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The dual Brunn–Minkowski inequality for log-volume of star bodies

Dandan Lai^{1*} and Hailin Jin¹

*Correspondence:

laidandanzkj@163.com

¹Department of Mathematics, Suzhou University of Science and Technology, Suzhou, 215009, China

Abstract

This paper aims to consider the dual Brunn–Minkowski inequality for log-volume of star bodies, and the equivalent Minkowski inequality for mixed log-volume.

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1 Introduction

The classical Brunn–Minkowski theory, also known as the theory of mixed volumes, is the core theory in convex geometric analysis. It originated with Minkowski when he combined his concept of mixed volume with the Brunn–Minkowski inequality. The Brunn–Minkowski theory has been extended to the L_p Brunn–Minkowski theory, which combines volume with a generalized vector addition of compact convex sets introduced by Fiery in the early 1990s (see [3]) and now called L_p addition. Thirty years after Fiery introduced his new L_p addition, the new L_p Brunn–Minkowski theory was born in Lutwak's papers (see [8, 9]).

Lutwak's dual Brunn–Minkowski theory, introduced in the 1970s, helped achieving a major breakthrough in the solution of the Busemann–Petty problem in the 1990s. In contrast to the Brunn–Minkowski theory, in the dual theory, convex bodies are replaced with star-shaped sets, and projections onto subspaces are replaced with intersections with subspaces.

The Orlicz–Brunn–Minkowski theory originated with the work of Lutwak, Yang, and Zhang [10, 11]. By introducing the Orlicz–Minkowski addition, Gardner, Hug, and Weil [5], and Xi, Jin, and Leng [15] proved the Orlicz–Brunn–Minkowski inequality and Orlicz–Minkowski inequality. It is a natural extension of the L_p -Brunn–Minkowski theory for $p \geq 1$. For dual Orlicz–Brunn–Minkowski theory, see [6, 16].

The well-known classic Minkowski problem is: given a finite Borel measure μ on S^{n-1} , what are the necessary and sufficient conditions on μ such that μ is the surface area measure $S(K, \cdot)$ of a convex body K on \mathbb{R}^n ? The Minkowski problem was first studied by Minkowski, who demonstrated both existence and uniqueness of solutions for the problem when given measure is either discrete or has a continuous density. Aleksandrov and Fenchel–Jensen independently solved the problem in 1938 for arbitrary measure. The L_p Minkowski problem, posed by Lutwak in 1993, has developed quickly. Recently, the dual

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Minkowski problem was introduced by Huang et al. [7]. In [7] new links were established between the Brunn–Minkowski theory and the dual Brunn–Minkowski theory by making critical use of the radial Gauss image.

In [2], Boroczky et al. introduced the Gauss image problem: Suppose that λ is a submeasure defined on the Lebesgue measurable subsets of S^{n-1} , and μ is a Borel submeasure on S^{n-1} . What are the necessary and sufficient conditions on λ and μ , so that there exists a convex body $K \in \mathcal{K}_o^n$ such that

$$\lambda(K, \cdot) = \mu$$

on the Borel subsets of S^{n-1} ? And if such a body exists, to what extent is it unique?

Let μ be a Borel measure on S^{n-1} . The log-volume $\mu(K)$ of a star body $K \in \mathcal{S}_o^n$ with respect to μ is defined by

$$\mu(K) = \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \rho_K(v) d\mu\right).$$

The log-volume $\mu(K)$ of a convex body K with respect to μ plays a very important role in solving the Gauss image problem.

In this paper, we establish the Brunn–Minkowski theory of the log-volume. Concretely, we prove the dual Brunn–Minkowski inequality for the log-volume $\mu(K)$ of the star body K , and the equivalent Minkowski inequality for mixed log-volume.

2 Preliminaries

In this section, some notations and some basic facts for convex bodies and star bodies are listed. More detailed references to the theory of these bodies can be found in [13].

\mathbb{R}^n denotes the usual n -dimensional Euclidean space with the usual inner product $\langle \cdot, \cdot \rangle$. S^{n-1} denotes the unit sphere of \mathbb{R}^n . A compact convex subset of \mathbb{R}^n with nonempty interior is called a convex body. The set of convex bodies in \mathbb{R}^n is denoted by \mathcal{K}^n , and the set of convex bodies in \mathbb{R}^n containing the origin in their interiors is denoted by \mathcal{K}_o^n .

If K is a compact convex subset of \mathbb{R}^n , then its support function $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$h_K(x) := \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

The support function is positively homogeneous of degree 1 and convex. Note that a compact convex subset of \mathbb{R}^n is uniquely determined by its support function.

Suppose that $K \subset \mathbb{R}^n$ is a compact star-shaped set with respect to the origin, that is, the line segment joining each point of K to the origin is contained completely in K . The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ of K is given by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The radial function is positively homogeneous of degree -1 , and a compact star-shaped set (with respect to the origin) is uniquely determined by its radial function. If ρ_K is positive and continuous, then K is called a star body (with respect to the origin). Write \mathcal{S}_o^n for the set of star bodies with respect to the origin in \mathbb{R}^n .

Two star bodies K and L are dilates (of one other) if $\rho_K(u) \setminus \rho_L(u)$ is independent of $u \in S^{n-1}$.

