# Generalizations of some integral inequalities related to Hardy type integral inequalities via ( $p, q$ )-calculus 

Suriyakamol Thongjob ${ }^{1}$ © ${ }^{(C)}$ Kamsing Nonlaopon ${ }^{1 *}$ © $\mathbb{C}$, Jessada Tariboon ${ }^{2}$ © and Sortiris K. Ntouyas ${ }^{3,4}$ ©

Correspondence: nkamsi@kku.ac.th ${ }^{1}$ Department of Mathematics, Khon Kaen University, 40002, Khon Kaen, Thailand
Full list of author information is available at the end of the article


#### Abstract

In this paper, we study generalizations of some integral inequalities related to Hardy type integral inequalities via ( $p, q$ )-calculus. Many results obtained in this paper provide extensions of existing results in the literature. Furthermore, some examples are given to illustrate the investigated results.


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## 1 Introduction

Quantum calculus, shortly $q$-calculus, is the study of calculus without limits. The history of $q$-calculus can be dated back to Euler, who first introduced $q$-calculus in the track of Newton's work on infinite series. In 1910, F. H. Jackson [1] defined the definite $q$-integral, which is known as the $q$-Jackson integral. It was the starting point of $q$-calculus in a systematic way. In recent years, $q$-calculus has been actively developed and many researchers have been increasingly interested in the topic of $q$-calculus due to applications of the $q$ calculus in mathematics and physics such as combinatorics, dynamical systems, fractals, number theory, orthogonal polynomials, special functions, mechanics and also for scientific problems in some applied areas, see [2-16] for more details.

The subject $(p, q)$-calculus is a generalization of $q$-calculus and it is two parameters quantum calculus. In 2013, P.N. Sadjang [17] studied the ( $p, q$ )-derivative, the $(p, q)$ integral, and obtained some of their properties and the fundamental of $(p, q)$-calculus. Recently, M. Tunç, and E. Göv [18] defined the ( $p, q$ )-derivative and ( $p, q$ )-integral on finite interval. The applications of $(p, q)$-calculus play important roles in physical sciences, number theory, orthogonal polynomials, see [19-23] for more details. Furthermore, they studied some properties of $(p, q)$-calculus and $(p, q)$-analogue of some important integral inequalities. The $(p, q)$-integral inequalities have been studied and rapidly developed during this period by many authors, see [24-31] and the references therein.

Mathematical inequalities have been applied in various branches of mathematics like analysis, differential equations, and geometry. One of the famous inequalities is the Hardy

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inequality. Let us just mention that in 1920, G. H. Hardy [32] presented the following famous inequality for $f$ being a non-negative integrable function and $s>1$ :
\[

$$
\begin{equation*}
\int_{0}^{\infty}\left(\frac{1}{x} \int_{0}^{x} f(t) d t\right)^{s} d x \leq\left(\frac{s}{s-1}\right)^{s} \int_{0}^{\infty} f^{s}(x) d x \tag{1.1}
\end{equation*}
$$

\]

which is now known as Hardy inequality. This inequality plays an important role in analysis and applications, see $[33,34]$ for more details.

The Hardy inequality has been studied by a large number of authors during the twentieth century. Over the last 20 years, a large number of papers have appeared in the literature which deals with the simple proofs, various generalizations and discrete analogue of Hardy inequality, see [35-39] for more details.
In 2014, L. Maligranda et al. [40] studied a $q$-analogue of Hardy inequality (1.1) and some related inequalities. It seems to be a huge new research area to study of these so called $q$-Hardy type inequalities. They obtained more general results on $q$-Hardy type inequalities. By taking $q \rightarrow 1$, we obtain classical results on Hardy inequality (1.1). Next, L.-E. Persson and S. Shaimardan [41] studied some $q$-analogue of Hardy type inequalities for the Riemann-Liouville fractional integral operator; see [42, 43] for more details.

In 1964, N. Levinson [44] presented the inequality respecting integration from $a$ to $b$ for $0<a<b<\infty, f$ is a non-negative integrable function and $s>1$, then

$$
\begin{equation*}
\int_{a}^{b}\left(\frac{1}{x} \int_{a}^{x} f(t) d t\right)^{s} d x \leq\left(\frac{s}{s-1}\right)^{s} \int_{a}^{b} f^{s}(x) d x . \tag{1.2}
\end{equation*}
$$

In 2012, W.T. Sulaiman [45] gave a generalization and improvement for inequalities similar to Hardy inequality in the sense when $f>0$ on $[a, b] \subset(0, \infty)$ and $0<k<1 \leq h$, as follows:

$$
\begin{equation*}
h \int_{a}^{b} \frac{1}{x^{h}}\left(\int_{a}^{x} f(t) d t\right)^{h} d x \leq(b-a)^{h} \int_{a}^{b}\left(\frac{f(x)}{x}\right)^{h} d x-\int_{a}^{b} \frac{(x-a)^{h}}{x^{h}} f^{h}(x) d x \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
k \int_{a}^{b} \frac{1}{x^{k}}\left(\int_{a}^{x} f(t) d t\right)^{k} d x \leq(1-a / b)^{k} \int_{a}^{b} f^{k}(x) d x-\frac{1}{b^{k}} \int_{a}^{b}(x-a)^{k} f^{k}(x) d x \tag{1.4}
\end{equation*}
$$

In 2013, B. Sroysang [46] presented a generalization for inequalities (1.3) and (1.4) with additional parameter $m$ in the sense when $f>0$ on $[a, b] \subset(0, \infty), 0<k<1 \leq h$ and $m>0$, as follows:

$$
\begin{equation*}
h \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t) d t\right)^{h} d x \leq(b-a)^{h} \int_{a}^{b} \frac{f^{h}(x)}{x^{m}} d x-\int_{a}^{b} \frac{(x-a)^{h}}{x^{m}} f^{h}(x) d x \tag{1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
k \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t) d t\right)^{k} d x \leq \frac{(b-a)^{k}}{b^{m}} \int_{a}^{b} f^{k}(x) d x-\frac{1}{b^{m}} \int_{a}^{b}(x-a)^{k} f^{k}(x) d x . \tag{1.6}
\end{equation*}
$$

