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Iterative algorithms of common solutions for a hierarchical fixed point problem, a system of variational inequalities, and a split equilibrium problem in Hilbert spaces

Yali Zhao[1*], Xin Liu[1] and Ruonan Sun[1]

*Correspondence:
yalizhao2000@163.com
[1]School of Mathematical Science, Bohai University, Liaoning, Jinzhou 121013, China

Abstract

In this paper, we suggest and analyze an iterative algorithm to approximate a common solution of a hierarchical fixed point problem for nonexpansive mappings, a system of variational inequalities, and a split equilibrium problem in Hilbert spaces. Under some suitable conditions imposed on the sequences of parameters, we prove that the sequence generated by the proposed iterative method converges strongly to a common element of the solution set of these three kinds of problems. The results obtained here extend and improve the corresponding results of the relevant literature.

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1 Introduction

Let H_1 and H_2 be two real Hilbert spaces, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. And let C_1 and C_2 be two nonempty closed convex subsets of H_1 and H_2, respectively. Recall that the mapping $T : C_1 \to C_1$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C_1$. We denote the fixed point set of T by $\mathrm{Fix}(T) = \{x \in C_1 : x = Tx\}$. If T is nonexpansive, then $\mathrm{Fix}(T)$ is nonempty, closed, and convex. Next, we consider the following three kinds of problems, which are paid attention to in our paper.

Problem 1 (Hierarchical fixed point problem (HFPP)) In 2006, Moudafi and Mainge [23] introduced and studied the following hierarchical fixed point problem (in short HFPP) for a nonexpansive mapping T with respect to another nonexpansive mapping S on C_1: Find $x \in \mathrm{Fix}(T)$ such that

$$\langle x - Sx, y - x\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T), \tag{1}$$

which amounts to saying that $x \in \mathrm{Fix}(T)$ satisfies the variational inequality depending on a given criterion S, namely, find $x \in C_1$ such that

$$0 \in (I - S)x + N_{\mathrm{Fix}(T)}(x),$$

where I is the identity mapping on C_1 and $N_{\mathrm{Fix}(T)}$ is the normal cone to $\mathrm{Fix}(T)$ at x defined by

$$N_{\mathrm{Fix}(T)}(x) = \begin{cases} \{u \in H_1 : \langle y - x, u\rangle \le 0, \forall y \in \mathrm{Fix}(T)\} & \text{if } x \in \mathrm{Fix}(T), \\ \emptyset & \text{otherwise.} \end{cases}$$

We know that the hierarchical fixed point problem links with some monotone variational inequalities and convex programming problems, see [39] and the references therein. In 2007, Moudafi [22] introduced the following Krasnoselski–Mann algorithm for solving HFPP (1):

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\bigl(\sigma_n S x_n + (1 - \sigma_n)Tx_n\bigr),$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two real sequences in (0,1).

On the other hand, in 2011, Ceng, Anasri, and Yao [8] proposed the following iterative method:

$$x_{n+1} = P_C\bigl[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)\bigl(T(y_n)\bigr)\bigr],$$

where U is a Lipschitzian mapping, and F is a Lipschitzian and strongly monotone mapping. Under some approximate assumptions, they proved that the sequence $\{x_n\}$ generated by the above iterative algorithm converges strongly to the unique solution of the variational inequality

$$\bigl\langle \rho U(x) - \mu F(x), y - x\bigr\rangle \ge 0, \quad \forall y \in \mathrm{Fix}(T). \tag{2}$$

Note that HFPP (2) is more general than HFPP (1).

Problem 2 (Split equilibrium problem (SEP)) Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem(in short, EP) for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \ge 0, \quad \forall y \in C, \tag{3}$$

which was introduced and studied by Blum and Oettli [3]. It contains many problems, such as fixed point problem, variational inequality problem, Nash equilibrium problem, optimization problem, and complementarity problem as special cases, see, e.g., [1, 2, 20, 31] and the references therein. In 1997, Combettes and Hirstoaga [15] introduced an iterative scheme of finding the best approximation to the initial data when a set of solutions (3) is nonempty and proved a strong convergence theorem. We denote the solution set of EP (3) by $EP(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}$.

Recently, Kazmi and Rizvi [21] considered the following split equilibrium problem (in short, SEP): Let $F_1 : C_1 \times C_1 \to R$ and $F_2 : C_2 \times C_2 \to R$ be two nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the SEP is to find $x^* \in C_1$ such that

$$F_1\left(x^*,x\right) \geq 0, \quad \forall x \in C_1 \tag{4}$$

and

$$F_2\left(y^*,y\right) \geq 0, \quad \forall y \in C_2, \tag{5}$$

where $y^* = Ax^* \in C_2$. The solution set of SEP (4)–(5) is denoted by $\Gamma = \{p \in EP(F_1) : Ap \in EP(F_2)\}$. This formalism is also the core of modeling of many inverse problems arising in phase retrieval and other real word problems, for example, in sensor networks in computerized tomography, in intensity-modulated radiation therapy treatment planning, and data compression, see, e.g., [5, 6, 12–14] and the references therein.

Problem 3 (System of variational inequalities (SVI)) Let C_1 be a nonempty closed convex subset of H_1 and $A, B : C_1 \to H_1$ be two mappings. Ceng, Wang, and Yao [11] considered the following problem which finds $(x^*, y^*) \in C_1 \times C_1$ such that

$$\begin{cases} \langle \lambda_1 Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C_1, \\ \langle \lambda_2 Bx^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C_1. \end{cases} \tag{6}$$

Problem (6) is called a general system of variational inequalities, where $\lambda_1 > 0$ and $\lambda_2 > 0$ are constants. In 2015, Jitsupa et al. [19] introduced the following system of variational inequalities in a Hilbert space H_1, that is, finding $x_i^* \in C_1 (i = 1, 2, \dots, N)$ such that

$$\begin{cases} \langle \lambda_N B_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, & \forall x \in C_1, \\ \langle \lambda_{N-1} B_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, & \forall x \in C_1, \\ \vdots & \\ \langle \lambda_2 B_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, & \forall x \in C_1, \\ \langle \lambda_1 B_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, & \forall x \in C_1, \end{cases} \tag{7}$$

which is called a more general system of variational inequalities, where $\lambda_i > 0$ and $B_i : C_1 \to H_1$ is a nonlinear mapping for all $i \in \{1, 2, \dots, N\}$. The solution set of SVI (7) is denoted by $GSVI(C_1, B_i)$.

In view of these different three kinds of problems, there are some new research results on numerical algorithm in the recent literature. Under the setting of uniformly convex Banach spaces, in [27–30], the Thakur three-step iterative process in the context of Suzuki-type nonexpansive mappings or generalized nonexpansive mappings enriched with property (E) was studied, and a comparative numerical experiment was performed with the visualization of some convergence behaviors. In [25], an S-iteration technique for finding common fixed points for nonself quasi-nonexpansive mappings was developed, and convergence properties of the proposed algorithm were analyzed. And in [17], a hybrid projection algorithm for a countable family of mappings was considered, and the strong convergence of the algorithm converging to the common fixed point of the mappings was given. Very recently, Dadashi and Postolache [18] constructed a forward–backward splitting algorithm for approximating a zero of the sum of an α-inverse strongly monotone operator and a maximal monotone operator. They proved the strong convergence theorem under mild conditions. Especially, they added a nonexpansive mapping in the algorithm and proved that the generated sequence converged strongly to a common element of the fixed point set of a nonexpansive mapping and the zero point set of the sum of monotone operators. They also applied their main result both to equilibrium problems and convex programming.