If $s > 0$, we have

$$\rho(sK, u) = s\rho(K, u) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

More generally, from the definition of the radial function, it follows immediately that for $A \in GL(n)$ the radial function of the image $AK = \{Ay : y \in K\}$ of K is given by

$$\rho(AK, u) = \rho(K, A^{-1}u) \quad \text{for all } x \in \mathbb{R}^n.$$

If K and L are star bodies, and $\alpha, \beta \geq 0$ (not both zero), then for $p \neq 0$, the radial p -combination $\alpha \cdot K \tilde{\tau}_p \beta \cdot L$ is a star body and is defined by (see [4])

$$\rho(\alpha \cdot K \tilde{\tau}_p \beta \cdot L, u)^p = \alpha\rho(K, u)^p + \beta\rho(L, u)^p, \quad u \in S^{n-1}.$$

The set \mathcal{K}_0^n can be endowed with the Hausdorff metric, which means the distance between $K, L \in \mathcal{K}^n$ is defined by

$$\|h_K - h_L\| = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

The set \mathcal{S}_0^n can be endowed with the radial metric, which means the distance between $K, L \in \mathcal{S}_0^n$ is defined by

$$\|\rho_K - \rho_L\| = \max_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

For each $K \in \mathcal{K}_0^n$, its polar body K^* is given by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It follows that $K^* \in \mathcal{K}_0^n$ and $K = (K^*)^*$. By the definition, there exists an important fact between K and K^* as follows:

$$h_K = \frac{1}{\rho_{K^*}} \quad \text{and} \quad \rho_K = \frac{1}{h_{K^*}}.$$

3 Properties of $\mu(K)$

Lemma 3.1 *Let μ be the spherical Lebesgue measure of S^{n-1} . If $K \in \mathcal{S}_0^n$, then for $A \in O(n)$, we have $\mu(AK) = \mu(K)$.*

Proof By the definition of $\mu_0(K)$, we have

$$\begin{aligned} \mu(AK) &= \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \rho_{AK}(u) d\mu(u)\right) \\ &= \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \rho_K(A^{-1}u) d\mu(A^{-1}u)\right) \\ &= \mu(K). \end{aligned}$$

□

Theorem 3.2 *If $K \in \mathcal{K}_o^n$, then*

$$\mu(K)\mu(K^*) \leq \mu(B)\mu(B^*).$$

When μ is a spherical Lebesgue measure of S^{n-1} , the equality holds if and only if K is an Euclidean ball.

Proof By the definition of K^* and the fact that $\frac{\rho_K(u)}{h_K(u)} \leq 1$, for all $u \in S^{n-1}$, we obtain

$$\begin{aligned} \mu(K)\mu(K^*) &= \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \rho_K(u) d\mu(u)\right) \cdot \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \rho_{K^*}(u) d\mu(u)\right) \\ &= \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log(\rho_K(u)\rho_{K^*}(u)) d\mu(u)\right) \\ &= \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log\left(\frac{\rho_K(u)}{h_K(u)}\right) d\mu(u)\right) \\ &\leq 1 = \mu(B)\mu(B^*). \end{aligned}$$

When μ is a spherical Lebesgue measure of S^{n-1} , then $\frac{\rho_K(u)}{h_K(u)} = 1$ for all $u \in S^{n-1}$ if and only if K is an Euclidean ball. □

4 L_p -Brunn–Minkowski inequality for $\mu(K)$ in the case $p \neq 0$

In this section, we establish the L_p -Minkowski inequality and L_p -Brunn–Minkowski inequality as follows.

Definition 4.1 ([4]) For arbitrary $p \in \mathbb{R} \setminus \{0\}$, $\alpha, \beta \geq 0$ (not both zero), the radial p -combination of $K, L \in \mathcal{S}_o^n$ is defined by

$$\rho(\alpha \cdot K \tilde{\tau}_p \beta \cdot L, \cdot)^p = \alpha \rho(K, \cdot)^p + \beta \rho(L, \cdot)^p.$$

Lemma 4.1 *Suppose $K, L \in \mathcal{S}_o^n$ and $\alpha, \beta \geq 0$ (not both zero). Then, for $A \in GL(n)$,*

$$A(\alpha \cdot K \tilde{\tau}_p \beta \cdot L) = \alpha \cdot AK \tilde{\tau}_p \beta \cdot AL.$$

Proof For $u \in S^{n-1}$, by Definition 4.1, we have

$$\begin{aligned} \rho_{\alpha \cdot AK \tilde{\tau}_p \beta \cdot AL}^p(u) &= a \rho_{AK}^p(u) + b \rho_{AL}^p(u) \\ &= \alpha \rho_K^p(A^{-1}u) + \beta \rho_L^p(A^{-1}u) \\ &= \rho_{\alpha \cdot K \tilde{\tau}_p \beta \cdot L}^p(A^{-1}u) \\ &= \rho_{A(\alpha \cdot K \tilde{\tau}_p \beta \cdot L)}^p(u). \end{aligned}$$

Thus, we obtain $A(\alpha \cdot K \tilde{\tau}_p \beta \cdot L) = \alpha \cdot AK \tilde{\tau}_p \beta \cdot AL$. □

Definition 4.2 For $p \neq 0$, we define the L_p -dual mixed log-volume $\mu_p(K, L)$ of $K, L \in \mathcal{S}_o^n$ by

$$\mu_p(K, L) = \frac{\mu(K)}{|\mu|} \int_{S^{n-1}} \left(\frac{\rho_L(u)}{\rho_K(u)}\right)^p d\mu(u).$$

We are ready to derive the variational formula of $\mu(K)$ for the radial p -sum.