In 2014, B. Sroysang [47] presented a new kind of Hardy inequality and obtained a direct generalization of the original Hardy inequality. Next, K. Mehrez [48] studied some
generalizations and new refined Hardy type inequalities by using Jensen's inequality and Chebyshev integral inequality, see [49-52] for more details.
In 2016, S. Wu, B. Sroysang and S. Li [53] investigated certain integral inequalities similar to the Hardy inequality. They generalized versions of some known results related to the Hardy inequality and gave some new integral inequalities of Hardy type by introducing a monotonous function and established the inequality for $\beta$ being a non-negative real number, as follows:

$$
\begin{equation*}
\int_{a}^{b}\left(\frac{F(x)}{g(x)}\right)^{p} d x \leq\left(\frac{p}{p-1}\right)^{p} \int_{a}^{b}\left((x-a+\beta) \frac{f(x)}{g(x)}\right)^{p} d x \tag{1.7}
\end{equation*}
$$

where $f \geq 0, g>0$ on $[a, b] \subseteq(0, \infty), p \geq 1,(x-a+\beta) / g(x)$ is non-increasing, and

$$
F(x)=\int_{a}^{x} f(t) d t,
$$

for $x \in[a, b]$.
Inspired by this ongoing study, we establish the generalization of some integral inequalities related to Hardy type integral inequalities via $(p, q)$-calculus. Many results obtained in this paper provide extensions of other results given in previous papers. Furthermore, we give some examples to illustrate the investigated results.

## 2 Preliminaries

Throughout this paper, let $[a, b] \subseteq \mathbb{R}$ be an interval and $0<q<p \leq 1$. The following definitions and theorems for $(p, q)$-calculus are given in [17, 18, 24-29].
First, we give some $(p, q)$-notation, which would appear in this paper. For any real number $n$, the $(p, q)$-analogue of $n$ is defined by

$$
\begin{equation*}
[n]_{p, q}=\frac{p^{n}-q^{n}}{p-q} \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
[-n]_{p, q}=-\frac{1}{(p q)^{n}}[n]_{p, q} . \tag{2.2}
\end{equation*}
$$

If $p=1$, then (2.1) reduces to

$$
\begin{equation*}
[n]_{q}=\frac{1-q^{n}}{1-q}, \tag{2.3}
\end{equation*}
$$

which is $q$-analogue of $n$.

Definition 2.1 ([24]) If $f:[a, b] \rightarrow \mathbb{R}$ is a continuous function, then the $(p, q)$-derivative of the function $f$ on $[a, b]$ at $x$ is defined by

$$
\begin{equation*}
{ }_{a} D_{p, q} f(x)=\frac{f(p x+(1-p) a)-f(q x+(1-q) a)}{(p-q)(x-a)}, \quad x \neq a . \tag{2.4}
\end{equation*}
$$

The function $f$ is said to be a $(p, q)$-differentiable function on $[a, b]$ if ${ }_{a} D_{p, q} f(x)$ exists for all $x \in[a, b]$.

It should be noted that

$$
{ }_{a} D_{p, q} f(a)=\lim _{x \rightarrow a} D_{p, q} f(x) .
$$

In Definition 2.1, if $a=0$, then ${ }_{0} D_{p, q} f=D_{p, q} f$ is defined by

$$
\begin{equation*}
D_{p, q} f(x)=\frac{f(p x)-f(q x)}{(p-q) x}, x \neq 0 . \tag{2.5}
\end{equation*}
$$

And, if $p=1$, then $D_{p, q} f(x)=D_{q} f(x)$, which is the $q$-derivative of the function $f$, and also if $q \rightarrow 1$ in (2.5), then it reduces to a classical derivative.

Example 2.1 Define function $f:[a, b] \rightarrow \mathbb{R}$ by $f(x)=2 x^{2}+2 x+c$, where $c$ is a constant. Then, for $x \neq a$, we have

$$
\begin{aligned}
{ }_{a} D_{p, q}\left(2 x^{2}+2 x+c\right)= & \frac{2[p x+(1-p) a]^{2}+2[p x+(1-p) a]+c}{(p-q)(x-a)} \\
& -\frac{2[q x+(1-q) a]^{2}+2[q x+(1-q) a]+c}{(p-q)(x-a)} \\
= & \frac{2(p+q) x^{2}-4 a x(p+q)+4 a x-4 a^{2}+2 a^{2}(p+q)+2(x-a)}{x-a} \\
= & 2(p+q)(x-a)+4 a+2 .
\end{aligned}
$$

Theorem 2.1 Iff, $g:[a, b] \rightarrow \mathbb{R}$ are continuous functions and $c, d$ are constants, then the following formulas hold:
(i) ${ }_{a} D_{p, q}[c f(x) \pm d g(x)]=c_{a} D_{p, q} f(x) \pm d_{a} D_{p, q} g(x)$;
(ii) ${ }_{a} D_{p, q}[f(x) g(x)]=f(p x+(1-p) a)_{a} D_{p, q} g(x)+g(q x+(1-q) a)_{a} D_{p, q} f(x)$;
(iii) ${ }_{a} D_{p, q}\left[\frac{f(x)}{g(x)}\right]=\frac{g(p x+(1-p) a)_{a} D_{p, q} f(x)-f(p x+(1-p) a) a D_{p, q} g(x)}{g(p x+(1-p) a) g(q x+(1-q) a)}$.

The proof of this theorem is given by [18].

Definition 2.2 ([24]) If $f:[a, b] \rightarrow \mathbb{R}$ is a continuous function and $0<a<b$, then the $(p, q)$-integral is defined by

$$
\begin{equation*}
\int_{a}^{b} f(x)_{a} d_{p, q} x=(p-q)(b-a) \sum_{k=0}^{\infty} \frac{q^{k}}{p^{k+1}} f\left(\frac{q^{k}}{p^{k+1}} b+\left(1-\frac{q^{k}}{p^{k+1}}\right) a\right) \tag{2.6}
\end{equation*}
$$

$f$ is said to be a $(p, q)$-integrable function on $[a, b]$ if $\int_{a}^{b} f(x)_{a} d_{p, q} x$ exists for all $x \in[a, b]$.
Moreover, if $c \in(a, b)$, then $(p, q)$-integral is defined by

$$
\begin{equation*}
\int_{c}^{b} f(x)_{a} d_{p, q} x=\int_{a}^{b} f(x)_{a} d_{p, q} x-\int_{a}^{c} f(x)_{a} d_{p, q} x . \tag{2.7}
\end{equation*}
$$