On the other hand, Ceng et al. [9] introduced a hybrid viscosity extragradient method for finding the common elements of the solution set of a general system of variational inequalities and the common fixed point set of a countable family of nonexpansive mappings and zero points of an accretive operator in real smooth Banach spaces. Moreover, they [10] proposed an implicit composite extragradient-like method based on the Mann iteration method, the viscosity approximation method, and the Korpelevich extragradient method for solving a general system of variational inequalities with a hierarchical variational inequality constraint for countably many uniformly Lipschitzian pseudocontractive mappings and an accretive operator in a real Banach space. In [36, 38], Yao, Postolache, and Yao suggested a projected type algorithm and an extragradient algorithm for finding the common solutions of two variational inequalities and the common element of the set of fixed points of a pseudocontractive operator and the set of solutions of the variational inequality problem in Hilbert spaces, respectively. In [35, 37], Yao et al. introduced iterative algorithms for solving a split variational inequality and a fixed point problem that requires finding a solution of a generalized variational inequality whose image is a fixed point of a pseudocontractive operator or a fixed point of two quasi-pseudocontractive operators under a nonlinear transformation in Hilbert spaces. In [33, 34], Yao et al. constructed iterative algorithms for solving the split feasibility problem and the fixed point problem, the split equilibrium problems and fixed point problems involved in the pseudocontractive mappings in Hilbert spaces and proved their strong convergence.

Inspired and motivated by the above research work, we suggest an iterative approximation method for finding an element of the common solution set of HFPP (2), SEP (4)–(5), and SVI (7) involved in nonexpansive mappings. To our best knowledge, there is no further study on finding the element of the common solution set of HFPP (2), SEP (4)–(5), and SVI (7). When the mappings take different types of cases, we can obtain a corollary on the common element of the set of fixed points of a nonexpansive mapping, the solution set of a variational inequality and an equilibrium problem. So, our results presented here are new and very interesting.

The paper is organized as follows. In Sect. 2, we recall some concepts and lemmas which are needed in proving our main results. In Sect. 3, we suggest an iterative algorithm for solving the three different kinds of problems and prove its strong convergence. At last, the conclusion is given.

2 Preliminaries

In this section, we list some fundamental results that are useful in the consequent analysis.

Let H be a real Hilbert space, C be a nonempty closed and convex subset of H.

Then, for all $x, y \in H$, the following inequalities hold:

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \qquad \|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle.$$

A function $F : C \times C \to R$ is called an equilibrium function if it satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;

(A3) $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x,y)$ for all $x, y, z \in C$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous;

(A5) Fix $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H and $x \in C \cap K$ such that

$$F(y,x) + \frac{1}{r}\langle y-x, x-z\rangle < 0, \quad \forall y \in C \backslash K.$$

Lemma 2.1 ([16]) *Assume that $F : C \times C \to R$ is an equilibrium function. For $r > 0$, define a mapping $R_{r,F} : H \to C$ as follows:*

$$R_{r,F}(x) = \left\{ z \in C : F(x,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(B1) *$R_{r,F}$ is single-valued;*

(B2) *Fix $(R_{r,F}) = EP(F)$ and $EP(F)$ is a nonempty closed and convex subset of C;*

(B3) *$R_{r,F}$ is a firmly nonexpansive mapping, i.e.,*

$$\left\| R_{r,F}(x) - R_{r,F}(y) \right\|^2 \le \left\langle R_{r,F}(x) - R_{r,F}(y), x-y \right\rangle, \quad \forall x, y \in H.C.$$

Lemma 2.2 *Let $F : C \times C \to R$ be an equilibrium function, and let $R_{r,F}$ be defined as in Lemma 2.1 for $r > 0$. Let $x, y \in H$ and $r_1, r_2 > 0$, then*

$$\left\| R_{r_2,F}(y) - R_{r_1,F}(x) \right\| \le \|y-x\| + \left| \frac{r_2 - r_1}{r_2} \right| \left\| R_{r_2,F}(y) - y \right\|.$$

Lemma 2.3 ([32]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \delta_n, \quad n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

$$\text{(i)} \quad \sum_{n=1}^{\infty} \alpha_n = \infty; \qquad \text{(ii)} \quad \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0 \quad \textit{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 *Let P_C denote the projection of H onto C. It is known that P_C is nonexpansive and the following inequalities hold*:

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \quad \forall x, y \in H,$$
$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C,$$
$$\big\|(x - y) - (P_C x - P_C y)\big\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Lemma 2.5 *If B is an α-inverse-strongly monotone mapping of C into H, and $\lambda \in [0, 2\alpha]$, then $I - \lambda B$ is a nonexpansive mapping.*

Proof For any $w, u \in C_1$, we have

$$\begin{aligned}
\big\|(I - \lambda B)w - (I - \lambda B)u\big\|^2 &= \big\|(w - u) - \lambda(Bw - Bu)\big\|^2 \\
&= \|w - u\|^2 - 2\lambda\langle Bw - Bu, w - u\rangle + \lambda^2\|Bw - Bu\|^2 \\
&\leq \|w - u\|^2 + \lambda(\lambda - 2\alpha)\|Bw - Bu\|^2 \\
&\leq \|w - u\|^2,
\end{aligned}$$

which implies that $I - \lambda B$ is nonexpansive, completing the proof. □

Lemma 2.6 ([7]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B_i : C \to H$ be an α_i-inverse-strongly monotone mapping, where $i \in \{1, 2, \dots, N\}$. Let $G : C \to C$ be a mapping defined by*

$$G(x) = P_C(I - \lambda_N B_N)P_C(I - \lambda_{N-1}B_{N-1}) \cdots P_C(I - \lambda_2 B_2)P_C(I - \lambda_1 B_1)x, \quad \forall x \in C.$$

If $\lambda_i \in [0, 2\alpha_i]$, $i = 1, 2, \dots, N$, then $G : C \to C$ is nonexpansive.

Proof Putting $T^i = P_C(I - \lambda_i B_i)P_C(I - \lambda_{i-1}B_{i-1}) \cdots P_C(I - \lambda_2 B_2)P_C(I - \lambda_1 B_1)$, $i = 1, 2, \dots, N$, and $T^0 = I$, where I is an identity mapping on C. Then $G = T^N$. For all $x, y \in C$, we have

$$\begin{aligned}
\big\|G(x) - G(y)\big\| &= \big\|T^N(x) - T^N(y)\big\| \\
&= \big\|P_C(I - \lambda_N B_N)T^{N-1}x - P_C(I - \lambda_N B_N)T^{N-1}y\big\| \\
&\leq \big\|(I - \lambda_N B_N)T^{N-1}x - (I - \lambda_N B_N)T^{N-1}y\big\| \\
&\leq \big\|T^{N-1}x - T^{N-1}y\big\| \\
&\vdots \\
&\leq \|x - y\|.
\end{aligned}$$

Then G is nonexpansive, which completes the proof. □

Lemma 2.7 ([8]) *Let $U : C \to H$ be a τ-Lipschitzian mapping, and let $F : C \to H$ be a k-Lipschitzian mapping and η-strongly monotone mapping, then, for $0 \leq \rho\tau < \mu\eta$, $\mu F - \rho U$*

is $(\mu\eta - \rho\tau)$-strongly monotone, i.e.,

$$\langle(\mu F - \rho U)x - (\mu F - \rho U)y, x - y\rangle \geq (\mu\eta - \rho\tau)\|x - y\|^2, \quad \forall x, y \in C.$$

Lemma 2.8 ([26]) *Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone mapping. In association with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda(x) = T(x) - \lambda\mu FT(x), \quad \forall x \in C.$$

Then T^λ is a contractive mapping with $\mu < \frac{2\eta}{k^2}$, that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\nu)\|x - y\|, \quad \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 2.9 ([24]) *Each Hilbert space H satisfies the Opial condition, that is, for any sequence $\{x_n\}$ with x_n converging weakly to x, the inequality $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$ holds for every $y \in H$ with $y \neq x$.*