Theorem 4.1 *Let $p \neq 0$. If $K, L \in \mathcal{S}_o^n$, then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K \tilde{+}_p \varepsilon \cdot L) - \mu(K)}{\varepsilon} = \frac{\mu_p(K, L)}{p}.$$

Proof Suppose $\varepsilon > 0, K, L \in \mathcal{S}_o^n$, and $u \in S^{n-1}$. It follows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{\log \rho_{K \tilde{+}_p \varepsilon \cdot L}^p(u) - \log \rho_K^p(u)}{\varepsilon} &= \rho_{K \tilde{+}_p \varepsilon \cdot L}^{-p}(u)|_{\varepsilon=0^+} \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{\rho_{K \tilde{+}_p \varepsilon \cdot L}^p(u) - \rho_K^p(u)}{\varepsilon} \\ &= \left(\frac{\rho_L(u)}{\rho_K(u)} \right)^p, \end{aligned}$$

uniformly on S^{n-1} .

Hence

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K \tilde{+}_p \varepsilon \cdot L) - \mu(K)}{\varepsilon} &= \frac{\mu(K)}{p} \lim_{\varepsilon \rightarrow 0^+} \frac{\frac{1}{|\mu|} \int_{S^{n-1}} (\log \rho_{K \tilde{+}_p \varepsilon \cdot L}^p(u) - \log \rho_K^p(u)) d\mu(u)}{\varepsilon} \\ &= \frac{\mu(K)}{p} \frac{1}{|\mu|} \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{\log \rho_{K \tilde{+}_p \varepsilon \cdot L}^p(u) - \log \rho_K^p(u)}{\varepsilon} d\mu(u) \\ &= \frac{\mu(K)}{p} \frac{1}{|\mu|} \int_{S^{n-1}} \left(\frac{\rho_L(u)}{\rho_K(u)} \right)^p d\mu(u). \end{aligned}$$

We complete the proof of Theorem 4.1. □

Lemma 4.2 *Suppose that μ is a spherical Lebesgue measure of S^{n-1} . Let $p \neq 0$ and $K, L \in \mathcal{S}_o^n$. Then, for $A \in O(n)$,*

$$\mu_p(AK, AL) = \mu_p(K, L).$$

Proof From Theorem 4.1 and Lemma 3.1, we have

$$\begin{aligned} \mu_p(AK, AL) &= p \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(AK \tilde{+}_p \varepsilon \cdot AL) - \mu(AK)}{\varepsilon} \\ &= p \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(A(K \tilde{+}_p \varepsilon \cdot L)) - \mu(K)}{\varepsilon} \\ &= p \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K \tilde{+}_p \varepsilon \cdot L) - \mu(K)}{\varepsilon} \\ &= \mu_p(K, L). \end{aligned} \quad \square$$

The following is the L_p -Minkowski inequality for dual mixed log-volume.

Theorem 4.2 *Let $p \neq 0$ and $K, L \in \mathcal{S}_o^n$. Then*

$$\mu_p(K, L) \geq \mu(L)^p \mu(K)^{1-p}.$$

When μ is a spherical Lebesgue measure of S^{n-1} , the equality holds if and only if K and L are dilates of each other.

Proof By the definitions of $\mu_p(K, L)$, $\mu(K)$, and $\mu(L)$, we have

$$\left(\frac{\mu_p(K, L)}{\mu(K)}\right)^{\frac{1}{p}} = \left(\int_{S^{n-1}} \frac{1}{|\mu|} \left(\frac{\rho_L(u)}{\rho_K(u)}\right)^p d\mu(u)\right)^{\frac{1}{p}}$$

and

$$\frac{\mu(L)}{\mu(K)} = \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u)\right).$$

(1) If $p > 0$, then we have

$$\left(\int_{S^{n-1}} \frac{1}{|\mu|} \left(\frac{\rho_L(u)}{\rho_K(u)}\right)^p d\mu(u)\right)^{\frac{1}{p}} \geq \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u)\right),$$

which implies that $\mu_p(K, L) \geq \mu(L)^p \mu(K)^{1-p}$.

(2) If $p < 0$, then we have

$$\left(\int_{S^{n-1}} \frac{1}{|\mu|} \left(\frac{\rho_L(u)}{\rho_K(u)}\right)^p d\mu(u)\right)^{\frac{1}{p}} \leq \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u)\right),$$

which also implies that $\mu_p(K, L) \geq \mu(L)^p \mu(K)^{1-p}$.

When μ is a spherical Lebesgue measure of S^{n-1} , the equality holds if and only if K and L are dilates of each other. □

Remark 4.1 When $p = 1$, the L_1 -dual mixed log-volume $\mu_1(K, L)$ is written as $\mu(K, L)$, so we have

$$\mu(K, L) \geq \mu(L).$$

When μ is a spherical Lebesgue measure of S^{n-1} , the equality holds if and only if K and L are dilates of each other.

Lemma 4.3 *Suppose $p \neq 0$ and $\mathcal{M} \in S^n$ such that $K, L \in \mathcal{M}$. If $\mu_p(M, K) = \mu_p(M, L)$ for all $M \in \mathcal{M}$, then $K = L$.*

Proof Set $M = K$, then we have

$$\mu(K) = \mu_p(K, K) = \mu_p(K, L) \geq \mu(L)^p \mu(K)^{1-p},$$

so, $1 \geq \frac{\mu(L)^p}{\mu(K)^p}$. Set $M = L$, we have

$$\mu(L) = \mu_p(L, L) = \mu_p(L, K) \geq \mu(K)^p \mu(L)^{1-p},$$

so, $1 \geq \frac{\mu(K)^p}{\mu(L)^p}$. Hence $\mu(K) = \mu(L)$, which implies that the inequalities above are all equalities. By the equality condition of L_p -dual mixed log-volume, we have K and L are dilates. Since $\mu(K) = \mu(L)$, we get $K = L$. □

By the L_p -Minkowski inequality for dual mixed log-volume, we have the following L_p -Brunn–Minkowski inequality for log-volume.