If $a=0$ in (2.6), then one can get the classical ( $p, q$ )-integral defined by

$$
\begin{equation*}
\int_{0}^{b} f(x) d_{p, q} x=(p-q) b \sum_{k=0}^{\infty} \frac{q^{k}}{p^{k+1}} f\left(\frac{q^{k}}{p^{k+1}} b\right) \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{a}^{b} f(x) d_{p, q} x=\int_{0}^{b} f(x) d_{p, q} x-\int_{0}^{a} f(x) d_{p, q} x \tag{2.9}
\end{equation*}
$$

Example 2.2 Define a function $f:[a, b] \rightarrow \mathbb{R}$ by $f(x)=x^{2}+2 x+c$, where $c$ is a constant. Then we have

$$
\begin{aligned}
\int_{a}^{b} f(x)_{a} d_{p, q} x= & \int_{a}^{b}\left(x^{2}+2 x+c\right)_{a} d_{p, q} x \\
= & (p-q)(b-a) \sum_{k=0}^{\infty} \frac{q^{k}}{p^{k+1}}\left(\frac{q^{k}}{p^{k+1}} b+\left(1-\frac{q^{k}}{p^{k+1}}\right) a\right)^{2} \\
& +2(p-q)(b-a) \sum_{k=0}^{\infty} \frac{q^{k}}{p^{k+1}}\left(\frac{q^{k}}{p^{k+1}} b+\left(1-\frac{q^{k}}{p^{k+1}}\right) a\right) \\
& +(p-q)(b-a) \sum_{k=0}^{\infty} \frac{q^{k}}{p^{k+1}}(c) \\
= & \frac{(b-a)^{3}}{[3]_{p, q}}+\frac{2(b-a)[a(b-a)+b-a(1-p-q)]}{[2]_{p, q}}+(b-a)\left(a^{2}+c\right) \\
= & \frac{(b-a)^{3}}{[3]_{p, q}}+\frac{2(a+1)(b-a)^{2}}{[2]_{p, q}}+(b-a)\left(a^{2}+2 a+c\right) .
\end{aligned}
$$

The proofs of the following theorems are given in [18].

Theorem 2.2 Iff, $g:[a, b] \rightarrow \mathbb{R}$ are continuous functions, $t \in[a, b]$ and $\alpha$ is a constant, then the following formulas hold:
(i) ${ }_{a} D_{p, q} \int_{a}^{t} f(x)_{a} d_{p, q} x=f(t)$;
(ii) $\int_{c}^{t}{ }_{a} D_{p, q} f(x)_{a} d_{p, q} x=f(t)-f(c)$ for $c \in(a, t)$;
(iii) $\int_{a}^{t}[f(x)+g(x)]_{a} d_{p, q} x=\int_{a}^{b} f(x)_{a} d_{p, q} x+\int_{a}^{b} g(x)_{a} d_{p, q} x$;
(iv) $\int_{a}^{t} \alpha f(x)_{a} d_{p, q} x=\alpha \int_{a}^{b} f(x)_{a} d_{p, q} x$;
(v) $\int_{a}^{t}(x-a)^{\alpha}{ }_{a} d_{p, q} x=\frac{(t-a)^{\alpha+1}}{[\alpha+1]_{p, q}}$;
(vi) $\int_{c}^{t} f(p x+(1-p) a)_{a} D_{p, q} g(x)_{a} d_{p, q} x=\left.(f g)(x)\right|_{c} ^{t}-\int_{c}^{t} g(q x+(1-q) a)_{a} D_{p, q} f(x)_{a} d_{p, q} x$.

Theorem 2.3 Iff, $g:[a, b] \rightarrow \mathbb{R}$ are continuous functions and $r>1$ with $1 / r+1 / s=1$, then

$$
\begin{equation*}
\int_{a}^{b}|f(x) g(x)|_{a} d_{p, q} x \leq\left(\int_{a}^{b}|f(x)|_{a}^{r} d_{p, q} x\right)^{1 / r}\left(\int_{a}^{b}|g(x)|_{a}^{s} d_{p, q} x\right)^{1 / s} \tag{2.10}
\end{equation*}
$$

## 3 Main results

In this section, we are going to establish the generalization of some integral inequalities related to Hardy type integral inequalities via $(p, q)$-calculus. The first result is presented as follows.

Theorem 3.1 Iff, $g:[a, b] \rightarrow \mathbb{R}$ are positive continuous functions and $0<r<1$ with $1 / r+$ $1 / s=1$, then

$$
\begin{equation*}
\int_{a}^{b}|f(x) g(x)|_{a} d_{p, q} x \geq\left(\int_{a}^{b}|f(x)|_{a}^{r} d_{p, q} x\right)^{1 / r}\left(\int_{a}^{b}|g(x)|_{a}^{s} d_{p, q} x\right)^{1 / s} \tag{3.1}
\end{equation*}
$$

Proof From Theorem 2.3, we get

$$
\begin{aligned}
\int_{a}^{b}|f(x)|^{r}{ }_{a} d_{p, q} x & \leq\left(\int_{a}^{b}\left(|f(x) g(x)|^{1 / k}\right)^{k}{ }_{a} d_{p, q} x\right)^{1 / k}\left(\int_{a}^{b}\left(|g(x)|^{-1 / k}\right)^{k^{\prime}}{ }_{a} d_{p, q} x\right)^{1 / k^{\prime}} \\
& =\left(\int_{a}^{b}|f(x) g(x)|_{a} d_{p, q} x\right)^{1 / k}\left(\int_{a}^{b}|g(x)|^{-k^{\prime} / k}{ }_{a} d_{p, q} x\right)^{1 / k^{\prime}}
\end{aligned}
$$

where $r=1 / k$ and $1 / k+1 / k^{\prime}=1$.
Consequently,

$$
\left(\int_{a}^{b}|f(x)|^{r}{ }_{a} d_{p, q} x\right)^{1 / r} \leq\left(\int_{a}^{b}|f(x) g(x)|_{a} d_{p, q} x\right)^{1 /(r k)}\left(\int_{a}^{b}|g(x)|^{-k^{\prime} \mid k}{ }_{a} d_{p, q} x\right)^{1 /\left(r k^{\prime}\right)},
$$

or

$$
\left(\int_{a}^{b}|f(x)|^{r}{ }_{a} d_{p, q} x\right)^{1 / r}\left(\int_{a}^{b}|g(x)|^{-k^{\prime} \mid k}{ }_{a} d_{p, q} x\right)^{-1 /\left(r k^{\prime}\right)} \leq \int_{a}^{b}|f(x) g(x)|_{a} d_{p, q} x .
$$

Therefore,

$$
\int_{a}^{b}|f(x) g(x)|_{a} d_{p, q} x \geq\left(\int_{a}^{b}|f(x)|^{r}{ }_{a} d_{p, q} x\right)^{1 / r}\left(\int_{a}^{b}|g(x)|_{a}^{s} d_{p, q} x\right)^{1 / s} .
$$

The proof is thus accomplished.