Lemma 2.10 ([4] Demiclosedness principle) *Let C be a closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, x_n converges weakly to $x, x_n - Tx_n \to 0$ implies $x = Tx$.*

3 Main results

Theorem 3.1 *For $i \in \{1,2\}$, let H_i be a real Hilbert space, C_i be a nonempty closed convex subset of H_i, let $F_i : C_i \times C_i \to R$ be an equilibrium function. Let $A : H_1 \to H_2$ be bounded linear operators with their adjoint operators A^*. Let B_i be ξ_i-inverse-strongly monotone, respectively, where $i \in \{1,2,\ldots,N\}$. Let $F : C_1 \to C_1$ be a k-Lipschitzian mapping and η-strongly monotone, and let $U : C_1 \to C_1$ be a τ-Lipschitzian mapping. Let $S, T : C_1 \to C_1$ be two nonexpansive mappings such that $\Theta = \Gamma \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T) \neq \emptyset$. For a given $x_0 \in C_1$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = R_{r_n,F_1}(x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n), \\ y_n = P_{C_1}(I - \lambda_N B_N)P_{C_1}(I - \lambda_{N-1}B_{N-1})\cdots P_{C_1}(I - \lambda_2 B_2)P_{C_1}(I - \lambda_1 B_1)u_n, \\ z_n = \beta_n Sx_n + (1 - \beta_n)y_n, \\ x_{n+1} = P_{C_1}[\alpha_n\rho U(x_n) + (I - \alpha_n\mu F)(T(z_n))], \end{cases} \tag{8}$$

where $\{r_n\} \subset (0,\infty), \gamma \in (0, 1/L_A), L_A$ is the spectral radius of the operators A^*A. Suppose that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$, $k \geq \eta$, $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k)^2}$, and $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n-1} - \alpha_n| < \infty$;

(ii) $\limsup_{n\to\infty}\frac{\beta_n}{\alpha_n} = 0$, $\beta_n \leq \alpha_n (n \geq 1)$ *and* $\sum_{n=1}^{\infty}|\beta_{n-1} - \beta_n| < \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty}|r_{n-1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ generated by (8) converges strongly to $w \in \Theta$.

Proof Let $p \in \Theta$, i.e., $p \in \Gamma$, that is, $p = R_{r_n,F_1}(p)$ and $Ap = R_{r_n,F_2}(Ap)$. For convenience, we split the proof into several steps.

Step 1. We show that $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ are bounded.

First, by (8) and the expansiveness of R_{r_n,F_1}, we estimate

$$\begin{aligned}
\|u_n - p\|^2 &= \left\|R_{r_n,F_1}\left(x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n\right) - p\right\|^2 \\
&= \left\|R_{r_n,F_1}\left(x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n\right) - R_{r_n,F_1}(p)\right\|^2 \\
&\le \left\|x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n - p\right\|^2 \\
&= \|x_n - p\|^2 + \gamma^2\left\|A^*(R_{r_n,F_2} - I)Ax_n\right\|^2 + 2\gamma\left\langle x_n - p, A^*(R_{r_n,F_2} - I)Ax_n\right\rangle. \qquad (9)
\end{aligned}$$

It follows from the definition of L_A that

$$\begin{aligned}
&\gamma^2\left\|A^*(R_{r_n,F_2} - I)Ax_n\right\|^2 \\
&\quad = \gamma^2\left\langle (R_{r_n,F_2} - I)Ax_n, AA^*(R_{r_n,F_2} - I)Ax_n\right\rangle \\
&\quad \le L_A\gamma^2\left\|(R_{r_n,F_2} - I)Ax_n\right\|^2. \qquad (10)
\end{aligned}$$

By using Lemma 2.4, we have

$$\begin{aligned}
&2\gamma\left\langle x_n - p, A^*(R_{r_n,F_2} - I)Ax_n\right\rangle \\
&\quad = 2\gamma\left\langle A(x_n - p), (R_{r_n,F_2} - I)Ax_n\right\rangle \\
&\quad = 2\gamma\left\langle A(x_n - p) + (R_{r_n,F_2} - I)Ax_n - (R_{r_n,F_2} - I)Ax_n, (R_{r_n,F_2} - I)Ax_n\right\rangle \\
&\quad = 2\gamma\left\{\left\langle R_{r_n,F_2}Ax_n - Ap, (R_{r_n,F_2} - I)Ax_n\right\rangle - \left\|(R_{r_n,F_2} - I)Ax_n\right\|^2\right\} \\
&\quad \le 2\gamma\left\{\frac{1}{2}\left\|(R_{r_n,F_2} - I)Ax_n\right\|^2 - \left\|(R_{r_n,F_2} - I)Ax_n\right\|^2\right\} \\
&\quad = -\gamma\left\|(R_{r_n,F_2} - I)Ax_n\right\|^2. \qquad (11)
\end{aligned}$$

From (9)–(11) and $\gamma \in (0, 1/L_A)$ it follows that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \gamma(L_A\gamma - 1)\left\|(R_{r_n,F_2} - I)Ax_n\right\|^2 \le \|x_n - p\|^2. \qquad (12)$$

It follows from (8), (12), and Lemma 2.6 that we have

$$\|y_n - p\| = \left\|T^N u_n - T^N p\right\| \le \|u_n - p\| \le \|x_n - p\|. \qquad (13)$$

Next, we prove that the sequence $\{x_n\}$ is bounded. Note $\beta_n \le \alpha_n$ for all $n \ge 1$. Put $V_n = \alpha_n\rho U(x_n) + (I - \alpha_n\mu F)(T(z_n))$,

from (8), we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|P_{C_1}\left[\alpha_n\rho U(x_n) + (I - \alpha_n\mu F)\left(T(z_n)\right)\right] - p\right\| \\
&\le \alpha_n\left\|\rho U(x_n) - \mu F(p)\right\| + \left\|(I - \alpha_n\mu F)\left(T(z_n)\right) - (I - \alpha_n\mu F)\left(T(p)\right)\right\| \\
&= \alpha_n\left\|\rho U(x_n) - \rho U(p) + (\rho U - \mu F)(p)\right\|
\end{aligned}$$

$$
\begin{aligned}
&+ \left\|(I-\alpha_n\mu F)\big(T(z_n)\big) - (I-\alpha_n\mu F)\big(T(p)\big)\right\| \\
&\le \alpha_n\rho\tau\|x_n - p\| + \alpha_n\left\|(\rho U - \mu F)(p)\right\| + (1-\alpha_n\nu)\|z_n - p\| \\
&\le \alpha_n\rho\tau\|x_n - p\| + \alpha_n\left\|(\rho U - \mu F)(p)\right\| \\
&\quad + (1-\alpha_n\nu)\left\|\beta_n Sx_n + (1-\beta_n)y_n - p\right\| \\
&\le \alpha_n\rho\tau\|x_n - p\| + \alpha_n\left\|(\rho U - \mu F)(p)\right\| \\
&\quad + (1-\alpha_n\nu)\big(\beta_n\|Sx_n - Sp\| + \beta_n\|Sp - p\| + (1-\beta_n)\|y_n - p\|\big) \\
&\le \alpha_n\rho\tau\|x_n - p\| + \alpha_n\left\|(\rho U - \mu F)(p)\right\| \\
&\quad + (1-\alpha_n\nu)\big(\beta_n\|x_n - p\| + \beta_n\|Sp - p\| + (1-\beta_n)\|x_n - p\|\big) \\
&\le \big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n - p\| + \alpha_n\left\|(\rho U - \mu F)(p)\right\| \\
&\quad + (1-\alpha_n\nu)\beta_n\|Sp - p\| \\
&\le \big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n - p\| + \alpha_n\left\|(\rho U - \mu F)(p)\right\| + \beta_n\|Sp - p\| \\
&\le \big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n - p\| + \alpha_n\big(\left\|(\rho U - \mu F)(p)\right\| + \|Sp - p\|\big) \\
&\le \big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n - p\| + \frac{\alpha_n(\nu-\rho\tau)}{\nu-\rho\tau}\big(\left\|(\rho U - \mu F)(p)\right\| + \|Sp - p\|\big) \\
&\le \max\left\{\|x_0 - p\|, \frac{1}{\nu-\rho\tau}\big(\left\|(\rho U - \mu F)(p)\right\| + \|Sp - p\|\big)\right\}. \qquad (14)
\end{aligned}
$$

So $\{x_n\}$ is bounded, and consequently we can deduce that $\{u_n\}$, $\{y_n\}, \{z_n\}$ are also bounded.

