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# Some Hermite–Hadamard and Opial dynamic inequalities on time scales

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**Abstract**

In this article, we are interested in some well-known dynamic inequalities on time scales. For this reason, we will prove some new Hermite–Hadamard (H-H) and Opial dynamic inequalities on time scales. The main results here will be derived via the dynamic integration by parts and chain rule formulas on time scales. In addition, we will extend and unify the inequalities for the convex functions.

**MSC:** 26D07; 26D10; 26D15; 26A33

**Keywords:** Time scales; H-H inequality; Steffensen inequalities; Opial inequality; Hölder's inequality

## 1 Introduction

In 1893, the H-H inequality was established for a convex function $\wp$ on a given interval $[d_1, d_2]$ in [1]:

$$\wp\left(\frac{d_1 + d_2}{2}\right) \le \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \wp(x)\, dx \le \frac{\wp(d_1) + \wp(d_2)}{2}. \tag{1.1}$$

The study of the H-H inequality have been attracted the attention of many scholars. In recent years, many refinements, generalizations, and extensions have been made to the inequality (1.1); we advise the interested reader to visit the published papers [2–8] and the references cited therein.

After the H-H inequality and in 1960, Opial [9] established another important integral inequality, called in the literature Opial's integral inequality, which is as follows:

$$\int_0^\mu \left|\wp(\mathbf{s})\wp'(\mathbf{s})\right| d\mathbf{s} \le \frac{\mu}{4} \int_0^\mu \left(\wp'(\mathbf{s})\right)^2 d\mathbf{s}, \tag{1.2}$$

where $\wp(\mathbf{s}) \in C^1[0, \mu]$ with $\wp(0) = \wp(\mu) = 0$ and $\wp(\mathbf{s}) > 0$ for $\mathbf{s} \in (0, \mu)$. A best possibility here is $\frac{\mu}{4}$. Inequality (1.2) with their extensions play a great role in analysis and its applications. The interest in inequality (1.2) comes from their mathematical structure. Many results concerning the generalizations and extensions of this inequality have been established; see [10–17].

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The inequalities (1.1) and (1.2) have been proved not only for the ordinary order but also for various fractional models, for example, the Riemann–Liouville model, the Atangana–Baleanu model, the tempered fractional model, the Caputo–Fabrizo model, and the conformable model; see [18–22] and the references therein.

The use of dynamic system to study the continuous and discrete times is well studied, especially for the real-world modeling issues. It is better to check if structures can be given that encourage us in integrating all dynamic systems at the same time to derive a superior and perspective comprehension of the contrasts between continuous and discrete domains. In fact, constructing a correlation between discrete and continuous situations is the primary aim of dynamic equations on time scales. It is well known that the theory of time scales was originated by Hilger in his Ph.D. thesis [23]. After that, this setting was evolved by many researchers, for more details refer to [24, 25].

Over the recent couple of years, there has been growing interest in the study of dynamic inequalities on time scales and this has become an important field in applied and pure mathematics; see for details [25–30].

This article is devoted to establishing some dynamic H-H and Opial inequalities on time scales. The obtained inequalities will extend some known integral inequalities, and extend and unify some continuous inequalities.

The article consists of five sections. Section 1 is for the introduction. In Sect. 2 we present basic concepts and preliminaries of time scale notations, and in Sect. 3 we discuss and derive some dynamic inequalities of H-H on time scales. Opial dynamic inequalities will be discussed in Sect. 4. Section 5 concludes the article finally.

## 2 Preliminaries

This section deals with recalling time scale notation and basic lemmas on Steffensen inequalities on time scales. Let R be the set of real numbers, then a time scale $T_0$ is a nonempty and closed subset of R. For $\iota \in T_0$, the forward and backward jump operators $\sigma, \rho : T_0 \to R$ are, respectively, defined by

$$\sigma(\iota) = \inf\{n \in T_0 : n > \iota\} \quad \text{and} \quad \rho(\iota) = \sup\{n \in T_0 : n < \iota\}.$$

We define the graininess function $\wp : T_0 \to [0, \infty)$ by $\wp(\iota) = \sigma(\iota) - \iota$. An element $\iota \in T_0$ is said to be left-dense if $\rho(\iota) = \iota$ and left-scattered if $\rho(\iota) < \iota$, and right-dense if $\sigma(\iota) = \iota$ and right-scattered if $\sigma(\iota) > \iota$. The set $T_0^k$ is defined to be $T_0$ if it has a left-scattered maximum $\wp_2$, then $T_0^k = T_0 - \{\wp_2\}$ otherwise $T_0^k = T_0$. For further information on these notions we refer the reader to Refs. [24, 25].