Theorem 4.3 *Suppose $p \neq 0, \alpha, \beta > 0$. If $K, L \in \mathcal{S}_o^n$, then*

$$\mu(\alpha \cdot K \tilde{\tau}_p \beta \cdot L)^p \geq \alpha \mu(K)^p + \beta \mu(L)^p.$$

When μ is a spherical Lebesgue measure of S^{n-1} , the equality holds if and only if K and L are dilates of each other.

Proof From the L_p -Minkowski inequality of $\mu_p(K, L)$, it follows that

$$\begin{aligned} \mu_p(Q, \alpha \cdot K \tilde{\tau}_p \beta \cdot L) &= \frac{\mu(Q)}{|\mu|} \int_{S^{n-1}} \left(\frac{\rho_{\alpha \cdot K \tilde{\tau}_p \beta \cdot L}(u)}{\rho_Q(u)} \right)^p d\mu(u) \\ &= \frac{\mu(Q)}{|\mu|} \int_{S^{n-1}} \frac{\alpha \rho_K^p(u) + \beta \rho_L^p(u)}{\rho_Q^p(u)} d\mu(u) \\ &= \alpha \frac{\mu(Q)}{|\mu|} \int_{S^{n-1}} \frac{\rho_K^p(u)}{\rho_Q^p(u)} d\mu(u) + \beta \frac{\mu(Q)}{|\mu|} \int_{S^{n-1}} \frac{\rho_L^p(u)}{\rho_Q^p(u)} d\mu(u) \\ &= \alpha \mu_p(Q, K) + \beta \mu_p(Q, L) \\ &\geq \alpha \mu(K)^p \mu(Q)^{1-p} + \beta \mu(L)^p \mu(Q)^{1-p} \\ &= (\alpha \mu(K)^p + \beta \mu(L)^p) \mu(Q)^{1-p}. \end{aligned}$$

Let $Q = \alpha \cdot K \tilde{\tau}_p \beta \cdot L$, we have $\mu_p(\alpha \cdot K \tilde{\tau}_p \beta \cdot L, \alpha \cdot K \tilde{\tau}_p \beta \cdot L) = \mu(\alpha \cdot K \tilde{\tau}_p \beta \cdot L)$, so

$$\mu(\alpha \cdot K \tilde{\tau}_p \beta \cdot L)^p \geq \alpha \mu(K)^p + \beta \mu(L)^p.$$

When μ is a spherical Lebesgue measure of S^{n-1} , by the equality condition of L_p -Minkowski inequality, we have that the equality holds if and only if K and L are dilates of each other. □

Theorem 4.4 *The L_p -Brunn–Minkowski inequality is equivalent to the L_p -Minkowski inequality.*

Proof Since we have proved the L_p -Brunn–Minkowski inequality by the L_p -Minkowski inequality, we only need to prove the L_p -Minkowski inequality by the L_p -Brunn–Minkowski inequality.

We first consider the case $p > 0$. From the variational formula of $\mu(K)$ and the L_p -Brunn–Minkowski inequality, for $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{p} \mu_p(K, L) &= \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K \tilde{\tau}_p \varepsilon \cdot L) - \mu(K)}{\varepsilon} \\ &\geq \lim_{\varepsilon \rightarrow 0^+} \frac{(\mu(K)^p + \varepsilon \mu(L)^p)^{\frac{1}{p}} - \mu(K)}{\varepsilon} \\ &= \mu(K) \lim_{\varepsilon \rightarrow 0^+} \frac{(1 + \varepsilon (\frac{\mu(L)}{\mu(K)})^p)^{\frac{1}{p}} - 1}{\varepsilon} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\mu(K)}{p} \left(\frac{\mu(L)}{\mu(K)} \right)^p \\
 &= \frac{1}{p} \mu(L)^p \mu(K)^{1-p}.
 \end{aligned}$$

Thus, we obtain $\mu_p(K, L) \geq \mu(L)^p \mu(K)^{1-p}$.

Now we consider the case $p < 0$. From the variational formula of $\mu(K)$ and the L_p -Brunn–Minkowski inequality, for $\varepsilon > 0$, we have

$$\begin{aligned}
 \frac{1}{p} \mu_p(K, L) &= \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K \tilde{\tau}_p \varepsilon \cdot L) - \mu(K)}{\varepsilon} \\
 &\leq \lim_{\varepsilon \rightarrow 0^+} \frac{(\mu(K)^p + \varepsilon \mu(L)^p)^{\frac{1}{p}} - \mu(K)}{\varepsilon} \\
 &= \mu(K) \lim_{\varepsilon \rightarrow 0^+} \frac{(1 + \varepsilon (\frac{\mu(L)}{\mu(K)})^p)^{\frac{1}{p}} - 1}{\varepsilon} \\
 &= \frac{\mu(K)}{p} \left(\frac{\mu(L)}{\mu(K)} \right)^p \\
 &= \frac{1}{p} \mu(L)^p \mu(K)^{1-p}.
 \end{aligned}$$

Thus, we obtain $\mu_p(K, L) \geq \mu(L)^p \mu(K)^{1-p}$. □

5 The log-Brunn–Minkowski inequality for $\mu(K)$

In the L_p -Brunn–Minkowski theory, L_p -Brunn–Minkowski inequality plays a core role. Among L_p -Brunn–Minkowski inequality for $p \geq 0$, the L_0 -Brunn–Minkowski inequality, also called the log-Brunn–Minkowski inequality, is stronger than any others (see [1]).