Step 2. We will show the following:

$$
\text{(a)} \quad \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0; \qquad \text{(b)} \quad \lim_{n\to\infty}\|u_n - x_n\| = 0; \qquad \text{(c)} \quad \lim_{n\to\infty}\|u_n - y_n\| = 0.
$$

Noting $u_n = R_{r_n,F_1}(x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n)$ and $u_{n-1} = R_{r_{n-1},F_1}(x_{n-1} + \gamma A^*(R_{r_{n-1},F_2} - I)Ax_{n-1})$, from Lemma 2.2, we have

$$
\begin{aligned}
&\|u_n - u_{n-1}\| \\
&\quad = \|R_{r_n,F_1}v_n - R_{r_{n-1},F_1}v_{n-1}\| \\
&\quad \le \left\|x_n - x_{n-1} + \gamma A^*\big[(R_{r_n,F_2} - I)Ax_n - (R_{r_{n-1},F_2} - I)Ax_{n-1})\big]\right\| \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right|\left\|R_{r_n,F_1}\big(x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n\big) - x_n - \gamma A^*(R_{r_n,F_2} - I)Ax_n\right\| \\
&\quad \le \left\|x_n - x_{n-1} - \gamma A^*A(x_n - x_{n-1})\right\| + \gamma\left\|A^*\right\|\|R_{r_n,F_2}Ax_n - R_{r_{n-1},F_2}Ax_{n-1}\| \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right|\delta_{n-1} \\
&\quad \le \left\{\|x_n - x_{n-1}\|^2 - 2\gamma\|Ax_n - Ax_{n-1}\|^2 + \gamma^2\|A\|^4\|x_n - x_{n-1}\|^2\right\}^{\frac{1}{2}} \\
&\qquad + \gamma\|A\|\left\{\|Ax_n - Ax_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\|R_{r_n,F_2}Ax_n - Ax_n\|\right\} + \left|1 - \frac{r_{n-1}}{r_n}\right|\delta_{n-1} \\
&\quad \le \big(1 - 2\gamma\|A\|^2 + \gamma^2\|A\|^4\big)^{\frac{1}{2}}\|x_n - x_{n-1}\| + \gamma\|A\|^2\|x_n - x_{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|1-\frac{r_{n-1}}{r_n}\right|\gamma\|A\|\sigma_{n-1}+\left|1-\frac{r_{n-1}}{r_n}\right|\delta_{n-1} \\
&=\|x_n-x_{n-1}\|+\left|1-\frac{r_{n-1}}{r_n}\right|\bigl(\gamma\|A\|\sigma_{n-1}+\delta_{n-1}\bigr),
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
&\delta_{n-1}=\bigl\|R_{r_n,F_1}\bigl(x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n\bigr)-\bigl(x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n\bigr)\bigr\|, \\
&\sigma_{n-1}=\|R_{r_n,F_2}Ax_n-Ax_n\|.
\end{aligned}
$$

So, from Lemma 2.6, we have

$$
\begin{aligned}
\|y_n-y_{n-1}\|&=\bigl\|G(u_n)-G(u_{n-1})\bigr\|\le\|u_n-u_{n-1}\| \\
&\le\|x_n-x_{n-1}\|+\left|1-\frac{r_{n-1}}{r_n}\right|\bigl(\gamma\|A\|\sigma_{n-1}+\delta_{n-1}\bigr).
\end{aligned}
\tag{16}
$$

Then from (16) we get

$$
\begin{aligned}
\|z_n-z_{n-1}\|&=\bigl\|\beta_nSx_n+(1-\beta_n)y_n-\beta_{n-1}Sx_{n-1}-(1-\beta_{n-1})y_{n-1}\bigr\| \\
&\le\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\bigl(\|Sx_{n-1}\|+\|y_{n-1}\|\bigr)+(1-\beta_n)\|y_n-y_{n-1}\| \\
&\le\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\bigl(\|Sx_{n-1}\|+\|y_{n-1}\|\bigr) \\
&\quad+(1-\beta_n)\left\{\|x_n-x_{n-1}\|+\left|1-\frac{r_{n-1}}{r_n}\right|\bigl(\gamma\|A\|\sigma_{n-1}+\delta_{n-1}\bigr)\right\} \\
&\le\|x_n-x_{n-1}\|+\left|1-\frac{r_{n-1}}{r_n}\right|\bigl(\gamma\|A\|\sigma_{n-1}+\delta_{n-1}\bigr) \\
&\quad+|\beta_n-\beta_{n-1}|\bigl(\|Sx_{n-1}\|+\|y_{n-1}\|\bigr).
\end{aligned}
\tag{17}
$$

Next, by Lemma 2.8, we estimate

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad=\bigl\|P_C[V_n]-P_C[V_{n-1}]\bigr\| \\
&\quad\le\bigl\|\alpha_n\rho\bigl(U(x_n)-U(x_{n-1})\bigr)+(\alpha_n-\alpha_{n-1})\rho U(x_{n-1})+(I-\alpha_n\mu F)\bigl(T(z_n)\bigr) \\
&\qquad-(I-\alpha_n\mu F)\bigl(T(z_{n-1})\bigr)+(I-\alpha_n\mu F)\bigl(T(z_{n-1})\bigr)-(I-\alpha_{n-1}\mu F)\bigl(T(z_{n-1})\bigr)\bigr\| \\
&\quad\le\alpha_n\rho\tau\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\bigl(\bigl\|\rho U(x_{n-1})\bigr\|+\bigl\|\mu F\bigl(T(z_{n-1})\bigr)\bigr\|\bigr) \\
&\qquad+(1-\alpha_n\nu)\|z_n-z_{n-1}\|.
\end{aligned}
\tag{18}
$$