**Definition 2.1** ([25]) Assume that $\wp : T_0 \to R$ is a real-valued function. Then we say $\wp$ is $\mathcal{RD}$-continuous on R if its left limit is finite at any left-dense point of $T_0$ and it is continuous on every right-dense point of $T_0$.

**Definition 2.2** ([25]) Assume that $\wp : T_0 \to R$ is a real-valued function. Then we say $\wp$ is $\mathcal{LD}$-continuous on R if its right limit is finite at any right-dense point of $T_0$ and it is continuous on every left-dense point of $T_0$.

**Theorem 2.1** ([25]) *Let $d_1, d_2 \in T_0$ with $d_1 < d_2$. Let $\wp_1, \wp_2 : [d_1, d_2]_{T_0} \to R$ be $\Delta$-integrable functions such that $\wp_1$ of one sign and decreasing and $0 \le \wp_2(\mathbf{s}) \le 1$ for each*

$\mathbf{s} \in [d_1, d_2]_{T_0}$. *Also, suppose that* $\vartheta_1, \vartheta_2 \in [d_1, d_2]_{T_0}$ *such that*

$$d_2 - \vartheta_1 \leq \int_{d_1}^{d_2} \wp_2(\mathbf{s})\Delta\mathbf{s} \leq \vartheta_2 - d_1, \quad \text{if } \wp_1(\mathbf{s}) > 0, \forall \mathbf{s} \in [d_1, d_2]_{T_0}, \tag{2.1}$$

$$\vartheta_2 - d_1 \leq \int_{d_1}^{d_2} \wp_2(\mathbf{s})\Delta\mathbf{s} \leq d_2 - \vartheta_1, \quad \text{if } \wp_1(\mathbf{s}) < 0, \forall \mathbf{s} \in [d_1, d_2]_{T_0}, \tag{2.2}$$

*then*

$$\int_{\vartheta_1}^{d_2} \wp_1(\mathbf{s})\Delta\mathbf{s} \leq \int_{d_1}^{d_2} \wp_1(\mathbf{s})\wp_2(\mathbf{s})\Delta\mathbf{s} \leq \int_{d_1}^{\vartheta_2} \wp_1(\mathbf{s})\Delta\mathbf{s}. \tag{2.3}$$

**Theorem 2.2** ([25]) *Let* $d_1, d_2 \in T_0$ *with* $d_1 < d_2$. *Let* $\wp_1, \wp_2 : [d_1, d_2]_{T_0} \to \mathbb{R}$ *be* $\nabla$-*integrable functions such that* $\wp_1$ *is of one sign and decreasing and* $0 \leq \wp_2(\mathbf{s}) \leq 1$ *for each* $\mathbf{s} \in [d_1, d_2]_{T_0}$. *Also, suppose that* $\vartheta_1, \vartheta_2 \in [d_1, d_2]_{T_0}$ *such that*

$$d_2 - \vartheta_1 \leq \int_{d_1}^{d_2} \wp_2(\mathbf{s})\nabla\mathbf{s} \leq \vartheta_2 - d_1, \quad \text{if } \wp_1(\mathbf{s}) > 0, \forall \mathbf{s} \in [d_1, d_2]_{T_0},$$

$$\vartheta_2 - d_1 \leq \int_{d_1}^{d_2} \wp_2(\mathbf{s})\nabla\mathbf{s} \leq d_2 - \vartheta_1, \quad \text{if } \wp_1(\mathbf{s}) < 0, \forall \mathbf{s} \in [d_1, d_2]_{T_0},$$

*then*

$$\int_{\vartheta_1}^{d_2} \wp_1(\mathbf{s})\nabla\mathbf{s} \leq \int_{d_1}^{d_2} \wp_1(\mathbf{s})\wp_2(\mathbf{s})\nabla\mathbf{s} \leq \int_{d_1}^{\vartheta_2} \wp_1(\mathbf{s})\nabla\mathbf{s}. \tag{2.4}$$

**Theorem 2.3** ([25]) *Let* $d_1, d_2 \in T_0$ *with* $d_1 < d_2$. *Let* $\wp_1, \wp_2 : [d_1, d_2]_{T_0} \to \mathbb{R}$ *be* $\Delta$-*integrable functions such that* $\wp_1$ *of one sign and decreasing and* $0 \leq \wp_2(\mathbf{s}) \leq 1$ *for each* $\mathbf{s} \in [d_1, d_2]_{T_0}$. *Suppose that*

$$\lambda := \int_{d_1}^{d_2} \wp_2(\mathbf{s})\Delta\mathbf{s} \quad \text{provided that } d_2 - \lambda, d_1 + \lambda \in T_0,$$

