Theorem 2.3 For the sake of brevity, we write

$$\sum_{j=-\infty}^{\infty} f(\mathbf{x})\chi_j(\mathbf{x}) = \sum_{j=-\infty}^{\infty} f_j(\mathbf{x}).$$

Since

$$\begin{aligned}
\|(H^{p,b}_{\Omega,\alpha}f)\chi_k\|^{q_2}_{L^{q_2}(\mathbb{Q}_p^n)} &= \int_{S_k} |\mathbf{x}|_p^{-q_2(n-\alpha)} \left| \int_{|\mathbf{y}|_p \le |\mathbf{x}|_p} \Omega(|\mathbf{y}|_p\mathbf{y}) f(\mathbf{y})(b(\mathbf{x}) - b(\mathbf{y}))\,d\mathbf{y} \right|^{q_2} d\mathbf{x} \\
&\le Cp^{-kq_2(n-\alpha)} \int_{S_k} \left(\int_{|\mathbf{y}|_p \le p^k} |\Omega(|\mathbf{y}|_p\mathbf{y}) f(\mathbf{y})(b(\mathbf{x}) - b(\mathbf{y}))|\,d\mathbf{y} \right)^{q_2} d\mathbf{x} \\
&= Cp^{-kq_2(n-\alpha)} \int_{S_k} \left(\sum_{j=-\infty}^{k} \int_{S_j} |f(\mathbf{y})\Omega(p^j\mathbf{y})(b(\mathbf{x}) - b(\mathbf{y}))|\,d\mathbf{y} \right)^{q_2} d\mathbf{x} \\
&\le Cp^{-kq_2(n-\alpha)} \int_{S_k} \left(\sum_{j=-\infty}^{k} \int_{S_j} |f(\mathbf{y})\Omega(p^j\mathbf{y})(b(\mathbf{x}) - b_{B_k})|\,d\mathbf{y} \right)^{q_2} d\mathbf{x} \\
&\quad + Cp^{-kq_2(n-\alpha)} \int_{S_k} \left(\sum_{j=-\infty}^{k} \int_{S_j} |f(\mathbf{y})\Omega(p^j\mathbf{y})(b(\mathbf{y}) - b_{B_k})|\,d\mathbf{y} \right)^{q_2} d\mathbf{x} \\
&= I + II. \qquad (2.1)
\end{aligned}$$

For $j,k \in \mathbb{Z}$ with $j \le k$, we get

$$\int_{S_j} |\Omega(p^j\mathbf{y})|^s\,d\mathbf{y} = \int_{|\mathbf{z}|_p=1} |\Omega(\mathbf{z})|^s p^{jn}\,d\mathbf{z} \le Cp^{kn}. \qquad (2.2)$$

Note that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{\alpha}{n}$ and $\frac{1}{q_1} + \frac{1}{s} + \frac{1}{t} = 1$, where $\frac{1}{t} = \frac{1}{q_1'} - \frac{1}{s}$. Applying Hölder's inequality we have

$$\begin{aligned}
I &\le Cp^{-kq_2(n-\alpha)} \int_{B_k} |b(\mathbf{x}) - b_{B_k}|^{q_2} \\
&\quad \times \left\{ \sum_{j=-\infty}^{k} \left(\int_{S_j} |f(\mathbf{y})|^{q_1}\,d\mathbf{y} \right)^{1/q_1} \left(\int_{S_j} |\Omega(p^j\mathbf{y})|^s\,d\mathbf{y} \right)^{1/s} p^{jn(1/q_1'-1/s)} \right\}^{q_2} d\mathbf{x} \\
&\le C\|b\|^{q_2}_{CMO^{q_2}(\mathbb{Q}_p^n)} p^{kn-kq_2(n-\alpha)} \left\{ \sum_{j=-\infty}^{k} p^{jn(1/q_1'-1/s)} p^{kn/s} \|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)} \right\}^{q_2} \\
&= C\|b\|^{q_2}_{CMO^{q_2}(\mathbb{Q}_p^n)} \left\{ \sum_{j=-\infty}^{k} p^{(j-k)n(1/q_1'-1/s)} \|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)} \right\}^{q_2}. \qquad (2.3)
\end{aligned}$$