From (17) and (18), we get

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad\le\alpha_n\rho\tau\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\bigl(\bigl\|\rho U(x_{n-1})\bigr\|+\bigl\|\mu F\bigl(T(z_{n-1})\bigr)\bigr\|\bigr) \\
&\qquad+(1-\alpha_n\nu)\left\{\|x_n-x_{n-1}\|+\left|1-\frac{r_{n-1}}{r_n}\right|\bigl(\gamma\|A\|\sigma_{n-1}+\delta_{n-1}\bigr)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + |\beta_n - \beta_{n-1}|\bigl(\|Sx_{n-1}\| + \|z_{n-1}\|\bigr)\Bigr\} \\
&\le \bigl(1-(\nu-\rho\tau)\alpha_n\bigr)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\bigl(\bigl\|\rho U(x_{n-1})\bigr\| + \bigl\|\mu F\bigl(T(z_{n-1})\bigr)\bigr\|\bigr) \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right|\bigl(\gamma\|A\|\sigma_{n-1} + \delta_{n-1}\bigr) + |\beta_n - \beta_{n-1}|\bigl(\|Sx_{n-1}\| + \|z_{n-1}\|\bigr) \\
&\le \bigl(1-(\nu-\rho\tau)\alpha_n\bigr)\|x_n - x_{n-1}\| + M\left(|\alpha_n - \alpha_{n-1}| + \frac{1}{\varepsilon}|r_{n-1} - r_n| + |\beta_n - \beta_{n-1}|\right), \qquad (19)
\end{aligned}
$$

where $M = \max\{\sup_{n\ge 1}(\|\rho U(x_{n-1})\| + \|\mu F(T(z_{n-1}))\|), \sup_{n\ge 1}(\gamma\|A\|\sigma_{n-1} + \delta_{n-1}), \sup_{n\ge 1}(\|Sx_{n-1}\| + \|z_{n-1}\|)\}$. And ε is a real number such that $0 < \varepsilon < r_n$. So, it follows from Conditions (i)–(iii) and Lemma 2.3 that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (20)
$$

Next, we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. In view of (8), (9), (12), and (13), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \bigl\langle P_C[V_n] - p, x_{n+1} - p\bigr\rangle \\
&= \bigl\langle P_C[V_n] - V_n, P_C[V_n] - p\bigr\rangle + \langle V_n - p, x_{n+1} - p\rangle \\
&\le \bigl\langle \alpha_n\bigl(\rho U(x_n) - \mu F(p)\bigr) + (I - \alpha_n\mu F)\bigl(T(z_n)\bigr) \\
&\quad - (I - \alpha_n\mu F)\bigl(T(p)\bigr), x_{n+1} - p\bigr\rangle \\
&= \bigl\langle \alpha_n\rho\bigl(U(x_n) - U(p)\bigr), x_{n+1} - p\bigr\rangle + \alpha_n\bigl\langle \rho U(p) - \mu F(p), x_{n+1} - p\bigr\rangle \\
&\quad + \bigl\langle (I - \alpha_n\mu F)\bigl(T(z_n)\bigr) - (I - \alpha_n\mu F)\bigl(T(p)\bigr), x_{n+1} - p\bigr\rangle \\
&\le \alpha_n\rho\tau\|x_n - p\|\|x_{n+1} - p\| + \alpha_n\bigl\langle \rho U(p) - \mu F(p), x_{n+1} - p\bigr\rangle \\
&\quad + (1 - \alpha_n\nu)\|z_n - p\|\|x_{n+1} - p\| \\
&\le \frac{\alpha_n\rho\tau}{2}\bigl(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\bigr) + \alpha_n\bigl\langle \rho U(p) - \mu F(p), x_{n+1} - p\bigr\rangle \\
&\quad + \frac{(1 - \alpha_n\nu)}{2}\bigl(\|z_n - p\|^2 - \|x_{n+1} - p\|^2\bigr) \\
&\le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - p\|^2 + \alpha_n\bigl\langle \rho U(p) - \mu F(p), x_{n+1} - p\bigr\rangle \\
&\quad + \frac{\alpha_n\rho\tau}{2}\|x_n - p\|^2 + \frac{(1 - \alpha_n\nu)}{2}\|z_n - p\|^2 \\
&\le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - p\|^2 + \alpha_n\bigl\langle \rho U(p) - \mu F(p), x_{n+1} - p\bigr\rangle \\
&\quad + \frac{\alpha_n\rho\tau}{2}\|x_n - p\|^2 + \frac{(1 - \alpha_n\nu)}{2}\bigl(\beta_n\|Sx_n - p\|^2 + (1 - \beta_n)\|y_n - p\|^2\bigr). \qquad (21)
\end{aligned}
$$

From the above inequality and (12), (13), we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \frac{\alpha_n\rho\tau}{1 + \alpha_n(\nu - \rho\tau)}\|x_n - p\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho\tau)}\bigl\langle \rho U(p) - \mu F(p), x_{n+1} - p\bigr\rangle \\
&\quad + \frac{(1 - \alpha_n\nu)\beta_n}{1 + \alpha_n(\nu - \rho\tau)}\|Sx_n - p\|^2 \\
&\quad + \frac{(1 - \alpha_n\nu)(1 - \beta_n)}{1 + \alpha_n(\nu - \rho\tau)}\bigl\{\|x_n - p\|^2 + \gamma(L_A\gamma - 1)\bigl\|(R_{r_n,F_2} - I)Ax_n\bigr\|^2\bigr\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\alpha_n \rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(p)-\mu F(p), x_{n+1}-p\rangle \\
&\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2 + \|x_n-p\|^2 \\
&\quad + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\bigl\{\gamma(L_A\gamma-1)\bigl\|(R_{r_n,F_2}-I)Ax_n\bigr\|^2\bigr\},
\end{aligned} \tag{22}$$

which means that

$$\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\bigl\{\gamma(1-L_A\gamma)\bigl\|(R_{r_n,F_2}-I)Ax_n\bigr\|^2\bigr\} \\
&\quad\leq \frac{\alpha_n \rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(p)-\mu F(p), x_{n+1}-p\rangle \\
&\quad\quad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2 + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\quad\leq \frac{\alpha_n \rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(p)-\mu F(p), x_{n+1}-p\rangle \\
&\quad\quad + \beta_n\|Sx_n-p\|^2 + \bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr)\|x_{n+1}-x_n\|.
\end{aligned} \tag{23}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, we obtain

$$\lim_{n\to\infty}\bigl\|(R_{r_n,F_2}-I)Ax_n\bigr\| = 0.$$

And since R_{r_n,F_1} is firmly nonexpansive, from (8) we get

$$\begin{aligned}
&\|u_n-p\|^2 \\
&\quad= \bigl\|R_{r_n,F_1}\bigl(x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n\bigr)-p\bigr\|^2 \\
&\quad= \bigl\|R_{r_n,F_1}\bigl(x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n\bigr)-R_{r_n,F_1}(p)\bigr\|^2 \\
&\quad\leq \langle u_n-p, x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n-p\rangle \\
&\quad= \frac{1}{2}\bigl\{\|u_n-p\|^2 + \bigl\|x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n-p\bigr\|^2 \\
&\quad\quad - \bigl\|u_n-p-\bigl[x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n-p\bigr]\bigr\|^2\bigr\} \\
&\quad= \frac{1}{2}\bigl\{\|u_n-p\|^2 + \bigl\|x_n+\gamma A^*(R_{r_n,F_2}-I)Ax_n-p\bigr\|^2 \\
&\quad\quad - \bigl\|u_n-x_n-\gamma A^*(R_{r_n,F_2}-I)Ax_n\bigr\|^2\bigr\} \\
&\quad= \frac{1}{2}\bigl\{\|u_n-p\|^2 + \|x_n-p\|^2 + 2\gamma\langle x_n-p, A^*(R_{r_n,F_2}-I)Ax_n\rangle \\
&\quad\quad + \gamma^2\bigl\|A^*(R_{r_n,F_2}-I)Ax_n\bigr\|^2 \\
&\quad\quad - \bigl[\|u_n-x_n\|^2 - 2\gamma\langle u_n-x_n, A^*(R_{r_n,F_2}-I)Ax_n\rangle + \gamma^2\bigl\|A^*(R_{r_n,F_2}-I)Ax_n\bigr\|^2\bigr]\bigr\} \\
&\quad= \frac{1}{2}\bigl\{\|u_n-p\|^2 + \|x_n-p\|^2 + 2\gamma\langle u_n-p, A^*(R_{r_n,F_2}-I)Ax_n\rangle - \|u_n-x_n\|^2\bigr\},
\end{aligned} \tag{24}$$

which implies that

$$\|u_n-p\|^2 \leq \|x_n-p\|^2 - \|u_n-x_n\|^2 + 2\gamma\bigl\|A(u_n-p)\bigr\|\bigl\|(R_{r_n,F_2}-I)Ax_n\bigr\|. \tag{25}$$