*then*

$$\int_{d_2-\lambda}^{d_2} \wp_1(\mathbf{s})\Delta\mathbf{s} \leq \int_{d_1}^{d_2} \wp_1(\mathbf{s})\wp_2(\mathbf{s})\Delta\mathbf{s} \leq \int_{d_1}^{d_1+\lambda} \wp_1(\mathbf{s})\Delta\mathbf{s}. \tag{2.5}$$

**Theorem 2.4** ([25]) *Let* $d_1, d_2 \in T_0$ *with* $d_1 < d_2$. *Let* $\wp_1, \wp_2 : [d_1, d_2]_{T_0} \to \mathbb{R}$ *be* $\nabla$-*integrable functions such that* $\wp_1$ *of one sign and decreasing and* $0 \leq \wp_2(\mathbf{s}) \leq 1$ *for each* $\mathbf{s} \in [d_1, d_2]_{T_0}$. *Suppose that*

$$\lambda := \int_{d_1}^{d_2} \wp_2(\mathbf{s})\nabla\mathbf{s} \quad \text{such that } d_2 - \lambda, d_1 + \lambda \in T_0,$$

*then*

$$\int_{d_2-\lambda}^{d_2} \wp_1(\mathbf{s})\nabla\mathbf{s} \leq \int_{d_1}^{d_2} \wp_1(\mathbf{s})\wp_2(\mathbf{s})\nabla\mathbf{s} \leq \int_{d_1}^{d_1+\lambda} \wp_1(\mathbf{s})\nabla\mathbf{s}. \tag{2.6}$$

**Theorem 2.5** ($\Delta$-integration by parts [24, 25]) *Let* $\wp_1, \wp_2 : [d_1, d_2]_{T_0} \to R$ *with* $\wp_1, \wp_2 \in \mathbb{C}_{\mathcal{RD}}$ *and* $d_1, d_2 \in T_0$. *Then the integration by parts in the sense of* $\Delta$ *is formulated as follows*:

$$\int_{d_1}^{d_2} \wp_1(\mathbf{s}) \wp_2^{\Delta}(\mathbf{s}) \Delta\mathbf{s} = \wp_1(\mathbf{s}) \wp_2(\mathbf{s})\big|_{d_1}^{d_2} - \int_{d_1}^{d_2} \wp_1^{\Delta}(\mathbf{s}) \wp_2^{\sigma}(\mathbf{s}) \Delta\mathbf{s}, \tag{2.7}$$

**Theorem 2.6** ($\nabla$-integration by parts [17, 24, 25]) *Let* $\wp_1, \wp_2 : [d_1, d_2]_{T_0} \to R$ *with* $\wp_1, \wp_2 \in \mathbb{C}_{\mathcal{LD}}$ *and* $d_1, d_2 \in T_0$. *Then the integration by parts in the sense of* $\nabla$ *is formulated as follows*:

$$\int_{d_1}^{d_2} \wp_1(\mathbf{s}) \wp_2^{\nabla}(\mathbf{s}) \nabla\mathbf{s} = \wp_1(\mathbf{s}) \wp_2(\mathbf{s})\big|_{d_1}^{d_2} - \int_{d_1}^{d_2} \wp_1^{\nabla}(\mathbf{s}) \wp_2^{\rho}(\mathbf{s}) \nabla\mathbf{s}. \tag{2.8}$$

**Definition 2.3** ([31]) Let $d_1, d_2 \in T_0$ with $d_1 < d_2$. A function $\wp : T_0 \to \mathbb{R}$ is said to be convex on $T_0$, if

$$\wp\big(\vartheta d_1 + (1 - \vartheta)d_2\big) \leq \vartheta \wp(d_1) + (1 - \vartheta)\wp(d_2)$$

holds for each $\vartheta \in T_{0[d_1, d_2]} \subseteq [0, 1]$.