Theorem 3.2 Iff is a non-negative function, $g$ is a positive function on $[a, b] \subseteq(0, \infty), \gamma$ is a positive real number, $r>1$, and $(x-a+\gamma) / g(x)$ is non-increasing, then

$$
\begin{equation*}
\int_{a}^{b} \frac{1}{g^{r}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \leq \frac{(p q)^{1-1 / r}}{[1-1 / r]_{p, q}^{r}} \int_{a}^{b} \frac{(x-a+\gamma)^{r}}{g^{r}(x)} f^{r}(x)_{a} d_{p, q} x . \tag{3.2}
\end{equation*}
$$

Proof From Theorem 2.3, we get

$$
\begin{aligned}
\int_{a}^{b} \frac{1}{g^{r}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \leq & \int_{a}^{b} g^{-r}(x) \int_{a}^{x}(t-a)^{1-1 / r} f^{r}(t)_{a} d_{p, q} t \\
& \times\left(\int_{a}^{x}(t-a)^{-1 / r} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x .
\end{aligned}
$$

By Theorem 2.2(v), we have

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{g^{r}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \quad \leq \frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} g^{-r}(x) \int_{a}^{x} f^{r}(t)(t-a)^{1-1 / r}(x-a)^{(1-1 / r)(r-1)}{ }_{a} d_{p, q} t_{a} d_{p, q} x \\
& \quad=\frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \int_{t}^{b} g^{-r}(x) f^{r}(t)(t-a)^{1-1 / r}(x-a)^{(1-1 / r)(r-1)}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
& \quad=\frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \int_{t}^{b} \frac{(x-a)^{r}}{g^{r}(x)} f^{r}(t)(t-a)^{1-1 / r}(x-a)^{1 / r-2}{ }_{a} d_{p, q} x_{a} d_{p, q} t
\end{aligned}
$$

$$
\leq \frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \int_{t}^{b} \frac{(x-a+\gamma)^{r}}{g^{r}(x)} f^{r}(t)(t-a)^{1-1 / r}(x-a)^{1 / r-2}{ }_{a} d_{p, q} x_{a} d_{p, q} t
$$

By the assumption that the function $(x-a+\gamma) / g(x)$ is non-increasing and Theorem 2.2(v), we have

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{g^{r}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \leq \frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \frac{(t-a+\gamma)^{r}}{g^{r}(t)} f^{r}(t)(t-a)^{1-1 / r} \int_{t}^{b}(x-a)^{1 / r-2}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
& \quad=\frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \frac{(t-a+\gamma)^{r}}{g^{r}(t)} f^{r}(t)\left(\frac{(b-a)^{1 / r-1}-(t-a)^{1 / r-1}}{[1 / r-1]_{p, q}}\right)(t-a)^{1-1 / r}{ }_{a} d_{p, q} t \\
& \quad=\frac{(p q)^{1-1 / r}}{[1-1 / r]_{p, q}^{r}} \int_{a}^{b} \frac{(t-a+\gamma)^{r}}{g^{r}(t)} f^{r}(t)\left(1-\frac{(t-a)^{1-1 / r}}{(b-a)^{1-1 / r}}\right){ }_{a} d_{p, q} t \\
& \quad \leq \frac{(p q)^{1-1 / r}}{[1-1 / r]_{p, q}^{r}} \int_{a}^{b} \frac{(t-a+\gamma)^{r}}{g^{r}(t)} f^{r}(t)_{a} d_{p, q} t .
\end{aligned}
$$

This proof is completed.

Corollary 3.1 Iff is a non-negative function, $\gamma$ is a positive real number, and $r>1$, then

$$
\begin{equation*}
\int_{a}^{b} \frac{1}{(x-a+\gamma)^{r}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \leq \frac{(p q)^{1-1 / r}}{[1-1 / r]_{p, q}^{r}} \int_{a}^{b} f^{r}(x)_{a} d_{p, q} x \tag{3.3}
\end{equation*}
$$

Remark 3.1 (1) If $p=1$, then (3.2) reduces to

$$
\begin{equation*}
\int_{a}^{b} \frac{1}{g^{r}(x)}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r}{ }_{a} d_{q} x \leq \frac{q^{1-1 / r}}{[1-1 / r]_{q}^{r}} \int_{a}^{b} \frac{(x-a+\gamma)^{r}}{g^{r}(x)} f^{r}(x)_{a} d_{q} x \tag{3.4}
\end{equation*}
$$

Also, if $q \rightarrow 1$, then (3.4) reduces to an inequality, which appeared in [53].
(2) If $0<\gamma<a$, we obtain the following inequality:

$$
\begin{align*}
\int_{a}^{b} \frac{1}{x^{r}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x & \leq \int_{a}^{b} \frac{1}{(x-a+\gamma)^{r}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \leq \frac{(p q)^{1-1 / r}}{[1-1 / r]_{p, q}^{r}} \int_{a}^{b} f^{r}(x)_{a} d_{p, q} x \tag{3.5}
\end{align*}
$$

Also, if $p=1$ and $q \rightarrow 1$, then (3.5) is reduced to (1.2).