So, from (21) and (25) we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1}-p\|^2+\alpha_n\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle \\
&\qquad+\frac{\alpha_n\rho\tau}{2}\|x_n-p\|^2+\frac{(1-\alpha_n\nu)}{2}\big(\beta_n\|Sx_n-p\|^2+(1-\beta_n)\|u_n-p\|^2\big) \\
&\quad\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1}-p\|^2+\alpha_n\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle+\frac{\alpha_n\rho\tau}{2}\|x_n-p\|^2 \\
&\qquad+\frac{(1-\alpha_n\nu)}{2}\big\{\beta_n\|Sx_n-p\|^2+(1-\beta_n)\big(\|x_n-p\|^2-\|u_n-x_n\|^2 \\
&\qquad+2\gamma\big\|A(u_n-p)\big\|\big\|(R_{r_n,F_2}-I)Ax_n\big\|\big)\big\},
\end{aligned}\tag{26}$$

which implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle \\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\big\{\|x_n-p\|^2-\|u_n-x_n\|^2 \\
&\qquad+2\gamma\big\|A(u_n-p)\big\|\big\|(R_{r_n,F_2}-I)Ax_n\big\|\big\} \\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle \\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\|x_n-p\|^2 \\
&\qquad+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\big\{-\|u_n-x_n\|^2+2\gamma\big\|A(u_n-p)\big\|\big\|(R_{r_n,F_2}-I)Ax_n\big\|\big\}.
\end{aligned}\tag{27}$$

Hence

$$\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\|u_n-x_n\|^2 \\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle \\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\qquad+\frac{2(1-\alpha_n\nu)(1-\beta_n)\gamma}{1+\alpha_n(\nu-\rho\tau)}\big\|A(u_n-p)\big\|\big\|(R_{r_n,F_2}-I)Ax_n\big\| \\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle \\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_{n+1}-x_n\|.
\end{aligned}\tag{28}$$

Since $\lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\beta_n = 0, \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, and $\lim_{n\to\infty}\|(R_{r_n,F_2} - I)Ax_n\| = 0$, we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0.$$

Then, by Lemma 2.5 and Lemma 2.6, we obtain

$$\begin{aligned}
&\|T^N u_n - T^N p\|^2 \\
&\quad = \|P_{C_1}(I-\lambda_N B_N)T^{N-1}u_n - P_{C_1}(I-\lambda_N B_N)T^{N-1}p\|^2 \\
&\quad \le \|(I-\lambda_N B_N)T^{N-1}u_n - (I-\lambda_N B_N)T^{N-1}p\|^2 \\
&\quad \le \|T^{N-1}u_n - T^{N-1}p\|^2 + \lambda_N(\lambda_N - 2\xi_N)\|B_N T^{N-1}u_n - B_N T^{N-1}p\|^2 \\
&\quad \le \|u_n - p\|^2 + \sum_{i=1}^{N}\lambda_i(\lambda_i - 2\xi_i)\|B_i T^{i-1}u_n - B_i T^{i-1}p\|^2 \\
&\quad \le \|x_n - p\|^2 + \sum_{i=1}^{N}\lambda_i(\lambda_i - 2\xi_i)\|B_i T^{i-1}u_n - B_i T^{i-1}p\|^2.
\end{aligned} \tag{29}$$

From (21), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n - p\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(p) - \mu F(p), x_{n+1} - p\rangle \\
&\qquad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n - p\|^2 \\
&\qquad + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\left\{\|x_n - p\|^2 + \sum_{i=1}^{N}\lambda_i(\lambda_i - 2\xi_i)\|B_i T^{i-1}u_n - B_i T^{i-1}p\|^2\right\} \\
&\quad \le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n - p\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(p) - \mu F(p), x_{n+1} - p\rangle \\
&\qquad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n - p\|^2 \\
&\qquad + \|x_n - p\|^2 + \frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\left\{\sum_{i=1}^{N}\lambda_i(\lambda_i - 2\xi_i)\|B_i T^{i-1}u_n - B_i T^{i-1}p\|^2\right\},
\end{aligned} \tag{30}$$

which implies that

$$\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\left\{\sum_{i=1}^{N}\lambda_i(2\xi_i - \lambda_i)\|B_i T^{i-1}u_n - B_i T^{i-1}p\|^2\right\} \\
&\quad \le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n - p\|^2 + \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(p) - \mu F(p), x_{n+1} - p\rangle \\
&\qquad + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad \le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n - p\|^2
\end{aligned}$$

$$+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle$$
$$+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|. \quad (31)$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\beta_n=0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we have

$$\lim_{n\to\infty}\left\|B_iT^{i-1}u_n-B_iT^{i-1}p\right\|=0.$$

By Lemma 2.4, we obtain

$$\begin{aligned}
&\|y_n-p\|^2\\
&\quad=\left\|T^Nu_n-T^Np\right\|^2\\
&\quad=\left\|P_C(I-\lambda_NB_N)T^{N-1}u_n-P_C(I-\lambda_NB_N)T^{N-1}p\right\|^2\\
&\quad\le\langle(I-\lambda_NB_N)T^{N-1}u_n-(I-\lambda_NB_N)T^{N-1}p,T^Nu_n-T^Np\rangle\\
&\quad=\frac{1}{2}\big(\|y_n-p\|^2+\left\|(I-\lambda_NB_N)T^{N-1}u_n-(I-\lambda_NB_N)T^{N-1}p\right\|^2\\
&\qquad-\left\|(I-\lambda_NB_N)T^{N-1}u_n-(I-\lambda_NB_N)T^{N-1}p-\big(T^Nu_n-T^Np\big)\right\|^2\big)\\
&\quad\le\frac{1}{2}\big(\|y_n-p\|^2+\left\|T^{N-1}u_n-T^{N-1}p\right\|^2\\
&\qquad-\left\|T^{N-1}u_n-T^Nu_n+T^Np-T^{N-1}p-\lambda_N\big(B_NT^{N-1}u_n-B_NT^{N-1}p\big)\right\|^2\big),
\end{aligned} \quad (32)$$

which implies

$$\begin{aligned}
&\|y_n-p\|^2\\
&\quad\le\left\|T^{N-1}u_n-T^{N-1}p\right\|^2\\
&\qquad-\left\|T^{N-1}u_n-T^Nu_n+T^Np-T^{N-1}p-\lambda_N\big(B_NT^{N-1}u_n-B_NT^{N-1}p\big)\right\|^2\\
&\quad=\left\|T^{N-1}u_n-T^{N-1}p\right\|^2-\left\|T^{N-1}u_n-T^Nu_n+T^Np-T^{N-1}p\right\|^2\\
&\qquad-\lambda_N^2\left\|B_NT^{N-1}u_n-B_NT^{N-1}p\right\|^2\\
&\qquad+2\lambda_N\langle T^{N-1}u_n-T^Nu_n+T^Np-T^{N-1}p,B_NT^{N-1}u_n-B_NT^{N-1}p\rangle\\
&\quad\le\left\|T^{N-1}u_n-T^{N-1}p\right\|^2-\left\|T^{N-1}u_n-T^Nu_n+T^Np-T^{N-1}p\right\|^2\\
&\qquad+2\lambda_N\left\|T^{N-1}u_n-T^Nu_n+T^Np-T^{N-1}p\right\|\left\|B_NT^{N-1}u_n-B_NT^{N-1}p\right\|.
\end{aligned} \quad (33)$$