## 3  Dynamic H-H inequalities

**Theorem 3.1** *Let* $\wp : [d_1, d_2]_{T_0} \to R$ *be convex and monotonic and* $d_1, d_2 \in T_0$ *with* $d_1 < d_2$. *Suppose that also* $\vartheta_1, \vartheta_2 \in [d_1, d_2]_{T_0}$, *then we have*

$$\wp\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \leq \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \wp^{\sigma}(\mathbf{s}) \Delta\mathbf{s} \leq \frac{2\wp(d_1) - \wp(\vartheta_1) - \wp(\vartheta_2) + 2\wp(d_2)}{2}, \tag{3.1}$$

*such that* $\frac{d_1 + d_2}{2}, \frac{\vartheta_1 + \vartheta_2}{2} \in T_0$ *and*

$$\vartheta_1, \vartheta_2 \geq \frac{d_1 + 3d_2}{4}, \quad \text{if } \wp \text{ is decreasing,}$$

$$\vartheta_1, \vartheta_2 \leq \frac{d_1 + 3d_2}{4}, \quad \text{if } \wp \text{ is increasing.}$$

*Proof* Suppose that $\wp$ is decreasing and convex. It follows that $\wp^{\Delta} \leq 0$. Set $\Psi_1 := -\wp^{\Delta}$, then it is clear that $\Psi_1$ is decreasing and $\Psi_1 \geq 0$. If we choose $\Psi_2(\mathbf{s}) := \frac{2(d_2 - \mathbf{s})}{d_2 - d_1}$, we see that $0 \leq \Psi_2(\mathbf{s}) \leq 1$ for each $\mathbf{s} \in [\frac{d_1 + d_2}{2}, d_2]$. Now, by making use of inequality (2.1) with $\wp_2(\mathbf{s}) = \Psi_2(\mathbf{s}) = \frac{2(d_2 - \mathbf{s})}{d_2 - d_1}$, we get

$$d_2 - \vartheta_1 \leq \frac{d_2 - d_1}{4} \leq \vartheta_2 - \frac{d_1 + d_2}{2}.$$

This implies that $\vartheta_1, \vartheta_2 \geq \frac{d_1 + 3d_2}{4}$. Thus, $\Psi_1$ and $\Psi_2$ satisfy the hypotheses in Theorem 2.1 and therefore

$$\int_{\vartheta_1}^{d_2} \Psi_1(\mathbf{s}) \Delta\mathbf{s} \leq \int_{\frac{d_1 + d_2}{2}}^{d_2} \Psi_1(\mathbf{s}) \Psi_2(\mathbf{s}) \Delta\mathbf{s} \leq \int_{\frac{d_1 + d_2}{2}}^{\vartheta_2} \Psi_1(\mathbf{s}) \Delta\mathbf{s}. \tag{3.2}$$

By using $\Delta$-integration by parts (Theorem 2.5), we have

$$\int_{\frac{d_1+d_2}{2}}^{d_2} \Psi_1(\mathbf{s})\Psi_2(\mathbf{s})\Delta\mathbf{s} = \wp\left(\frac{d_1+d_2}{2}\right) - \frac{2}{d_2-d_1}\int_{\frac{d_1+d_2}{2}}^{d_2} \wp^\sigma(\mathbf{s})\Delta\mathbf{s}.$$

Then, by making use of the above and the fact $\int_{x_1}^{x_2} \Psi_1(\mathbf{s})\Delta\mathbf{s} = \wp(x_1) - \wp(x_2)$ in the inequality (3.2), we get

$$\wp(\vartheta_1) - \wp(d_2) \le \wp\left(\frac{d_1+d_2}{2}\right) - \frac{2}{d_2-d_1}\int_{\frac{d_1+d_2}{2}}^{d_2} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \wp\left(\frac{d_1+d_2}{2}\right) - \wp(\vartheta_2).$$

This simplifies to

$$\wp(\vartheta_2) \le \frac{2}{d_2-d_1}\int_{\frac{d_1+d_2}{2}}^{d_2} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \frac{\wp(d_1)+\wp(d_2)}{2} + \wp(d_2) - \wp(\vartheta_1), \tag{3.3}$$

where we used the convexity of $\wp$ to get $\wp(\frac{d_1+d_2}{2}) \le \frac{\wp(d_1)+\wp(d_2)}{2}$.

On the other hand, if we choose $\Psi_3(\mathbf{s}) := \frac{2(\mathbf{s}-d_1)}{d_2-d_1}$, we see that $0 \le \Psi_3(\mathbf{s}) \le 1$ for each $\mathbf{s} \in [d_1, \frac{d_1+d_2}{2}]$. Again, by making use of inequality (2.1) for the new $\Psi_3(\mathbf{s})$, we find

$$\frac{d_1+d_2}{2} - \vartheta_1 \le \frac{d_2-d_1}{4} \le \vartheta_2 - d_1.$$

This implies that $\vartheta_1, \vartheta_2 \ge \frac{3d_1+d_2}{4} < \frac{d_1+3d_2}{4}$. Thus, $\Psi_1$ and $\Psi_3$ satisfy the hypotheses in Theorem 2.1. Then, by using the same technique as used above, we can deduce

$$\wp(\vartheta_1) \le \frac{2}{d_2-d_1}\int_{d_1}^{\frac{d_1+d_2}{2}} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \wp(d_1) + \wp\left(\frac{d_1+d_2}{2}\right) - \wp(\vartheta_2).$$