Lemma 2.1 will be helpful for estimating II. Thus

$$\begin{aligned} II &\le Cp^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}\int_{S_j}\left|f(\mathbf{y})\Omega(p^j\mathbf{y})\left(b(\mathbf{y})-b_{B_j}\right)\right|d\mathbf{y}\right)^{q_2}d\mathbf{x} \\ &\quad + C\|b\|^{q_2}_{CMO^1(\mathbb{Q}_p^n)}p^{-kq_2(n-\alpha)}\int_{S_k}\left(\sum_{j=-\infty}^{k}(k-j)\int_{S_j}\left|f(\mathbf{y})\Omega(p^j\mathbf{y})\right|d\mathbf{y}\right)^{q_2}d\mathbf{x} \\ &= I_1 + II_2. \end{aligned} \tag{2.4}$$

We use Hölder's inequality to estimate I_1. We have

$$\begin{aligned} I_1 &\le Cp^{-kq_2(n-\alpha)}\int_{S_k}\left\{\sum_{j=-\infty}^{k}\left(\int_{S_j}\left|b(\mathbf{y})-b_{B_j}\right|^t d\mathbf{y}\right)^{1/t}\right. \\ &\quad \times\left.\left(\int_{S_j}\left|\Omega(p^j\mathbf{y})\right|^s d\mathbf{y}\right)^{1/s}\left(\int_{S_j}\left|f(\mathbf{y})\right|^{q_1}d\mathbf{y}\right)^{1/q_1}\right\}^{q_2}d\mathbf{x} \\ &\le \|b\|^{q_2}_{CMO^t(\mathbb{Q}_p^n)}\sum_{j=-\infty}^{k}\left\{p^{-kn/q_1'}p^{kn/s}p^{jn/t}\left(\frac{1}{|B_j|_H}\int_{B_j}\left|b(\mathbf{y})-b_{B_j}\right|^t\right)^{1/t}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right\}^{q_2} \\ &= C\|b\|^{q_2}_{CMO^t(\mathbb{Q}_p^n)}\left\{\sum_{j=-\infty}^{k}p^{(j-k)n(1/q_1'-1/s)}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right\}^{q_2}. \end{aligned} \tag{2.5}$$

In a similar fashion we can estimate II_2. Using Hölder's inequality we have

$$\begin{aligned} II_2 &\le C\|b\|^{q_2}_{CMO^1(\mathbb{Q}_p^n)}p^{-kq_2(n-\alpha)} \\ &\quad \times\int_{S_k}\left\{\sum_{j=-\infty}^{k}(k-j)\left(\int_{S_j}\left|f(\mathbf{y})\right|^{q_1}d\mathbf{y}\right)^{1/q_1}\left(\int_{S_j}\left|\Omega(p^j\mathbf{y})\right|^s d\mathbf{y}\right)^{1/s}p^{jn/t}\right\}^{q_2}d\mathbf{x} \\ &= C\|b\|^{q_2}_{CMO^1(\mathbb{Q}_p^n)}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n(1/q_1'-1/s)}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{q_2}. \end{aligned} \tag{2.6}$$

From (2.3), (2.5) and (2.6) together with the Jensen inequality, we have

$$\begin{aligned} &\left\|H^{p,b}_{\Omega,\alpha}f\right\|_{\dot{K}^{\beta,r_2}_{q_2}(\mathbb{Q}_p^n)} \\ &\quad = \left(\sum_{k=-\infty}^{\infty}p^{k\beta r_2}\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^{r_2}_{L^{q_2}(\mathbb{Q}_p^n)}\right)^{1/r_2} \\ &\quad \le \left(\sum_{k=-\infty}^{\infty}p^{k\beta r_1}\left\|\left(H^{p,b}_{\Omega,\alpha}f\right)\chi_k\right\|^{r_1}_{L^{q_2}(\mathbb{Q}_p^n)}\right)^{1/r_1} \\ &\quad \le C\|b\|_{CMO^{q_2}(\mathbb{Q}_p^n)}\left(\sum_{k=-\infty}^{\infty}p^{k\beta r_1}\left(\sum_{j=-\infty}^{k}p^{(j-k)n/t}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{r_1}\right)^{1/r_1} \\ &\qquad + C\|b\|_{CMO^t(\mathbb{Q}_p^n)}\left(\sum_{k=-\infty}^{\infty}p^{k\beta r_1}\left(\sum_{j=-\infty}^{k}p^{(j-k)n/t}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{r_1}\right)^{1/r_1} \end{aligned}$$