By induction and (12), we have

$$\begin{aligned}
\|y_n-p\|^2\le{}&\|x_n-p\|^2-\sum_{i=1}^N\left\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\right\|^2\\
&+\sum_{i=1}^N2\lambda_i\left\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\right\|\left\|B_iT^{i-1}u_n-B_iT^{i-1}p\right\|.
\end{aligned} \quad (34)$$

It follows from (21) and (34) that we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle\\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\Biggl\{\|x_n-p\|^2\\
&\qquad-\sum_{i=1}^{N}\bigl\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\bigr\|^2\\
&\qquad+\sum_{i=1}^{N}2\lambda_i\bigl\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\bigr\|\bigl\|B_iT^{i-1}u_n-B_iT^{i-1}p\bigr\|\Biggr\}\\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle\\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\|x_n-p\|^2\\
&\qquad+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\Biggl\{-\sum_{i=1}^{N}\bigl\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\bigr\|^2\\
&\qquad+\sum_{i=1}^{N}2\lambda_i\bigl\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\bigr\|\bigl\|B_iT^{i-1}u_n-B_iT^{i-1}p\bigr\|\Biggr\},
\end{aligned}\tag{35}$$

which implies

$$\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\Biggl\{\sum_{i=1}^{N}\bigl\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\bigr\|^2\Biggr\}\\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle\\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-p\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(p)-\mu F(p),x_{n+1}-p\rangle\\
&\qquad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-p\|^2+\bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr)\|x_n-x_{n+1}\|\\
&\qquad+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\Biggl\{\sum_{i=1}^{N}2\lambda_i\bigl\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\bigr\|\\
&\qquad\times\bigl\|B_iT^{i-1}u_n-B_iT^{i-1}p\bigr\|\Biggr\}.
\end{aligned}\tag{36}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\beta_n=0$ and $\lim_{n\to\infty}\|B_iT^{i-1}u_n-B_iT^{i-1}p\|^2=0$, we have

$$\lim_{n\to\infty}\bigl\|T^{i-1}u_n-T^iu_n+T^ip-T^{i-1}p\bigr\|=0.\tag{37}$$

From (37), we obtain

$$\|u_n - y_n\| = \left\| T^0 u_n - T^N u_n \right\| \leq \sum_{i=1}^{N} \left\| T^{i-1} u_n - T^i u_n + T^i p - T^{i-1} p \right\|, \tag{38}$$

which means $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Note $\lim_{n\to\infty} \|u_n - x_n\| = 0$, $\lim_{n\to\infty} \|u_n - y_n\| = 0$, then we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since $T(x_n) \in C_1$, we have

$$\begin{aligned}
\left\|x_n - T(x_n)\right\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T(x_n)\right\| \\
&= \|x_n - x_{n+1}\| + \left\|P_{C_1}[V_n] - P_{C_1}\left[T(x_n)\right]\right\| \\
&\leq \|x_n - x_{n+1}\| + \left\|\alpha_n(\rho U(x_n) - \mu F\left(T(y_n)\right) + T(y_n) - T(x_n)\right\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \left\|\rho U(x_n) - \mu F(T(y_n)\right\| + \|y_n - x_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \left\|\rho U(x_n) - \mu F(T(y_n)\right\| + \left\|\beta_n S x_n + (1-\beta_n) y_n - x_n\right\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n \left\|\rho U(x_n) - \mu F(T(y_n)\right\| \\
&\quad + \beta_n \|S x_n - x_n\| + (1-\beta_n)\|y_n - x_n\|.
\end{aligned}$$

Noting that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \|x_n - y_n\| = 0$,and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Step 3. We show that $z \in F(T)$. Assume that $z \notin F(T)$. Since x_{n_i} converges weakly to z and $Tz \neq z$, by Lemma 2.9, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \|x_{n_i} - z\| \\
&\quad < \liminf_{n\to\infty} \|x_{n_i} - Tz\| \leq \liminf_{n\to\infty} \left(\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tz\|\right) \leq \liminf_{n\to\infty} \|x_{n_i} - z\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $z \in F(T)$. To prove the convergence of the sequence $\{x_n\}$, we need to prove the following conclusion, that is, the sequence $\{x_n\}$ generated by (8) converges strongly to w, which is the unique solution of the variational inequality

$$\left\langle \rho U(w) - \mu F(w), x - w \right\rangle \leq 0, \quad \forall x \in \Theta.$$

In fact, noting that $u_n = R_{r_n,F_1}(x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n$ and

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\left\langle y - u_n, \gamma A^*(R_{r_n,F_2} - I)Ax_n\right\rangle \geq 0, \quad \forall y \in C_1.$$

From the monotonicity of F_1, we have

$$-\frac{1}{r_n}\left\langle y - u_n, \gamma A^*(R_{r_n,F_2} - I)Ax_n\right\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n), \quad \forall y \in C_1,$$

and

$$-\frac{1}{r_{n_i}}\left\langle y - u_{n_i}, \gamma A^*(R_{r_{n_i},F_2} - I)Ax_{n_i}\right\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F_1(y, u_{n_i}), \quad \forall y \in C_1.$$

Since $\|u_n - x_n\| \to 0$, $\|(R_{r_n,F_2} - I)Ax_n\| \to 0$, we get $\{u_{n_i}\}$ converges weakly to z. By (A4), we know $F_1(y,z) \leq 0$, $\forall y \in C_1$. Let $y_t = ty + (1-t)z$, $t \in (0,1]$, it follows from $y \in C_1$, $z \in C_1$ and the convexity of C_1 that $F_1(y_t,z) \leq 0$. So, from (A1), (A3), and (A4), we have

$$0 = F_1(y_t,y_t) \leq tF_1(y_t,y) + (1-t)F_1(y_t,z) \leq F_1(y_t,y).$$

Therefore $F_1(z,y) \geq 0, \forall y \in C_1$. This is $z \in EP(F_1)$.

Next we show that $Az \in EP(F_2)$, since $\|u_n - x_n\| \to 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to z, and since A is a bounded linear operator, $\{Ax_{n_k}\}$ converges weakly to Az. Setting $\varpi_{n_k} = Ax_{n_k} - R_{r_{n_k},F_2}Ax_{n_k}$, it follows tfrom $\lim_{n\to\infty} \|(R_{r_n,F_2} - I)Ax_n\| = 0$ that $\lim_{k\to\infty} \varpi_{n_k} = 0$. By Lemma 2.1, we have

$$F_2(Ax_{n_k} - \varpi_{n_k}, y) + \frac{1}{r_{n_k}}\bigl\langle y - (Ax_{n_k} - \varpi_{n_k}), (Ax_{n_k} - \varpi_{n_k}) - Ax_{n_k}\bigr\rangle \geq 0, \quad \forall y \in C_2.$$

Since F_2 is upper semicontinuous in the first argument, taking limsup to the above inequality as $k \to \infty$, we have $F_2(Az,y) \geq 0$, $\forall y \in C_2$, which means that $Az \in EP(F_2)$, so $z \in \Gamma$. Next, we claim that $z \in \operatorname{Fix}(G)$. From Lemma 2.6, we know $G = T^N$ is nonexpansive, and

$$\|y_n - Gy_n\| = \bigl\|T^N u_n - T^N y_n\bigr\| \leq \|u_n - y_n\|.$$

It follows from $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ that $\lim_{n\to\infty} \|y_n - Gy_n\| = 0$. Furthermore, we get

$$\begin{aligned}\|x_n - Gx_n\| &\leq \|x_n - y_n\| + \|y_n - Gy_n\| + \|Gy_n - Gx_n\| \\ &\leq 2\|x_n - y_n\| + \|y_n - Gy_n\|,\end{aligned}$$

which implies $\lim_{n\to\infty} \|x_n - Gx_n\| = 0$. Then, by Lemma 2.10, we obtain $z \in \operatorname{Fix}(G)$. Thus, we have $z \in \Theta$. Observe that the constants satisfy $0 \leq \rho\tau < \nu$ and $k \geq \eta$, from Lemma 2.7, the operator $\mu F - \rho U$ is $\mu\eta - \rho\tau$ strongly monotone. Then we get the uniqueness of the solution of the variational inequality and denote it by $w \in \Theta$.