By convexity of $\wp$, it follows that

$$\wp(\vartheta_1) \le \frac{2}{d_2-d_1}\int_{d_1}^{\frac{d_1+d_2}{2}} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \wp(d_1) + \frac{\wp(d_1)+\wp(d_2)}{2} - \wp(\vartheta_2). \tag{3.4}$$

Adding inequalities (3.3) and (3.4) together and simplifying the result we get

$$\frac{\wp(\vartheta_2)+\wp(\vartheta_1)}{2} \le \frac{1}{d_2-d_1}\int_{d_1}^{d_2} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \frac{2\wp(d_1)+2\wp(d_2)}{2} - \frac{\wp(\vartheta_2)+\wp(\vartheta_1)}{2}.$$

Again, by using the convexity of $\wp$ (we see $\wp(\frac{\vartheta_2+\vartheta_1}{2}) \le \frac{\wp(\vartheta_2)+\wp(\vartheta_1)}{2}$) for the last inequality and rearranging the terms, we get the desired result. $\square$

**Corollary 3.1** *Theorem* 3.1 *with* $\vartheta_1 = \vartheta_2 = \frac{d_1+3d_2}{4}$ *gives the new inequality*

$$\wp\left(\frac{d_1+3d_2}{4}\right) \le \frac{1}{d_2-d_1}\int_{d_1}^{d_2} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \frac{2\wp(d_1)+2\wp(d_2)}{2} - \wp\left(\frac{d_1+3d_2}{4}\right). \tag{3.5}$$

**Theorem 3.2** *Let $\wp : [d_1, d_2]_{T_0} \to R$ be convex and monotonic and $d_1, d_2 \in T_0$ with $d_1 < d_2$, then we have*

$$\wp\left(\frac{d_1 + d_2}{2}\right) \le \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \frac{2\wp(d_1) + 2\wp(d_2)}{2} - \wp\left(\frac{d_1 + d_2}{2}\right), \qquad (3.6)$$

*such that $\frac{d_1 + d_2}{2} \in T_0$.*

*Proof* Suppose that $\wp$ is decreasing and convex. It follows that $\wp^\Delta \le 0$. Let $\Psi_1 := -\wp^\Delta$, then we see that $\Psi_1$ is decreasing and $\Psi_1 \ge 0$. If we choose $\Psi_2(\mathbf{s}) := \frac{d_2 - \mathbf{s}}{d_2 - d_1}$, we see that $0 \le \Psi_2(\mathbf{s}) \le 1$ for each $\mathbf{s} \in [d_1, d_2]$. Moreover,

$$\lambda := \int_{d_1}^{d_2} \Psi_2(\mathbf{s})\Delta\mathbf{s} = \frac{d_2 - d_1}{2}.$$

It follows that $d_1 + \lambda = d_2 - \lambda = \frac{d_1 + d_2}{2} \in T_0$. Thus, $\Psi_1$ and $\Psi_2$ satisfy the hypotheses in Theorem 2.3 and therefore inequality (2.5) holds true for $\Psi_1 = -\wp^\Delta$ and $\Psi_2(\mathbf{s}) = \frac{d_2 - \mathbf{s}}{d_2 - d_1}$.

$$\int_{\frac{d_1 + d_2}{2}}^{d_2} \Psi_1(\mathbf{s})\Delta\mathbf{s} \le \int_{d_1}^{d_2} \Psi_1(\mathbf{s})\Psi_2(\mathbf{s})\Delta\mathbf{s} \le \int_{d_1}^{\frac{d_1 + d_2}{2}} \Psi_1(\mathbf{s})\Delta\mathbf{s}.$$

By making use of integration by parts and the fact $\int_{x_1}^{x_2} \Psi_1(\mathbf{s})\Delta\mathbf{s} = \wp(x_1) - \wp(x_2)$, we can deduce

$$\wp\left(\frac{d_1 + d_2}{2}\right) - \wp(d_2) \le \wp(d_1) - \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \wp^\sigma(\mathbf{s})\Delta\mathbf{s} \le \wp(d_1) - \wp\left(\frac{d_1 + d_2}{2}\right),$$

which rearranges to the desired result.  □

The above results can be obtained for the $\nabla$ case by using Theorems 2.2 and 2.4, respectively.