$$+ C\|b\|_{CMO^1(\mathbb{Q}_p^n)}\left(\sum_{k=-\infty}^{\infty} p^{k\beta r_1}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n/t}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{r_1}\right)^{1/r_1}$$
$$= J.$$

For brevity, we may choose $\|b\|_{CMO^{\max\{q_2,t\}}(\mathbb{Q}_p^n)} = 1$. Consequently,

$$J \le C\left(\sum_{k=-\infty}^{\infty} p^{k\beta r_1}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n/t}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{r_1}\right)^{1/r_1}.$$

Case 1: When $0 < r_1 \le 1$, we have

$$\begin{aligned}
J^{r_1} &= C\sum_{k=-\infty}^{\infty} p^{k\beta r_1}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)n/t}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{r_1}\\
&= C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k} p^{j\beta}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}(k-j)p^{(j-k)(n/t-\beta)}\right)^{r_1}\\
&\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k} p^{j\beta r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}(k-j)^{r_1}p^{(j-k)(n/t-\beta)r_1}\\
&= C\sum_{k=-\infty}^{\infty} p^{j\beta r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}\sum_{k=j}^{\infty}(k-j)^{r_1}p^{(j-k)(n/t-\beta)r_1}\\
&= C\|f\|_{\dot{K}_{q_1}^{\beta,r_1}(\mathbb{Q}_p^n)}^{r_1}.
\end{aligned}$$

Case 2: When $r_1 > 1$, applying Hölder's inequality we get

$$\begin{aligned}
J^{r_1} &= C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k} p^{j\beta}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}(k-j)p^{(j-k)(n/t-\beta)}\right)^{r_1}\\
&\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k} p^{j\beta r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}p^{(j-k)(n/t-\beta)r_1/2}\\
&\quad\times\left(\sum_{j=-\infty}^{k}(k-j)^{r_1'}p^{(j-k)(n/t-\beta)r_1'/2}\right)^{r_1/r_1'}\\
&= C\sum_{k=-\infty}^{\infty} p^{j\beta r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}\sum_{k=j}^{\infty}p^{(j-k)(n/t-\beta)r_1/2}\\
&= C\|f\|_{\dot{K}_{q_1}^{\beta,r_1}(\mathbb{Q}_p^n)}^{r_1}.
\end{aligned}$$

The proof of Theorem 2.3 is thus completed. □

Theorem 2.4 *Let $0 < r_1 \le r_2 < \infty$, $1 \le q_1, q_2 < \infty$. Also, let $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\alpha}{n}$, $q_1' < s < \infty$, $\frac{1}{q_1'} - \frac{1}{t} = \frac{1}{s}$, and $\lambda > 0$. If $\beta < \frac{n}{t} + \lambda$, then the inequality*

$$\|H_{\Omega,\alpha}^{p,b}f\|_{M\dot{K}_{r_2,q_2}^{\beta,\lambda}(\mathbb{Q}_p^n)} \le C\|f\|_{M\dot{K}_{r_1,q_1}^{\beta,\lambda}(\mathbb{Q}_p^n)},$$

holds for all $\Omega \in L^s(S_0(\mathbf{0}))$, $b \in CMO^{\max\{q_2,t\}}(\mathbb{Q}_p^n)$ and $f \in L_{\mathrm{loc}}^{q_1}(\mathbb{Q}_p^n)$.