The main purpose of this section is to establish the dual forms of the log-Minkowski inequality and the log-Brunn–Minkowski inequality as follows. In fact, we found that these inequalities are all equalities.

We first give the log radial combination of two star bodies. It was introduced in [14].

Definition 5.1 ([14]) Let K and L be two star bodies in \mathbb{R}^n and $0 \leq \lambda \leq 1$, then the log radial combination $(1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L$ of K, L is defined by

$$\rho_{(1-\lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L}(u) = \rho_K(u)^{1-\lambda} \rho_L(u)^\lambda, \quad \forall u \in S^{n-1}.$$

In particular, if $\lambda = 0$, then $(1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L = K$. If $\lambda = 1$, then $(1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L = L$.

From the definition of the log radial combination, we have the following two lemmas.

Lemma 5.1 Let $0 \leq \lambda \leq 1$. If $K, L \in \mathcal{S}_o^n$, then $(1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L \in \mathcal{S}_o^n$.

Lemma 5.2 Let $0 \leq \lambda \leq 1$. If $K, L \in \mathcal{S}_o^n$, then for $A \in GL(n)$,

$$A((1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L) = (1 - \lambda) \cdot AK \tilde{\tau}_0 \lambda \cdot AL.$$

Now we give the definition of dual log mixed log-volume $\mu_0(K, L)$ of $K, L \in \mathcal{S}_o^n$.

Definition 5.2 Let $K, L \in S^n_o$, the dual log mixed log-volume $\mu_0(K, L)$ of K, L is defined by

$$\mu_0(K, L) = \mu(K) \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u)\right).$$

The following is the variational formula of $\mu(K)$ for the log radial combination.

Theorem 5.1 Let $K, L \in S^n_o$. Then

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mu((1 - \varepsilon) \cdot K \tilde{\tau}_0 \varepsilon \cdot L) - \mu(K)}{\varepsilon} = \mu(K) \log \frac{\mu_0(K, L)}{\mu(K)}.$$

Proof Suppose $\varepsilon > 0, K, L \in S^n_o$, and $u \in S^{n-1}$. It follows that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\log \rho_{(1-\lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L}(u) - \log \rho_K(u)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon \log \frac{\rho_L(u)}{\rho_K(u)}}{\varepsilon} = \log \frac{\rho_L(u)}{\rho_K(u)},$$

uniformly on S^{n-1} .

Therefore, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \frac{\mu((1 - \varepsilon) \cdot K \tilde{\tau}_0 \varepsilon \cdot L) - \mu(K)}{\varepsilon} \\ &= \mu(K) \lim_{\varepsilon \rightarrow 0^+} \frac{\exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} (\log \rho_{(1-\lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L}(u) - \log \rho_K(u)) d\mu(u)\right) - 1}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{S^{n-1}} \frac{\log \rho_{(1-\lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L}(u) - \log \rho_K(u)}{\varepsilon} d\mu(u) \\ &= \frac{\mu(K)}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u) \\ &= \mu(K) \log \frac{\mu_0(K, L)}{\mu(K)}. \end{aligned} \quad \square$$

Lemma 5.3 Suppose that μ is a spherical Lebesgue measure of S^{n-1} . If $K, L \in S^n_o$ and $A \in O(n)$, then

$$\mu_0(AK, AL) = \mu_0(K, L).$$

Proof From Theorem 5.1, Lemma 5.2, and Lemma 3.1, we have

$$\begin{aligned} \log \frac{\mu_0(AK, AL)}{\mu(K)} &= \frac{1}{\mu(K)} \lim_{\varepsilon \rightarrow 0^+} \frac{\mu((1 - \varepsilon) \cdot AK \tilde{\tau}_0 \varepsilon \cdot AL) - \mu(AK)}{\varepsilon} \\ &= \frac{1}{\mu(K)} \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(A(K \tilde{\tau}_0 \varepsilon \cdot L)) - \mu(K)}{\varepsilon} \\ &= \frac{1}{\mu(K)} \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K \tilde{\tau}_0 \varepsilon \cdot L) - \mu(K)}{\varepsilon} \\ &= \log \frac{\mu_0(K, L)}{\mu(K)}. \end{aligned}$$

So, $\mu_0(AK, AL) = \mu_0(K, L)$. □

Theorem 5.2 *If $K, L \in \mathcal{S}_o^n$, then $\mu_0(K, L) = \mu(L)$.*

Proof By the definitions of $\mu(K)$ and $\mu_0(K, L)$, we have

$$\begin{aligned} \frac{\mu(L)}{\mu(K)} &= \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u)\right) \\ &= \frac{\mu_0(K, L)}{\mu(K)}. \end{aligned}$$

Therefore, we have $\mu_0(K, L) = \mu(L)$. □

Theorem 5.3 *Let $0 \leq \lambda \leq 1$. If $K, L \in \mathcal{S}_o^n$, then*

$$\mu((1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L) = \mu(K)^{1-\lambda} \mu(L)^\lambda.$$

Proof For $0 \leq \lambda \leq 1$, we obtain

$$\begin{aligned} &\log \mu((1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L) \\ &= \frac{1}{|\mu|} \int_{S^{n-1}} \log(\rho_K(u)^{1-\lambda} \rho_L(u)^\lambda) d\mu(u) \\ &= (1 - \lambda) \frac{1}{|\mu|} \int_{S^{n-1}} \log \rho_K(u) d\mu(u) + \lambda \frac{1}{|\mu|} \int_{S^{n-1}} \log \rho_L(u) d\mu(u) \\ &= (1 - \lambda) \log \mu(K) + \lambda \log \mu(L) \\ &= \log(\mu(K)^{1-\lambda} \mu(L)^\lambda). \end{aligned}$$