**Theorem 3.3** *Let $\wp : [d_1, d_2]_{T_0} \to R$ be convex and monotonic and $d_1, d_2 \in T_0$ with $d_1 < d_2$. Suppose that also $\vartheta_1, \vartheta_2 \in [d_1, d_2]_{T_0}$, then we have*

$$\wp\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \le \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \wp^\rho(\mathbf{s})\nabla\mathbf{s} \le \frac{2\wp(d_1) - \wp(\vartheta_1) - \wp(\vartheta_2) + 2\wp(d_2)}{2}, \qquad (3.7)$$

*such that $\frac{d_1 + d_2}{2}, \frac{\vartheta_1 + \vartheta_2}{2} \in T_0$ and*

$$\vartheta_1, \vartheta_2 \ge \frac{d_1 + 3d_2}{4}, \quad \text{if } \wp \text{ is decreasing,}$$

$$\vartheta_1, \vartheta_2 \le \frac{d_1 + 3d_2}{4}, \quad \text{if } \wp \text{ is increasing.}$$

**Corollary 3.2** *Theorem 3.3 with $\vartheta_1 = \vartheta_2 = \frac{d_1 + 3d_2}{4}$ gives the new inequality*

$$\wp\left(\frac{d_1 + 3d_2}{4}\right) \le \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \wp^\rho(\mathbf{s})\nabla\mathbf{s} \le \frac{2\wp(d_1) + 2\wp(d_2)}{2} - \wp\left(\frac{d_1 + 3d_2}{4}\right). \qquad (3.8)$$

**Theorem 3.4** *Let* $\wp : [d_1, d_2]_{T_0} \to R$ *be convex and monotonic and* $d_1, d_2 \in T_0$ *with* $d_1 < d_2$, *then we have*

$$\wp\left(\frac{d_1 + d_2}{2}\right) \leq \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \wp^\rho(\mathbf{s})\nabla\mathbf{s} \leq \frac{2\wp(d_1) + 2\wp(d_2)}{2} - \wp\left(\frac{d_1 + d_2}{2}\right), \qquad (3.9)$$

*such that* $\frac{d_1 + d_2}{2} \in T_0$.

## 4 Dynamic Opial inequalities

**Theorem 4.1** *Let* $0, \mu \in T_0$. *For a delta differentiable and increasing function* $\wp :$ $[0, \mu]_{T_0} \to R$ *with* $\wp(0) = 0$, *then*

$$\left|\wp(\mu)\right|\left(\left|\wp(\mu)\right| - \left|\wp\left(\frac{\mu}{2}\right)\right|\right) \leq \int_0^\mu \left|\wp(\mathbf{s})\right|\left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s} \leq \left|\wp(\mu)\right|\left|\wp\left(\frac{\mu}{2}\right)\right|, \qquad (4.1)$$

*with equality when* $\wp(\mathbf{s}) = c\mathbf{s}$, *for some* $c \in R$.

*Proof* Let $\Psi_1(\mathbf{s}) := \left|\wp^\Delta(\mathbf{s})\right|$, then $\Psi_1(\mathbf{s}) \geq 0$ for all $\mathbf{s} \in [0, \mu]$. Taking $\Psi_2(\mathbf{s}) := \frac{|\wp(\mathbf{s})|}{|\wp(\mu)|}$ and since $\wp(\mathbf{s})$ is an increasing function on $[0, \mu]$, we see that $0 \leq \Psi_2(\mathbf{s}) \leq 1$ for each $\mathbf{s} \in [0, \mu]$. Thus, $\Psi_1$ and $\Psi_2$ satisfy the hypotheses in Theorem 2.3. Hence

$$\int_{\frac{\mu}{2}}^\mu \Psi_1(\mathbf{s})\Delta\mathbf{s} \leq \int_0^\mu \Psi_1(\mathbf{s})\Psi_2(\mathbf{s})\Delta\mathbf{s} \leq \int_0^{\frac{\mu}{2}} \Psi_1(\mathbf{s})\Delta\mathbf{s}.$$

So,

$$\int_{\frac{\mu}{2}}^\mu \left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s} \leq \frac{1}{|\wp(\mu)|} \int_0^\mu \left|\wp(\mathbf{s})\right|\left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s} \leq \int_0^{\frac{\mu}{2}} \left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s}.$$