Proof of Theorem 2.4 From the proof of Theorem 2.3 and

$$\big\|\big(H^{p,b}_{\Omega,\alpha}f\big)\chi_k\big\|_{L^{q_2}(\mathbb{Q}_p^n)}\le C\sum_{j=-\infty}^{k}(k-j)p^{\frac{(j-k)n}{t}}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)},$$

together with the definition of a Morrey–Herz space, the Jensen inequality, $\beta < n/t+\lambda$, $\lambda > 0$ and $1 < r_1 < \infty$, it follows that

$$\begin{aligned}
&\big\|H^{p,b}_{\Omega,\alpha}f\big\|_{M\dot{K}^{\beta,\lambda}_{r_2,q_2}(\mathbb{Q}_p^n)}\\
&=\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{k\beta r_2}\big\|\big(H^{p,b}_{\Omega,\alpha}f\big)\chi_k\big\|^{r_2}_{L^{q_2}(\mathbb{Q}_p^n)}\right)^{1/r_2}\\
&\le\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{k\beta r_1}\big\|\big(H^{p,b}_{\Omega,\alpha}f\big)\chi_k\big\|^{r_1}_{L^{q_2}(\mathbb{Q}_p^n)}\right)^{1/r_1}\\
&\le C\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{k\beta r_1}\left(\sum_{j=-\infty}^{k}(k-j)p^{\frac{(j-k)n}{t}}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{r_1}\right)^{1/r_1}\\
&\le C\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k}p^{k\beta}(k-j)p^{\frac{(j-k)n}{t}}p^{-j\beta}p^{j\lambda}p^{j\lambda}\right.\right.\\
&\quad\left.\left.\times\left(\sum_{l=-\infty}^{j}p^{l\beta r_1}\|f_j\|^{r_1}_{L^{q_1}(\mathbb{Q}_p^n)}\right)^{1/r_1}\right)^{r_1}\right)^{1/r_1}\\
&\le C\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{k\lambda r_1}\left(\sum_{j=-\infty}^{k}(k-j)p^{(j-k)(n/t-\beta+\lambda)}\|f\|_{M\dot{K}^{\beta,\lambda}_{r_1,q_1}(\mathbb{Q}_p^n)}\right)^{r_1}\right)^{1/r_1}\\
&\le C\|f\|_{M\dot{K}^{\beta,\lambda}_{r_1,q_1}(\mathbb{Q}_p^n)}.
\end{aligned}$$

□

3 Lipschitz estimates for commutators of *p*-adic rough fractional Hardy operator

The current section deals with the boundedness for the commutators of p-adic rough fractional Hardy operator on homogeneous p-adic Herz-type spaces by considering the symbol function from Lipschitz space. We open the discussion for this section from the following lemma.

Lemma 3.1 *Suppose* $f\in\Lambda_\delta(\mathbb{Q}_p^n)$ *and* $0<\delta<1$, *then*

$$\big|f(\mathbf{x})-f(\mathbf{y})\big|\le|\mathbf{x}-\mathbf{y}|_p^\delta\|f\|_{\Lambda_\delta(\mathbb{Q}_p^n)}.$$

Proof Proof immediately follows from Definition 1.6. □

Theorem 3.2 *Let* $1\le q_1, q_2<\infty$, $0<r_1\le r_2<\infty$. *Also, let* $\frac{1}{q_1}-\frac{1}{q_2}=\frac{\delta+\alpha}{n}$, $q_1'<s<\infty$, $\frac{1}{q_1'}-\frac{1}{t}=\frac{1}{s}$, *and* $0<\delta<1$. *If* $\beta<n(\frac{1}{q_1'}-\frac{1}{s})$, *then the inequality*

$$\big\|H^{p,b}_{\Omega,\alpha}f\big\|_{\dot{K}^{\beta,r_2}_{q_2}(\mathbb{Q}_p^n)}\le C\|f\|_{\dot{K}^{\beta,r_1}_{q_1}(\mathbb{Q}_p^n)}$$

holds for all $\Omega\in L^s(S_{\mathbf{0}}(\mathbf{0}))$, $b\in\Lambda_\delta(\mathbb{Q}_p^n)$, *and* $f\in L^{q_1}_{\mathrm{loc}}(\mathbb{Q}_p^n)$.