Therefore, we have $\mu((1 - \lambda) \cdot K \tilde{\tau}_0 \lambda \cdot L) = \mu(K)^{1-\lambda} \mu(L)^\lambda$. □

6 Dual Orlicz–Brunn–Minkowski inequality for $\mu(K)$

Let Φ be the set of strictly increasing functions $\phi : (0, \infty) \rightarrow (0, \infty)$ which are continuously differentiable on $(0, \infty)$ with positive derivative and satisfy that $\phi(\infty) = \infty$ and that $\log \circ \phi^{-1}$ is concave. Notice that whenever $\phi \in \Phi$ is convex, the composite function $\log \circ \phi^{-1}$ is concave. The collection of convex functions from Φ shall be denoted by \mathcal{C} . There are many fundamental examples of the functions $\phi \in \Phi$. Convex examples in Φ include the power function $\phi(t) = t^p$ with $p \geq 1$; the logistic function $\phi(t) = t + 2 \log(1 + e^{-t})$; the Laplace function $\phi(t) = e^{-t}$, and so on. Nonconvex examples of Φ include $\phi(t) = t^p$ with $0 < p < 1$ and $\phi(t) = \frac{1}{q} \log(1 + t)$ with $0 < q < 1$ (see [12]).

Let Ψ be the set of strictly decreasing functions $\psi : (0, \infty) \rightarrow (0, \infty)$ which are continuously differentiable on $(0, \infty)$ with negative derivative and satisfy that $\psi(0^+) = \infty$, $\psi(\infty) = 0$ and that $\log \circ \psi^{-1}$ is convex. Notice that $\psi(t) = t^p$ with $p < 0$ belong to Ψ .

Definition 6.1 Let $\phi \in \Phi \cup \Psi$ and $\alpha, \beta \geq 0$ (not both zero), the Orlicz radial combination $\alpha \cdot K \tilde{\tau}_\phi \beta \cdot L$ of $K, L \in \mathcal{S}_o^n$ is defined by

$$\alpha \phi\left(\frac{\rho(K, u)}{\rho(\alpha \cdot K \tilde{\tau}_\phi \beta \cdot L, u)}\right) + \beta \phi\left(\frac{\rho(L, u)}{\rho(\alpha \cdot K \tilde{\tau}_\phi \beta \cdot L, u)}\right) = \phi(1).$$

From Definition 6.1, we have the following lemma.

Lemma 6.1 *Suppose $\phi \in \Phi \cup \Psi$ and $\alpha, \beta \geq 0$ (not both zero). If $K, L \in \mathcal{S}_o^n$, $A \in GL(n)$, then $A(\alpha \cdot K \tilde{\tau}_\phi \beta \cdot L) = \alpha \cdot AK \tilde{\tau}_\phi \beta \cdot AL$.*

Lemma 6.2 *Suppose $\phi \in \Phi \cup \Psi$. If $K, L \in \mathcal{S}_o^n$, then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\rho_{K \tilde{\tau}_\phi \varepsilon \cdot L}(u) - \rho_K(u)}{\varepsilon} = \frac{\rho_K(u)}{\phi'_r(1)} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),$$

uniformly for all $u \in S^{n-1}$.

Proof Let $\rho_{K_\varepsilon}(u) = \rho_{K \tilde{\tau}_\phi \varepsilon \cdot L}(u)$. Then $\rho_{K_\varepsilon}(u) \rightarrow \rho_K(u)$ uniformly on S^{n-1} as $\varepsilon \rightarrow 0^+$. By the definition of $K \tilde{\tau}_\phi \varepsilon \cdot L$, we have

$$\frac{\rho_K(u)}{\rho_{K_\varepsilon}(u)} = \phi^{-1}(\phi(1) - \varepsilon \phi\left(\frac{\rho_L(u)}{\rho_{K_\varepsilon}(u)}\right)).$$

Let $s = \phi^{-1}(\phi(1) - \varepsilon \phi\left(\frac{\rho_L(u)}{\rho_{K_\varepsilon}(u)}\right))$. Then we have $\frac{\rho_{K_\varepsilon}(u) - \rho_K(u)}{\rho_{K_\varepsilon}(u)} = 1 - s$. Note that $s \rightarrow 1$ as $\varepsilon \rightarrow 0^+$. Hence, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{\rho_{K_\varepsilon}(u) - \rho_K(u)}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} \rho_{K_\varepsilon}(u) \cdot \frac{\rho_{K_\varepsilon}(u) - \rho_K(u)}{\varepsilon \rho_{K_\varepsilon}(u)} \\ &= \lim_{\varepsilon \rightarrow 0^+} \rho_{K_\varepsilon}(u) \cdot \phi\left(\frac{\rho_L(u)}{\rho_{K_\varepsilon}(u)}\right) \cdot \frac{\frac{\rho_{K_\varepsilon}(u) - \rho_K(u)}{\rho_{K_\varepsilon}(u)}}{\phi(1) - (\phi(1) - \varepsilon \phi\left(\frac{\rho_L(u)}{\rho_{K_\varepsilon}(u)}\right))} \\ &= \rho_K(u) \cdot \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{1 - s}{\phi(1) - \phi(s)} \\ &= \frac{\rho_K(u)}{\phi'_r(1)} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right). \end{aligned}$$

Since $\rho_{K_\varepsilon}(u) \rightarrow \rho_K(u)$ uniformly on S^{n-1} as $\varepsilon \rightarrow 0^+$, we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\rho_{K \tilde{\tau}_\phi \varepsilon \cdot L}(u) - \rho_K(u)}{\varepsilon} = \frac{\rho_K(u)}{\phi'_r(1)} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),$$

uniformly for all $u \in S^{n-1}$. □

Remark 6.1 The ideal of the proof of Theorem 6.2 is introduced by [16].