Theorem 3.3 Iff is a non-negative function, $g$ is a positive function on $[a, b] \subseteq(0, \infty), \gamma$ is a positive real number, $r>1,0 \leq r-m<1-1 / r$, and $(x-a+\gamma) / g(x)$ is non-increasing, then

$$
\begin{align*}
\int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r} d_{p, q} x \leq & \frac{(p q)^{m+1-r-1 / r}}{[1-1 / r]_{p, q}^{r-1}[m+1-r-1 / r]_{p, q}} \\
& \times \int_{a}^{b} \frac{(x-a+\gamma)^{r}}{g^{m}(x)} f^{r}(x)_{a} d_{p, q} x \tag{3.6}
\end{align*}
$$

Proof From Theorem 2.3, we get

$$
\begin{aligned}
\int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r} d_{p, q} x \leq & \int_{a}^{b} g^{-m}(x) \int_{a}^{x}(t-a)^{1-1 / r} f^{r}(t)_{a} d_{p, q} t \\
& \times\left(\int_{a}^{x}(t-a)^{-1 / r} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x
\end{aligned}
$$

By Theorem 2.2(v), we have

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \quad \leq \frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} g^{-m}(x) \int_{a}^{x} f^{r}(t)(t-a)^{1-1 / r}(x-a)^{(1-1 / r)(r-1)}{ }_{a} d_{p, q} t_{a} d_{p, q} x \\
& \quad=\frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \int_{t}^{b} g^{-m}(x) f^{r}(t)(t-a)^{1-1 / r}(x-a)^{(1-1 / r)(r-1)}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
& \quad=\frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \int_{t}^{b} \frac{(x-a)^{m}}{g^{m}(x)} f^{r}(t)(t-a)^{1-1 / r}(x-a)^{r+1 / r-2-m_{a}} d_{p, q} x_{a} d_{p, q} t \\
& \quad \leq \frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \int_{t}^{b} \frac{(x-a+\gamma)^{m}}{g^{m}(x)} f^{r}(t)(t-a)^{1-1 / r}(x-a)^{r+1 / r-2-m}{ }_{a} d_{p, q} x_{a} d_{p, q} t .
\end{aligned}
$$

By the assumption that the function $(x-a+\gamma) / g(x)$ is non-increasing and Theorem 2.2(v), we have

$$
\begin{aligned}
\int_{a}^{b} & \frac{1}{g^{r}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
\leq & \frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \frac{(t-a+\gamma)^{m}}{g^{m}(t)} f^{r}(t)(t-a)^{1-1 / r} \int_{t}^{b}(x-a)^{r+1 / r-2-m}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
= & \frac{1}{[1-1 / r]_{p, q}^{r-1}} \int_{a}^{b} \frac{(t-a+\gamma)^{m}}{g^{m}(t)} f^{r}(t) \\
& \times\left(\frac{(b-a)^{r+1 / r-1-m}-(t-a)^{r+1 / r-1-m}}{[r+1 / r-1-m]_{p, q}}\right) \times(t-a)^{1-1 / r}{ }_{a} d_{p, q} t \\
= & \frac{(p q)^{m+1-r-1 / r}}{[1-1 / r]_{p, q}^{r-1}[m+1-r-1 / r]_{p, q}} \int_{a}^{b} \frac{(t-a+\gamma)^{m}}{g^{m}(t)} f^{r}(t) \\
& \times\left((t-a)^{r-m}-\frac{(t-a)^{1-1 / r}}{(b-a)^{1-1 / r+m-r}}\right)_{a} d_{p, q} t \\
\leq & \frac{(p q)^{m+1-r-1 / r}}{[1-1 / r]_{p, q}^{r-1}[m+1-r-1 / r]_{p, q}} \int_{a}^{b} \frac{(t-a+\gamma)^{m}}{g^{m}(t)}(t-a+\gamma)^{r-m} f^{r}(t)_{a} d_{p, q} t \\
= & \frac{(p q)^{m+1-r-1 / r}}{[1-1 / r]_{p, q}^{r-1}[m+1-r-1 / r]_{p, q}} \int_{a}^{b} \frac{(t-a+\gamma)^{r}}{g^{m}(t)} f^{r}(t)_{a} d_{p, q} t .
\end{aligned}
$$

Hence, the inequality (3.6) is established.

Remark 3.2 If $p=1$, then (3.6) reduces to a generalization of $q$-Hardy inequality as

$$
\begin{align*}
\int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r}{ }_{a} d_{q} x \leq & \frac{q^{m+1-r-1 / r}}{[1-1 / r]_{q}^{r-1}[m+1-r-1 / r]_{q}} \\
& \times \int_{a}^{b} \frac{(x-a+\gamma)^{r}}{g^{m}(x)} f^{r}(x)_{a} d_{q} x . \tag{3.7}
\end{align*}
$$

Theorem 3.4 Iff is a positive function on $[a, b] \subseteq(0, \infty), r \geq 1$, and $m>0$, then

$$
\begin{align*}
{[r]_{p, q} \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \leq } & (b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{x^{m}}{ }_{a} d_{p, q} x \\
& -\int_{a}^{b} \frac{(x-a)^{r}}{x^{m}} f^{r}(x)_{a} d_{p, q} x . \tag{3.8}
\end{align*}
$$

Proof From Theorem 2.3, we get

$$
\begin{aligned}
\int_{a}^{b} & \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \leq \int_{a}^{b} x^{-m}\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)\left(\int_{a}^{x}{ }_{a} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x \\
& =\int_{a}^{b} x^{-m}\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)(x-a)^{r-1}{ }_{a} d_{p, q} x \\
& =\int_{a}^{b} \int_{t}^{b} x^{-m} f^{r}(t)(x-a)^{r-1}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
& \leq \int_{a}^{b} t^{-m} f^{r}(t)\left(\int_{t}^{b}(x-a)^{r-1}{ }_{a} d_{p, q} x\right){ }_{a} d_{p, q} t .
\end{aligned}
$$

By Theorem 2.2(v), we obtain

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \quad=\int_{a}^{b} t^{-m} f^{r}(t)\left(\frac{(b-a)^{r}-(t-a)^{r}}{[r]_{p, q}}\right){ }_{a} d_{p, q} t \\
& \quad=\frac{1}{[r]_{p, q}}\left((b-a)^{r} \int_{a}^{b} \frac{f^{r}(t)}{t^{m}} a d_{p, q} t-\int_{a}^{b} \frac{(t-a)^{r}}{t^{m}} f^{r}(t)_{a} d_{p, q} t\right)
\end{aligned}
$$

The inequality (3.8) is proved.

Remark 3.3 If $p=1$, then (3.8) reduces to

$$
\begin{align*}
& {[r]_{q} \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r} d_{q} x} \\
& \quad \leq(b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{x^{m}}{ }^{a} d_{q} x-\int_{a}^{b} \frac{(x-a)^{r}}{x^{m}} f^{r}(x)_{a} d_{q} x . \tag{3.9}
\end{align*}
$$

Also, if $q \rightarrow 1$, then (3.9) reduces to an inequality, which appeared in [46].