Last, we show that $x_n \to w$. Note that

$$\begin{aligned}&\limsup_{n\to\infty}\bigl\langle \rho U(w) - \mu F(w), x_n - w\bigr\rangle \\ &\quad= \limsup_{i\to\infty}\bigl\langle \rho U(w) - \mu F(w), x_{n_i} - w\bigr\rangle = \bigl\langle \rho U(w) - \mu F(w), z - w\bigr\rangle \leq 0,\end{aligned}$$

and

$$\begin{aligned}&\|x_{n+1} - w\|^2 \\ &\quad= \bigl\langle P_C[V_n] - w, x_{n+1} - w\bigr\rangle \\ &\quad= \bigl\langle P_C[V_n] - V_n, P_C[V_n] - w\bigr\rangle + \langle V_n - w, x_{n+1} - w\rangle \\ &\quad\leq \bigl\langle \alpha_n\bigl(\rho U(x_n) - \mu F(w)\bigr) + (I - \alpha_n\mu F)\bigl(T(z_n)\bigr) - (I - \alpha_n\mu F)\bigl(T(w)\bigr), x_{n+1} - w\bigr\rangle \\ &\quad= \bigl\langle \alpha_n\rho\bigl(U(x_n) - U(w)\bigr), x_{n+1} - w\bigr\rangle + \alpha_n\bigl\langle \rho U(w) - \mu F(w), x_{n+1} - w\bigr\rangle\end{aligned}$$

$$
\begin{aligned}
&+\big\langle (I-\alpha_n\mu F)\big(T(z_n)\big)-(I-\alpha_n\mu F)\big(T(w)\big),x_{n+1}-w\big\rangle \\
&\le \alpha_n\rho\tau\|x_n-w\|\|x_{n+1}-w\|+\alpha_n\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+(1-\alpha_n\nu)\|z_n-w\|\|x_{n+1}-w\| \\
&\le \alpha_n\rho\tau\|x_n-w\|\|x_{n+1}-w\|+\alpha_n\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+(1-\alpha_n\nu)\big\{\beta_n\|Sx_n-Sw\|+\beta_n\|Sw-w\|+(1-\beta_n)\|y_n-w\|\big\}\|x_{n+1}-w\| \\
&\le \alpha_n\rho\tau\|x_n-w\|\|x_{n+1}-w\|+\alpha_n\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+(1-\alpha_n\nu)\big\{\beta_n\|x_n-w\|+\beta_n\|Sw-w\|+(1-\beta_n)\|x_n-w\|\big\}\|x_{n+1}-w\| \\
&=\big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n-w\|\|x_{n+1}-w\|+\alpha_n\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+(1-\alpha_n\nu)\beta_n\|Sw-w\|\|x_{n+1}-w\| \\
&\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\big(\|x_n-w\|^2+\|x_{n+1}-w\|^2\big)+\alpha_n\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+(1-\alpha_n\nu)\beta_n\|Sw-w\|\|x_{n+1}-w\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-w\|^2 &\le \frac{1-\alpha_n(\nu-\rho\tau)}{1+\alpha_n(\nu-\rho\tau)}\|x_n-w\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+\frac{2(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sw-w\|\|x_{n+1}-w\| \\
&\le \big(1-\alpha_n(\nu-\rho\tau)\big)\|x_n-w\|^2 \\
&+\frac{2\alpha_n(\nu-\rho\tau)}{1+\alpha_n(\nu-\rho\tau)}\bigg\{\frac{1}{\nu-\rho\tau}\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+\frac{(1-\alpha_n\nu)\beta_n}{\alpha_n(\nu-\rho\tau)}\|Sw-w\|\|x_{n+1}-w\|\bigg\}.
\end{aligned}
$$

Let $\sigma_n=\|x_n-w\|^2,\phi_n=\alpha_n(\nu-\rho\tau)$ and

$$
\begin{aligned}
\varphi_n=&\frac{2\alpha_n(\nu-\rho\tau)}{1+\alpha_n(\nu-\rho\tau)}\bigg\{\frac{1}{\nu-\rho\tau}\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+\frac{(1-\alpha_n\nu)\beta_n}{\alpha_n(\nu-\rho\tau)}\|Sw-w\|\|x_{n+1}-w\|\bigg\}.
\end{aligned}
$$

Then the above inequality turns into the following:

$$
\sigma_{n+1}=(1-\phi_n)\sigma_n+\varphi_n.
$$

From Conditions (i) and (ii) of Theorem 3.1, we have

$$
\phi_n\to 0(n\to\infty)\quad\text{and}
$$

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{\varphi_n}{\phi_n}=\limsup_{n\to\infty}&\frac{2}{1+\alpha_n(\nu-\rho\tau)}\bigg\{\frac{1}{\nu-\rho\tau}\big\langle \rho U(w)-\mu F(w),x_{n+1}-w\big\rangle \\
&+\frac{(1-\alpha_n\nu)\beta_n}{\alpha_n(\nu-\rho\tau)}\|Sw-w\|\|x_{n+1}-w\|\bigg\}\le 0.
\end{aligned}
$$

Then all conditions in Lemma 2.3 are satisfied, thus we can get $\sigma_n \to 0$ $(n \to \infty)$, that is, $x_n \to w$ $(n \to \infty)$. This completes the proof. □

Corollary 3.1 *For $i \in \{1, 2\}$, let H_i be a real Hilbert space, C_i be a nonempty closed convex subset of H_i, let $F_i : C_i \times C_i \to R$ be an equilibrium function. Let $A : H_1 \to H_2$ be bounded linear operators with their adjoint operators A^*. Let B_1 be ξ_1-inverse-strongly monotone. Let $F : C_1 \to C_1$ be a k-Lipschitzian mapping and be η-strongly monotone, and let $U : C_1 \to C_1$ be a τ-Lipschitzian mapping. Let $S, T : C_1 \to C_1$ be two nonexpansive mappings such that $\Theta = \Gamma \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T) \neq \emptyset$. For given $x_0 \in C_1$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = R_{r_n,F_1}(x_n + \gamma A^*(R_{r_n,F_2} - I)Ax_n), \\ y_n = P_{C_1}(I - \lambda_1 B_1)u_n, \\ z_n = \beta_n Sx_n + (1 - \beta_n)y_n, \\ x_{n+1} = P_{C_1}[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(z_n))], \end{cases} \tag{39}$$

where $\{r_n\} \subset (0, \infty)$, $\gamma \in (0, 1/L_A)$, L_A is the spectral radius of the operators A^*A. Suppose that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$, $k \geq \eta$, $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k)^2}$, and $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$;

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$, *and* $\beta_n \leq \alpha_n (n \geq 1)$, $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ generated by (39) converges strongly to $w \in \Theta$.

Proof Putting $N = 1$ in Theorem 3.1, we can conclude the desired conclusion directly. □

4 Conclusion

In this paper, we considered a hierarchical fixed point problem (2), a split equilibrium problem (4)–(5), and a system of variational inequalities (7) in Hilbert spaces. An iterative algorithm for finding the common element of the solution sets of the three kinds of problems is presented. Strong convergence of the proposed algorithm is proved. The results presented here are new and very interesting.

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