Definition 6.2 Suppose $\phi \in \Phi \cup \Psi$. The Orlicz dual mixed log-volume $\mu_\phi(K, L)$ of $K, L \in \mathcal{S}_o^n$ is defined by

$$\mu_\phi(K, L) = \frac{\mu(K)}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) d\mu(u).$$

The following is the variational formula of $\mu(K)$ for the Orlicz radial sum.

Theorem 6.1 *Suppose $\phi \in \Phi \cup \Psi$. If $K, L \in \mathcal{S}_o^n$, then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K \tilde{\tau}_\phi \varepsilon \cdot L) - \mu(K)}{\varepsilon} = \frac{\mu_\phi(K, L)}{\phi'_r(1)}.$$

Proof Suppose $\varepsilon > 0, K, L \in \mathcal{S}_o^n$, and $u \in S^{n-1}$. By Lemma 6.2, it follows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{\log \rho_{K\tilde{\tau}_\phi \varepsilon \cdot L}(u) - \log \rho_K(u)}{\varepsilon} &= \rho_{K\tilde{\tau}_\phi \varepsilon \cdot L}^{-1}(u)|_{\varepsilon=0^+} \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{\rho_{K\tilde{\tau}_\phi \varepsilon \cdot L}(u) - \rho_K(u)}{\varepsilon} \\ &= \frac{1}{\phi'_r(1)} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right), \end{aligned}$$

uniformly on S^{n-1} .

Hence

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K\tilde{\tau}_\phi \varepsilon \cdot L) - \mu(K)}{\varepsilon} &= \mu(K) \frac{1}{|\mu|} \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{\log \rho_{K\tilde{\tau}_\phi \varepsilon \cdot L}(u) - \log \rho_K(u)}{\varepsilon} d\mu(u) \\ &= \frac{\mu(K)}{\phi'_r(1)} \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) d\mu(u). \quad \square \end{aligned}$$

Lemma 6.3 *Suppose μ is a spherical Lebesgue measure of S^{n-1} . If $\phi \in \Phi \cup \Psi, A \in O(n)$, and $K, L \in \mathcal{S}_o^n$, then $\mu_\phi(AK, AL) = \mu_\phi(K, L)$.*

Proof From Theorem 6.1, we have

$$\begin{aligned} \mu_\phi(AK, AL) &= \phi'_r(1) \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(AK\tilde{\tau}_\phi \varepsilon \cdot AL) - \mu(AK)}{\varepsilon} \\ &= \phi'_r(1) \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(A(K\tilde{\tau}_\phi \varepsilon \cdot L)) - \mu(K)}{\varepsilon} \\ &= \phi'_r(1) \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K\tilde{\tau}_\phi \varepsilon \cdot L) - \mu(K)}{\varepsilon} \\ &= \mu_\phi(K, L). \end{aligned}$$

Thus, we obtain $\mu_\phi(AK, AL) = \mu_\phi(K, L)$. □

The following is the dual Orlicz–Minkowski inequality for the mixed log-volume.

Theorem 6.2 *Let $\phi \in \Phi \cup \Psi$ and $K, L \in \mathcal{S}_o^n$. Then*

$$\mu_\phi(K, L) \geq \mu(K) \phi\left(\frac{\mu(L)}{\mu(K)}\right).$$

When μ is a spherical Lebesgue measure of S^{n-1} , the equality holds if and only if K and L are dilates.

Proof If $\phi \in \Phi$, then ϕ and ϕ^{-1} are increasing. Since ϕ^{-1} is log-concave, by Jensen's inequality, we have

$$\begin{aligned} \log \circ \phi^{-1}\left(\frac{\mu_\phi(K, L)}{\mu(K)}\right) &= \log \circ \phi^{-1}\left(\frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) d\mu(u)\right) \\ &\geq \left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u)\right) = \log \frac{\mu(L)}{\mu(K)}. \end{aligned}$$

Thus, by ϕ^{-1} is increasing, we have $\mu_\phi(K, L) \geq \mu(K) \phi\left(\frac{\mu(L)}{\mu(K)}\right)$.

If $\phi \in \Psi$, then ϕ and ϕ^{-1} are decreasing. Since ϕ^{-1} is log-convex, we have

$$\begin{aligned} \log \circ \phi^{-1} \left(\frac{\mu_\phi(K, L)}{\mu(K)} \right) &= \log \circ \phi^{-1} \left(\frac{1}{|\mu|} \int_{S^{n-1}} \phi \left(\frac{\rho_L(u)}{\rho_K(u)} \right) d\mu(u) \right) \\ &\leq \left(\frac{1}{|\mu|} \int_{S^{n-1}} \log \frac{\rho_L(u)}{\rho_K(u)} d\mu(u) \right) \\ &= \log \frac{\mu(L)}{\mu(K)}. \end{aligned}$$

Thus, by ϕ^{-1} is decreasing, we have $\mu_\phi(K, L) \geq \mu(K)\phi\left(\frac{\mu(L)}{\mu(K)}\right)$.