Theorem 3.5 Iff is a positive function on $[a, b] \subseteq(0, \infty), 0<r<1$, and $m>0$, then

$$
\begin{align*}
{[r]_{p, q} \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \geq } & \frac{(b-a)^{r}}{b^{m}} \int_{a}^{b} f^{r}(x)_{a} d_{p, q} x \\
& -\frac{1}{b^{m}} \int_{a}^{b}(x-a)^{r} f^{r}(x)_{a} d_{p, q} x . \tag{3.10}
\end{align*}
$$

Proof From Theorem 3.1, we get

$$
\begin{aligned}
\int_{a}^{b} & \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \geq \int_{a}^{b} x^{-m}\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)\left(\int_{a}^{x}{ }_{a} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x \\
& =\int_{a}^{b} x^{-m}\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)(x-a)^{r-1}{ }_{a} d_{p, q} x \\
& =\int_{a}^{b} \int_{t}^{b} x^{-m} f^{r}(t)(x-a)^{r-1}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
& \geq b^{-m} \int_{a}^{b} f^{r}(t)\left(\int_{t}^{b}(x-a)^{r-1}{ }_{a} d_{p, q} x\right){ }_{a} d_{p, q} t .
\end{aligned}
$$

By Theorem 2.2(v), we obtain

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)_{a}^{r} d_{p, q} \\
& \quad=b^{-m} \int_{a}^{b} f^{r}(t)\left(\frac{(b-a)^{r}-(t-a)^{r}}{[r]_{p, q}}\right){ }_{a} d_{p, q} t \\
& \quad=\frac{(b-a)^{r}}{b^{m}} \int_{a}^{b} f^{r}(t)_{a} d_{p, q} t-\frac{1}{b^{m}} \int_{a}^{b}(t-a)^{r} f^{r}(t)_{a} d_{p, q} t .
\end{aligned}
$$

This completes the proof.

Remark 3.4 If $p=1$, then (3.10) reduces to

$$
\begin{align*}
& {[r]_{q} \int_{a}^{b} \frac{1}{x^{m}}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r}{ }_{a} d_{q} x} \\
& \quad \geq \frac{(b-a)^{r}}{b^{m}} \int_{a}^{b} f^{r}(x)_{a} d_{q} x-\frac{1}{b^{m}} \int_{a}^{b}(x-a)^{r} f^{r}(x)_{a} d_{q} x . \tag{3.11}
\end{align*}
$$

Also, if $q \rightarrow 1$, then (3.11) reduces to an inequality, which appeared in [46].

Theorem 3.6 Iff, $g$ are positive functions on $[a, b] \subseteq(0, \infty)$ such that $g$ is non-decreasing, $r \geq 1$, and $m>0$, then

$$
\begin{align*}
{[r]_{p, q} \int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \leq } & (b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{g^{m}(x)}{ }^{a} d_{p, q} x \\
& -\int_{a}^{b} \frac{(x-a)^{r}}{g^{m}(x)} f^{r}(x)_{a} d_{p, q} x \tag{3.12}
\end{align*}
$$

Proof From Theorem 2.3, we get

$$
\begin{aligned}
\int_{a}^{b} & \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
\leq & \int_{a}^{b} g^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right) \\
& \times\left(\int_{a}^{x}{ }_{a} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x \\
= & \int_{a}^{b} g^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)(x-a)^{r-1}{ }_{a} d_{p, q} x \\
= & \int_{a}^{b} \int_{t}^{b} g^{-m}(x) f^{r}(t)(x-a)^{r-1}{ }_{a} d_{p, q} x_{a} d_{p, q} t
\end{aligned}
$$

By the assumption of the function $g$ and Theorem 2.2(v), we obtain

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \quad \leq \int_{a}^{b} g^{-m}(t) f^{r}(t)\left(\int_{t}^{b}(x-a)^{r-1}{ }_{a} d_{p, q} x\right){ }_{a} d_{p, q} t \\
& \quad=\int_{a}^{b} g^{-m}(t) f^{r}(t)\left(\frac{(b-a)^{r}-(t-a)^{r}}{[r]_{p, q}}\right)_{a} d_{p, q} t \\
& \quad=\frac{1}{[r]_{p, q}}\left((b-a)^{r} \int_{a}^{b} \frac{f^{r}(t)}{g^{m}(t)^{a}} d_{p, q} t-\int_{a}^{b} \frac{(t-a)^{r}}{g^{m}(t)} f^{r}(t)_{a} d_{p, q} t\right),
\end{aligned}
$$

which finishes the proof.
Remark 3.5 If $p=1$, then (3.12) reduces to

$$
\begin{align*}
& {[r]_{q} \int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r}{ }_{a} d_{q} x} \\
& \quad \leq(b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{g^{m}(x)}{ }^{2} d_{q} x-\int_{a}^{b} \frac{(x-a)^{r}}{g^{m}(x)} f^{r}(x)_{a} d_{q} x . \tag{3.13}
\end{align*}
$$

Also, if $q \rightarrow 1$, then (3.13) reduces to the generalization of (1.5), which appeared in [53].
Theorem 3.7 Iff, $g$ are positive functions on $[a, b] \subseteq(0, \infty)$ such that $g$ is non-decreasing, $0<r<1$, and $m>0$, then

$$
\begin{align*}
& {[r]_{p, q} \int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)_{a}^{r} d_{p, q} x} \\
& \quad \geq(b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{g^{m}(b)}{ }^{a} d_{p, q} x-\int_{a}^{b} \frac{(x-a)^{r}}{g^{m}(b)} f^{r}(x)_{a} d_{p, q} x . \tag{3.14}
\end{align*}
$$

Proof From Theorem 3.1, we get

$$
\int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r} d_{p, q} x
$$

$$
\begin{aligned}
\geq & \int_{a}^{b} g^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right) \\
& \times\left(\int_{a}^{x}{ }_{a} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x \\
= & \int_{a}^{b} g^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)(x-a)^{r-1}{ }_{a} d_{p, q} x \\
= & \int_{a}^{b} \int_{t}^{b} g^{-m}(x) f^{r}(t)(x-a)^{r-1}{ }_{a} d_{p, q} x_{a} d_{p, q} t .
\end{aligned}
$$

By the assumption of the function $g$ and Theorem 2.2(v), we obtain

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \quad \geq g^{-m}(b) \int_{a}^{b} f^{r}(t)\left(\int_{t}^{b}(x-a)^{r-1}{ }_{a} d_{p, q} x\right){ }_{a} d_{p, q} t \\
& \quad=g^{-m}(b) \int_{a}^{b} f^{r}(t)\left(\frac{(b-a)^{r}-(t-a)^{r}}{[r]_{p, q}}\right)_{a} d_{p, q} t \\
& \quad=\frac{1}{[r]_{p, q}}\left((b-a)^{r} \int_{a}^{b} \frac{f^{r}(t)}{g^{m}(b)} a d_{p, q} t-\int_{a}^{b} \frac{(t-a)^{r}}{g^{m}(b)} f^{r}(t)_{a} d_{p, q} t\right)
\end{aligned}
$$

The proof is accomplished.