By making use of integration by parts and the fact

$$\int_{a_1}^{a_2} \Psi_1(\mathbf{s})\Delta\mathbf{s} = \left|\wp(a_2)\right| - \left|\wp(a_1)\right|,$$

we get

$$\left|\wp(\mu)\right| - \left|\wp\left(\frac{\mu}{2}\right)\right| \leq \frac{1}{|\wp(\mu)|} \int_0^\mu \left|\wp(\mathbf{s})\right|\left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s} \leq \left|\wp\left(\frac{\mu}{2}\right)\right| - \left|\wp(0)\right|. \qquad (4.2)$$

Multiplying inequality (4.2) on both sides by the factor $|\wp(\mu)| > 0$ and from the condition $\wp(0) = 0$ we obtain the desired result (4.1). Now, let $\wp(\mathbf{s}) = c\mathbf{s}$ for some $c \in R$. Then $\wp^\Delta(\mathbf{s}) = c$ and it is easy to check that equality holds in (4.1). The proof is complete.  □

**Theorem 4.2** *Let* $d_1, d_2 \in T_0$ *and* $d_1 < d_2$. *Assume that* $\wp, \wp^\sigma, \wp^\Delta \in \mathbb{C}_{\mathcal{RD}}([d_1, d_2]_{T_0}, R)$ *and* $p > 1$. *Then*

$$\left(\int_{d_1}^{d_2} \left|\wp(\mathbf{s}) + \wp^\sigma(\mathbf{s})\right|^p \left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s}\right)^{\frac{1}{p}} \leq \left(\int_{d_1}^{d_2} \left|\wp(\mathbf{s})\right|^p \left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s}\right)^{\frac{1}{p}}$$
$$+ \left(\int_{d_1}^{d_2} \left|\wp^\sigma(\mathbf{s})\right|^p \left|\wp^\Delta(\mathbf{s})\right|\Delta\mathbf{s}\right)^{\frac{1}{p}}. \qquad (4.3)$$

*Proof* Note

$$\int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s}$$

$$= \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^{p-1} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right| \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s}$$

$$\leq \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^{p-1} \left| \wp(\mathbf{s}) \right| \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s}$$

$$+ \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^{p-1} \left| \wp^\sigma(\mathbf{s}) \right| \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s}.$$

Applying the Hölder inequality, we get

$$\int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s}$$

$$\leq \left( \int_{d_1}^{d_2} \left( \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^{p-1} \right)^q \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{q}} \left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{p}}$$

$$+ \left( \int_{d_1}^{d_2} \left( \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^{p-1} \right)^q \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{q}} \left( \int_{d_1}^{d_2} \left| \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{p}}$$

$$= \left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{q}}$$

$$\times \left[ \left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{p}} + \left( \int_{d_1}^{d_2} \left| \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{p}} \right].$$

Therefore

$$\left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{p}}$$

$$= \left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{1 - \frac{1}{q}}$$

$$\leq \left[ \left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{p}} + \left( \int_{d_1}^{d_2} \left| \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta \mathbf{s} \right)^{\frac{1}{p}} \right],$$

which is the desired inequality (4.3). The proof is completed.                    □

By making use of Theorem 4.1 and the well-known inequality

$$|d_1 + d_2|^p \leq 2^{p-1} \left( |d_1|^p + |d_2|^p \right), \quad p \geq 1,$$

we can obtain the following result.

**Corollary 4.1** *Let* $0, \mu \in T_0$. *Assume that* $\wp, \wp^\sigma, \wp^\Delta \in \mathbb{C}_{\mathcal{RD}}([0, \mu]_{T_0}, R)$. *For a delta differentiable and increasing function* $\wp : [0, \mu]_{T_0} \to R$ *with* $\wp(0) = \wp^\sigma(0) = 0$ *and* $p \geq 1$, *then*

$$\int_0^\mu \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta\mathbf{s} \leq 2^{p-1} \left| \wp\left( \frac{\mu}{2} \right) \right| \left( \left| \wp(\mu) \right|^p + \left| \wp^\sigma(\mu) \right|^p \right). \tag{4.4}$$

By making use of Theorems 4.1 and 4.2, we can deduce the following inequality.

**Corollary 4.2** *Let* $0, \mu \in T_0$. *Assume that* $\wp, \wp^\sigma, \wp^\Delta \in \mathbb{C}_{\mathcal{RD}}([0, \mu]_{T_0}, R)$. *For a delta differentiable and increasing function* $\wp : [0, \mu]_{T_0} \to R$ *with* $\wp(0) = \wp^\sigma(0) = 0$ *and* $p > 1$, *then*

$$\left( \int_0^\mu \left| \wp(\mathbf{s}) + \wp^\sigma(\mathbf{s}) \right|^p \left| \wp^\Delta(\mathbf{s}) \right| \Delta\mathbf{s} \right)^{\frac{1}{p}} \leq \left| \wp\left( \frac{\mu}{2} \right) \right|^{\frac{1}{p}} \left( \left| \wp(\mu) \right| + \left| \wp^\sigma(\mu) \right| \right). \tag{4.5}$$

The above results can be obtained for the $\nabla$ case by using Theorems 2.2 and 2.4, respectively.