Proof of Theorem 3.2 By Hölder's inequality along with Lemma 3.1, we have

$$\begin{aligned}
&\big\|\big(H^{p,b}_{\Omega,\alpha}f\big)\chi_k\big\|^{q_2}_{L^{q_2}(\mathbb{Q}_p^n)}\\
&= \int_{S_k}|\mathbf{x}|_p^{-q_2(n-\alpha)}\bigg|\int_{|\mathbf{y}|_p\le|\mathbf{x}|_p}\Omega\big(|\mathbf{y}|_p\mathbf{y}\big)f(\mathbf{y})\big(b(\mathbf{x})-b(\mathbf{y})\big)\,d\mathbf{y}\bigg|^{q_2}d\mathbf{x}\\
&\le Cp^{-kq_2(n-\alpha)}\int_{S_k}\bigg(\int_{|\mathbf{y}|_p\le p^k}\big|\Omega\big(|\mathbf{y}|_p\mathbf{y}\big)f(\mathbf{y})\big(b(\mathbf{x})-b(\mathbf{y})\big)\big|\,d\mathbf{y}\bigg)^{q_2}d\mathbf{x}\\
&\le Cp^{-kq_2(n-\alpha)}\|b\|^{q_2}_{\Lambda_\delta(\mathbb{Q}_p^n)}\int_{S_k}\bigg(\sum_{j=-\infty}^{k}\int_{S_j}\big|\Omega\big(p^j\mathbf{y}\big)f(\mathbf{y})\big||\mathbf{x}-\mathbf{y}|_p^{\delta}\,d\mathbf{y}\bigg)^{q_2}d\mathbf{x}\\
&\le Cp^{-kq_2(n-\alpha-\delta)}\|b\|^{q_2}_{\Lambda_\delta(\mathbb{Q}_p^n)}\int_{S_k}\bigg(\sum_{j=-\infty}^{k}\int_{S_j}\big|\Omega\big(p^j\mathbf{y}\big)f(\mathbf{y})\big|\,d\mathbf{y}\bigg)^{q_2}d\mathbf{x}\\
&\le C\|b\|^{q_2}_{\Lambda_\delta(\mathbb{Q}_p^n)}p^{-kq_2(n-\alpha-\delta)+kn}\bigg(\sum_{j=-\infty}^{k}\bigg(\int_{S_j}\big|\Omega\big(p^j\mathbf{y}\big)\big|^s\,d\mathbf{y}\bigg)^{1/s}\\
&\quad\times\bigg(\int_{S_j}\big|f(\mathbf{y})\big|^{q_1}\,d\mathbf{y}\bigg)^{1/q_1}\bigg(\int_{S_j}d\mathbf{y}\bigg)^{1-1/q-1/s}\bigg)^{q_2}\\
&= I.
\end{aligned}\tag{3.1}$$

By virtue of (2.2), inequality (3.1) takes the following form:

$$\begin{aligned}
I &\le C\|b\|^{q_2}_{\Lambda_\delta(\mathbb{Q}_p^n)}p^{-kq_2(n-\alpha-\delta)+kn}\bigg(\sum_{j=-\infty}^{k}p^{kn/s+jn(1/q_1'-1/s)}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\bigg)^{q_2}\\
&\le C\|b\|^{q_2}_{\Lambda_\delta(\mathbb{Q}_p^n)}\bigg(\sum_{j=-\infty}^{k}p^{(j-k)n(1/q'-1/s)}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\bigg)^{q_2}.
\end{aligned}$$

For the sake of brevity, we take $\|b\|^{q_2}_{\Lambda_\delta(\mathbb{Q}_p^n)}=1$. Now, by definition of Herz spaces and the Jensen inequality, it follows that

$$\begin{aligned}
\big\|H^{p,b}_{\Omega,\alpha}f\big\|^{r_1}_{\dot{K}^{\beta,r_2}_{q_2}(\mathbb{Q}_p^n)} &= \bigg(\sum_{k=-\infty}^{\infty}p^{k\beta r_2}\big\|\big(H^{p,b}_{\Omega,\alpha}f\big)\chi_k\big\|^{r_2}_{L^{q_2}(\mathbb{Q}_p^n)}\bigg)^{r_1/r_2}\\
&\le \sum_{k=-\infty}^{\infty}p^{k\beta r_1}\big\|\big(H^{p,b}_{\Omega,\alpha}f\big)\chi_k\big\|^{r_1}_{L^{q_2}(\mathbb{Q}_p^n)}\\
&\le C\sum_{k=-\infty}^{\infty}p^{k\beta r_1}\bigg(\sum_{j=-\infty}^{k}p^{(j-k)n(1/q_1'-1/s)}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\bigg)^{r_1}\\
&= C\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k}p^{j\beta}p^{(j-k)(n/q_1'-n/s-\beta)}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}\bigg)^{r_1}.
\end{aligned}$$