Remark 3.6 If $p=1$, then (3.14) reduces to

$$
\begin{align*}
& {[r]_{q} \int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r}{ }_{a} d_{q} x} \\
& \quad \geq(b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{g^{m}(b)}{ }^{a} d_{q} x-\int_{a}^{b} \frac{(x-a)^{r}}{g^{m}(b)} f^{r}(x)_{a} d_{q} x \tag{3.15}
\end{align*}
$$

Also, if $q \rightarrow 1$, then (3.15) reduces to the generalization of (1.6), which appeared in [53].

Theorem 3.8 Let $f, g$ be positive functions on $[a, b] \subseteq(0, \infty), r \geq 1, m>0$ and

$$
\begin{equation*}
G(x)=\int_{0}^{x} g(t)_{a} d_{p, q} t \tag{3.16}
\end{equation*}
$$

If the function $G$ is non-decreasing, then

$$
\begin{align*}
& {[r]_{p, q} \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x} \\
& \quad \leq(b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{G^{m}(x)} a d_{p, q} x-\int_{a}^{b} \frac{(x-a)^{r}}{G^{m}(x)} f^{r}(x)_{a} d_{p, q} x . \tag{3.17}
\end{align*}
$$

Proof From Theorem 2.3, we get

$$
\int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r} d_{p, q} x
$$

$$
\begin{equation*}
\leq \int_{a}^{b} G^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)\left(\int_{a}^{x}{ }_{a} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x . \tag{3.18}
\end{equation*}
$$

By the assumption of the function $G(x)$, we have

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \quad \leq \int_{a}^{b} G^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)(x-a)^{r-1}{ }_{a} d_{p, q} x \\
& \quad=\int_{a}^{b} \int_{t}^{b} G^{-m}(x) f^{r}(t)(x-a)^{r-1}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
& \quad \leq \int_{a}^{b} G^{-m}(t) f^{r}(t)\left(\int_{t}^{b}(x-a)^{r-1}{ }_{a} d_{p, q} x\right){ }_{a} d_{p, q} t
\end{aligned}
$$

By Theorem 2.2(v), we obtain

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)_{a}^{r} d_{p, q} x \\
& \quad=\int_{a}^{b} G^{-m}(t) f^{r}(t)\left(\frac{(b-a)^{r}-(t-a)^{r}}{[r]_{p, q}}\right){ }_{a} d_{p, q} t \\
& \quad=\frac{1}{[r]_{p, q}}\left((b-a)^{r} \int_{a}^{b} \frac{f^{r}(t)}{G^{m}(t)^{a}} d_{p, q} t-\int_{a}^{b} \frac{(t-a)^{r}}{G^{m}(t)} f^{r}(t)_{a} d_{p, q} t\right),
\end{aligned}
$$

which gives the required inequality.
Remark 3.7 If $p=1$, then (3.17) reduces to

$$
\begin{align*}
& {[r]_{q} \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r}{ }_{a} d_{q} x} \\
& \quad \leq(b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{G^{m}(x)} a d_{q} x-\int_{a}^{b} \frac{(x-a)^{r}}{G^{m}(x)} f^{r}(x)_{a} d_{q} x . \tag{3.19}
\end{align*}
$$

Also, if $q \rightarrow 1$, then (3.19) reduces to the generalization of (1.5).

Theorem 3.9 Let $f, g$ be positive functions on $[a, b] \subseteq(0, \infty), 0<r<1, m>0$ and let $G(x)$ be defined by (3.16). If the function $G$ is non-decreasing, then

$$
\begin{align*}
{[r]_{p, q} \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r} d_{p, q} x \geq } & (b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{G^{m}(b)}{ }^{a} d_{p, q} x \\
& -\int_{a}^{b} \frac{(x-a)^{r}}{G^{m}(b)} f^{r}(x)_{a} d_{p, q} x \tag{3.20}
\end{align*}
$$

Proof From Theorem 3.1, we get

$$
\begin{align*}
\int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \geq & \int_{a}^{b} G^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right) \\
& \times\left(\int_{a}^{x}{ }_{a} d_{p, q} t\right)^{r-1}{ }_{a} d_{p, q} x \tag{3.21}
\end{align*}
$$

By the assumption of the function $G(x)$, we have

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x \\
& \quad \geq \int_{a}^{b} G^{-m}(x)\left(\int_{a}^{x} f^{r}(t)_{a} d_{p, q} t\right)(x-a)^{r-1}{ }_{a} d_{p, q} x \\
& \quad=\int_{a}^{b} \int_{t}^{b} x^{m} G^{-m}(x) f^{r}(t)(x-a)^{r-1}{ }_{a} d_{p, q} x_{a} d_{p, q} t \\
& \quad \geq \int_{a}^{b} G^{-m}(b) f^{r}(t)\left(\int_{t}^{b}(x-a)^{r-1}{ }_{a} d_{p, q} x\right){ }_{a} d_{p, q} t
\end{aligned}
$$

By Theorem 2.2(v), we obtain

$$
\begin{aligned}
& \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r} d_{p, q} x \\
& \quad=\int_{a}^{b} G^{-m}(b) f^{r}(t)\left(\frac{(b-a)^{r}-(t-a)^{r}}{[r]_{p, q}}\right)_{a} d_{p, q} t \\
& \quad=\frac{1}{[r]_{p, q}}\left((b-a)^{r} \int_{a}^{b} \frac{f^{r}(t)}{G^{m}(b)} a d_{p, q} t-\int_{a}^{b} \frac{(t-a)^{r}}{G^{m}(b)} f^{r}(t)_{a} d_{p, q} t\right),
\end{aligned}
$$

provided the left and right sides are finite.

Remark 3.8 If $p=1$, then (3.20) reduces to

$$
\begin{align*}
& {[r]_{q} \int_{a}^{b} \frac{1}{G^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{q} t\right)^{r} d_{q} x} \\
& \quad \geq(b-a)^{r} \int_{a}^{b} \frac{f^{r}(x)}{G^{m}(b)} a d_{q} x-\int_{a}^{b} \frac{(x-a)^{r}}{G^{m}(b)} f^{r}(x)_{a} d_{q} x . \tag{3.22}
\end{align*}
$$

Also, if $q \rightarrow 1$, then (3.22) reduces to the generalization of (1.6).