**Theorem 4.3** *Let* $0, \mu \in T_0$. *For a nabla differentiable and increasing function* $\wp : [0, \mu]_{T_0} \to R$ *with* $\wp(0) = 0$, *then*

$$\left| \wp(\mu) \right| \left( \left| \wp(\mu) \right| - \left| \wp\left( \frac{\mu}{2} \right) \right| \right) \leq \int_0^\mu \left| \wp(\mathbf{s}) \right| \left| \wp^\nabla(\mathbf{s}) \right| \nabla\mathbf{s} \leq \left| \wp(\mu) \right| \left| \wp\left( \frac{\mu}{2} \right) \right|, \tag{4.6}$$

*with equality when* $\wp(\mathbf{s}) = c\mathbf{s}$, *for some* $c \in R$.

**Theorem 4.4** *Let* $d_1, d_2 \in T_0$ *and* $d_1 < d_2$. *Assume that* $\wp, \wp^\rho, \wp^\nabla \in \mathbb{C}_{\mathcal{LD}}([d_1, d_2]_{T_0}, R)$ *and* $p > 1$. *Then*

$$\left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) + \wp^\rho(\mathbf{s}) \right|^p \left| \wp^\nabla(\mathbf{s}) \right| \nabla\mathbf{s} \right)^{\frac{1}{p}} \leq \left( \int_{d_1}^{d_2} \left| \wp(\mathbf{s}) \right|^p \left| \wp^\nabla(\mathbf{s}) \right| \nabla\mathbf{s} \right)^{\frac{1}{p}}$$
$$+ \left( \int_{d_1}^{d_2} \left| \wp^\rho(\mathbf{s}) \right|^p \left| \wp^\nabla(\mathbf{s}) \right| \nabla\mathbf{s} \right)^{\frac{1}{p}}. \tag{4.7}$$

By making use of Theorems 4.3 and 4.4, we can deduce the following inequalities, respectively.

**Corollary 4.3** *Let* $0, \mu \in T_0$. *Assume that* $\wp, \wp^\rho, \wp^\nabla \in \mathbb{C}_{\mathcal{LD}}([0, \mu]_{T_0}, R)$. *For a nabla differentiable and increasing function* $\wp : [0, \mu]_{T_0} \to R$ *with* $\wp(0) = \wp^\rho(0) = 0$ *and* $p > 1$, *then*

$$\left( \int_0^\mu \left| \wp(\mathbf{s}) + \wp^\rho(\mathbf{s}) \right|^p \left| \wp^\nabla(\mathbf{s}) \right| \nabla\mathbf{s} \right)^{\frac{1}{p}} \leq \left| \wp\left( \frac{\mu}{2} \right) \right|^{\frac{1}{p}} \left( \left| \wp(\mu) \right| + \left| \wp^\rho(\mu) \right| \right). \tag{4.8}$$

**Corollary 4.4** *Let* $0, \mu \in T_0$. *Assume that* $\wp, \wp^\rho, \wp^\nabla \in \mathbb{C}_{\mathcal{LD}}([0, \mu]_{T_0}, R)$. *For a nabla differentiable and increasing function* $\wp : [0, \mu]_{T_0} \to R$ *with* $\wp(0) = \wp^\rho(0) = 0$ *and* $p \geq 1$, *then*

$$\int_0^\mu \left| \wp(\mathbf{s}) + \wp^\rho(\mathbf{s}) \right|^p \left| \wp^\nabla(\mathbf{s}) \right| \nabla\mathbf{s} \leq 2^{p-1} \left| \wp\left( \frac{\mu}{2} \right) \right| \left( \left| \wp(\mu) \right|^p + \left| \wp^\rho(\mu) \right|^p \right). \tag{4.9}$$

## 5  Conclusion

In this article, by making use of the well-known dynamic inequalities, a dynamic version of integration by parts and chain rule formulas, we obtained some useful dynamic H-H and Opial inequalities on time scales. The derived inequalities generalize some well-known dynamic inequalities in the literature. For this purpose, the reader can see corollaries and remarks after each theorem of the main results.

**Availability of data and materials**
No data were used to support this study.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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