Case 1: If $0 < r_1 \leq 1$, then

$$\begin{aligned}\left\|H_{\Omega,\alpha}^{p,b}f\right\|_{\dot{K}_{q_2}^{\beta,r_2}(\mathbb{Q}_p^n)}^{r_1} &\leq C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}p^{j\beta r_1}p^{(j-k)(n/q_1'-n/s-\beta)r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}\\ &= C\sum_{j=-\infty}^{\infty}p^{j\beta r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}\sum_{k=j}^{\infty}p^{(j-k)(n/q_1'-n/s-\beta)r_1}\\ &\leq C\|f\|_{\dot{K}_{q_2}^{\beta,r_1}(\mathbb{Q}_p^n)}^{r_1}.\end{aligned}$$

Case 2: When $r_1 > 1$, applying Hölder's inequality, we have

$$\begin{aligned}\left\|H_{\Omega,\alpha}^{p,b}f\right\|_{\dot{K}_{q_2}^{\beta,r_2}(\mathbb{Q}_p^n)}^{r_1} &\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k}p^{j\beta}p^{(j-k)(n/q_1'-n/s-\beta)}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}\right)^{r_1}\\ &\leq C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k}p^{j\beta r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}p^{(j-k)(n/q_1'-n/s-\beta)r_1/2}\\ &\quad\times\left(\sum_{j=-\infty}^{k}p^{(j-k)(n/q_1'-n/s-\beta)r_1'/2}\right)^{r_1/r_1'}\\ &\leq C\sum_{j=-\infty}^{\infty}p^{j\beta r_1}\|f_j\|_{L^{q_1}(\mathbb{Q}_p^n)}^{r_1}\sum_{k=j}^{\infty}p^{(j-k)(n/q_1'-n/s-\beta)r_1/2}\\ &\leq C\|f\|_{\dot{K}_{q_1}^{\beta,r_1}(\mathbb{Q}_p^n)}^{r_1}.\end{aligned}$$

□

Theorem 3.3 *Let* $1 \leq q_1, q_2 < \infty$, $0 < r_1 \leq r_2 < \infty$. *Also, let* $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\delta+\alpha}{n}$, $s > q_1'$, $\frac{1}{q_1'} - \frac{1}{t} = \frac{1}{s}$, $\lambda \geq 0$ *and* $0 < \delta < 1$. *If* $n(\frac{1}{q_1'} - \frac{1}{s}) + \lambda > \beta$, *then the inequality*

$$\left\|H_{\Omega,\alpha}^{p,b}f\right\|_{M\dot{K}_{r_2,q_2}^{\beta,\lambda}(\mathbb{Q}_p^n)} \leq C\|f\|_{M\dot{K}_{r_1,q_1}^{\beta,\lambda}(\mathbb{Q}_p^n)},$$

holds for all $\Omega \in L^s(S_{\mathbf{0}}(\mathbf{0}))$, $b \in \Lambda_\delta(\mathbb{Q}_p^n)$, *and* $f \in L_{\mathrm{loc}}^{q_1}(\mathbb{Q}_p^n)$.

Proof of Theorem 3.3 The proof follows from standard analysis performed in our previous theorems. So, we omit the details. □

Acknowledgements
Researchers supporting Project number (RSP-2020/33), King Saud University, Riyadh, Saudi Arabia.

Funding
No specific funding received for this work.

Availability of data and materials
Data sharing not applicable to this article as no data-sets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Formal analysis, NS, AH; investigation, NS, AH; resources, IK, AA, NNH; funding acquisition, AA, NNH; supervision, AH, IK. All authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 December 2020 Accepted: 10 June 2021 Published online: 21 July 2021

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