When μ is a spherical Lebesgue measure of S^{n-1} , by the equality condition of Jensen’s inequality, if the equality holds, then there is $\lambda > 0$ such that $\rho_L(u) = \lambda\rho_K(u)$ for all $u \in S^{n-1}$, that means K and L are dilates. Conversely, if K and L are dilates, it is easy to check that the equality holds. □

By the dual Orlicz–Minkowski inequality for mixed log-volume, we get the following dual Orlicz–Brunn–Minkowski inequality for log-volume.

Theorem 6.3 *Suppose $\alpha, \beta > 0$ and $K, L \in S^n_o$. If $\phi \in \Phi \cup \Psi$, then*

$$\alpha\phi\left(\frac{\mu(K)}{\mu(\alpha \cdot K \tilde{\tau}_\phi \beta \cdot L)}\right) + \beta\phi\left(\frac{\mu(L)}{\mu(\alpha \cdot K \tilde{\tau}_\phi \beta \cdot L)}\right) \leq \phi(1).$$

When μ is the spherical Lebesgue measure of S^{n-1} , the equality holds if and only if K and L are dilates.

Proof Let $K_\phi = \alpha \cdot K \tilde{\tau}_\phi \beta \cdot L$. From Definition 6.1 and Theorem 6.2, it follows that

$$\begin{aligned} \phi(1) &= \frac{1}{|\mu|} \int_{S^{n-1}} \phi(1) d\mu(u) \\ &= \frac{1}{|\mu|} \int_{S^{n-1}} \left[\alpha\phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right) + \beta\phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right) \right] d\mu(u) \\ &= \alpha \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right) d\mu + \beta \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right) d\mu(u) \\ &= \alpha \frac{\mu_\phi(K_\phi, K)}{\mu(K_\phi)} + \beta \frac{\mu_\phi(K_\phi, L)}{\mu(K_\phi)} \\ &\geq \alpha\phi\left(\frac{\mu(K)}{\mu(K_\phi)}\right) + \beta\phi\left(\frac{\mu(L)}{\mu(K_\phi)}\right). \end{aligned}$$

When μ is a spherical Lebesgue measure of S^{n-1} , by the equality condition of Theorem 6.2, we get that the equality holds if and only if K and L are dilates. □

Now we show that the dual Orlicz–Minkowski inequality for the mixed log-volume and the dual Orlicz–Brunn–Minkowski inequality are equivalent.

Theorem 6.4 *The dual Orlicz–Minkowski inequality for the mixed log-volume is equivalent to the dual Orlicz–Brunn–Minkowski inequality for the log-volume.*

Proof We have proved the dual Orlicz–Brunn–Minkowski inequality by the dual Orlicz–Minkowski inequality. Thus, we only need to prove the dual Orlicz–Minkowski inequality by the dual Orlicz–Brunn–Minkowski inequality.

For $\varepsilon \geq 0$, let $K_\varepsilon = K \tilde{+}_{\phi, \varepsilon} L$, by the Orlicz–Brunn–Minkowski inequality, the following function

$$G(\varepsilon) = \phi\left(\frac{\mu(K)}{\mu(K_\varepsilon)}\right) + \varepsilon \phi\left(\frac{\mu(L)}{\mu(K_\varepsilon)}\right) - \phi(1)$$

is nonpositive. Then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{G(\varepsilon) - G(0)}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} \frac{\phi\left(\frac{\mu(K)}{\mu(K_\varepsilon)}\right) + \varepsilon \phi\left(\frac{\mu(L)}{\mu(K_\varepsilon)}\right) - \phi(1)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{\phi\left(\frac{\mu(K)}{\mu(K_\varepsilon)}\right) - \phi(1)}{\varepsilon} + \phi\left(\frac{\mu(L)}{\mu(K_\varepsilon)}\right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{\phi\left(\frac{\mu(K)}{\mu(K_\varepsilon)}\right) - \phi(1)}{\frac{\mu(K)}{\mu(K_\varepsilon)} - 1} \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{\frac{\mu(K)}{\mu(K_\varepsilon)} - 1}{\varepsilon} + \phi\left(\frac{\mu(L)}{\mu(K_\varepsilon)}\right). \end{aligned}$$

Let $s = \frac{\mu(K)}{\mu(K_\varepsilon)}$ and note that $s \rightarrow 1^+$ as $\varepsilon \rightarrow 0^+$. Consequently,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\phi\left(\frac{\mu(K)}{\mu(K_\varepsilon)}\right) - \phi(1)}{\frac{\mu(K)}{\mu(K_\varepsilon)} - 1} = \lim_{\varepsilon \rightarrow 0^+} \frac{\phi(s) - \phi(1)}{s - 1} = \phi'_r(1),$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{\frac{\mu(K)}{\mu(K_\varepsilon)} - 1}{\varepsilon} &= - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\mu(K_\varepsilon)} \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(K_\varepsilon) - \mu(K)}{\varepsilon} \\ &= - \frac{1}{\mu(K)} \cdot \frac{1}{\phi'_r(1)} \cdot \mu_\phi(K, L). \end{aligned}$$

From $G(\varepsilon)$ is nonpositive, we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{G(\varepsilon) - G(0)}{\varepsilon} = - \frac{\mu_\phi(K, L)}{\mu(K)} + \phi\left(\frac{\mu(L)}{\mu(K)}\right) \leq 0.$$

Hence, we have $\mu_\phi(K, L) \geq \mu(K)\phi\left(\frac{\mu(L)}{\mu(K)}\right)$. □

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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