## 4 Examples

In the following, we will give examples to illustrate our main results.

Example 4.1 Define the continuous functions $f, g:[1,3] \rightarrow \mathbb{R}$ by $f(x)=x^{4}$ and $g(x)=1 / x$. Applying Theorem 3.1 with $p=2 / 3, q=1 / 3, r=1 / 2$ and $s=-1$, the left side of (3.1) becomes

$$
\begin{aligned}
\int_{a}^{b}|f(x) g(x)|_{a} d_{p, q} x & =\int_{1}^{3}\left|x^{4}(1 / x)\right|_{1} d_{2 / 3,1 / 3} x \\
& =\int_{1}^{3} x^{3}{ }_{1} d_{2 / 3,1 / 3} x \\
& =\frac{(3-1)^{4}}{[4]_{2 / 3,1 / 3}}+\frac{3(1)(3-1)^{3}}{[3]_{2 / 3,1 / 3}}+\frac{3(1)^{2}(3-1)^{2}}{[2]_{2 / 3,1 / 3}}+1^{3}(3-1) \\
& \approx 73.65714286 .
\end{aligned}
$$

For the right side of (3.1), one has

$$
\begin{aligned}
& \left(\int_{a}^{b}|f(x)|_{a}^{r} d_{p, q} x\right)^{1 / r}\left(\int_{a}^{b}|g(x)|_{a}^{s} d_{p, q} x\right)^{1 / s} \\
& \quad=\left(\int_{1}^{3}\left(x^{4}\right)^{1 / 2}{ }_{1} d_{2 / 3,1 / 3} x\right)^{2}\left(\int_{1}^{3}(1 / x)^{-1}{ }_{1} d_{2 / 3,1 / 3} x\right)^{-1} \\
& \quad=\left(\int_{1}^{3} x^{2}{ }_{1} d_{2 / 3,1 / 3} x\right)^{2}\left(\int_{1}^{3} x_{1} d_{2 / 3,1 / 3} x\right)^{-1} \\
& \quad=\left[\frac{(3-1)^{3}}{[3]_{2 / 3,1 / 3}}+\frac{2(1)(3-1)^{2}}{[2]_{2 / 3,1 / 3}}+1^{2}(3-1)\right]^{2}\left[\frac{(3-1)[3-1(1-2 / 3-1 / 3)]}{[2]_{2 / 3,1 / 3}}\right]^{-1} \\
& \quad \approx 68.58503401 .
\end{aligned}
$$

It is clear that $73.65714286 \geq 68.58503401$, which confirms the result described in Theorem 3.1.

Example 4.2 Define functions $f, g:[1,3] \rightarrow \mathbb{R}$ by $f(x)=x$ and $g(x)=x-1 / 6$. Then $f$ is non-negative function and $g$ is a positive function on [1,3]. Applying Theorem 3.3 with $p=1 / 2, q=1 / 3, r=m=2$ and $\gamma=5 / 6$, the left side of (3.6) becomes

$$
\begin{aligned}
\int_{a}^{b} \frac{1}{g^{m}(x)}\left(\int_{a}^{x} f(t)_{a} d_{p, q} t\right)^{r}{ }_{a} d_{p, q} x & =\int_{1}^{3} \frac{1}{(x-1 / 6)^{2}}\left(\int_{1}^{x} t_{1} d_{1 / 2,1 / 3} t\right)^{2}{ }_{1} d_{1 / 2,1 / 3} x \\
& =\int_{1}^{3} \frac{1}{(x-1 / 6)^{2}} \frac{(x-1)^{2}(x-1 / 6)^{2}}{(5 / 6)^{2}}{ }_{1} d_{1 / 2,1 / 3} x \\
& =\frac{36}{25} \frac{(3-1)^{3}}{[2+1]_{1 / 2,1 / 3}} \\
& =\frac{10,368}{475} \\
& \approx 21.82736842
\end{aligned}
$$

by Theorem 2.2(v).
For the right side of (3.6), one has

$$
\begin{aligned}
\frac{(p q)^{m+1-r-1 / r}}{[1-1 / r]_{p, q}^{r-1}[m+1-r-1 / r]_{p, q}} & \int_{a}^{b} \frac{(x-a+\gamma)^{r}}{g^{m}(x)} f^{r}(x)_{a} d_{p, q} x \\
& =\frac{(1 / 6)^{2+1-2-1 / 2}}{[1-1 / 2]_{1 / 2,1 / 3}^{2-1}[2+1-2-1 / 2]_{1 / 2,1 / 3}} \int_{1}^{3} \frac{(x-1+5 / 6)^{2}}{(x-1 / 6)^{2}} x^{2}{ }_{1} d_{1 / 2,1 / 3} x \\
& =\frac{(1 / 6)^{1 / 2}}{[1 / 2]_{1 / 2,1 / 3}^{2}} \int_{1}^{3} x^{2}{ }_{1} d_{1 / 2,1 / 3} x \\
& =\frac{(1 / 6)^{1 / 2}}{[1 / 2]_{1 / 2,1 / 3}^{2}}\left[\frac{(3-1)^{3}}{[3]_{1 / 2,1 / 3}}+\frac{2(3-1)^{2}}{[2]_{1 / 2,1 / 3}}+1^{2}(3-1)\right] \\
& \approx 40.46697790 .
\end{aligned}
$$

It is clear that $21.82736842 \leq 40.46697790$, which confirms the result described in Theorem 3.3.

## 5 Conclusion

In the present paper, we use $(p, q)$-calculus to establish new integral inequalities related to Hardy type integral inequalities. Many existing results in the literature are reduced to special cases of our results when $p=1$ and $q \rightarrow 1$. The results of this paper are new and significantly contribute to the existing literature on the topic. In addition, we shall study these results in fractional $(p, q)$-calculus and conformable fractional $(p, q)$-calculus in the future.

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## Author details

${ }^{1}$ Department of Mathematics, Khon Kaen University, 40002, Khon Kaen, Thailand. ${ }^{2}$ Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, 10800, Bangkok, Thailand. ${ }^{3}$ Department of Mathematics, University of Ioannina, 45110, loannina, Greece. ${ }^{4}$ Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, 21588, Jeddah, Saudi Arabia.

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